Topological **M**ethods in **N**onlinear **A**nalysis
Volume 54, No. 1, 2019, 9–27
DOI: 10.12775/TMNA.2019.020

L^p-PULLBACK ATTRACTORS FOR NON-AUTONOMOUS REACTION-DIFFUSION EQUATIONS WITH DELAYS

Kaixuan Zhu — Yongqin Xie — Feng Zhou

Abstract. In this paper, we consider the non-autonomous reaction-diffusion equations with hereditary effects and the nonlinear term f satisfying the polynomial growth of arbitrary order $p-1$ ($p \geq 2$). The delay term may be driven by a function with very weak assumptions, namely, just measurability. We extend the asymptotic *a priori* estimate method (see [29]) to our problem and establish a new existence theorem for the pullback attractors in $C_{L^p(\Omega)}$ ($p > 2$) (see Theorem 2.12), which generalizes the results obtained in [12].

1. Introduction

Delay differential equations (DDE for short) are considered as mathematical models to describe the dynamics of events occurring in the past. For this reason DDE are receiving extensive attention and are widely applied to describe physical and chemical processes, engineering systems, biological and/or communication systems, etc. (see [18]). In the field of mathematics, one pays much attention to the well-posedness and long-time behaviour of solutions for the DDE. For the well-posedness of solutions and dynamical behaviour about DDE, there exists

2010 *Mathematics Subject Classification.* 35K57, 35B40, 35B41.
Key words and phrases. Reaction-diffusion equations; delays; pullback attractors.
This work was partly supported by the NSFC (Grant No. 11601522), the Fundamental Research Funds for the Central Universities of China (Grant No. 17CX02036A) and the Province Natural Science Foundation of Hu'nan (Grants Nos. 2018JJ2416, 2019JJ50399).

rich literature, see for instance [3], [4], [8]–[17], [19], [20], [23], [24], [27], [28] and references therein.

Now we state our problem properly. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary, we consider the asymptotic behaviour of the solutions for the following reaction-diffusion equation with delays:

$$\begin{cases} \partial_t u - \Delta u = f(u) + g(t, u_t) + k(t) & \text{in } \Omega \times (\tau, \infty), \\ u(x,t) = 0 & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau + \theta) = \phi(x, \theta) & \text{for } x \in \Omega,\ \theta \in [-h, 0], \end{cases} \tag{1.1}$$

where $\tau \in \mathbb{R}$, g is a operator acting on the solutions containing some hereditary characteristic (assumptions on g are given below), $k(\,\cdot\,) \in L^2_{\rm loc}(\mathbb{R}; L^2(\Omega))$ is the time-dependent external force term, $\phi \in C([-h, 0]; L^2(\Omega))$ is the initial datum, $h(> 0)$ is the length of the delay effects, and for each $t \geq \tau$, we denote by u_t the function defined in $[-h, 0]$ with $u_t(\theta) = u(t + \theta)$ for $\theta \in [-h, 0]$.

We will denote by C_X the Banach space $C([-h, 0]; X)$, equipped with the sup-norm. For an element $u \in C_X$, its norm will be written as

$$\|u\|_{C_X} = \max_{t \in [-h,0]} \|u(t)\|_X.$$

As for the operator g, similarly as in [12], we will assume that $g(\,\cdot\,,\,\cdot\,)\colon \mathbb{R} \times C_{L^2(\Omega)} \to L^2(\Omega)$, and:

(I) for all $\xi \in C_{L^2(\Omega)}$, the function $\mathbb{R} \ni t \mapsto g(t, \xi) \in L^2(\Omega)$ is measurable;

(II) $g(t, 0) = 0$ for all $t \in \mathbb{R}$;

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$ and $\xi, \eta \in C_{L^2(\Omega)}$, it holds

$$\|g(t, \xi) - g(t, \eta)\|_2 \leq L_g \|\xi - \eta\|_{C_{L^2(\Omega)}}.$$

For the nonlinearity $f \in C(\mathbb{R}; \mathbb{R})$, we make the following classical assumptions (e.g. see [1], [25], [26]):

$$(f(u) - f(v))(u - v) \leq l(u - v)^2, \tag{1.2}$$

$$-c_0 - c_1|u|^p \leq f(u)u \leq c_0 - c_2|u|^p, \quad p \geq 2 \tag{1.3}$$

for some positive constants c_0, c_1, c_2 and all $u, v \in \mathbb{R}$.

For the non-autonomous reaction-diffusion equations with delays, in [12], the authors have obtained the well-posedness of solutions by applying the Faedo–Galerkin methods. Then, they have verified the existence of the pullback attractors in $C_{L^2(\Omega)}$ by employing the energy methods (see [2] for details).

In this paper, we consider the existence of the pullback attractors in $C_{L^p(\Omega)}$ $(p > 2)$ for the non-autonomous reaction-diffusion equations with delays. For our problem, we will confront two main difficulties when verifying the compactness of the process $\{U(t, \tau)\}_{t \geq \tau}$. One difficulty is that the nonlinearity f satisfies the polynomial growth of arbitrary order $p - 1$ $(p \geq 2)$, which leads to the fact that the Sobolev embedding is no longer compact. The other difficulty is that our problem contains delay term $g(t, u_t)$, which makes C_X as the phase space rather than X. In the Banach space C_X, the already existing methods and techniques for verifying the compactness of the process $\{U(t,\tau)\}_{t\geq\tau}$ are no longer valid. In order to overcome these difficulties, we extend the asymptotic *a priori* estimate method (see [29]) to our problem and establish a new existence theorem for the pullback attractors in $C_{L^p(\Omega)}$ $(p > 2)$ (see Theorem 2.12), which generalizes the results obtained in [12].

The outline of the paper is as follows. In Section 2, we give some notions and results about pullback attractors and establish the existence theorem for the pullback attractors in $C_{L^p(\Omega)}$ $(p > 2)$; In Section 3, we verify the existence of the pullback attractors in $C_{L^p(\Omega)}$ $(p > 2)$ for the process $\{U(t,\tau)\}_{t\geq\tau}$ generated by equation (1.1) by applying the existence theorem established in Section 2 (see Theorems 3.7 and 3.8).

2. Preliminaries and abstract results

2.1. Preliminaries. In this subsection, we first give some basic notions and abstract results about pullback attractors (see [5], [6], [7] for details).

Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process (or a two-parameter semigroup) on a metric space X, i.e. a family $\{U(t,\tau) : \infty < \tau \leq t < +\infty\}$ of mappings $U(t,\tau)\colon X \to X$, such that $U(\tau,\tau)x = x$ for all $x \in X$ and

$$U(t,\tau) = U(t,s)U(s,\tau) \quad \text{for all } \tau \leq s \leq t.$$

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of X.

Definition 2.1. The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{D} \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in X.

Definition 2.2. It is said that $\widehat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geq\tau}$ if for any $t \in \mathbb{R}$ and any $\widehat{D} \in \mathcal{D}$, there exists a $\tau_0 = \tau_0(t, \widehat{D}) \leq t$ such that

$$U(t,\tau)D(\tau) \subset B(t) \quad \text{for all } \tau \leq \tau_0(t, \widehat{D}).$$

Definition 2.3. A family $\widehat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ in X if

(a) $\mathscr{A}(t)$ is compact in X for all $t \in \mathbb{R}$;

(b) $\widehat{\mathscr{A}}$ is pullback $\mathcal{D}$-attracting in X, i.e.

$$\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau)D(\tau), \mathscr{A}(t)) = 0,$$

for all $\widehat{D} \in \mathcal{D}$ and all $t \in \mathbb{R}$;

(c) $\widehat{\mathscr{A}}$ is invariant, i.e. $U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t)$ for any $-\infty < \tau \le t < +\infty$.

Being similar to that in [29], we have the following definition and results.

Definition 2.4. Let X be a Banach space and $\{U(t,\tau)\}_{t\ge\tau}$ be a process on X. We call that $\{U(t,\tau)\}_{t\ge\tau}$ is a norm-to-weak continuous process on X, if $\{U(t,\tau)\}_{t\ge\tau}$ satisfies:

(a) $U(\tau,\tau)x = x$ (the identity),
(b) $U(t,\tau) = U(t,s)U(s,\tau)$ for all $\tau \le s \le t$,
(c) $U(t,\tau)x_n \rightharpoonup U(t,\tau)x$ if $x_n \to x$ in X.

Lemma 2.5. *Let X, Y be two Banach spaces, X^*, Y^* be their dual spaces, respectively. Assume that X is dense in Y, the injection $i\colon X \to Y$ is continuous, its adjoint $i^*\colon Y^* \to X^*$ is dense, and $\{U(t,\tau)\}_{t\ge\tau}$ is a norm-to-weak continuous process on Y. Then $\{U(t,\tau)\}_{t\ge\tau}$ is a norm-to-weak continuous process on X if and only if for any $\tau \in \mathbb{R}$, $\{U(t,\tau)\}_{t\ge\tau}$ maps compact subsets of X into bounded subsets of X.*

Lemma 2.6. *Let $\{U(t,\tau)\}_{t\ge\tau}$ be a norm-to-weak continuous process on Banach space X. Suppose $\{U(t,\tau)\}_{t\ge\tau}$ satisfies the following assumptions*:

(a) *$\{U(t,\tau)\}_{t\ge\tau}$ has a pullback $\mathcal{D}$-absorbing set $\widehat{B}_0$ in X,*
(b) *$\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $\widehat{B}_0$.*

Then, the family $\widehat{\mathscr{A}} = \{\mathscr{A}(t); t \in \mathbb{R}\}$ defined by $\mathscr{A}(t) = \Lambda(\widehat{B}_0, t)$ is a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\ge\tau}$, where

$$\Lambda(\widehat{D}, t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)D(\tau)}^{X},$$

for all $t \in \mathbb{R}$ and for any $\widehat{D} \in \mathcal{D}$. In addition, $\widehat{\mathscr{A}}$ satisfies

$$\mathscr{A}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}} \Lambda(\widehat{D}, t)}^{X},$$

for all $t \in \mathbb{R}$, and $\widehat{\mathscr{A}}$ is minimal in the sense that if $\widehat{C} = \{C(t);\ t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of X and

$$\lim_{\tau\to-\infty} \operatorname{dist}_X(U(t,\tau), C(t)) = 0, \quad \textit{for all } t \in \mathbb{R},$$

then $\mathscr{A}(t) \subset C(t)$ for any $t \in \mathbb{R}$.

In the sequel, we shall need the following lemma, which belongs to the family of Gronwall type lemmas, see [21], [22] for details.

Lemma 2.7. *Let for some $\lambda > 0$, $\tau \in \mathbb{R}$ and, for $s > \tau$,*

$$y'(s) + \lambda y(s) \le h_1(s),$$

where the functions y, y', h_1 are assumed to be locally integrable and y, h_1 nonnegative on the interval $t < s < t + r$, for some $t \geq \tau$. Then

$$y(t+r) \leq e^{-\lambda r/2}\,\frac{2}{r}\int_t^{t+r/2} y(s)\,ds + e^{-\lambda(t+r)}\int_t^{t+r} e^{\lambda s} h_1(s)\,ds.$$

Remark 2.8. In the following, in this paper, we always assume that the structure of the pullback $\mathcal{D}$-attractors is as that in Lemma 2.6, and $\mathcal{D}$ is a nonempty class of parameterized sets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_{L^2(\Omega)})$.

2.2. Abstract results. In this subsection, we will give some abstract results, which are similar to those in [29] and used to verify the existence of the pullback $\mathcal{D}$-attractors in $C_{L^p(\Omega)}$.

Lemma 2.9. *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a process on $C_{L^p(\Omega)}$ $(p \geq 1)$ and have a pullback $\mathcal{D}$-absorbing set $\widehat{B} = \{B(t) : t \in \mathbb{R}\}$ in $C_{L^p(\Omega)}$. Then, for any $\varepsilon > 0$ and any set $\widehat{D} \in \mathcal{D}$ in $C_{L^p(\Omega)}$, there exist $\tau_0 = \tau_0(\varepsilon, \widehat{D}) \leq t$ and $M = M(\varepsilon, \widehat{D})$ such that*

$$m(\Omega(|U(t+\theta,\tau)u(\tau)| \geq M)) \leq \varepsilon \quad \textit{for any } u(\tau) \in D(\tau) \textit{ and } \tau \leq \tau_0(\varepsilon, \widehat{D}),$$

where $m(\Omega_0)$ denotes the Lebesgue measure of $\Omega_0 \subset \Omega$ and

$$\Omega(|U(t+\theta,\tau)u(\tau)| \geq M) \triangleq \{x \in \Omega : |u(t+\theta)| \geq M\} \quad \textit{with } \theta \in [-h, 0].$$

Proof. The process $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback $\mathcal{D}$-absorbing set $\widehat{B} = \{B(t) : t \in \mathbb{R}\}$ in $C_{L^p(\Omega)}(p > 0)$, then there exists a $\rho(t) > 0$ (which only depends on t), such that for any set $\widehat{D} \in \mathcal{D}$ in $C_{L^p(\Omega)}$, we can find a $\tau_0 = \tau_0(\varepsilon, \widehat{D})$, such that

$$\max_{\theta\in[-h,0]} \|U(t+\theta,\tau)u(\tau)\|_p^p \leq \rho(t) \quad \text{for any } u(\tau) \in D(\tau) \text{ and } \tau \leq \tau_0.$$

Therefore,

$$\begin{aligned}
\rho(t) &\geq \max_{\theta\in[-h,0]} \int_\Omega |U(t+\theta,\tau)u(\tau)|^p\,dx \\
&\geq \max_{\theta\in[-h,0]} \int_{\Omega(|U(t+\theta,\tau)u(\tau)|\geq M)} |U(t+\theta,\tau)u(\tau)|^p\,dx \\
&\geq \max_{\theta\in[-h,0]} \int_{\Omega(|U(t+\theta,\tau)u(\tau)|\geq M)} M^p\,dx \\
&= M^p \cdot m(\Omega(|U(t+\theta,\tau)u(\tau)| \geq M)),
\end{aligned}$$

which implies that $m(\Omega(|U(t+\theta,\tau)u(\tau)| \geq M)) \leq \varepsilon$ if we choose M large enough such that $M \geq (\rho(t)/\varepsilon)^{1/p}$. □

Lemma 2.10. *Let $\widehat{D} = \{D(t) : t \in \mathbb{R}\}$, then for any $\varepsilon > 0$, $D(t)$ has a finite ε-net in $C_{L^p(\Omega)}$ $(p > 0)$ if there exists a positive constant $M = M(\varepsilon, \widehat{D})$ which depends on ε and $\widehat{D}$, such that*

(a) *$D(t)$ has a finite $(3M)^{(q-p)/q}(\varepsilon/2)^{p/q}$-net in $C_{L^q(\Omega)}$ for some q $(q > 0)$,*

(b) *for all* $u(t) \in D(t)$, $\theta \in [-h, 0]$,

$$\left(\max_{\theta\in[-h,0]} \int_{\Omega(u(t+\theta)\geq M)} |u(t+\theta)|^p \right)^{1/p} < 2^{-(2p+2)/p}\varepsilon. \tag{2.1}$$

Proof. When $q \geq p$, the conclusion is obvious, so we just need to verify the case of $q < p$. When $q < p$, then it follows from the assumptions that for any fixed $\varepsilon > 0$, $D(t)$ has a finite $(3M)^{(q-p)/q}(\varepsilon/2)^{p/q}$-net in $C_{L^q(\Omega)}$, i.e., there exist $u_1(t), u_2(t), \ldots, u_k(t) \in D(t)$ such that, for any $u(t) \in D(t)$, we can find some $u_i(t)$ $(1 \leq i \leq k)$ satisfying

$$\begin{aligned} \|u(t+\theta) - u_i(t+\theta)\|_q^q &\leq \max_{\theta\in[-h,0]} \|u(t+\theta) - u_i(t+\theta)\|_q^q \\ &= \max_{\theta\in[-h,0]} \|u_t(\theta) - u_{it}(\theta)\|_q^q < (3M)^{(q-p)}\left(\frac{\varepsilon}{2}\right)^p. \end{aligned}$$

Then, we have

$$\begin{aligned} \|u(t+\theta) - u_i(t+\theta)\|_p^p &= \int_{\Omega(|u(t+\theta)-u_i(t+\theta)|\geq 3M)} |u(t+\theta) - u_i(t+\theta)|^p \, dx \\ &\quad + \int_{\Omega(|u(t+\theta)-u_i(t+\theta)|\leq 3M)} |u(t+\theta) - u_i(t+\theta)|^p \, dx, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_{\Omega(|u(t+\theta)-u_i(t+\theta)|\leq 3M)} |u(t+\theta) - u_i(t+\theta)|^p \, dx \\ &\leq (3M)^{p-q} \int_{\Omega(|u(t+\theta)-u_i(t+\theta)|\leq 3M)} |u(t+\theta) - u_i(t+\theta)|^q \, dx \\ &\leq (3M)^{p-q} \cdot (3M)^{(q-p)}\left(\frac{\varepsilon}{2}\right)^p = \left(\frac{\varepsilon}{2}\right)^p. \end{aligned} \tag{2.3}$$

On the other hand, set

$$\begin{aligned} \Omega_1 &= \Omega\left(|u(t+\theta)| \geq \frac{3M}{2}\right) \cap \Omega\left(|u_i(t+\theta)| \leq \frac{3M}{2}\right), \\ \Omega_2 &= \Omega\left(|u(t+\theta)| \leq \frac{3M}{2}\right) \cap \Omega\left(|u_i(t+\theta)| \geq \frac{3M}{2}\right), \\ \Omega_3 &= \Omega\left(|u(t+\theta)| \geq \frac{3M}{2}\right) \cap \Omega\left(|u_i(t+\theta)| \geq \frac{3M}{2}\right), \end{aligned} \tag{2.4}$$

then we have

$$\Omega(|u(t+\theta) - u_i(t+\theta)| \geq 3M) \subset \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

From (2.4) we know that $|u(t+\theta) - u_i(t+\theta)| \le 2|u(t+\theta)|$ in Ω_1 and $|u(t+\theta) - u_i(t+\theta)| \le 2|u_i(t+\theta)|$ in Ω_2, combining with (2.1), we have

$$
\begin{aligned}
(2.5)\quad & \int_{\Omega(|u(t+\theta)-u_i(t+\theta)|\ge 3M)} |u(t+\theta) - u_i(t+\theta)|^p\,dx \\
&\le \int_{\Omega_1} |u(t+\theta) - u_i(t+\theta)|^p\,dx \\
&\quad + \int_{\Omega_2} |u(t+\theta) - u_i(t+\theta)|^p\,dx + \int_{\Omega_3} |u(t+\theta) - u_i(t+\theta)|^p\,dx \\
&\le 2^p \int_{\Omega_1} |u(t+\theta)|^p\,dx + 2^p \int_{\Omega_2} |u_i(t+\theta)|^p\,dx \\
&\quad + 2^p \int_{\Omega_3} |u(t+\theta)|^p\,dx + 2^p \int_{\Omega_3} |u_i(t+\theta)|^p\,dx \\
&\le 2^p \bigg(\int_{\Omega(|u(t+\theta)|\ge M)} |u(t+\theta)|^p\,dx + \int_{\Omega(|u_i(t+\theta)|\ge M)} |u_i(t+\theta)|^p\,dx \bigg) \\
&\quad + 2^p \bigg(\int_{\Omega(|u(t+\theta)|\ge M)} |u(t+\theta)|^p\,dx + \int_{\Omega(|u_i(t+\theta)|\ge M)} |u_i(t+\theta)|^p\,dx \bigg) \\
&\le 2^{p+2} \cdot 2^{-(2p+2)} \varepsilon^p = \left(\frac{\varepsilon}{2}\right)^p.
\end{aligned}
$$

Substituting (2.3) and (2.5) into (2.2), we can deduce that

$$\max_{\theta\in[-h,0]} \|u(t+\theta) - u_i(t+\theta)\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $D(t)$ has a finite ε-net in $C_{L^p(\Omega)}$. □

LEMMA 2.11. *Let $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_{L^p(\Omega)})$ $(p \ge 1)$. If $D(t)$ has a finite ε-net in $C_{L^p(\Omega)}$, then there exists a positive $M = M(\varepsilon, \widehat{D})$ such that, for any $u(t) \in D(t)$, the following estimate holds*

$$\max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\ge M)} |u(t+\theta)|^p \le 2^{p+1}\varepsilon^p.$$

PROOF. Since $D(t)$ has a finite ε-net in $C_{L^p(\Omega)}$, then there exist $u_1(t), \ldots,$ $u_k(t)$ in $D(t)$, such that for any $u(t) \in D(t)$, we can find some $u_i(t)$ $(1 \le i \le k)$ satisfying

$$\max_{\theta\in[-h,0]} \int_{\Omega} |u(t+\theta) - u_i(t+\theta)|^p \le \varepsilon^p. \tag{2.6}$$

At the same time, for the fixed $\varepsilon > 0$, there exists a $\delta_0 > 0$, such that for each $u_i(t) \in D(t)$ $(1 \le i \le k)$, we have

$$\max_{\theta\in[-h,0]} \int_{\Omega_0} |u_i(t+\theta)|^p\,dx \le \varepsilon^p, \tag{2.7}$$

provided that $m(\Omega_0) < \delta_0$ $(\Omega_0 \subset \Omega)$.

On the other hand, since $D(t) \subset C_{L^p(\Omega)}$, then for the given $\delta_0 > 0$ above, there exist $M > 0$ and $\theta \in [-h, 0]$, such that $m(\Omega(|u(t+\theta)| \geq M)) < \delta_0$ holds for each $u(t) \in D(t)$.

Combining (2.6) and (2.7), we immediately obtain that

$$\begin{aligned}
&\max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta)|^p dx \\
&\quad = \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta) - u_i(t+\theta) + u_i(t+\theta)|^p \, dx \\
&\quad \leq 2^p \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta) - u_i(t+\theta)|^p \, dx \\
&\qquad + 2^p \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u_i(t+\theta)|^p \, dx \leq 2^{p+1}\varepsilon^p. \qquad \square
\end{aligned}$$

Being similar to that in [29], we have the following results, which is useful to verify the existence of the pullback $\mathcal{D}$-attractors in $C_{L^p(\Omega)}$ $(p > 2)$.

Theorem 2.12. *Let $\{U(t,\tau)\}_{t\geq\tau}$ be a norm-to-weak continuous process on $C_{L^2(\Omega)}$ and $C_{L^p(\Omega)}$ $(p > 2)$, respectively. Suppose that $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback $\mathcal{D}$-attractor in $C_{L^2(\Omega)}$, then $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback $\mathcal{D}$-attractor in $C_{L^p(\Omega)}$ provided that the following conditions hold*:

(a) *$\{U(t,\tau)\}_{t\geq\tau}$ has a pullback $\mathcal{D}$-absorbing set $\widehat{B}_p$ in $C_{L^p(\Omega)}$,*
(b) *for any $\varepsilon > 0$, $\tau \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exist a positive constant $M = M(\varepsilon, \widehat{D})$ and $\tau_1 = \tau_1(\varepsilon, \widehat{D})$ such that*

$$\max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta)|^p < \varepsilon$$

for any $u(\tau) \in D(\tau)$ and $\tau \leq \tau_1$.

Proof. We divide the proof into three steps.

Step 1. By Lemma 2.6, we first verify the process $\{U(t,\tau)\}_{t\geq\tau}$ is pullback $\mathcal{D}$-asymptotically compact in $C_{L^p(\Omega)}$. Then it is sufficient to prove that for any $t \in \mathbb{R}$, $\widehat{B}_p \in \mathcal{D}$, any sequence $\tau_n \to -\infty$ and $x_n \in B_p(\tau_n)$, the sequence $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $C_{L^p(\Omega)}$, which is equivalent to prove that for any $\varepsilon > 0$, $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ has a finite ε-net in $C_{L^p(\Omega)}$.

In fact, from the assumption that $\{U(t,\tau)\}_{n=1}^{\infty}$ has a pullback $\mathcal{D}$-attractor in $C_{L^2(\Omega)}$, we know that there exists a τ_2, which depends on ε and $\widehat{D}$ such that $\{U(t,\tau_n)x_n| \ \tau_n \leq \tau_2\}$ has a finite $(3M)^{(2-p)/2}(\varepsilon/2)^{p/2}$-net in $C_{L^2(\Omega)}$. Let $\tau_3 = \min\{\tau_1, \tau_2\}$, then from Lemma 2.10, we know that $\{U(t,\tau_n)x_n : \tau_n \leq \tau_3\}$ has a finite ε-net in $C_{L^p(\Omega)}$. Since $\tau_n \to -\infty$, we obtain that $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ has a finite ε-net in $C_{L^p(\Omega)}$, too. By the arbitrariness of ε, we know that $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $C_{L^p(\Omega)}$.

Step 2. Secondly, we prove the pullback $\mathcal{D}$-attractor is invariant in $C_{L^p(\Omega)}$. Set

$$\mathscr{A}(t) = \bigcap_{s\le t} \overline{\bigcup_{\tau\le s} U(t,\tau)B_p(\tau)}^{ws}, \quad \text{for all } t\in\mathbb{R},$$

where $\overline{A}^{ws}$ denotes the closure of A with respect to the weak sequence. By the above process of proof we know that $\mathscr{A}(t)$ is nonempty and compact.

Now, we claim that

$$x \in \mathscr{A}(t) \;\Leftrightarrow \text{there exist } \tau_n \to -\infty \text{ and } \{x_n\} \subset B_p(\tau_n) \text{ such that } U(t,\tau_n)x_n \rightharpoonup x. \tag{2.8}$$

In fact, for any $x \in \mathscr{A}(\tau)$, there exist $\tau_n \to -\infty$ and $x_n \in B_p(\tau_n)$ such that

$$U(\tau,\tau_n)x_n \rightharpoonup x.$$

On the other hand, from the proof process of Step 1, we know $\{U(\tau,\tau_n)x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{U(\tau,\tau_{n_k})x_{n_k}\}_{k=1}^{\infty}$ such that $U(\tau,\tau_{n_k})x_{n_k} \to x$. Combining it with the norm-to-weak continuity of $\{U(t,\tau)\}_{t\ge\tau}$, we have

$$U(t,\tau_{n_k})x_{n_k} = U(t,\tau)U(\tau,\tau_{n_k})x_{n_k} \rightharpoonup U(t,\tau)x.$$

Then by (2.8), we know that $U(t,\tau)x \in \mathscr{A}(t)$, which implies that

$$U(t,\tau)\mathscr{A}(\tau) \subset \mathscr{A}(t). \tag{2.9}$$

On the contrary, for any $x \in \mathscr{A}(t)$, by (2.8) again, there exist $\tau_n \to -\infty$ and $x_n \in B_p(\tau_n)$ such that $U(t,\tau_n)x_n \rightharpoonup x$. By the proof process of Step 1 again, we know that $\{U(\tau,\tau_n)x_n\}_{n=1}^{\infty}$ has a subsequence $\{U(\tau,\tau_{n_k})x_{n_k}\}_{k=1}^{\infty}$, which converges to some point y in $C_{L^p(\Omega)}$, that is $U(\tau,\tau_{n_k})x_{n_k} \to y$, which induces that $y \in \mathscr{A}(\tau)$. By the norm-to-weak continuity of the process $\{U(t,\tau)\}_{t\ge\tau}$ again, we obtain

$$x \leftharpoonup U(t,\tau_{n_k})x_{n_k} = U(t,\tau)U(\tau,\tau_{n_k})x_{n_k} \rightharpoonup U(t,\tau)y.$$

Therefore $U(t,\tau)y = x$, which implies that

$$\mathscr{A}(t) \subset U(t,\tau)\mathscr{A}(\tau) \quad \text{for any } t \ge \tau. \tag{2.10}$$

Together with (2.9) and (2.10), we immediately obtain that

$$U(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t).$$

Step 3. Finally, we will prove that $\mathscr{A}(t)$ pullback attracts every sets $\widehat{D} \in \mathcal{D}$ of $C_{L^p(\Omega)}$ with the $C_{L^p(\Omega)}$-norm. In fact, since $B_p(\tau)$ pullback absorbs every sets $\widehat{D} \in \mathcal{D}$ of $C_{L^p(\Omega)}$, we only need to verify that

$$\operatorname{dist}(U(t,\tau)B_p(\tau), \mathscr{A}(t)) \to 0 \quad \text{as } \tau \to -\infty.$$

Assume on the contrary that this is not true, then there exist some $\varepsilon_0 > 0$, $\tau_n \to -\infty$ and $x_n \in B_p(\tau)$, such that

$$\operatorname{dist}(U(t,\tau_n)x_n, \mathscr{A}(t)) \ge \varepsilon_0. \tag{2.11}$$

Thanks to the proof process of Step 1 again, we know that $\{U(t,\tau_n)x_n\}_{n=1}^{\infty}$ has a subsequence $\{U(t,\tau_{n_k})x_{n_k}\}_{k=1}^{\infty}$, which satisfies $U(t,\tau_{n_k})x_{n_k} \to x$, and, by (2.8), we know that $x \in \mathscr{A}(t)$, which contradicts (2.11). $\square$

3. Pullback attractors for equation (1.1)

3.1. Existence and uniqueness results. In this subsection, we will show the well-posedness of solutions for equation (1.1). We first define the weak solutions, which is similar to that in [12], as follows.

Definition 3.1. A weak solution of equation (1.1) is a function

$$u \in C([\tau-h,T];L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega)), \quad \text{for all } T>\tau,$$

with $u(t)=\phi(t-\tau)$ for all $t\in[\tau-h,\tau]$ and for all $\varphi \in H_0^1(\Omega)\cap L^p(\Omega)$, it satisfies

$$\frac{d}{dt}[(u(t),\varphi)+(\nabla u(t),\nabla\varphi)]+(\nabla u(t),\nabla\varphi) = (f(u(t)),\varphi)+(g(t,u_t),\varphi)+(k(t),\varphi),$$

almost everywhere in $(\tau,+\infty)$.

The following theorem gives the existence and uniqueness of solutions, which can be obtained by the Faedo–Galerkin methods (see [12]). Here we only state the results.

Lemma 3.2. *Let f satisfy* (1.2)–(1.3), *$g(t,u_t)$ subject to assumptions* (I)–(III), *$k(\,\cdot\,)\in L^2_{\mathrm{loc}}(\mathbb{R};H^{-1}(\Omega))$ and $\phi\in C_{L^2(\Omega)}$ given. Then, for any $\tau\in\mathbb{R}$ and $T>\tau$, there exists a unique solution $u(\,\cdot\,)=u(\,\cdot\,;\tau,\phi)$ for equation* (1.1), *which satisfies*

$$u \in C([\tau-h,T];L^2(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)) \cap L^p(\tau,T;L^p(\Omega)),$$

and the mapping $\phi\to u(t)$ is continuous in $C_{L^2(\Omega)}$.

By Lemma 3.2 we can define the process $\{U(t,\tau)\}_{t\geq\tau}$ in $C_{L^2(\Omega)}$ as

$$U(t,\tau)\colon U(t,\tau)\phi = u(t), \tag{3.1}$$

where $u(t)$ is the solution of equation (1.1).

3.2. Some estimates. At first, we give the following estimates, which will be helpful to prove the existence of the pullback attractors in $C_{L^p(\Omega)}$.

Lemma 3.3. *Let f satisfy* (1.2)–(1.3), *$g(t,u_t)$ subject to assumptions* (I)–(III), *$k(\,\cdot\,)\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\Omega))$, $\tau\in\mathbb{R}$ and $\phi\in C_{L^2(\Omega)}$ given. Then, the weak solutions $u(t)$ of equation* (1.1) *satisfies*:

$$\|u_t\|^2_{C_{L^2(\Omega)}} \leq e^{\lambda_1 h-\delta_2(t-\tau)}\|u_\tau\|^2_{C_{L^2(\Omega)}} + \frac{2c_0|\Omega|e^{\lambda_1 h}}{\delta_2} + \frac{e^{\lambda_1 h}}{\delta_2}\, e^{-\delta_2 t}\int_{-\infty}^{t} e^{\delta_2 s}\|k(s)\|_2^2\,ds, \tag{3.2}$$

for all $t\geq\tau$, where $\delta_2=\lambda_1-L_g e^{\lambda_1 h/2}$.

Proof. Multiplying (1.1) by $u(t)$ and integrating over $x \in \Omega$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 = (f(u), u) + (g(t, u_t), u) + (k(t), u).$$

Thanks to (1.3), assumption (III), the Hölder and Young inequalities, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \|\nabla u\|_2^2 &\le c_0|\Omega| - c_2\|u\|_p^p + \|g(t, u_t)\|_2\|u\|_2 + \|k(t)\|_2\|u\|_2 \\ &\le c_0|\Omega| - c_2\|u\|_p^p + \frac{L_g^2}{2\delta_1}\|u_t\|_{C_{L^2(\Omega)}}^2 + \frac{1}{2\delta_2}\|k(t)\|_2^2 + \frac{\delta_1+\delta_2}{2}\|u\|_2^2.\end{aligned}$$

Let $\delta_1 + \delta_2 = \lambda_1$, we can get that

$$\frac{d}{dt}\|u\|_2^2 + \lambda_1\|u\|_2^2 + 2c_2\|u\|_p^p \le 2c_0|\Omega| + \frac{L_g^2}{\delta_1}\|u_t\|_{C_{L^2(\Omega)}}^2 + \frac{1}{\delta_2}\|k(t)\|_2^2. \tag{3.3}$$

Furthermore,

$$\frac{d}{dt}\|u\|_2^2 + \lambda_1\|u\|_2^2 \le 2c_0|\Omega| + \frac{L_g^2}{\delta_1}\|u_t\|_{C_{L^2(\Omega)}}^2 + \frac{1}{\delta_2}\|k(t)\|_2^2. \tag{3.4}$$

Multiplying (3.4) by $e^{\lambda_1 t}$ and integrating it in $[\tau, t]$, we obtain

$$\begin{aligned}e^{\lambda_1 t}\|u(t)\|_2^2 \le{}& e^{\lambda_1 \tau}\|u(\tau)\|_2^2 + 2c_0|\Omega|\int_\tau^t e^{\lambda_1 s}\,ds \\ &+ \frac{L_g^2}{\delta_1}\int_\tau^t e^{\lambda_1 s}\|u_s\|_{C_{L^2(\Omega)}}^2\,ds + \frac{1}{\delta_2}\int_\tau^t e^{\lambda_1 s}\|k(s)\|_2^2\,ds.\end{aligned}$$

In particular, putting $t + \theta$ instead of t, we deduce that

$$\begin{aligned}e^{\lambda_1 t}\|u_t\|_{C_{L^2(\Omega)}}^2 \le{}& e^{\lambda_1(h+\tau)}\|\phi\|_{C_{L^2(\Omega)}}^2 + 2c_0|\Omega|e^{\lambda_1 h}\int_\tau^t e^{\lambda_1 s}\,ds \\ &+ \frac{L_g^2 e^{\lambda_1 h}}{\delta_1}\int_\tau^t e^{\lambda_1 s}\|u_s\|_{C_{L^2(\Omega)}}^2\,ds + \frac{e^{\lambda_1 h}}{\delta_2}\int_\tau^t e^{\lambda_1 s}\|k(s)\|_2^2\,ds.\end{aligned}$$

By the Gronwall lemma, it yields

$$\begin{aligned}e^{\lambda_1 t}\|u_t\|_{C_{L^2(\Omega)}}^2 \le{}& e^{\lambda_1(h+\tau)}e^{L_g^2 e^{\lambda_1 h}(t-\tau)/\delta_1}\|\phi\|_{C_{L^2(\Omega)}}^2 \\ &+ 2c_0|\Omega|e^{\lambda_1 h}e^{L_g^2 e^{\lambda_1 h}t/\delta_1}\int_\tau^t e^{(\lambda_1 - L_g^2 e^{\lambda_1 h}/\delta_1)s}\,ds \\ &+ \frac{e^{\lambda_1 h}}{\delta_2}e^{L_g^2 e^{\lambda_1 h}t/\delta_1}\int_\tau^t e^{(\lambda_1 - L_g^2 e^{\lambda_1 h}/\delta_1)s}\|k(s)\|_2^2\,ds.\end{aligned}$$

Let $\delta_1 = L_g e^{\lambda_1 h/2}$, then $\delta_2 = \lambda_1 - L_g e^{\lambda_1 h/2}$, and

$$\begin{aligned}\|u_t\|^2_{C_{L^2(\Omega)}} \leq & e^{\lambda_1 h} e^{-\delta_2(t-\tau)} \|\phi\|^2_{C_{L^2(\Omega)}} \\ & + 2c_0|\Omega| e^{\lambda_1 h} e^{-\delta_2 t} \int_\tau^t e^{\delta_2 s}\, ds \\ & + \frac{e^{\lambda_1 h}}{\delta_2} e^{-\delta_2 t} \int_\tau^t e^{\delta_2 s} \|k(s)\|_2^2 ds \\ \leq & e^{\lambda_1 h - \delta_2(t-\tau)} \|\phi\|^2_{C_{L^2(\Omega)}} \\ & + \frac{2c_0|\Omega| e^{\lambda_1 h}}{\delta_2} + \frac{e^{\lambda_1 h}}{\delta_2} e^{-\delta_2 t} \int_{-\infty}^t e^{\delta_2 s} \|k(s)\|_2^2\, ds. \end{aligned} \tag{3.5}$$
□

Here we will assume that

$$\delta_2 = \lambda_1 - L_g e^{\lambda_1 h/2} > 0, \tag{3.6}$$

$$\int_{-\infty}^t e^{\delta_2 s} \|k(s)\|_2^2\, ds < \infty. \tag{3.7}$$

Remark 3.4. In (3.7), due to the test function is $|u|^{p-2}u$ (see Theorem 3.7), we suppose that

$$\int_{-\infty}^t e^{\delta_2 s} \|k(s)\|_2^2\, ds < \infty$$

rather than

$$\int_{-\infty}^t e^{\delta_2 s} \|k(s)\|^2_{H^{-1}(\Omega)}\, ds < \infty,$$

which is different from that in [12].

Now, we give the following definition.

Definition 3.5. For any $\delta_2 > 0$, we will denote by $\mathcal{D}_{\delta_2}$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_{L^2(\Omega)})$ such that

$$\lim_{\tau \to -\infty} \Big(e^{\delta_2 \tau} \sup_{u \in D(\tau)} \|u\|^2_{C_{L^2(\Omega)}} \Big) = 0.$$

Moreover, we have the following lemma (see [12] for details), which gives the existence of pullback attractors in $C_{L^2(\Omega)}$.

Lemma 3.6. *Let f satisfy* (1.2)–(1.3), *$g(t, u_t)$ subject to assumptions* (I)–(III), *$k(\cdot) \in L^2_{\rm loc}(\mathbb{R}, H^{-1}(\Omega))$, $\tau \in \mathbb{R}$, and $\phi \in C_{L^2(\Omega)}$ given. $\{U(t,\tau)\}_{t\geq\tau}$ is the process defined by* (3.1). *Then $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback attractor in $C_{L^2(\Omega)}$, which is compact in $C_{L^2(\Omega)}$ and pullback attracts every set $\widehat{D} \in \mathcal{D}_{\delta_2}$ with the $C_{L^2(\Omega)}$-norm.*

3.3. Pullback attractors in $C_{L^p(\Omega)}$. In this subsection, we will establish the existence of the pullback attractors in $C_{L^p(\Omega)}$.

At first, we give the asymptotic *a priori* estimate of the process $\{U(t,\tau)\}_{t\geq\tau}$ with respect to $C_{L^p(\Omega)}$-norm, which plays a crucial role in the proof of the existence of the pullback attractors in $C_{L^p(\Omega)}$.

Theorem 3.7. *For any $\varepsilon > 0$, $\tau \in \mathbb{R}$ and $\widehat{D} \in \mathcal{D}$, there exists a positive constant $M = M\big(\varepsilon, \widehat{D}\big)$ and $\tau_1 = \tau_1\big(\varepsilon, \widehat{D}\big)$ such that*

$$\max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta)|^p \leq C\varepsilon \tag{3.8}$$

for any $u(\tau) \in D(\tau)$ and $\tau \leq \tau_1$, where the constant C is independent of M, τ_1 and ε.

Proof. Multiplying (1.1) by $|(u - M_1)_+|^{p-2}(u - M_1)_+$ and integrating over Ω, we arrive at

$$\frac{1}{p}\frac{d}{dt}\|(u-M_1)_+\|_p^p + (p-1)\int_\Omega |\nabla(u-M_1)_+|^2|(u-M_1)_+|^{p-2}\,dx \tag{3.9}$$
$$= \int_\Omega f(u)(u-M_1)_+^{p-1}\,dx + \int_\Omega g(t,u_t)(u-M_1)_+^{p-1}\,dx + \int_\Omega k(t)(u-M_1)_+^{p-1}\,dx,$$

where $(u - M_1)_+$ denotes the positive part of $u - M_1$, that is

$$(u-M_1)_+ = \begin{cases} u - M_1 & \text{for } u \geq M_1, \\ 0 & \text{for } u < M_1. \end{cases}$$

In view of (1.3), for $M = M(c_0, c_2, p)$ large enough, we can deduce that

$$f(u) \leq -\frac{c_2}{2}|u|^{p-1} \quad \text{for } u \geq M.$$

In consequence,

$$\begin{aligned} f(u)(u-M_1)_+^{p-1} &\leq -\frac{c_2}{2}|u|^{p-1}(u-M_1)_+^{p-1} \\ &= -\frac{c_2}{4}|u|^{p-1}(u-M_1)_+^{p-1} - \frac{c_2}{4}|u|^{p-1}(u-M_1)_+^{p-1} \\ &\leq -\frac{c_2}{4}|u|^{p-1}(u-M_1)_+^{p-1} - \frac{c_2}{4}(u-M_1)_+^{2p-2}. \end{aligned} \tag{3.10}$$

Moreover, by assumption (III), the Hölder and Young inequalities, we have

$$\int_\Omega g(t,u_t)(u-M_1)_+^{p-1}\,dx \leq \frac{2L_g^2}{c_2}\|u_t\|_{C_{L^2(\Omega)}}^2 + \frac{c_2}{8}\|(u-M_1)_+\|_{2p-2}^{2p-2}, \tag{3.11}$$

$$\int_\Omega k(t)(u-M_1)_+^{p-1}\,dx \leq \frac{2}{c_2}\|k(t)\|_2^2 + \frac{c_2}{8}\|(u-M_1)_+\|_{2p-2}^{2p-2}. \tag{3.12}$$

Substituting (3.10)–(3.12) into (3.9), we obtain that

$$\frac{1}{p}\frac{d}{dt}\|(u-M_1)_+\|_p^p + \frac{c_2}{4}\int_\Omega |u|^{p-1}(u-M_1)_+^{p-1} \leq \frac{2L_g^2}{c_2}\|u_t\|_{C_{L^2(\Omega)}}^2 + \frac{2}{c_2}\|k(t)\|_2^2.$$

Notice that $u \geq M_1$ and $u \geq u - M_1$, we can get that

$$(3.13)\quad \frac{d}{dt}\|(u-M_1)_+\|_p^p + \frac{c_2 p}{4}\, M_1^{p-2}\int_\Omega (u-M_1)_+^p \leq \frac{2pL_g^2}{c_2}\|u_t\|^2_{C_{L^2(\Omega)}} + \frac{2p}{c_2}\|k(t)\|_2^2.$$

On the other hand, taking $|(u+M_1)_-|^{p-2}(u+M_1)_-$ instead of $|(u-M_1)_+|^{p-2}\cdot(u-M_1)_+$ in the preceding proof, we deduce similarly that

$$(3.14)\quad \frac{d}{dt}\|(u+M_1)_-\|_p^p + \frac{c_2 p}{4}\, M_1^{p-2}\int_\Omega (u+M_1)_-^p \leq \frac{2pL_g^2}{c_2}\|u_t\|^2_{C_{L^2(\Omega)}} + \frac{2p}{c_2}\|k(t)\|_2^2,$$

where $(u+M_1)_-$ denotes the negative part of $u+M_1$, that is

$$(u+M_1)_- = \begin{cases} u+M_1 & \text{if } u \leq -M_1, \\ 0 & \text{if } u > -M_1. \end{cases}$$

Combining with (3.13) and (3.14), we obtain that

$$(3.15)\quad \frac{d}{dt}\|(|u(t)|-M_1)_+\|_p^p + \frac{c_2 p}{4} M_1^{p-2}\int_\Omega (|u(t)|-M_1)_+^p \leq \frac{2pL_g^2}{c_2}\|u_t\|^2_{C_{L^2(\Omega)}} + \frac{2p}{c_2}\|k(t)\|_2^2.$$

Setting $\alpha = c_2 p M_1^{p-2}/4$, $\beta = 2pL_g^2/c_2$, $\gamma = 2p/c_2$, then applying Lemma 2.7 to (3.15) with $r = 1$, we deduce that

$$(3.16)\quad \|(|u(t+1)|-M_1)_+\|_p^p \leq e^{-\alpha/2}\int_t^{t+1/2}\int_\Omega (|u(s)|-M_1)_+^p\,dx\,ds + \beta e^{-\alpha(t+1)}\int_t^{t+1} e^{\alpha s}\|u_s\|^2_{C_{L^2(\Omega)}}\,ds + \gamma e^{-\alpha(t+1)}\int_t^{t+1} e^{\alpha s}\|k(s)\|_2^2\,ds = I_1 + I_2 + I_3.$$

In the following, we will estimate each term on the right hand of (3.16). At first, we have

$$(3.17)\qquad I_1 \leq e^{-\alpha/2}2^{p+1}\bigg(\int_t^{t+1/2}\|u(s)\|_p^p\,ds + \frac{1}{2}\, M_1^p|\Omega|\bigg).$$

Integrating (3.3) with respect to t from t to $t+1/2$, we have

$$(3.18)\quad \int_t^{t+1/2}\|u(s)\|_p^p\,ds \leq \frac{c_0|\Omega|}{2c_2} + \frac{1}{2c_2}\|u(t)\|_2^2 + \frac{L_g^2}{2c_2\delta_1}\int_t^{t+\frac{1}{2}}\|u_s\|^2_{C_{L^2(\Omega)}}\,ds + \frac{1}{2c_2\delta_2}\int_t^{t+1/2}\|k(s)\|_2^2\,ds.$$

Moreover, thanks to (3.5), from (3.17) and (3.18), we know that there exists a constant $N_0 = N_0\big(\lambda_1, h, \delta_1, \delta_2, c_0, c_2, |\Omega|, \|\phi\|^2_{C_{L^2(\Omega)}}, \int_t^{t+1/2} \|k(s)\|_2^2\, ds\big)$ such that

$$I_1 \le e^{-\alpha/2} 2^{p+1}\left(N_0 + \frac{1}{2} M_1^p |\Omega\right), \quad \text{where } \alpha = \frac{c_2 p}{4} M_1^{p-2}. \tag{3.19}$$

Therefore, for any given $\varepsilon > 0$, let $M_1 = M_1(\varepsilon)$ large enough, we can get that

$$I_1 \le \frac{\varepsilon}{4}. \tag{3.20}$$

Secondly, from (3.5) we know that

$$\begin{aligned} I_2 &= \beta e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s} \|u_s\|^2_{C_{L^2(\Omega)}}\, ds \\ &\le \beta e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s} \left(e^{\lambda_1 h - \delta_2 (s-\tau)} \|\phi\|^2_{C_{L^2(\Omega)}} + \frac{2c_0 |\Omega| e^{\lambda_1 h}}{\delta_2} \right) ds \\ &\quad + \beta e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s} \left(\frac{1}{\delta_2} e^{\lambda_1 h - \delta_2 s} \int_{-\infty}^s e^{\delta_2 s_1} \|k(s_1)\|_2^2 ds_1 \right) ds \\ &\le \beta \left(\|\phi\|^2_{C_{L^2(\Omega)}} + \frac{2c_0|\Omega|}{\delta_2} \right) e^{\lambda_1 h} e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s}\, ds \\ &\quad + \frac{\beta}{\delta_2} e^{\lambda_1 h} e^{-\alpha(t+1)} \int_t^{t+1} e^{(\alpha - \delta_2)s}\, ds \int_{-\infty}^{t+1} e^{\delta_2 s} \|k(s)\|_2^2\, ds. \end{aligned} \tag{3.21}$$

Obviously,

$$e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s}\, ds = \frac{1}{\alpha} e^{-\alpha(t+1)} \big(e^{\alpha(t+1)} - e^{\alpha t} \big) \le \frac{1}{\alpha} \tag{3.22}$$

and

$$\begin{aligned} e^{-\alpha(t+1)} \int_t^{t+1} e^{(\alpha-\delta_2)s}\, ds &= e^{-\delta_2(t+1)} e^{-(\alpha-\delta_2)(t+1)} \int_t^{t+1} e^{(\alpha-\delta_2)s}\, ds \\ = \frac{1}{\alpha - \delta_2} e^{-\delta_2(t+1)} e^{-(\alpha-\delta_2)(t+1)} \big(e^{(\alpha-\delta_2)(t+1)} - e^{(\alpha-\delta_2)t} \big) &\le \frac{1}{\alpha-\delta_2} e^{-\delta_2(t+1)}. \end{aligned} \tag{3.23}$$

Combining with (3.21)–(3.23), for the given $\varepsilon > 0$ (as that in (3.20)), we can let $M_1 = M_1(\varepsilon)$ large enough, such that

$$I_2 \le \frac{\varepsilon}{4}. \tag{3.24}$$

Finally, we can take any $\delta \in (0, 1)$ such that

$$\begin{aligned} I_3 &= \gamma e^{-\alpha(t+1)} \int_t^{t+1} e^{\alpha s} \|k(s)\|_2^2\, ds \\ &= \gamma e^{-\alpha(t+1)} \int_t^{t+1-\delta} e^{\alpha s} \|k(s)\|_2^2\, ds + \gamma e^{-\alpha(t+1)} \int_{t+1-\delta}^{t+1} e^{\alpha s} \|k(s)\|_2^2\, ds \\ &\le \gamma e^{-\alpha(t+1)} \int_t^{t+1-\delta} e^{(\alpha-\delta_2)s} e^{\delta_2 s} \|k(s)\|_2^2\, ds + \gamma \int_{t+1-\delta}^{t+1} \|k(s)\|_2^2\, ds \end{aligned} \tag{3.25}$$

$$\leq \gamma e^{-\alpha\delta} e^{-\delta_2(t+1-\delta)} \int_t^{t+1-\delta} e^{\delta_2 s}\|k(s)\|_2^2\,ds + \gamma \int_{t+1-\delta}^{t+1} \|k(s)\|_2^2\,ds$$
$$\leq \gamma e^{-\alpha\delta} \int_{-\infty}^{t+1} e^{\delta_2 s}\|k(s)\|_2^2\,ds + \gamma \int_{t+1-\delta}^{t+1} \|k(s)\|_2^2 ds.$$

From (3.25), we can choose $\delta \in (0,1)$ small enough such that

$$\gamma \int_{t+1-\delta}^{t+1} \|k(s)\|_2^2\,ds \leq \frac{\varepsilon}{4},$$

then for the given $\delta \in (0,1)$ above, let $M_1 = M_1(\varepsilon)$ large enough such that

$$\gamma e^{-\alpha\delta} \int_{-\infty}^{t+1} e^{\delta_2 s}\|k(s)\|_2^2\,ds \leq \frac{\varepsilon}{4}.$$

Hence,

$$I_3 \leq \frac{\varepsilon}{2}. \tag{3.26}$$

Combining with (3.20), (3.24) and (3.26), we conclude that

$$\|(|u(t+1)| - M_1)_+\|_p^p \leq \varepsilon.$$

In particular, replacing $t+1$ by t, we get

$$\|(|u(t)| - M_1)_+\|_p^p \leq \varepsilon.$$

Therefore,

$$\begin{aligned} \int_{\Omega(|u(t)|\geq 2M_1)} |u(t)|^p\,dx &= \int_{\Omega(|u(t)|\geq 2M_1)} ((|u(t)| - M_1) + M_1)^p\,dx \\ &\leq 2^p \bigg(\int_{\Omega(|u(t)|\geq 2M_1)} (|u(t)| - M_1)^p\,dx + \int_{\Omega(|u(t)|\geq 2M_1)} M_1^p \bigg) \\ &\leq 2^{p+1} \int_{\Omega(|u(t)|\geq 2M_1)} (|u(t)| - M_1)^p\,dx \leq 2^{p+1}\varepsilon \end{aligned} \tag{3.27}$$

as $|u(t)| - M_1 \geq M_1$ for $|u(t)| \geq 2M_1$. Setting $M = 2M_1$, $C = 2^{p+1}$, and putting $t+\theta$ instead of t, we can deduce that

$$\max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta)|^p\,dx \leq C\varepsilon,$$

which complete the proof. □

Then, we have the following theorem, which states the existence of the pullback attractors in $C_{L^p(\Omega)}$.

THEOREM 3.8. *Let f satisfy (1.2)–(1.3), $g(t,u_t)$ subject to assumptions (I)–(III), $k(\,\cdot\,) \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\Omega))$, $\tau \in \mathbb{R}$, and $\phi \in C_{L^2(\Omega)}$ given. $\{U(t,\tau)\}_{t\geq\tau}$ is the process defined by (3.1). Then $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback attractor in $C_{L^p(\Omega)}$, which is compact in $C_{L^p(\Omega)}$ and pullback attracts every set $\widehat{D} \in \mathcal{D}_{\delta_2}$ with the $C_{L^p(\Omega)}$-norm.*

Proof. Firstly, we check that the process $\{U(t,\tau)\}_{t\geq\tau}$ is norm-to-weak continuous in $C_{L^p(\Omega)}$. In fact, by Lemma 2.5, it is sufficient to show that $\{U(t,\tau)\}_{t\geq\tau}$ maps compact sets of $C_{L^p(\Omega)}$ into bounded sets of $C_{L^p(\Omega)}$.

Let $\widehat{B}_1 = \{B_1(t) : t \in \mathbb{R}\}$ be a family of compact sets of $C_{L^p(\Omega)}$. By the continuity of $\{U(t,\tau)\}_{t\geq\tau}$ in $C_{L^2(\Omega)}$, we know that $U(t,\tau)B_1(\tau)$ is a bounded set in $C_{L^2(\Omega)}$. Then, for any $t > \tau$, $u(\tau) \in B_1(\tau)$, and M_1 as in Theorem 3.7, combining with (3.27), (3.16)–(3.19), (3.24) and (3.26), we immediately get that

$$\begin{aligned}
&\int_\Omega |U(t+1,\tau)u(\tau)|^p\,dx \\
&\qquad = \int_{\Omega(|u(t+1)|\leq 2M_1)} |u(t+1)|^p\,dx + \int_{\Omega(|u(t+1)|\geq 2M_1)} |u(t+1)|^p\,dx \\
&\qquad \leq (2M_1)^p|\Omega| + 2^{p+1}\int_{\Omega(|u(t+1)|\geq 2M_1)} (|u(t+1)| - M_1)^p\,dx \\
&\qquad \leq (2M_1)^p|\Omega| + 2^{2p+2}\left(N_0 + \frac{1}{2}\,M_1^p|\Omega|\right),
\end{aligned}$$

therefore, putting $t+\theta$ instead of $t+1$, we can deduce that

$$\begin{aligned}
\|u_t(\theta)\|^p_{C_{L^p(\Omega)}} &= \max_{\theta\in[-h,0]} \int_\Omega |u(t+\theta)|^p\,dx \\
&\leq \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\leq 2M_1)} |u(t+\theta)|^p\,dx \\
&\quad + \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq 2M_1)} |u(t+\theta)|^p\,dx \\
&\leq (2M_1)^p|\Omega| + 2^{2p+2}\left(N_0 + \frac{1}{2}\,M_1^p|\Omega|\right),
\end{aligned}$$

which complete the proof of the norm-to-weak continuity.

Secondly, from

$$\begin{aligned}
\|u_t(\theta)\|^p_{C_{L^p(\Omega)}} &= \max_{\theta\in[-h,0]} \int_\Omega |u(t+\theta)|^p\,dx \\
&\leq \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\geq M)} |u(t+\theta)|^p\,dx \\
&\quad + \max_{\theta\in[-h,0]} \int_{\Omega(|u(t+\theta)|\leq M)} |u(t+\theta)|^p\,dx \leq C\varepsilon + M^p|\Omega|,
\end{aligned}$$

we know that the process $\{U(t,\tau)\}_{t\geq\tau}$ has a pullback $\mathcal{D}$-absorbing sets $\widehat{B}_p$ in $C_{L^p(\Omega)}$. Together with Lemma 3.6 and (3.8) in Theorem 3.7, we know that the conditions in Theorem 2.12 are all satisfied, and immediately obtain the existence of the pullback $\mathcal{D}$-attractors $\mathscr{A}$ in $C_{L^p(\Omega)}$; that is, $\mathscr{A}$ is compact in $C_{L^p(\Omega)}$ and pullback attracts every set $\widehat{D} \in \mathcal{D}_{\delta_2}$ with the $C_{L^p(\Omega)}$-norm. □

Acknowledgments. The authors would like to thank the referee for his/her helpful comments and suggestions.

References

[1] A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[2] J.M. Ball, *Global attractors for damped semilinear wave equations*, Discrete Contin. Dyn. Syst. **10** (2004), 31–52.

[3] T. Caraballo, X.Y. Han and P.E. Kloeden, *Nonautonomous chemostats with variable delays*, SIAM J. Math. Anal. **47** (2015), 2178–2199.

[4] T. Caraballo, P.E. Kloeden and P. Marín-Rubio, *Numerical and finite delay approximations of attractors for logistic differential-integral equations with infinite delay*, Discrete Contin. Dyn. Syst. **19** (2007), 177–196.

[5] A.N. Carvalho, J.A. Langa and J.C. Robinson, *Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems*, Appl. Math. Sci., Vol. 182, Springer, New York, 2013.

[6] T. Caraballo, G. Łukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Anal. **64** (2006) 484–498.

[7] T. Caraballo, G. Łukaszewicz and J. Real, *Pullback attractors for non-autonomous* 2D-*Navier–Stokes equations in some unbounded domains*, C. R. Math. Acad. Sci. Paris **342** (2006), 263–268.

[8] T. Caraballo, P. Marín-Rubio and J. Valero, *Autonomous and non-autonomous attractors for differential equations with delays*, J. Differential Equations **208** (2005), 9–41.

[9] T. Caraballo, P. Marín-Rubio and J. Valero, *Attractors for differential equations with unbounded delays*, J. Differential Equations **239** (2007), 311–342.

[10] T. Caraballo and J. Real, *Attractors for* 2D-*Navier–Stokes models with delays*, J. Differential Equations **205** (2004), 271–297.

[11] T. Caraballo, J. Real and A.M. Márquez, *Three-dimensional system of globally modified Navier–Stokes equations with delay*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **20** (2010), 2869–2883.

[12] J. García-Luengo and P. Marín-Rubio, *Reaction-diffusion equations with non-autonomous force in H^{-1} and delays under measurability conditions on the driving delay term*, J. Math. Anal. Appl. **417** (2014), 80–95.

[13] J. García-Luengo, P. Marín-Rubio and G. Planas, *Attractors for a double time-delayed* 2D-*Navier–Stokes model*, Discrete Contin. Dyn. Syst. **34** (2014), 4085–4105.

[14] J. García-Luengo, P. Marín-Rubio and J. Real, *Pullback attractors for* 2D *Navier–Stokes equations with delays and their regularity*, Adv. Nonlinear Stud. **13** (2013), 331–357.

[15] J. García-Luengo, P. Marín-Rubio and J. Real, *Regularity of pullback attractors and attraction in H^1 in arbitrarily large finite intervals for* 2D *Navier–Stokes equations with infinite delay*, Discrete Contin. Dyn. Syst. **34** (2014), 181–201.

[16] J. García-Luengo, P. Marín-Rubio and J. Real, *Some new regularity results of pullback attractors for* 2D *Navier–Stokes equations with delays*, Commun. Pure Appl. Anal. **14** (2015), 1603–1621.

[17] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[18] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer–Verlag, New York, 1993.

[19] P.E. Kloeden, *Upper semi continuity of attractors of delay differential equations in the delay*, Bull. Austral. Math. Soc. **73** (2006), 299–306.

[20] P.E. KLOEDEN AND P. MARÍN-RUBIO, *Equi-attraction and the continuous dependence of attractors on time delays*, Discrete Contin. Dyn. Syst. Ser. B **9** (2008), 581–593.

[21] G. ŁUKASZEWICZ, *On pullback attractors in L^p for nonautonomous reaction-diffusion equations*, Nonlinear Anal. **73** (2010), 350–357.

[22] G. ŁUKASZEWICZ, *On pullback attractors in H_0^1 for nonautonomous reaction-diffusion equations*, Internat. J. Bifur. Chaos **20** (2010), 2637–2644.

[23] P. MARÍN-RUBIO, A.M. MÁRQUEZ-DURÁN AND J. REAL, *Three dimensional system of globally modified Navier–Stokes equations with infinite delays*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), 655–673.

[24] P. MARÍN-RUBIO, A.M. MÁRQUEZ-DURÁN AND J. REAL, *Pullback attractors for globally modified Navier–Stokes equations with infinite delays*, Discrete Contin. Dyn. Syst. **31** (2011), 779–796.

[25] J.C. ROBINSON, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge, Cambridge University Press, 2001.

[26] R. TEMAM, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer–Verlag, New York, 1997.

[27] Y.J. WANG AND P.E. KLOEDEN, *The uniform attractor of a multi-valued process generated by reaction-diffusion delay equations on an unbounded domain*, Discrete Contin. Dyn. Syst. **34** (2014), 4343–4370.

[28] F. WU AND P.E. KLOEDEN, *Mean-square random attractors of stochastic delay differential equations with random delay*, Discrete Contin. Dyn. Syst. Ser. B **18** (2013), 1715–1734.

[29] C.K. ZHONG, M.H. YANG AND C.Y. SUN, *The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equations*, J. Differential Equations **223** (2006), 367–399.

Manuscript received December 24, 2017

accepted June 5, 2018

KAIXUAN ZHU (corresponding author)
Hu'nan Province Cooperative Innovation Center
for the Construction and Development
of Dongting Lake Ecological Economic Zone
College of Mathematics and Physics Science
Hu'nan University of Arts and Science
Changde, 415000, P.R. CHINA

E-mail address: zhukx12@163.com

YONGQIN XIE
School of Mathematics and Statistics
Changsha University of Science and Technology
Changsha, 410114, P.R. CHINA

E-mail address: xieyq@csust.edu.cn

FENG ZHOU
College of Science
China University of Petroleum (East China)
Qingdao, 266580, P.R. CHINA

E-mail address: zhoufeng13@upc.edu.cn