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MARKOV PERFECT EQUILIBRIA IN OLG MODELS WITH RISK SENSITIVE AGENTS

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Abstract. In this paper, we present an overlapping generation model (OLG for short) of resource extraction with a random production function and an altruism having both paternalistic and non-paternalistic features. All generations are risk-sensitive with a constant coefficient of absolute risk aversion. The preferences are represented by a possibly dynamic inconsistent dynamic recursive utility function with non-cooperating generations. Under general conditions on the aggregator and transition probability, we examine the existence and the uniqueness of a recursive utility function and the existence of a stationary mixed Markov Perfect Nash Equilibria.

1. Introduction

Over fifty years ago Phelps and Pollak [44] postulated a model of optimal economic growth without Ramsey assumption of perfect altruism of generations which we now call *overlapping generations model* (OLG for short). From a game-theoretic point of view the OLG model is an infinite horizon dynamic game with countably many identical short-lived players. The player represents a generation which lives for one period. Each generation derives utility from its own consumption and all successors. Arrow [4] and Dasgupta [19] quickly took over this

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project by proposing an alternative model in which the preference of generations are represented by the utility based on Rawlsian "just savings principle".

In OLG models, the sequence of generations may be viewed as the same agent whose preferences change over time. The agent is sophisticated and chooses a policy taking into account the fact that a tomorrow's policy will be chosen with respect to other preferences than the today's policy. Because of that, the agent plays the game with itself and the today's optimal policy is usually not consistent with the plan for tomorrow. This property is called *dynamic inconsistency* and was considered in the seminal papers by Strotz [46] and Kydland and Prescott [33]. In turn, the opposite term *dynamic consistency* is the axiom formulated by Koopmans [31]. Koopmans shows that the preferences are represented by a dynamic recursive utility function where the today's utility is connected with the tomorrow's utility by a function called an *aggregator*. Kreps and Porteus [32] and Epstein and Zin [22] extended the Koopmans model by defining the recursive utilities on the set of lotteries. Similar models of recursive utilities have been further considered by Weil [50], Marinacci and Montrucchio [37]. The set of such recursive utilities includes the standard deterministic discounting utility as a special instance.

In standard Markov Decision Processes and in some OLG models in [44] the today's utility is aggregated with the future utility by an affine aggregator. But the affine has limited applicability to numerous economic problems. To name just a few arguments, observe that when the aggregator is affine then the *elasticity of intertemporal substitution* (EIS) is equal to the inverse of the risk aversion coefficient. As a result, the standard utility formulation cannot explain many important puzzles in the literature (e.g. the equity premium puzzle postulated by Mehra and Prescott [40] in the literature on asset pricing). A non-linear aggregator, however, can separate EIS and risk aversion coefficient (see [22], as well as [32]).

Finally, in most dynamic models the expected utility is used to parametrize the future utility. But the expectation has a limited applicability. There is evidence that some decision-makers prefer to know the realization of uncertainty as quickly as possible, while other prefer to know in later on. This cannot be captured by the standard expected utility model (see Kreps and Porteus [32] as well as Chew and Epstein [17]). Generally speaking, the expectation does not take into account the *risk-sensitivity* of decision-makers. The risk-sensitivity may denote the *risk-aversion*, i.e. the decision maker prefers the assured output to any lottery having the same expectation, or the *risk-loving* property if the agent prefers the random output to the corresponding assured output. If the preferences are represented by the standard expectation, then the decision maker is risk indifferent. To take into account the risk sensitivity of agents, Weil [50] postulated a kind of *quasilinear mean* called the *entropic risk measure*. In the literature, there are some modifications of dynamic models toward the risk-sensitive agents. For example, the MDP with the risk-sensitive agent is considered by di-Masi and Stettner [18] and Bäuerle and Jaśkiewicz [12]. The zero-sum game with risk-sensitive agents is considered by Bäuerle and Rieder [13]. In turn, in Basu and Ghosh [11] and in Klompstra [29] is considered the model of the non-zero sum stochastic game. The OLG models with risk-sensitive agents are considered in Epstein and Zin [22], Weil [50], Marinacci and Montrucchio [37], Jaśkiewicz and Nowak [26] and in Balbus et al. [6].

The main problem is the existence of Markov Perfect Nash Equilibria (MPE for short) in OLG models. Until now many authors have continued this research by proposing some modifications and extensions. For a survey of the existing literature on this topic the reader is referred to Kohlberg [30], Leininger [34], Harris and Laibson [23], where the transition function is deterministic, and to some models with a random transition function, for example Alj and Haurie [2], Amir [3], Nowak [42, 43], Balbus and Nowak [8], Balbus et al. [10], or Balbus et al. [6], [7].

In this paper, we consider a dynamic consumption in an OLG model where the state space is uncountable. For simplicity, we assume there is no population growth inside the generation and we normalize the size of each generation to the unity. In OLG models presented in this paper, each generation has a capital inherited from the previous generation and has to decide on the consumption level of this capital, and the remaining part is an investment for future generations. The capital level of the next generation is a lottery whose distribution depends on the investment level. The today's overall utility has the *paternalistic* feature, i.e. does depend on the *consumption* of the immediate successor, and the *non-paternalistic* feature, i.e. does depend on the *overall utility* of the immediate successor. Observe that the utilities by Epstein and Zin [22], Weil [50], and Marinacci and Montrucchio [37] have non-paternalistic features only. In turn, the following utilities with risk-sensitive agents like in Jaśkiewicz and Nowak [26], Balbus et al. [6] have paternalistic features only. The utility having both paternalistic and non-paternalistic features was introduced first by Hori [25] in a dynamic allocation model between two coexisting generations.

Central issues in this paper are the existence, the uniqueness and the global attractivity of a recursive utility function. Then we are able to prove the existence of a stationary and mixed MPE. Even if we have the existence and some desired properties of the recursive utility function, the existence of an MPE has a positive answer only in some special cases (e.g. Dutta and Sundaram [21], Nowak [41], Balbus et al. [9] and He and Sun [24], Harris and Laibson [23], Nowak [42], Balbus et al. [6] and the references therein). The problem with the existence is highlighted in Levy and Mc Lennan [36] where there is an example of the standard stochastic game having no Nash equilibrium when the state space is uncountable. In the case of risk-sensitive agents, there is a problem with finding a pure Nash equilibrium even if the state space is countable (Basu and Ghosh [11] prove the existence of mixed equilibria only). The existence of pure Nash equilibria in the case of uncountable state space is possible (see Bäuerle and Rieder [13] or Jaśkiewicz and Nowak [26]).

In this paper, the recursive utility of the present generation is connected with the utility and consumption of the future generation by the non-linear aggregator, having Blackwell ([1]) property. Moreover, the utility and consumption are parametrized by the entropic risk measure by Weil [50]. This makes the model more general than in Bäuerle and Jaśkiewicz [12], Jaśkiewicz and Nowak [26], and Asienkiewicz and Jaśkiewicz [5]. The point is that in [12], [5] the aggregators have an additive form, and the utility in [26] has the paternalistic feature only. By similar reason, our model extends also the models with the hyperbolic discounting like Harris and Laibson [23], Balbus et al. [10] and even some models with utilities having both paternalistic and non-paternalistic features like Hori [25] and Doepke and Zilibotti [20]. But as in Nowak [43] and Basu and Ghosh [11], we prove the existence of mixed MPE only. The assumption on transition probability in this paper is rather similar to those from the following models: Brock and Mirman [16], Dutta and Sundaram [21], Balbus et al. [6] and the references therein. The transition in this paper is different than in Nowak [43] and He and Sun [24] because it allows the lack of norm continuity at 0.

This paper is organized as follows. Section 2 presents the basic terminology and abbreviations. Section 3 presents current knowledge and theorems on local contractions which are used for proving main theorems in this paper. Section 4 presents a mathematical formulation of the model. Basic assumptions and relations to some other results are in Section 5. Main results are in Sections 6 and in Section 7. In Sections 6 we show that if we have a policy, then we uniquely determine an overall utility function for each generation which is called dynamic recursive utility. The existence of a stationary MPE can be found in Section 7. All proofs of technical lemmas are in Appendix A. Appendix B includes the auxiliary results.

2. Basic notations

Let X and Y be topological spaces. Following notations are used in the entire paper:

([1]) According Marinacci and Montrucchio [37] terminology the Blackwell property of the function means the uniform contractivity with respect to future expected utility.

- Let $f\colon X \to X$ be a function; then by $f^{(n)}(\,\cdot\,)$, we denote the n-th iterate of f, i.e. $f^{(n)}(\,\cdot\,) = \underbrace{f \circ \ldots \circ f(\,\cdot\,)}_{n \text{ times}}$.
- By $\mathrm{C}(X)$ we denote the set of real valued continuous functions.
- $\mathrm{Bor}(X, Y)$ is the collection of Borel functions from X into Y, where $\mathrm{Bor}(X)$ is the collection of Borel real valued functions on X. Furthermore, if $X = \mathbb{R}_+$ then $\mathrm{Bor}_0(X)$ is the set of nonnegative Borel measurable function vanishing at 0.
- By $\mathcal{B}(X)$ we denote the collection of Borel subsets of X.
- $\Delta(X)$ means the set of all probability measures on Borel subsets of X.
- For each $x \in X$, the $\delta_x \in \Delta(X)$ means the *Dirac delta* at x, i.e. a unit point mass at x. That is $\delta_x(B) = 1$ for each Borel set B including x, in particular $\delta_x(\{x\}) = 1$.
- If $\eta \in \mathrm{Bor}(X, \Delta(Y))$, then for each $x \in X$ we denote $\eta(\,\cdot \mid x) := \eta(\,\cdot\,)(x)$, i.e. the image of x under η.
- If X and Y are both ordered sets, then $f\colon X \to Y$ is said to be *increasing* if $x_1 \le x_2$ in X implies that $f(x_1) \le f(x_2)$ in Y. f is said to be *strictly increasing* if $x_1 < x_2$ in X implies $f(x_1) < f(x_2)$ in Y.
- For $\mu \in \Delta(X)$, $L_\infty(X, \mu)$ is the space of all μ-essentially bounded functions (more formally we consider a quotient space of equivalence classes of functions equal μ-almos everywhere).
- X^∞ is a set of all sequences of elements of X. We usually endow X^∞ with the standard product topology.
- By $\overline{x} \in X^\infty$ we denote the sequence $(x_t)_{t=1}^\infty$, and by x^t we denote $(x_\tau)_{\tau=t}^\infty$.

3. Local contractions

Let X be a topological space and assume that there is a sequence of subsets

$$X_1 \subset \ldots X_j \subset \ldots \subset X \quad \text{and} \quad X = \bigcup_{j=1}^{\infty} X_j.$$

For each $f\colon X \to \mathbb{R}$ we put $||f||_j := \sup_{x \in X_j} |f(x)|$.

Let $F(X)$ be a set of real valued functions f on X such that $||f||_j < \infty$ for each $j \in \mathbb{N}$. Then $\{||\cdot||_j : j \in \mathbb{N}\}$ is a family of seminorms on $F(X)$. Let

$$\mathbf{M} := \Big\{\overline{m} \in \mathbb{R}_+^\infty : 0 < m_1 < m_2 < \ldots \text{ and } \lim_{j \to \infty} m_j = \infty\Big\}.$$

Fix arbitrary $\kappa > 1$ and each $\overline{m} \in \mathbf{M}$ we define

$$F_{\overline{m}}(X) := \left\{ f \in F(X) : \sum_{j=1}^{\infty} \frac{||f||_j}{\kappa^j m_j} < \infty \right\}.$$

Clearly $(F_{\overline{m}}(X), ||\cdot||)$ is a normed space, with $||f|| := \sum_{j=1}^{\infty} ||f||_j / \kappa^j m_j$.

Lemma 3.1 (see [39]). *Assume that*

(a) *For all $j \in \mathbb{N}$ the normed space $(F(X_j), ||\cdot||_j)$ of all restrictions of $f \in F(X)$ to X_j is a Banach space.*
(b) *If for each $j \in \mathbb{N}$, $f_j \in F(X_j)$ and $f_{j+1}(x) = f_j(x)$ for all $x \in X_j$, then f defined by $f(x) := f_j(x)$ for $x \in X_j$ belongs to $F(X)$.*

Then, $F_{\overline{m}}(X)$ is a Banach space.

Remark 3.2. Observe that if $F(X)$ is the set of all continuous functions on X, and

$$X = \bigcup_{j=1}^{\infty} \mathrm{Int}(K_j),$$

then all conditions of Lemma 3.1 are satisfied. Similarly, if (X, Σ) is a measurable space, and all X_j are measurable sets (i.e. $X_j \in \Sigma$) then all assumptions of Lemma 3.1 are satisfied. Hence in both cases $(F_{\overline{m}}(X), ||\cdot||)$ is a Banach space. For more details, see Remark 1 in [39].

Following Rincón–Zapatero and Rodríguez–Palmero [47] we introduce the following definition.

Definition 3.3. Let $G \subset F_{\overline{m}}(X)$. The operator $\Psi\colon G \to G$ is a 1-*local contraction* (1-LC for short) if there is a constant $\gamma \in]0,1[$, such that for each $j \in \mathbb{N}$ and each pair $(f, g) \in G \times G$, the following inequality holds:

$$||\Psi(f) - \Psi(g)||_j \leq \gamma ||f - g||_{j+1}.$$

Let $r := \sup_{j\in\mathbb{N}}(m_{j+1}/m_j)$ and assume $r < \infty$. The following theorem summarizes Theorem 2 in [47] ([2]) and Proposition 2 in [39].

Theorem 3.4 (see also [39], [47], [48], [38]). *Let G be a closed subset of $F_{\overline{m}}(X)$. Suppose that $\Psi\colon G \to G$ is a* 1-LC *with a constant $\gamma \in]0,1[$. Let $\alpha := \gamma r \kappa$, and assume $\alpha \in]0,1[$. Then Ψ is a contraction mapping with the contraction coefficient α. As a result, if $F_{\overline{m}}(X)$ is a Banach space, then Ψ has a unique fixed point $f^* \in G$ and, for each $f_0 \in G$, $\lim_{n\to\infty} \left\|\Psi^{(n)}(f_0) - f^*\right\| = 0$.*

4. The model

Let us define an OLG model in which

(1) $S := [0, \infty[$ is the space of capital level,
(2) $A(s) = [0, s]$ is the set of possible consumption levels for the current generation when its current capital is $s \in S$,
(3) $q\colon S \to \Delta(S)$ is a Borel measurable transition probability.

([2]) In fact in [47] $F(X)$ is the set of continuous functions.

In this model, generations are labeled by $\mathbb{N}$. Each generation t owns a level of the capital $s_t \in S$. The generation t selects the consumption level $c_t \in [0, s_t]$ and the remaining part $i_t = s_t - c_t$ is an investment level for the next generation $t+1$. The capital of $t+1$ is a random variable s_{t+1} whose distribution is $q(\,\cdot \mid i_t)$. The *Markov policy* of generation t is a Borel measurable transition $\pi_t \in \mathrm{Bor}(S, \Delta(S))$ such that $\pi_t(A(s)|s) = 1$ for each $s \in S$.

If π_t is a Markov policy for generation t, then $\overline{\pi} := (\pi_t)_{t=1}^{\infty}$ is said to be a *Markov profile.* By $\pi^t := (\pi_\tau)_{\tau=t}^{\infty}$ we denote a profile from generation t onward. Let Π be a set of all Markov policies for a single generation, and Π^∞ a set of all Markov profiles. We are going to construct a recursive utility function for the generation having risk-sensitive parameters. The Markov profile $\overline{\pi} \in \Pi^\infty$ is *stationary* if there is a profile $\pi \in \Pi$ such that $\pi_t = \pi$ for each $t \in \mathbb{N}$. Then we will identify the profile $\overline{\pi}$ with the policy π. Let $s_t \in S$ be a capital for t, and $c_t \in [0, s_t]$ its consumption level. Furthermore, by $(c_t, \overline{\pi})$ we denote the profile such that the t generation selects the consumption c_t and the generations from $t+1$ onward use the profile $\overline{\pi} \in \Pi^\infty$. For $\pi \in \Pi$ we denote (c_t, π) as the profile such that the t generation selects the consumption c_t and the generations from $t+1$ onward use the stationary profile π.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $f \in L_\infty(\Omega, P)$. According to Weil [50], we introduce a following definition:

Definition 4.1. The *entropic risk measure* of f is

$$\mathcal{E}^{\nu,P}(f) := \begin{cases} -\nu^{-1} \ln \left(\displaystyle\int_\Omega e^{-\nu f(\omega)} P(d\omega) \right) & \text{whenever } \nu \neq 0, \\ E^P(f) & \text{otherwise.} \end{cases}$$

Here $E^P(\,\cdot\,)$ is the standard expectation operator, i.e.

$$E^P(f) := \int_\Omega f(\omega) P(d\omega).$$

Let f, g be random variables from the set $L^\infty(\Omega, P)$ and $a \in \mathbb{R}$. We have four basic properties of the risk measure:

1. *monotonicity*, i.e. if $f \leq g$ P-almost everywhere, then $\mathcal{E}^{\nu,P}(f) \leq \mathcal{E}^{\nu,P}(g)$,
2. *constant preserving*, i.e. $\mathcal{E}^{\nu,P}(a) = a$,
3. *translation invariance*, i.e. $\mathcal{E}^{\nu,P}(f + a) = \mathcal{E}^{\nu,P}(f) + a$,
4. *Jensen inequality*, i.e. if $\nu > 0$ ($\nu < 0$) then $\mathcal{E}^{\nu,P}(f) \leq (\geq) E^P(f)$.

The property (4) above justifies why we call $\mathcal{E}^{\nu,P}(\,\cdot\,)$ a risk measure. Namely, if $\nu < 0$, then the agent prefers the assured output $E^P(f)$ to any lottery f having the same average value. In such case the agent is *risk-averse.* If $\nu > 0$, the agent prefers the lottery f to $E^P(f)$ and then is *risk-loving.* Finally, if $\nu = 0$, then the agent is *risk-indifferent.* The parameter ν is a coefficient of *absolute* *risk-aversion* (or *risk-coefficient* for short) and measures the risk-sensitivity of the agent.

Now we adapt the notations of $\mathcal{E}$ to our needs. Let $\eta \in \mathrm{Bor}(S, \Delta(S))$, $s \in S$, and $f \in L_\infty(S, \eta(\cdot\,|\,s))$. We denote

$$\mathcal{E}_s^{\nu,\eta}(f) := \mathcal{E}^{\nu,\eta(\cdot\,|\,s)}(f) = -\nu^{-1} \ln \left(\int_S e^{-\nu f(s')} \eta(ds'|s) \right)$$

where $\nu \neq 0$ is a fixed number. If $\nu = 0$ then the operator $\mathcal{E}_s^{\nu,\eta}$ is simply expectation with respect to $\eta(\cdot\,|\,s)$:

$$\mathcal{E}_s^{\nu,\eta}(f) := E_s^\eta(f) := E^{\eta(\cdot\,|\,s)}(f) = \int_S f(s')\eta(ds'|s).$$

We define some operators:

Definition 4.2. The operator $\mathrm{AV}\colon \Pi \to \mathrm{Bor}(S)$ defined as

$$\mathrm{AV}_s^\nu(\pi) = \mathcal{E}_s^{\nu,q}\left(\int_{A(s)} c'\pi(dc'|\cdot) \right)$$

is said to be the *quasilinear average consumption value*. We denote $\mathrm{AV}_s^0(\pi) = \mathrm{AV}_s(\pi)$.

Let π be a Markov profile and $s \in S$. The preferences of generation t are represented by the so-called *dynamic recursive utility function*. The formal definition below

Definition 4.3. The *dynamic recursive utility function* (DRU – for short) is the sequence of the functions $U^* := (U_t^*)_{t=1}^\infty$ such that

(a) $U_t^* : S \times \Pi^\infty \to \mathbb{R}_+$ and $U_t^*(\cdot\,, \pi^t) \in Bor(S)$ for each $\overline{\pi} \in \Pi^\infty$ and each $t \in \mathbb{N}$,

(b) U^* satisfies the following recursive equations

$$U_t^*(s, \pi^t) = \int_S V\left(c, \mathrm{AV}_{s-c}^{\nu_1}(\pi_t), \mathcal{E}_{s-c}^{\nu_2,q}\big(U_{t+1}^*(\cdot\,, \pi^{t+1})\big)\right)\pi_t(dc|s), \tag{4.1}$$

where $\nu_1, \nu_2 \in \mathbb{R}$ are risk-coefficients of the current generation. Call ν_1 as a risk-sensitivity from the consumption of the next generation, and ν_2 a risk-sensitivity from the utility of the next generation.

A comment on equation (4.1) is in order. The utility from today's consumption depends on the utility from tomorrow by V, the so-called *aggregator*. This term has been originally postulated by Koopmans [31] in order to provide a property of dynamic consistency. Observe however that the DRU has been modified in such a way that the dynamic consistency does not occur. Therefore, our model is more general than in [31]. Moreover, our model can be viewed as a bridge between paternalistic models, for example Kohlberg [30], Amir [3], Leininger [34], Nowak [42], Balbus et al. [10] and the references therein, and non-paternalistic models like Ray [45], Balbus et al. [7] since the today's utility does depend on both the utility of the successor and his/her consumption. Finally, observe that due to the randomness of the transition function our model encompasses that by Hori [25] as a special case which has also paternalistic and non-paternalistic features.

Remark 4.4. The utility U_t^*, if exists, has both paternalistic and non-paternalistic features (see Ray [45], Balbus et al. [7], Hori [25] for more discussion). The paternalistic feature is parametrized in the second argument of V and the non-paternalistic feature is parametrized in the last argument in equation (4.1). Observe that the utilities by Koopmans [31] or Epstein and Zin [22] have non-paternalistic feature only. For more motivation on this model, the reader is referred to the parent-child model by Doepke and Zilibotti [20]. In that paper, there are many generations living two periods. The preferences of any agent evolve during his/her life. The agent is sophisticated and makes decision today taking into consideration the fact that his/her preferences tomorrow will be different. As a result, it would be not natural if the agent ignores the future consumption or the future utility. Hence the utility has both paternalistic and non-paternalistic features.

Having the DRU, we introduce the definition of Markov Perfect Equilibria.

Definition 4.5. The profile $\overline{\pi}^* \in \Pi^\infty$ is said to be a *Markov Perfect Equilibrium* (MPE for short) if for each $t \in \mathbb{N}$, $s \in S$, and $c \in A(s)$ it holds

$$U_t^*(s, \pi^t) \geq U_t^*\big(s, \big(c, \pi^{t+1}\big)\big).$$

A comment is in order. The term of MPE is in fact a classic term from game theory. It is a profile which means that for no generation its unilateral deviation is profitable.

In this paper, we focus attention on Markov policies. In general, the policy may depend on the history including past actions and capitals. But the most natural interpretation has a Markov profile.

Remark 4.6. The Markov profile has the natural interpretation that the decision maker make decisions depending on the present situation, as if he/she forgot about the past.

5. Basic assumptions

Let $\overline{m} := (m_j)_{j=1}^\infty$ and $\overline{\xi} := (\xi_j)_{j=1}^\infty$ be sequences from $\mathbf{M}$. Put $S_j = [0, \xi_j]$ for $j \in \mathbb{N}$. Clearly $S = \bigcup_{j=1}^\infty S_j$. Suppose that there exists an upper bound on the ratio m_{j+1}/m_j ($j \in \mathbb{N}$).

Put $r := \sup_{j\in\mathbb{N}} (m_{j+1}/m_j)$. Let $\kappa \in]1,\infty[$ and $\gamma \in]0,1[$ be given, and suppose that $\alpha = \kappa\gamma r \in]0,1[$. We are going to prove that there exists a unique DRU such that

$$\sup_{s\in S_j,\,\overline{\pi}\in\Pi^\infty,\,t\in\mathbb{N}} \left|U_t\big(s,\pi^t\big)\right| \le m_j \quad \text{for each } j \in \mathbb{N}.$$

The following assumptions we need for proving the existence and the uniqueness of recursive utility function U_t^* that is bounded on $S_j \times \Pi^\infty$ by m_j for each $j \in \mathbb{N}$ and $t \in \mathbb{N}$.

Assumption 5.1 (Aggregator). The aggregator $V\colon S \times S \times \mathbb{R}_+ \to \mathbb{R}_+$ obeys the following assumptions:

(a) V is jointly Borel measurable, increasing in all arguments and

$$V(0,0,0) = 0.$$

(b) V is a Blackwell aggregator with a constant γ, that is, for any $c, c' \in S$, $U_1, U_2 \in \mathbb{R}$

$$|V(c,c',U_1) - V(c,c',U_2)| \le \gamma |U_1 - U_2|.$$

(c) The following boundness condition holds:

$$V(\xi_j, \xi_{j+1}, m_{j+1}) \le m_j \quad \text{for all } j \in \mathbb{N}.$$

The following assumption assures us that regardless on investment levels, the evolution of the capital will be no faster than the sequence $(\xi_j)_{j=1}^\infty$.

Assumption 5.2 (Transition probability). The transition probability q is Borel measurable and obeys the following assumption: if $j \in \mathbb{N}$ and $s \in S_j$, then $q(S_{j+1}|s) = 1$.

For proving the existence of stationary MPE, we accept the following assumption:

Assumption 5.3 (Transition probability — further assumptions). The transition probability q obey the following assumptions:

(a) The state 0 is an absorbing for the transition q, i.e. $q(\{0\}|0) = 1$ and $q(\cdot\,|\,s_n) \to^w \delta_0(\cdot)$ as $n \to \infty$ (3).
(b) There exists $\mu \in \Delta(S)$ such that $\mu(\{0\}) = 0$, and the function $\rho\colon S\times S \to \mathbb{R}_+$ such that, for any $s \in S \setminus \{0\}$

(3) the $\to^w$ means the standard convergence on the space of probability measure, i.e. $\eta_n \to^w \eta$ if and only if for each continuous and bounded real valued function f the convergence

$$\int_S f(s)\eta_n(ds) \to \int_S f(s)\eta(ds) \quad \text{as } n \to \infty$$

holds.

(i) $\rho(s,\,\cdot\,)$ is the Radon–Nikodym derivative of $q(\,\cdot\,|\,s)$ with respect to μ that is

$$q(K|s) = \int_K \rho(s,s')\,\mu(ds') \quad \text{for each } K \in \mathcal{B}(S),$$

(ii) ρ satisfies the norm continuity condition at every $s>0$, i.e.

$$\lim_{i\to s} \int_S |\rho(i,s') - \rho(s,s')|\,\mu(ds') = 0.$$

Remark 5.4. Assumption on the norm continuity of ρ in Assumption 5.3 can be found in [43] and [24]. Observe however that we only require the norm continuity condition at every $s>0$ and the weak continuity at 0. This is a crucial generalization since the norm continuity at all $s \in S$ is not to reconciled with the condition that 0 is an absorbing state of q, i.e. $q(\{0\}|0) = 1$ which is required in the most growth models. As an example, let us consider the extension of the fish wars in [35] and suppose that the current investment i determines the next output stock s' in the following formula $s' = s^p\zeta$. Here $p \in\,]0,1[$ and ζ a random variable uniformly distributed on $[0,1]$. Then $q(\,\cdot\,|\,s)$ is a uniform distribution on $[0,s^p]$ for $s>0$ and $q(\,\cdot\,|\,0) = \delta_0(\,\cdot\,)$. Then the Assumption 5.3 is satisfied. Indeed, it is easy to see that $q(\,\cdot\,|\,s) \xrightarrow{w} \delta_0(\,\cdot\,)$ as $s \downarrow 0$. Observe that for $s>0$ the density has a form

$$\rho(s,s') = \frac{1}{s^p}\,\mathbf{1}_{[0,s^p]}(s'),$$

and μ is the Lebesgue measure. Let $s>0$ and $i \downarrow s$. Then

$$\begin{aligned}\int_S |\rho(s,s') - \rho(i,s')|\,ds' &= \int_S \left|\frac{1}{s^p}\,\mathbf{1}_{[0,s^p]}(s') - \frac{1}{i^p}\,\mathbf{1}_{[0,i^p]}(s')\right| ds' \\ &= 1 - \frac{s^p}{i^p} + (i-s)\frac{1}{s^p} \to 0 \quad \text{as } i \downarrow s > 0.\end{aligned}$$

Similar result we have if $i \uparrow s$.

Remark 5.5. Alternatively, we may replace Assumption 5.3 by the norm continuty on all S and allow $q(\{0\}|0) < 1$. Then we obtain the same results. Such model would be similar as that in Karatzas et al. [28].

6. Existence of dynamic recursive utility function

For proving that there exists a unique DRU, we apply the results from Section 3. Let $X := S \times \Pi^\infty \times \mathbb{N}$ and $X_j := S_j \times \Pi^\infty \times \mathbb{N}$ for $j \in \mathbb{N}$. Let $\kappa > 1$ be as in Section 5. We follow all other notations from Section 3. For each $j \in \mathbb{N}$ and $f\colon X \to \mathbb{R}$ let us define

$$||f||_j := \sup_{x\in X_j} |f(x)| := \sup_{s\in S_j,\,\pi\in\Pi^\infty,\,t\in\mathbb{N}} |f(s,\pi,t)|.$$

Furthermore, let

$$F(X) := \big\{ f\colon X \to \mathbb{R} : f(\,\cdot\,, \overline{\pi}, t) \in \mathrm{Bor}_0(S) \\ \text{for each } t \in \mathbb{N},\ \overline{\pi} \in \Pi^\infty \text{ and } ||f||_j < \infty \text{ for each } j \in \mathbb{N} \big\}.$$

Obviously, each $||\cdot||_j$ is a seminorm on $F(X)$. Furthermore, define

$$||f|| := \sum_{j=1}^{\infty} \frac{||f||_j}{m_j \kappa^j}$$

and put

$$\begin{aligned} F_{\overline{m}}(X) &:= \{ f \in F(X) : ||f|| < \infty \}, \\ G &:= \{ f \in F_{\overline{m}}(X) : ||f||_j \le m_j \text{ for each } j \in \mathbb{N} \}. \end{aligned} \tag{6.1}$$

Clearly $(F_{\overline{m}}(X), ||\cdot||)$ is a normed space. Let d be the standard metric induced by $||\cdot||$, i.e. $d(f,g) := ||f-g||$ for $f, g \in F_{\overline{m}}(X)$. We need the following result for proving that the potential DRU is placed in a complete metric space.

Proposition 6.1. $(F_{\overline{m}}(X), ||\cdot||)$ *is a Banach space, and* G *is a closed subset of* $F_{\overline{m}}(X)$.

Proof. We show that $(F_{\overline{m}}(X), ||\cdot||)$ is a Banach space. Let $F_{\overline{m}}(X_j)$ be the set of all restrictions of $F_{\overline{m}}(X)$ to X_j. More formally $f \in F_{\overline{m}}(X_j)$ if there exists a $\widetilde{f} \in F(X)$ such that $\widetilde{f} = f|_{X_j}$. Then, $(F_{\overline{m}}(X_j), ||\cdot||_j)$ is a Banach space. For each $j \in \mathbb{N}$ choose $f_j \in F_{\overline{m}}(X_j)$ such that the collection $(f_j)_{j=1}^{\infty}$ satisfies the following condition: $f_{j+1}(x) = f_j(x)$ for each $x = (s, \pi, t) \in X_j = S_j \times \Pi^\infty \times \mathbb{N}$. We can define $f(s,\pi,t) := f_j(s,\pi,t)$ for each $s \in S_j$ and $(\pi, t) \in \Pi^\infty \times \mathbb{N}$. Put $\mathbf{f}_j(s,\pi,t) := f_j(s,\pi,t)$ if $s \in S_j$ and $(\pi,t) \in \Pi^\infty \times \mathbb{N}$, and $\mathbf{f}_j(s) = 0$ otherwise. Since S_j is obviously Borel measurable, hence $\mathbf{f}_j(\,\cdot\,,\pi,t) \in \mathrm{Bor}(S)$ for each $(\pi,t) \in \Pi^\infty \times \mathbb{N}$. Observe that $f(\,\cdot\,,\pi,t) = \lim_{j\to\infty} \mathbf{f}_j(\,\cdot\,,\pi,t)$ for each $(\pi,t) \in \Pi^\infty \times \mathbb{N}$. Hence, $f(\,\cdot\,,\pi,t) \in \mathrm{Bor}(S)$ and consequently $f \in F_{\overline{m}}(X)$. By Lemma 3.1, we deduce that $(F_{\overline{m}}(X), ||\cdot||)$ is a Banach space.

It is routine to verify that G is a closed subset of $F_{\overline{m}}(X)$, hence we omit this proof. □

We construct a DRU as a unique fixed point of the following operator

$$\Psi(f)(s,\overline{\pi},t) := \int_{A(s)} V\big(c, \mathrm{AV}^{\nu_1}_{s-c}(\pi_{t+1}), \mathcal{E}^{\nu_2,q}_{s-c}\big(f(\,\cdot\,,\pi^{t+1}, t+1)\big)\big)\pi_t(dc|s)$$

where $f \in G$. We state the following lemma for proving that Ψ is in fact a contraction mapping.

Lemma 6.2. Ψ *maps* (G,d) *into itself and is a contraction mapping with a constant* α.

The proof of this lemma is in Appendix A. We use the local contraction argument due to Rincón-Zapatero and Rodríguez-Palmero [47], [48]. We have the following result.

Theorem 6.3. *Let Assumptions* 5.1 *and* 5.2 *be satisfied. Then there exists a recursive utility function* $U_t^*(s,\pi) = f^*(s,\pi,t)$ *such that* $f^* \in G$ *and* f^* *is a unique fixed point of* Ψ. *Furthermore, for each* $f_0 \in G$,

$$\lim_{n\to\infty} \big|\big|\Psi^{(n)}(f_0) - f^*\big|\big|_j = 0 \quad \text{for each } j \in \mathbb{N}. \tag{6.2}$$

Proof. From Lemma 6.2 the operator Ψ maps G into itself and it is a contraction mapping with a constant α. The metric space (G,g) is complete, since G is a closed subset of $F_{\overline{m}}(X)$. Moreover, by Proposition 6.1, $F_{\overline{m}}(X)$ is a Banach space. Hence, there exists $f^* \in G$ which is a unique fixed point of Ψ. Moreover, $d(\Psi^{(n)}(f_0), f^*) \to 0$ as $n \to 0$ for arbitrary $f_0 \in G$. By definition of d, it follows that the equity in (6.2) holds. □

7. Existence of Markov perfect equilibria

In this section, we make an additional assumption that $\nu_1 = 0$. Put $\nu_2 := \nu$. Let Π^μ be a quotient space of equivalence classes of functions $\pi \in \Pi$ equal μ-almost everywhere. Since all sets $A(s)$ are compact, Π^μ is compact and metrizable when endowed with the weak-star topology. For the details, we refer the reader to Chapter IV in [49]. Here, we only mention that a sequence π_n converges to π in Π^μ if and only if, for every $w\colon S\times S \to \mathbb{R}$ such that $w(s,\cdot)$ is continuous on $A(s)$ for each $s \in S$, $w(\cdot,c)$ is measurable for each $c \in A(s)$ and

$$s \to \max_{c\in A(s)} |w(s,c)|$$

is μ-integrable over S, we have

$$\lim_{n\to\infty} \int_S \int_{A(s)} w(s,c)\pi_n(dc|s)\,\mu(ds) = \int_S \int_{A(s)} w(s,c)\pi(dc|s)\,\mu(ds).$$

Notation 7.1. Let $\to^*$ denote the convergence in Π^μ. Let $[\pi]_\mu$ be an equivalence class containing $\pi \in \Pi$. In the most notations in this paper, we drop the $[\,\cdot\,]_\mu$ and identify each element of this class with the representative element.

We are going to prove that there exists a stationary MPE in the set of mixed policies.

Theorem 7.2. *Let* $\nu_1 = 0$ *and let* $\nu_2 := \nu$ *be arbitrary. Then, under Assumptions* 5.1–5.3 *there exists a stationary* MPE.

We will prove this fact for $\nu \neq 0$ only. The proof in the case $\nu = 0$ proceeds in a similar way. Before we present a formal proof, we will present a mathematical background and formulate some lemmas. Let us define the set of all possible DRU which can be obtained whenever all generations use the stationary MPE:

$$\mathcal{Z} := \Big\{ g \in C(S \times \Pi^{\mu}) : \sup_{s \in S_j,\, \pi \in \Pi^{\mu}} |g(s,\pi)| \le m_j \text{ for each } j \in \mathbb{N},$$

$$g(s,\pi) \in \mathrm{Bor}_0(S) \text{ for all } \pi \in \Pi^{\mu} \text{ and } s \to \sup_{\pi \in \Pi^{\mu}} g(s,\pi) \text{ is continuous at } 0 \Big\}.$$

Then we define

$$\Lambda(g)(s,\pi) := \max_{c \in A(s)} V(c, AV_{s-c}(\pi), \mathcal{E}_{s-c}(g(\,\cdot\,,\pi))).$$

The operator Λ returns the best possible utility for the current generation whenever all other generations use the stationary policy π and $g(\cdot,\pi)$ is the utility function for the next generation. It is easy to see that $\mathcal{Z}$ can be viewed as a subset of G defined in equation (6.1). In the lemma below, we also show that $\mathcal{Z}$ is a closed set.

Lemma 7.3. *$\mathcal{Z}$ can be embedded in the set G. Moreover, $\mathcal{Z}$ is a closed subset of G. As a result $(\mathcal{Z}, d)$ is a complete metric space, where d is the metric inherited from G.*

See Appendix A for the proof of this lemma. In the next lemma we prove that Λ maps $\mathcal{Z}$ into itself.

Lemma 7.4. *Λ maps $\mathcal{Z}$ into itself.*

The proof of this lemma can be found in Appendix A. For proving this lemma, we use the lemmas in Appendix B which state the joint continuity of operators $(s,\pi) \in S \times \Pi^{\mu} \to AV_s(\pi)$ and $(s,\pi) \in S \times \Pi^{\mu} \to \mathcal{E}_s(g(\,\cdot\,,\pi))$ for any $g \in \mathcal{Z}$. The proof of the next lemma, based on the local contraction stuff is also technical and is included in Appendix A.

Lemma 7.5. *Λ is a d-contraction mapping with a constant α. As a result, Λ has a unique fixed point $g^* \in \mathcal{Z}$.*

Let g^* be a unique fixed point of Λ which exist from Lemma 7.5. Now we construct a correspondence whose fixed point is MPE. Let us define

$$\Gamma(\pi)(s) := \arg\max_{c \in A(s)} V\big(c, AV_{s-c}(\pi), \mathcal{E}^{\nu,q}_{s-c}(g^*(\,\cdot\,,\pi))\big)$$

and let

$$\Phi(\pi) := \big\{ \pi' \in \Pi^{\mu} : \pi'(\Gamma(\pi)(s)\,|\,s) = 1 \text{ for } \mu\text{-a.a. } s \in S \big\}$$

be such a correspondence. We shall prove that there exists at least one $\widetilde{\pi} \in \Pi^{\mu}$ that is $\widetilde{\pi} \in \Phi(\widetilde{\pi})$. Next, we shall show that MPE is equal $\widetilde{\pi}$ μ-a.e. For proving the existence of such $\widetilde{\pi}$, we shall use the standard Kakutani–Fan–Gliksberg Fixed Point Theorem (see Corollary 17.55 in [1] for example). It is easy to see that Φ

is convex-valued. In the following lemmas below, we shall show that Φ satisfies all other conditions of Kakutani–Fan–Gliksberg Fixed Point Theorem. For the proofs of these lemmas the reader is referred to Appendix A.

Lemma 7.6. $\Phi(\pi) \neq \emptyset$ *for each* $\pi \in \Pi^\mu$.

Lemma 7.7. $\Phi(\pi)$ *is an upper semicontinuous correspondence.*

Proof of Theorem 7.2. By Lemma 7.6, the correspondence Φ has non-empty convex values. By Lemma 7.7, Φ has closed graph; consequently, by the Kakutani–Fan–Gliksberg Theorem (Corollary 17.55 in [1]) Φ has a fixed point $\widetilde{\pi} \in \Phi(\widetilde{\pi})$. Hence, $\widetilde{\pi}(\Gamma(\widetilde{\pi})(s)\,|\,s) = 1$ for μ-a.a. $s \in S$. Let

$$\widetilde{S} := \big\{s \in S : \widetilde{\pi}(\Gamma(\widetilde{\pi})(s)\,|\,s) = 1\big\}.$$

Clearly $\mu(\widetilde{S}) = 1$. Observe that by the Measurable Maximum Theorem (see Theorem 18.19 in [1]) there is a Borel measurable selection $\gamma(s) \in \Gamma(\widetilde{\pi})(s)$ for all $s \in S$. Put $\pi^*(\,\cdot\,|\,s) = \widetilde{\pi}(\,\cdot\,|\,s)$ for $s \in \widetilde{S}$ and $\pi^*(\,\cdot\,|\,s) = \delta_{\gamma(s)}(\,\cdot\,)$ otherwise. Clearly since $\widetilde{\pi}(\,\cdot\,|\,s) = \pi^*(\,\cdot\,|\,s)$ for μ-a.a. $s \in S$, hence by Assumption 5.3, we have that $\Gamma(\pi^*)(s) = \Gamma(\widetilde{\pi})(s)$ for all $s \in S$. As a result $\pi^*(\Gamma(\pi^*)(s)|s) = 1$ for all $s \in S$. Hence, $\pi^* \in \Pi$ is a MPE, and by Theorem 6.3, $g^*(s, \pi^*) = U^*(s, \pi^*, t)$ for each $t \in \mathbb{N}$. Hence $g^*(s, \pi^*)$ is an equilibrium value. □

Appendix A

Proof of Lemma 6.2. Let $f \in G$. First we show that $s \to \Psi(f)(s, \overline{\pi}, t)$ is Borel measurable for each $(\overline{\pi}, t) \in \Pi^\infty \times \mathbb{N}$. By definition of $\mathcal{A}_i^{\nu_1}(\pi_{t+1})$ and Assumption 5.2, it follows that $i \in S \to \mathcal{A}_i^{\nu_1}(\pi_{t+1})$ is Borel measurable. By the same argument, $i \in S \to \mathcal{E}_i^{\nu_2,q} f(\,\cdot\,, \pi^{t+1}, t+1)$ is Borel measurable. As a result,

$$(s, c) \in S \times S \to V\big(c, \mathcal{A}_{s-c}^{\nu_1}(\pi_{t+1}), \mathcal{E}_{s-c}^{\nu_2,q} f\big(\,\cdot\,, \pi^{t+1}, t+1\big)\big)$$

is jointly Borel measurable as well. Consequently $s \in S \to \Psi(f)(s, \overline{\pi}, t)$ is Borel measurable. Now we show that $|\Psi(f)(s, \overline{\pi}, t)| \le m_j$ for each t, π, $s \in S_j$ and $j \in \mathbb{N}$. Let $s \in S_j$ and $c \in [0, s]$ be given. Hence $c \in S_j$. Then, by Assumption 5.2 it follows that $q(S_{j+1}|s) = 1$. Hence,

$$\mathcal{A}_{s-c}^{\nu_1}(\pi_{t+1}) \le \xi_{j+1} \quad \text{and} \quad \mathcal{E}_i^{\nu_2,q} f(\,\cdot\,, \pi^{t+1}, t+1) \le m_{j+1}.$$

Furthermore, by the points (a) and (c) of Assumption 5.1 we have

$$V\big(c, \mathcal{A}_{s-c}^{\nu_1}(\pi_{t+1}), \mathcal{E}_{s-c}^{\nu_2,q} f(\,\cdot\,, \pi^{t+1}, t+1)\big) \le V(\xi_j, \xi_{j+1}, m_{j+1}) \le m_j.$$

Integrating both sides over $\pi_t(\,\cdot\,|\,s)$ and taking a supremum over $s \in S_j$ we have then $||\Psi(f)||_j < m_j$. Consequently, $\Psi(f) \in G$.

Finally, we show that Ψ is a contraction mapping, by proving that is 1-LC with γ. Let $f_1, f_2 \in G$, $j \in \mathbb{N}$, $\overline{\pi} \in \Pi^\infty$, $t \in \mathbb{N}$ and $s \in S_j$. By Assumptions 5.2 and 5.1 (b) we have

$$\text{(A.1)} \quad |\Psi(f_1)(s,\overline{\pi},t) - \Psi(f_1)(s,\overline{\pi},t)| \leq \sup_{s\in S_j} \Big|\mathcal{E}_s^{\nu^2,q} f_1\big(\,\cdot\,,\pi^{t+1},t+1\big) - \mathcal{E}_s^{\nu^2,q} f_2\big(\,\cdot\,,\pi^{t+1},t+1\big)\Big|.$$

To finish the proof that Ψ is 1-LC, observe that

$$\text{(A.2)} \quad \sup_{s\in S_j} \Big|\mathcal{E}_s^{\nu^2,q} f_1\big(\,\cdot\,,\pi^{t+1},t+1)\big) - \mathcal{E}_s^{\nu^2,q} f_2\big(\,\cdot\,,\pi^{t+1},t+1\big)\Big| \leq \sup_{s'\in S_{j+1},\,\overline{\pi},\,t\in\mathbb{N}} |f_1(s',\overline{\pi},t+1) - f_2(s',\overline{\pi},t+1)|.$$

The inequality in (A.2) follows from the fact that $f \in \mathrm{Bor}(S_{j+1}) \to \mathcal{E}_s^{\nu^2,q} f$ obeys Blackwell's conditions ([4]). Combining (A.1) and (A.2), we have the thesis that Ψ is 1-LC with γ. Hence, by Lemma 3.1 and Proposition 3.4 Ψ is a contraction mapping. □

Proof of Lemma 7.3. Observe that $\mathcal{Z}$ can be written as follows

$$\begin{aligned}\mathcal{Z} = \{f \in G:\ & \text{there is } g \in C(S\times \Pi^\mu) \\ & \text{such that } f(s,\pi,t) = g(s,[\pi]_\mu) \text{ for each } (s,\pi,t) \in S\times\Pi\times\mathbb{N}, \\ & s \in S \to g(s,[\pi]_\mu) \text{ is continuous at } 0\}.\end{aligned}$$

We show that $\mathcal{Z}$ is closed. By Proposition 6.1, (G,d) is a complete metric space. Suppose that $f_n \to f$ as $n \to \infty$ in (G,d), and each of f_n belongs to $\mathcal{Z}$. We show that $f \in \mathcal{Z}$. It is easy to see that the limiting function f does not depend on t and the selection from the class $[\pi]_\mu$. Let $f_n(s,\pi,t) = g_n(s,\pi)$ and $f(s,\pi,t) = g(s,\pi)$ (drop the symbol from $[\,\cdot\,]_\mu$ for short). We first need to show that g is continuous as a function on $S\times\Pi^\mu$. But it is clear since if $f_n \to f$ in (G,d) means that for each $j \in \mathbb{N}$ the convergence $||f_n - f||_j \to 0$ holds as $n \to \infty$. Since $||f_n - f||_j = ||g_n - g||_j$, and each of g_n is continuous, hence g is continuous as well. Now we show that $\sup_{\pi\in\Pi^\mu} g(s,\pi)$ is continuous at $s = 0$. Let $\varepsilon > 0$ be given, and let $j \in \mathbb{N}$ be arbitrarily fixed. Let $n_0 \in \mathbb{N}$ be such that $||g_{n_0} - g||_j < \varepsilon$. Then for any $s \in S_j$ we have

$$0 \leq \sup_{\pi\in\Pi^\mu} g(s,\pi) \leq ||g_{n_0} - g||_j + \sup_{\pi\in\Pi^\mu} g_{n_0}(s,\pi) < \varepsilon + \sup_{\pi\in\Pi^\mu} g_{n_0}(s,\pi).$$

Since $g_{n_0} \in \mathcal{Z}$, hence after taking a limit $s \downarrow 0$, we conclude that the last part in the inequality no greater than arbitrary $\varepsilon > 0$. Hence is 0. Consequently $f \in \mathcal{Z}$

([4]) Since $\mathcal{E}_s^{\nu^2,q}(\,\cdot\,)$ increasing and constant preserving, hence is Lipshitz continuous with a constant 1. It follows from Theorem 3.5.3 in [1].

and hence $\mathcal{Z}$ is a closed subset of G. As a result, $(\mathcal{Z}, d)$ is a complete metric space. □

Proof of Lemma 7.4. Let $g \in \mathcal{Z}$ and $\pi \in \Pi^\mu$. Similarly as in the proof of Lemma 6.2, we can prove that $\Lambda(g) \in G$. We show that the expression $\Lambda(g)(s, \pi') = \Lambda(g)(s, \pi)$ is equal for each selection $\pi' \in [\pi]_\mu$. Let π' be such selection. Then $S_0 := \{s \in S : \pi(\,\cdot\,|\,s) = \pi'(\,\cdot\,|\,s)\}$ is a μ-full set, i.e. $\mu(S_0) = 1$. Then, for all $s \in S$,

$$\begin{aligned}
\mathrm{AV}^q_s(\pi') &= \int_S \int_{A(s)} c'\pi(dc'|s)\rho(s, s')\,\mu(ds') \\
&= \int_{S_0} \int_{A(s)} c'\pi'(dc'|s)\rho(s, s')\,\mu(ds') = \int_{S_0} \int_{A(s)} c'\pi(dc'|s)\rho(s, s')\,\mu(ds') \\
&= \int_S \int_{A(s)} c'\pi(dc'|s)\rho(s, s')\mu(ds') = AV^q_s(\pi).
\end{aligned}$$

Furthermore, $g(s, \pi) = g(s, \pi')$ for all $s \in S$, hence

$$\begin{aligned}
\mathcal{E}^{\nu,q}_s(g(\,\cdot\,, \pi')) &= -\nu^{-1} \ln\left(\int_S e^{-\nu g(s',\pi')} \rho(s, s')\,\mu(ds') \right) \\
&= -\nu^{-1} \ln\left(\int_S e^{-\nu g(s',\pi)} \rho(s, s')\,\mu(ds') \right) = \mathcal{E}^{\nu,q}_s(g(\,\cdot\,, \pi)).
\end{aligned}$$

Hence, $\Lambda(g)(s, \pi') = \Lambda(g)(s, \pi)$. To finish the proof, we only need to verify that $\Lambda(g)$ is continuous on $S \times \Pi^\mu$. By Lemmas B.2 and B.3 in Appendix B we conclude that

$$(\pi, s, c) \in \Pi^\mu \times \mathrm{Gr}(A) \to V\big(c, \mathrm{AV}_{s-c}(\pi), \mathcal{E}^{\nu,q}_{s-c}(g(\,\cdot\,, \pi))\big)$$

is continuous and can be easily extended to a continuous function on $\Pi^\mu \times S \times S$. Hence, by Berge Maximum Theorem (Theorem 17.31 in [1]) we have the desired continuity of $\Lambda(g)$.

To complete the proof, we show that $s \in S \to \sup_{\pi \in \Pi^\mu} \Lambda(g)(s, \pi)$ is continuous at 0. Obviously $\Lambda(g)(0, \pi) = 0$ for all $\pi \in \Pi^\mu$. For $s > 0$, we have

$$1 \geq \int_S e^{-\nu g(s',\pi)} \rho(s, s')\,\mu(ds') \geq \int_S e^{-\nu \sup_{\pi \in \Pi^\mu} g(s',\pi)} \rho(s, s')\,\mu(ds') := \mathcal{K}(s).$$

By Assumption 5.3 and Lemma B.1 in Appendix B, we conclude that $\mathcal{K}(\,\cdot\,)$ tends to 1 as $s \downarrow 0$. Hence

$$0 \leq \mathcal{E}^\nu_s(g(s, \pi)) \leq -\nu^{-1} \ln(\mathcal{K}(s)) \to 0 \quad \text{as } s \downarrow 0,$$

and hence

$$0 \leq \Lambda(g)(s, \pi) \leq V\left(s, \int_S s'\rho(s, s')\,\mu(ds'), \ln(\mathcal{K}(s))\right) \to 0 \quad \text{as } s \downarrow 0.$$

Consequently, $\Lambda(g) \in \mathcal{Z}$. □

Proof of Lemma 7.5. By Lemma 7.3, $\mathcal{Z}$ is a closed subset of G and $G \subset F_{\overline{m}}(X)$. By Proposition 6.1, $F_{\overline{m}}(X)$ is a Banach space. Since by Lemma 7.4 the mapping Λ transforms $\mathcal{Z}$ into itself, hence applying Theorem 3.4 we only need to show that Λ is 1-LC on $\mathcal{Z}$. Let $g_1, g_2 \in \mathcal{Z}$, $\pi \in \Pi^\mu$ and $s \in S_j$. By Assumption 5.2, we have

$$\begin{aligned} \mathcal{E}_s^{\nu,q}(g_i(\,\cdot\,,\pi)) &= -\nu^{-1} \ln \left(\int_S e^{-\nu g_i(s',\pi)} q(ds'|\,s) \right) \\ &= -\nu^{-1} \ln \left(\int_{S_{j+1}} e^{-\nu g_i(s',\pi)} q(ds'|\,s) \right). \end{aligned} \tag{A.3}$$

We have also that $g_1(s') \le g_2(s') + ||g_1 - g_2||_{j+1}$ for each $s' \in S_j$. Hence, by (A.3), the monotonicity and the constant invariance of $\mathcal{E}_s^{\nu,q}$, we have

$$\begin{aligned} \mathcal{E}_s^{\nu,q}(g_1(\,\cdot\,,\pi)) &\le -\nu^{-1} \ln \left(\int_{S_{j+1}} e^{-\nu g_2(s',\pi) - \nu ||g_1 - g_2||_{j+1}} q(ds'|\,s) \right) \\ &= -\nu^{-1} \ln \left(\int_{S_{j+1}} e^{-\nu g_2(s',\pi)} q(ds'|\,s) \right) + ||g_1 - g_2||_{j+1}. \end{aligned} \tag{A.4}$$

By changing the role between $i = 1$ and $i = 2$ in (A.4), we have

$$\left| \mathcal{E}_s^{\nu,q}(g_1(\,\cdot\,,\pi)) - \mathcal{E}_s^{\nu,q}(g_2(\,\cdot\,,\pi)) \right| \le ||g_1 - g_2||_{j+1}. \tag{A.5}$$

We have

$$\begin{aligned} \Lambda(g_1)(s,\pi) - \Lambda(g_2)(s,\pi) \le\, & V\big(c_0, \mathrm{AV}^q_{s-c_0}(\pi), \mathcal{E}^{\nu,q}_{s-c_0}(g_1(\,\cdot\,,\pi))\big) \\ & - V\big(c_0, \mathrm{AV}^q_{s-c_0}(\pi), \mathcal{E}^{\nu,q}_{s-c_0}(g_2(\,\cdot\,,\pi))\big), \end{aligned} \tag{A.6}$$

where $c_0 \in \arg \max\limits_{c \in A(s)} V\big(c, \mathrm{AV}^q_{s-c}(\pi), \mathcal{E}^{\nu,q}_{s-c}(g_1(\,\cdot\,,\pi))\big)$. By Assumption 5.1 and inequality (A.6), we have

$$\Lambda(g_1)(s,\pi) - \Lambda(g_2)(s,\pi) \le \gamma \left| \mathcal{E}^{\nu,q}_{s-c_0}(g_1(\,\cdot\,,\pi)) - \mathcal{E}^{\nu,q}_{s-c_0}(g_2(\,\cdot\,,\pi)) \right|. \tag{A.7}$$

Combining (A.5) and (A.7), we have

$$\Lambda(g_1)(s,\pi) - \Lambda(g_2)(s,\pi) \le \gamma ||g_1 - g_2||_{j+1}.$$

By changing the roles between g_1 and g_2, and taking a supremum over $s \in S_j$ and $\pi \in \Pi$, we have

$$||\Lambda(g_1) - \Lambda(g_2)||_j \le \gamma ||g_1 - g_2||_{j+1},$$

hence Λ is 1-LC with γ. □

Proof of Lemma 7.6. Let $\pi \in \Pi^\mu$ be given. Combining Assumptions 5.1 and 5.2 as well as Lemmas B.2 and B.3 in Appendix B, we can conclude that

$$\mathrm{Gr}(A) \ni (s,c) \mapsto V(c, AV_{s-c}(\pi), \mathcal{E}^{\nu,q}_{s-c}(g^*(\,\cdot\,,\pi))).$$

is Carathéodory. Moreover, this function above can be easily extended to whole $S \times S$ and is still Carathéodory. By the Measurable Maximum Theorem (Theorem 18.19 in [1]) there exists a Borel measurable selector from $\Gamma(\pi)(s)$, i.e. there is Borel measurable $\gamma\colon S \to S$ such that $\gamma(s) \in \Gamma(\pi)(s)$ ([5]). Hence, $s \to \gamma(s)$ (more precisely, $s \to \delta_{\gamma(s)}(\,\cdot\,)$) is the element of $\Phi(\pi)$. □

Proof of Lemma 7.7. Suppose that $\pi_n \to^* \pi$ as $n \to \infty$ and $\pi'_n \in \Phi(\pi_n)$ for all $n \in \mathbb{N}$. We shall show that the limit point of π'_n belongs to $\Phi(\pi)$. Since Π^μ is compact and metrizable, hence may assume without loss of generality that $\pi'_n \to^* \pi'$. Put

$$\begin{aligned} w_n(s,c) &:= V(c, AV_{s-c}(\pi_n), \mathcal{E}^\nu_{s-c}(g^*(\,\cdot\,,\pi_n))),\\ w(s,c) &:= V(c, AV_{s-c}(\pi), \mathcal{E}^\nu_{s-c}(g^*(\,\cdot\,,\pi))). \end{aligned}$$

We claim that, for each j,

$$\max_{s\in S_j, c\in A(s)} |w_n(s,c) - w(s,c)| \to 0 \quad \text{as } n \to \infty. \tag{A.8}$$

Now, let S^0 be a subset of S such that $s \in S^0$ if and only if $\pi'_n(\Gamma(\pi_n)(s)|s) = 1$ for all $n \in \mathbb{N}$. Clearly $\mu(S^0) = 1$. Observe that, by Lemmas B.2 and B.3 in Appendix B, the functions w_n and w are both jointly continuous. Hence, there is $s_n \in S_j$ and $c_n \in A(s_n)$ that realize the above maximum. Without loss of generality assume $(s_n, c_n) \to (s_0, c_0)$ where $s_0 \in S_j$ and $c_0 \in A(s_0)$. Applying Lemmas B.2 and B.3 in Appendix B, we have $w_n(s_n, c_n) \to w(s_0, c_0)$ and $w(s_n, c_n) \to w(s_0, c_0)$. As a result, the convergence in (A.8) holds.

Pick any $s \in S^0$. Then the equation $\pi'_n(\Gamma(s)|s) = 1$ holds. Therefore, by definition of g^* and w_n, we have

$$g^*(s,\pi_n) = \int_{A(s)} w_n(s,c)\pi'_n(dc\,|\,s).$$

Observe that, since $g \in \mathcal{Z}$, hence $g^*(s,\pi)$ must be continuous at π. Therefore,

$$\lim_{n\to\infty} g^*(s,\pi_n) = g^*(s,\pi). \tag{A.9}$$

On the other hand, since $\mu(S^0) = 1$, hence for each $j \in \mathbb{N}$ we have

$$\begin{aligned} &\bigg| \int_{S^0\cap S_j} \int_{A(s)} w_n(s,c)\pi'_n(dc\,|\,s)\,\mu(ds) \\ &\qquad - \int_{S_j\cap S^0} \int_{A(s)} w(s,c)\pi'(dc\,|\,s)\,\mu(ds) \bigg| \\ &\quad \le \int_{S_j} \int_{A(s)} |w_n(s,c) - w(s,c)|\pi'_n(dc\,|\,s)\,\mu(ds) \end{aligned} \tag{A.10}$$

([5]) This result can be obtained more directly using Kuratowski and Ryll-Nardzewski Selection Theorem (see Theorem 18.13 in [1]).

$$+\int_{S_j}\int_{A(s)} w(s,c)\pi_n'(dc|\,s)\,\mu(ds)$$
$$-\int_{S_j}\int_{A(s)} w(s,c)\pi'(dc|\,s)\,\mu(ds).$$

Since $\pi_n' \to^* \pi$, hence we have

$$\lim_{n\to\infty}\int_S\int_{A(s)} w(s,c)\pi_n'(dc|\,s)\,\mu(ds) = \int_S\int_{A(s)} w(s,c)\pi'(dc|\,s)\,\mu(ds). \tag{A.11}$$

Moreover, by (A.8), we have

$$\int_S\int_{A(s)} |w_n(s,c) - w(s,c)|\pi_n'(dc|\,s)\mu(ds) \leq \sup_{s\in S_j, c\in A(s)} |w_n(s,c) - w(s,c)| \to 0 \quad \text{as } n \to 0. \tag{A.12}$$

Combining (A.11) and (A.12), we have that the left hand side in expression (A.10) tends to 0. As a result

$$\int_{S_j}\int_{A(s)} w(s,c)\pi'(dc|\,s)\,\mu(ds) = \int_{S_j} g^*(s,\pi)\,\mu(ds)$$

for each $j \in \mathbb{N}$. Since $g^*(s,\pi) = \max_{c\in A(s)} w(s,c)$, hence there exists a μ -full set $S^1 \subset S$ such that, for $s \in S^1$, it holds

$$\int_{A(s)} w(s,c)\pi'(dc|\,s) = g^*(s,\pi),$$

and hence $\pi'(\Gamma(\pi)(s)\,|\,s) = 1$ for $s \in S^1$. Consequently $\pi' \in \Phi(\pi)$, and finally we conclude that Φ is upper semicontinuous □

Appendix B

The following lemma is rather easy and may be proved in many ways.

Lemma B.1. *Let $S = [0,\infty[$ and let $f\colon S \to \mathbb{R}$ be bounded Borel measurable and continuous at* 0. *Let $\eta_n \in \Delta(S)$ be a sequence and $\eta_n \xrightarrow{w} \delta_0$ as $n \to \infty$. Then*

$$\int_S f(s)\,\eta_n(ds) \to f(0) \quad \text{as } n \to \infty.$$

Proof. By the Skorokhod Representation Theorem (see Theorem 6.7 in [15]) there exists a probability space $(\Omega, \mathcal{F}, P)$ and the sequence of random variables g_n whose distribution is η_n such that $g_n(\omega) \to 0$ for P-almost all $\omega \in \Omega$. Then, by the standard Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_S f(\omega)\,\eta_n(d\omega) = \lim_{n\to\infty}\int_S f(g_n(\omega))P(d\omega) = f(0).$$ □

We need the following auxilliary results that are crucial for proving that Λ maps $\mathcal{Z}$ into itself. We make Assumptions 5.2 and 5.3.

Lemma B.2. *Let* $s_n \to s$ *in* S *and* $\pi_n \xrightarrow{*} \pi$ *in* Π^μ *as* $n \to \infty$. *Then* $\mathrm{AV}_{s_n}(\pi_n) \to \mathrm{AV}_s(\pi)$ *as* $n \to \infty$.

Proof. In the first step suppose that $s > 0$. Let $j \in \mathbb{N}$ be such that $s < \xi_j$. Then $s_n \in S_j$ for all but finitely many $n \in \mathbb{N}$. We define

$$y_n := \int_S \int_{A(s')} c' \big|\rho(s_n, s') - \rho(s, s')\big| \pi_n(dc'|\, s')\, \mu(ds').$$

By Assumptions 5.2 and 5.3, we have

$$0 \le y_n \le \xi_{j+1} \int_S \big|\rho(s_n, s') - \rho(s, s')\big|\, \mu(ds') \to 0 \quad (\text{as } n \to \infty). \tag{B.1}$$

By definition of $\xrightarrow{*}$, we immediately have

$$AV_s(\pi_n) \to AV_s(\pi) \quad (\text{as } n \to \infty). \tag{B.2}$$

Obviously

$$|AV_{s_n}(\pi_n) - AV_s(\pi)| \le y_n + |AV_s(\pi_n) - AV_n(\pi)|. \tag{B.3}$$

Taking a limit $n \to \infty$ in (B.3) and combining with (B.1) and (B.2), we have the desired convergence.

Now suppose that $s = 0$. Clearly $AV_0(\pi) = 0$. Then

$$0 \le \int_S \int_{A(s')} c' \pi_n(dc'|\, s') \rho(s_n, s')\, \mu(ds') \le \int_S s' \rho(s_n, s')\, \mu(ds') \to 0 \quad \text{as } n \to \infty.$$

The last convergence follows by Assumption 5.3. As a result, $\mathrm{AV}_{s_n}(\pi) \to 0$. □

Lemma B.3. *Let* $g\colon S \times \Pi^\mu \to \mathbb{N}$ *be a jointly continuous function, such that* $||g||_j < \infty$ *for each* $j \in \mathbb{N}$, *the function* $s \in S \to \sup_{\pi \in \Pi^\mu} g(s, \pi)$ *is continuous at* 0, *and* $g(0, \pi) = 0$ *for all* $\pi \in \Pi^\mu$. *Then the function*

$$(s, \pi) \in S \times \Pi^{\mu,\infty} \to \mathcal{E}_s^{\nu,q}(g(\,\cdot\,, \pi)).$$

is continuous.

Proof. Let $s_n \to s$ in S and $\pi_n \xrightarrow{*} \pi$ in Π^μ as $n \to \infty$.

In the first step, suppose that $s > 0$. Let j be an integer such that $s \in S_j$ and $s_n \in S_j$ for all but finitely many $n \in \mathbb{N}$. Let us denote $\widetilde{g}_n(s') := g(s, \pi_n)$ and $\widetilde{g}(s') := g(s', \pi)$ for each $s' \in S$. We show that

$$\lim_{n\to\infty} \mathcal{E}_{s_n}^{\nu,q}(\widetilde{g}_n) = \mathcal{E}_s^{\nu,q}(\widetilde{g}).$$

Obviously, we have

$$\mathcal{E}_{s_n}^{\nu,q}(\widetilde{g}_n) - \mathcal{E}_s^{\nu,q}(\widetilde{g}) = \big(\mathcal{E}_{s_n}^{\nu,q}(\widetilde{g}_n) - \mathcal{E}_s^{\nu,q}(\widetilde{g}_n)\big) + \big(\mathcal{E}_s^{\nu,q}(\widetilde{g}_n) - \mathcal{E}_s^{\nu,q}(\widetilde{g})\big). \tag{B.4}$$

We show that both terms in the brackets on the right hand side in (B.4) converge to 0 as $n \to \infty$. In fact, we only need to show the convergence in the first bracket. The proof of the convergence in the second bracket is straightforward. We have

$$\text{(B.5)} \quad \mathcal{E}^{\nu,q}_{s_n}(\widetilde{g}_n) - \mathcal{E}^{\nu,q}_{s}(\widetilde{g}_n) = -\nu^{-1} \ln \left(1 + \frac{\int_{A(s)} e^{-\nu \widetilde{g}_n(s')} (\rho(s_n, s') - \rho(s, s'))\, \mu(ds')}{\int_{A(s)} e^{-\nu \widetilde{g_n}(s')} \rho(s, s')\, \mu(ds')} \right).$$

We show that

$$\text{(B.6)} \quad \lim_{n\to\infty} \int_{A(s)} e^{-\nu \widetilde{g_n}(s')} \rho(s_n, s')\, \mu(ds') = \int_{A(s)} e^{-\nu \widetilde{g_n}(s')} \rho(s, s')\, \mu(ds').$$

By Assumption 5.3, we have

$$\left| \int_{A(s)} e^{-\nu \widetilde{g_n}(s')} (\rho(s_n, s') - \rho(s, s'))\, \mu(ds') \right| \leq \int_{A(s)} |\rho(s_n, s') - \rho(s, s')|\, \mu(ds') \to 0$$

as $n \to \infty$. Hence we have (B.6). Consequently,

$$\text{(B.7)} \quad \lim_{n\to\infty} \int_{A(s)} e^{-\nu \widetilde{g}_n(s')} (\rho(s_n, s') - \rho(s, s'))\, \mu(ds') = 0.$$

Combining (B.5), (B.6) and (B.7), we have that the expression in (B.4) tends to 0 as $n \to \infty$, and the proof of the first step is complete.

Let us suppose $s = 0$. We only need to show that $\mathcal{E}^{\nu,q}_{s_n}(\widetilde{g}_n) \to 0$ as $n \to \infty$. For each n, we have

$$\text{(B.8)} \quad 1 > \int_S e^{-\gamma \widetilde{g}_n(s')} \rho(s_n, s')\, \mu(ds') \geq \int_S e^{-\gamma \sup_{\pi \in \Pi^\mu} g(s', \pi)} \rho(s_n, s')\, \mu(ds').$$

Using Assumption 5.3 and Lemma B.1 we have

$$\lim_{n\to\infty} \int_S e^{-\gamma \sup_{\pi \in \Pi^\mu} g(s', \pi)} \rho(s_n, s')\, \mu(ds') = 1.$$

Hence, by (B.8),

$$\lim_{n\to\infty} \int_S e^{-\gamma g(s', \pi_n)} \rho(s_n, s')\, \mu(ds') = 1.$$

Therefore, we eventually have the desired convergence. □

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