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Nicolaus Copernicus University in Toruń

POSITIVE LEAST ENERGY SOLUTIONS FOR COUPLED NONLINEAR CHOQUARD EQUATIONS WITH HARDY–LITTLEWOOD–SOBOLEV CRITICAL EXPONENT

Song You — Qingxuan Wang — Peihao Zhao

Abstract. In this paper, we study the existence and nonexistence of positive least energy solutions of the following coupled nonlinear Schrödinger equations with Choquard type nonlinearities:

$$\begin{cases} -\Delta u+\nu_1 u=\mu_1\bigg(\dfrac{1}{|x|^4}*u^2\bigg)u+\beta\bigg(\dfrac{1}{|x|^4}*v^2\bigg)u, & x\in\Omega,\\ -\Delta v+\nu_2 v=\mu_2\bigg(\dfrac{1}{|x|^4}*v^2\bigg)v+\beta\bigg(\dfrac{1}{|x|^4}*u^2\bigg)v, & x\in\Omega,\\ u,v\geq 0 \quad \text{in } \Omega, \qquad u=v=0 \quad \text{on } \partial\Omega. \end{cases}$$

Here $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain, $-\lambda_1(\Omega)<\nu_1,\nu_2<0,\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta,H_0^1(\Omega))$, $\mu_1,\mu_2>0$ and $\beta\neq 0$ is a coupling constant. We show that the critical nonlocal elliptic system has a positive least energy solution under appropriate conditions on parameters via variational methods. For the case in which $\nu_1=\nu_2$, we obtain the classification of the positive least energy solutions. Moreover, the asymptotic behaviors of the positive least energy solutions as $\beta\to 0$ are studied.

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1. Introduction

In this paper, we consider solitary wave solutions of the time-dependent coupled nonlinear Schrödinger equations with Choquard type nonlinearities in the following form (see [16], [37]):

$$
\begin{cases}
-i\,\dfrac{\partial}{\partial t}\,\Phi_1 = \Delta\Phi_1 + \mu_1\big(V(x) * |\Phi_1|^2\big)\Phi_1 + \beta\big(V(x) * |\Phi_2|^2\big)\Phi_1, & \\
 & x \in \Omega,\ t > 0, \\
-i\,\dfrac{\partial}{\partial t}\,\Phi_2 = \Delta\Phi_2 + \mu_2\big(V(x) * |\Phi_2|^2\big)\Phi_2 + \beta\big(V(x) * |\Phi_1|^2\big)\Phi_2, & \\
 & x \in \Omega,\ t > 0, \\
\Phi_j = \Phi_j(x,t) \in \mathbb{C}, & j = 1,2, \\
\Phi_j(x,t) = 0, \quad x \in \partial\Omega, & t > 0,\ j = 1,2,
\end{cases}
\tag{1.1}
$$

where $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, i is the imaginary unit, $\mu_1, \mu_2 > 0$, and $\beta \neq 0$ is a coupling constant which describes the scattering length of the attractive or repulsive interaction, $V(x)$ is the response function which possesses information on the mutual interaction between the particles. The problem (1.1) appears in many physical problem, especially in nonlinear optics. Physically, the solution Φ_j denotes the j-th component of the beam in Kerr-like photorefractive media (see [24], [25]). The positive constant μ_j indicate the self-focusing in the j-th components of the beam. The coupling constant β is the interaction between the two components of the beam. The problem (1.1) also arises in the basic quantum chemistry model of small number of electrons interacting with static nucleii which can be approximated by Hartree or Hartree–Fock minimization problems (see [17], [21], [15]).

To obtain solitary wave solutions of system (1.1), we set $\Phi_1(x,t) = e^{i\nu_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\nu_2 t}v(x)$. Then system (1.1) is reduced to the following elliptic system

$$
\begin{cases}
-\Delta u + \nu_1 u = \mu_1\big(V(x) * u^2\big)u + \beta\big(V(x) * v^2\big)u, & x \in \Omega, \\
-\Delta v + \nu_2 v = \mu_2\big(V(x) * v^2\big)v + \beta\big(V(x) * u^2\big)v, & x \in \Omega, \\
u, v \geq 0 \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega. &
\end{cases}
\tag{1.2}
$$

If the response function is a Dirac-delta function, i.e. $V(x) = \delta(x)$, then (1.2) turns to be the following semilinear elliptic system with local nonlinearities:

$$
\begin{cases}
-\Delta u + \nu_1 u = \mu_1 u^3 + \beta u v^2, & x \in \Omega, \\
-\Delta v + \nu_2 v = \mu_2 v^3 + \beta v u^2, & x \in \Omega, \\
u, v \geq 0 \quad \text{in } \Omega, \qquad u = v = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{1.3}
$$

Here, $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant. The existence and multiplicity of solutions to (1.3) have been the subject of extensive mathematical studies in recent years, for example, [2], [3], [5]–[11], [19], [20], [22], [23], [28]–[30], [35], and references therein.

In this paper we consider the system (1.2) with a response function of Riesz potential, i.e. $V(x) = |x|^{-\mu}$, then (1.2) is reduced to the following nonlocal elliptic system:

$$
\begin{cases}
-\Delta u + \nu_1 u = \mu_1\bigg(\dfrac{1}{|x|^\mu} * u^2\bigg)u + \beta\bigg(\dfrac{1}{|x|^\mu} * v^2\bigg)u, & x \in \Omega,\\
-\Delta v + \nu_2 v = \mu_2\bigg(\dfrac{1}{|x|^\mu} * v^2\bigg)v + \beta\bigg(\dfrac{1}{|x|^\mu} * u^2\bigg)v, & x \in \Omega,\\
u, v \geq 0 \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega.
\end{cases} \tag{1.4}
$$

Here, $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\mu \in (0, N) \cap (0, 4]$, $\mu_1, \mu_2 > 0$ and $\beta \neq 0$ is a coupling constant.

Recently, Wang [33] proved the existence of multiple nontrivial solutions of (1.4) with perturbations. In [34] Wang and Shi studied the existence of positive ground state solutions and various qualitative properties of ground state solutions are shown for system (1.4) with $N = 3$, $\mu = 1$. The paper [38] proved the existence and nonexistence of positive least energy solutions for system (1.4) with $N \geq 3$, $0 < \mu < 4$.

Note that the papers mentioned above deal with the subcritical case. In the present paper we investigate system (1.4) for the critical case with $N \geq 5$, $\mu = 4$, that is

$$
\begin{cases}
-\Delta u + \nu_1 u = \mu_1\bigg(\dfrac{1}{|x|^4} * u^2)u + \beta\bigg(\dfrac{1}{|x|^4} * v^2\bigg)u, & x \in \Omega,\\
-\Delta v + \nu_2 v = \mu_2\bigg(\dfrac{1}{|x|^4} * v^2\bigg)v + \beta\bigg(\dfrac{1}{|x|^4} * u^2\bigg)v, & x \in \Omega,\\
u, v \geq 0 \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{1.5}
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Recently, Chen and Zou [8] show that system (1.3) has a positive least energy solution for negative β, positive small β and positive large β when $N = 4$ in the critical case. Based on the above facts, a nature question is whether the critical nonlocal system (1.5) has a nontrivial least energy solution. The present paper is devoted to this aspect and partially answers this question.

The starting point of the variational approach to the problem (1.5) is the following classical Hardy–Littlewood–Sobolev inequality (see [18]), which leads to a new type of critical problem with nonlocal nonlinearities driven by Riesz potential.

Proposition 1.1. *Let $p, r > 1$ and $0 < \mu < N$ with $1/p + \mu/N + 1/r = 2$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(p, r, \mu, N)$, such* *that*

$$\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu}\,dx\,dy\right| \le C(p,r,\mu,N)|f|_p|h|_r, \tag{1.6}$$

where $|\cdot|_q$ *denotes the* $L^q(\mathbb{R}^N)$*-norm for* $q\in[1,\infty]$*. If* $p=r=2N/(2N-\mu)$*, then*

$$C(p,r,\mu,N)=C(N,\mu)=\pi^{\mu/2}\,\frac{\Gamma(N/2-\mu/2)}{\Gamma(N-\mu/2)}\left\{\frac{\Gamma(N/2)}{\Gamma(N)}\right\}^{-1+\mu/N}. \tag{1.7}$$

In this case there is equality in (1.6) *if and only if* $f\equiv(\mathrm{const.})h$ *and*

$$h(x)=A\big(\gamma^2+|x-a|^2\big)^{-(2N-\mu)/2}$$

for some $A\in\mathbb{C}$*,* $0\ne\gamma\in\mathbb{R}$ *and* $a\in\mathbb{R}^N$*.*

Notice that, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,dx\,dy$$

is well defined if $|u|^q\in L^t(\mathbb{R}^N)$ for some $t>1$ satisfying $2/t+\mu/N=2$. Thus, for $u\in H^1(\mathbb{R}^N)$, by Sobolev embedding theorems, we see that $2\le tq\le 2N/(N-2)$, that is

$$\frac{2N-\mu}{N}\le q\le\frac{2N-\mu}{N-2}.$$

Here, $(2N-\mu)/N$ is called the lower critical exponent and $2^*_\mu=(2N-\mu)/(N-2)$ is the upper critical exponent due to the Hardy–Littlewood–Sobolev inequality. It is easy to see that $2^*_\mu=2$ when $\mu=4$. In this sense we can call the problem (1.5) a critical nonlocal elliptic system.

Assume that $f,g\in H^1_0(\Omega)$, as [18] we define

$$D(f,g):=\int_\Omega\int_\Omega\frac{f(x)g(y)}{|x-y|^4}\,dx\,dy. \tag{1.8}$$

The following lemma is important for considering (1.5) and the proof is given in [18].

Lemma 1.2. *If* $D(|f|,|f|)<\infty$*, then* $D(f,f)\ge0$*, there is equality if and only if* $f\equiv0$*. Moreover, if* $D(|g|,|g|)<\infty$*, then*

$$|D(f,g)|^2\le D(f,f)D(g,g). \tag{1.9}$$

Suppose that $u\in H^1_0(\Omega)$, then by Proposition 1.1 we have

$$D\big(u^2,u^2\big)\le C|u^2|^2_{N/(N-2)}=C|u|^4_{2N/(N-2)}. \tag{1.10}$$

For any $\beta\in\mathbb{R}$ the system (1.5) possesses a trivial solution $(0,0)$ and a pair of semi-trivial solutions with one component being zero. These solutions have the form $(\omega_1,0)$ or $(0,\omega_2)$, where ω_i is the positive least energy solution of (see [12])

$$-\Delta u+\lambda u=\mu\bigg(\frac{1}{|x|^4}*u^2\bigg)u,\quad u\in H^1_0(\Omega), \tag{1.11}$$

with $(\lambda,\mu)=(\nu_1,\mu_1)$ for ω_1, and $(\lambda,\mu)=(\nu_2,\mu_2)$ for ω_2 respectively. The existence of solutions to (1.11) has received great interest recently, see [26], [32], [12]–[14], [1] and references therein.

We look for solutions of (1.5) which are different from the preceding ones. A solution (u,v) nontrivial if both $u\not\equiv 0$ and $v\not\equiv 0$. We call a nontrivial solution (u,v) positive if both $u>0$ and $v>0$. We say a solution (u,v) of (1.5) is a least energy solution if (u,v) is nontrivial and $E(u,v)\le E(\phi,\psi)$ for any other nontrivial solution (ϕ,ψ) of (1.5). Define $H:=H_0^1(\Omega)\times H_0^1(\Omega)$. It is well known that solutions of (1.5) correspond to the critical points of the C^1 functional $E\colon H\to\mathbb{R}$ given by

$$
\begin{aligned}
E(u,v)&=\frac{1}{2}\int_\Omega\big(|\nabla u|^2+\nu_1u^2+|\nabla v|^2+\nu_2v^2\big)\\
&\quad-\frac{1}{4}\int_\Omega\mu_1\bigg(\frac{1}{|x|^4}*u^2\bigg)u^2+2\beta\bigg(\frac{1}{|x|^4}*u^2\bigg)v^2+\mu_2\bigg(\frac{1}{|x|^4}*v^2\bigg)v^2.
\end{aligned}\tag{1.12}
$$

From (1.9) and (1.10), we infer that E is well defined in H, and so we have to assume that $N\ge 5$ in this paper.

As in [19], we define

$$
\begin{aligned}
\mathcal{N}=\bigg\{&(u,v)\in H,\ u\not\equiv 0,\ v\not\equiv 0,\\
&\int_\Omega|\nabla u|^2+\nu_1u^2=\int_\Omega\mu_1\bigg(\frac{1}{|x|^4}*u^2\bigg)u^2+\beta\bigg(\frac{1}{|x|^4}*v^2\bigg)u^2,\\
&\int_\Omega|\nabla v|^2+\nu_2v^2=\int_\Omega\mu_2\bigg(\frac{1}{|x|^4}*v^2\bigg)v^2+\beta\bigg(\frac{1}{|x|^4}*v^2\bigg)u^2\bigg\}.
\end{aligned}
$$

Then any nontrivial solution of (1.5) belongs to $\mathcal{N}$. We set

$$
A:=\inf_{(u,v)\in\mathcal{N}}E(u,v)=\inf_{(u,v)\in\mathcal{N}}\frac{1}{4}\int_\Omega|\nabla u|^2+\nu_1u^2+|\nabla v|^2+\nu_2v^2.\tag{1.13}
$$

Now, we list our main results. First we consider the special case $-\lambda_1(\Omega)<\nu_1=\nu_2=\nu<0$, where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition, with corresponding eigenfunction $\phi>0$. Let ω be any a positive least energy solution of (1.11) with $(\lambda,\mu)=(\nu,1)$. Then we have the following two theorems.

THEOREM 1.3. *Assume that $-\lambda_1(\Omega)<\nu_1=\nu_2=\nu<0$ and $N\ge 5$.*

(a) *If $0<\beta<\min\{\mu_1,\mu_2\}$ or $\beta>\max\{\mu_1,\mu_2\}$, then A is attained by $\big(\sqrt{\pi_1}\omega,\sqrt{\pi_2}\omega\big)$, where $\pi_1,\pi_2>0$ satisfying*

$$
\begin{cases}\mu_1\pi_1+\beta\pi_2=1,\\ \beta\pi_1+\mu_2\pi_2=1.\end{cases}\tag{1.14}
$$

Therefore, $\big(\sqrt{\pi_1}\omega,\sqrt{\pi_2}\omega\big)$ is a positive least energy solution of (1.5).

(b) *If* $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ *and* $\mu_1 \neq \mu_2$, *then* (1.5) *does not have any nontrivial nonnegative solution.*

Moreover, we obtain the classification of the positive least energy solutions.

Theorem 1.4. *Assume that assumptions in Theorem* 1.3 *hold, and let* $0 < \beta < \min\{\mu_1, \mu_2\}$ *or* $\beta > \max\{\mu_1, \mu_2\}$. *Suppose that* (u, v) *is any a nontrivial least energy solution of* (1.5), *then* $(u, v) = (\sqrt{\pi_1}\omega, \sqrt{\pi_2}\omega)$, *where* (π_1, π_2) *satisfies* (1.14).

We mention that by Lemma 1.2 and (3.5), the proofs of Theorema 1.3 and 1.4 are similar to that of Theorems 1.3 and 1.4 in [38] respectively, and so we omit it.

Now, let us consider the general case $-\lambda_1(\Omega) < \nu_1, \nu_2 < 0$. Without loss of generality, we may assume that $\nu_2 \leq \nu_1$. Our second result is more general, where we also deal with the case in which $\beta < 0$.

Theorem 1.5. *Assume that* $-\lambda_1(\Omega) < \nu_2 \leq \nu_1 < 0$ *and* $N \geq 5$.

(a) *There exists* $\overline{\beta} > 0$, *such that* (1.5) *has a positive least energy solution* (u, v) *with* $E(u, v) = A$ *for any* $\beta \in (-\overline{\beta}, 0)$.
(b) *There exists* $\beta_0 \in (0, \min\{\mu_1, \mu_2\})$, *such that* (1.5) *has a positive least energy solution* (u, v) *with* $E(u, v) = A$ *for any* $\beta \in (0, \beta_0)$.
(c) *There exists* $\beta_1 \in (\max\{\mu_1, \mu_2\}, +\infty)$, *such that* (1.5) *has a positive least energy solution* (u, v) *with* $E(u, v) = A$ *for any* $\beta \in (\beta_1, +\infty)$.
(d) *If* $\mu_1 \leq \beta \leq \mu_2$ *and* $\mu_1 < \mu_2$, *then* (1.5) *does not have any nontrivial nonnegative solution.*

In fact, we can give an accurate definition of $\overline{\beta}$, β_0 in Lemma 3.1 and β_1 in Lemma 3.4, but do not give it here in order to avoid introducing heavy notations at this stage.

We should point out that the loss of compactness due to the Hardy–Littlewood–Sobolev upper critical exponent makes it difficult to obtain the existence of nontrivial solutions. In order to obtain the existence of nontrivial least energy solutions, we need to estimate the least energy and give an accurate upper bound of the least energy; see Lemmas 3.1 and 3.4. The idea of the proof mainly follows from [8]. However, because of the nonlocal nature of the critical Choquard equations where the convolution type nonlinearities are totally determined by the behavior on the domain Ω, the method in [8] cannot be used directly, and some new techniques are needed for our proof (see Section 3).

Remark 1.6. Here, we only obtain existence of least energy solution for $\beta \in (-\overline{\beta}, 0)$ due to the nonlocal nature of the critical Choquard equations. We do not know whether system (1.5) has a least energy solution for $\beta \in (-\infty, -\overline{\beta}]$.

Remark 1.7. If $\Omega = \mathbb{R}^N$ and (u, v) is any a solution of (1.5), then by the Pohožaev Identity (see Appendix) and $E'(u, v)(u, v) = 0$, it is easy to get that

$\int_{\mathbb{R}^N} \nu_1 u^2 + \nu_2 v^2 = 0$, where E is a related functional defined in (1.12). This implies that $(u, v) \equiv (0, 0)$ if $\nu_1\nu_2 > 0$. Therefore, in the sequel we assume that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

Remark 1.8. If Ω is star-shaped with respect to some x_1 and $\nu_1, \nu_2 \geq 0$, then using the Pohožaev Identity (see Appendix) and $E'(u, v)(u, v) = 0$, it is easy to see that

$$0 \leq \int_{\partial\Omega} \big(|\nabla u|^2 + |\nabla v|^2\big)((x - x_1) \cdot \mathbf{n}) = -2 \int_{\Omega} \nu_1 u^2 + \nu_2 v^2 \leq 0,$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$. This yields that $(u, v) \equiv (0, 0)$. This is one reason that we need the assumption $\nu_1, \nu_2 < 0$. Moreover, this assumption is also required in the proof of Lemma 3.1 in section 3. On the other hand, assume that $\beta > 0$. We multiply the equation for u in (1.5) by the first eigenfunction ϕ and integrate over Ω, which yields

$$(\nu_1 + \lambda_1(\Omega)) \int_{\Omega} u\phi = \int_{\Omega} \left(\mu_1\left(\frac{1}{|x|^4} * u^2\right)u\phi + \beta\left(\frac{1}{|x|^4} * v^2\right)u\phi\right) > 0.$$

Thus, we have to assume that $\nu_1, \nu_2 > -\lambda_1(\Omega)$ if we want to obtain nontrivial nonnegative solutions of (1.5).

Finally, we study the asymptotic behavior of the positive least energy solutions in the case $\beta \to 0$. Then we have the following result.

Theorem 1.9. *Assume that $-\lambda_1(\Omega) < \nu_2 \leq \nu_1 < 0$ and $N \geq 5$. Let β_n, $n \in \mathbb{N}$, be a sequence with $0 < |\beta_n| < \min\{\overline{\beta}, \beta_0\}$, $\beta_n \to 0$ as $n \to +\infty$, and (u_n, v_n) be the positive least energy solutions of* (1.5) *with $\beta = \beta_n$ which exists by Theorem* 1.5. *Then, passing to a subsequence, $(u_n, v_n) \to (\widehat{u}, \widehat{v})$ strongly in $H_0^1(\Omega) \times H_0^1(\Omega)$ as $n \to +\infty$, where $\widehat{u}$ is a positive least energy solution of*

$$-\Delta u + \nu_1 u = \mu_1\left(\frac{1}{|x|^4} * u^2\right)u, \quad u \in H_0^1(\Omega),$$

and $\widehat{v}$ is a positive least energy solution of

$$-\Delta v + \nu_2 v = \mu_2\left(\frac{1}{|x|^4} * v^2\right)v, \quad v \in H_0^1(\Omega).$$

The paper is organized as follows. In Section 2, we consider the limit problem. In Section 3, we give the proof of Theorem 1.5. In Section 4, we investigate the the asymptotic behavior of the positive least energy solutions. In Section 5, we establish the Pohožaev type identity.

We give some notations here. Throughout this paper, we denote the norm of L^q by

$$|u|_q = \left(\int_{\Omega} |u|^q \, dx\right)^{1/q},$$

the norm of $H_0^1(\Omega)$ by

$$\|u\| = |\nabla u|_2,$$

the norm of H by

$$\|(u,v)\|^2 = \|u\|^2 + \|v\|^2$$

and positive constants (possibly different in different places) by C, C_1, C_2.

2. The limit problem

First of all, by [12] we know

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq S_{H,L}\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^4}\,dx\,dy\bigg)^{1/2}, \tag{2.1}$$

for all $u \in D^{1,2}(\mathbb{R}^N)$, where $S_{H,L}$ denotes the best constant defined by

$$S_{H,L} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{\bigg(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^4}\,dx\,dy\bigg)^{1/2}}, \tag{2.2}$$

and $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : |\nabla u| \in L^2(\mathbb{R}^N)\}$ with norm

$$\|u\|_{D^{1,2}} := \bigg(\int_{\mathbb{R}^N} |\nabla u|^2\,dx\bigg)^{1/2}.$$

Proposition 2.1 (see [12]). *The constant $S_{H,L}$ defined in* (2.2) *is achieved if and only if*

$$u(x) = C\bigg(\frac{b}{b^2+|x-a|^2}\bigg)^{(N-2)/2},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0,\infty)$ are parameters. What's more,

$$S_{H,L} = \frac{S}{C(N,4)^{1/2}},$$

where $C(\,\cdot\,,\,\cdot\,)$ is defined in (1.7) *and S is the best Sobolev constant.*

Let

$$U(x) := \frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}}$$

be a minimizer for S (see [36]), then

$$\widetilde{U}(x) = S^{(4-N)/4}C(N,4)^{-1/2}\frac{[N(N-2)]^{(N-2)/4}}{(1+|x|^2)^{(N-2)/2}} \tag{2.3}$$

is the unique minimizer for $S_{H,L}$ and satisfies

$$-\Delta u = \bigg(\frac{1}{|x|^4} * u^2\bigg)u, \quad \text{in } \mathbb{R}^N,$$

with

$$\int_{\mathbb{R}^N} \big|\nabla \widetilde{U}\big|^2\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big|\widetilde{U}(x)\big|^2\big|\widetilde{U}(y)\big|^2}{|x-y|^4}\,dx\,dy = S_{H,L}^2. \tag{2.4}$$

Moreover, for every open subset Ω of $\mathbb{R}^N$,

$$S_{H,L}(\Omega) := \inf_{u\in D_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\bigg(\int_\Omega\int_\Omega \frac{|u(x)|^2|u(y)|^2}{|x-y|^4}\,dx\,dy\bigg)^{1/2}},$$

and $S_{H,L}(\Omega)$ is never achieved except $\Omega=\mathbb{R}^N$ (see [12]).

Since the nonlinearity and the coupling terms are both critical in (1.5), the existence of nontrivial least energy solutions to (1.5) depends heavily on the least energy of the following limit system

$$\begin{cases} -\Delta u = \mu_1\bigg(\dfrac{1}{|x|^4} * u^2\bigg)u + \beta\bigg(\dfrac{1}{|x|^4} * v^2\bigg)u, & x\in\mathbb{R}^N,\\ -\Delta v = \mu_2\bigg(\dfrac{1}{|x|^4} * v^2\bigg)v + \beta\bigg(\dfrac{1}{|x|^4} * u^2\bigg)v, & x\in\mathbb{R}^N,\\ u,v\in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{2.5}$$

Note that (2.5) has semi-trivial solutions $\big(\mu_1^{-1/2}\widetilde{U},0\big)$ and $\big(0,\mu_2^{-1/2}\widetilde{U}\big)$. Here, we are only interested in nontrivial solutions of (2.5).

Define $D := D^{1,2}(\mathbb{R}^N)\times D^{1,2}(\mathbb{R}^N)$ and a C^1 functional $I\colon D\to\mathbb{R}$ given by

$$\begin{aligned} I(u,v) = {}& \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \\ & - \frac{1}{4}\int_{\mathbb{R}^N} \mu_1\bigg(\frac{1}{|x|^4} * u^2\bigg)u^2 + 2\beta\bigg(\frac{1}{|x|^4} * u^2\bigg)v^2 + \mu_2\bigg(\frac{1}{|x|^4} * v^2\bigg)v^2. \end{aligned}$$

As in [19], we consider the set

$$\begin{aligned} \mathcal{M} = \bigg\{&(u,v)\in D,\ u\neq 0,\ v\neq 0,\\ &\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} \mu_1\bigg(\frac{1}{|x|^4} * u^2\bigg)u^2 + \beta\bigg(\frac{1}{|x|^4} * u^2\bigg)v^2,\\ &\int_{\mathbb{R}^N} |\nabla v|^2 = \int_{\mathbb{R}^N} \mu_2\bigg(\frac{1}{|x|^4} * v^2\bigg)v^2 + \beta\bigg(\frac{1}{|x|^4} * u^2\bigg)v^2\bigg\}. \end{aligned} \tag{2.6}$$

Then any nontrivial solution of (2.5) belongs to $\mathcal{M}$. We set

$$A_0 := \inf_{(u,v)\in\mathcal{M}} I(u,v) = \inf_{(u,v)\in\mathcal{M}} \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2. \tag{2.7}$$

For any $(u,v)\in\mathcal{M}$, by (2.6) and Lemma 1.2, it is standard to see that

$$A_0 = \inf_{(u,v)\in\mathcal{M}} \frac{1}{4}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 \geq C > 0. \tag{2.8}$$

Then we have the following theorem, which plays an important role in the proof of Theorem 1.5.

Theorem 2.2.

(a) *If* $\beta < 0$, *then* A_0 *is not attained and* $A_0 = \big(\mu_1^{-1} + \mu_2^{-1}\big)S_{H,L}^2/4$.

(b) *If* $0 < \beta < \min\{\mu_1, \mu_2\}$ *or* $\beta > \max\{\mu_1, \mu_2\}$, *then* A_0 *is attained by* $\big(\sqrt{\pi_1}\,\widetilde{U}, \sqrt{\pi_2}\,\widetilde{U}\big)$ *and* $A_0 = (\pi_1 + \pi_2)S_{H,L}^2/4$, *where* $\pi_1, \pi_2 > 0$ *is defined in* (1.14). *Therefore,* $\big(\sqrt{\pi_1}\,\widetilde{U}, \sqrt{\pi_2}\,\widetilde{U}\big)$ *is a positive least energy solution of* (2.5).

(c) *If* $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$ *and* $\mu_1 \neq \mu_2$, *then* (2.5) *does not have a nontrivial nonnegative solution.*

Before proceeding, repeating the proof of Lemma 2.3 in [34], we have

Proposition 2.3. *If* A_0 (*resp.* A) *is attained by a couple* $(u, v) \in \mathcal{M}$ (*resp.* $(u, v) \in \mathcal{N}$), *then this couple is a critical point of* I (*resp.* E), *provided* $-\infty < \beta < \sqrt{\mu_1\mu_2}$.

Now, we turn to prove Theorem 2.2. Similarly to the proof of Theorem 1.3, we see that (c) in Theorem 2.2 holds. It remains to prove (a) and (b) of Theorem 2.2.

Proof of (a) in Theorem 2.2. By (2.3) we see that $\omega_{\mu_i} := \mu_i^{-1/2}\widetilde{U}$ satisfies equation

$$-\Delta u = \mu_i\bigg(\frac{1}{|x|^4} * u^2\bigg)u \quad \text{in } \mathbb{R}^N.$$

Let $e_1 = (1, \ldots, 0) \in \mathbb{R}^N$ and

$$(u_1(x), v_\varepsilon(x)) = (\omega_{\mu_1}(x), \omega_{\mu_2}(x + \varepsilon e_1)).$$

Then $v_\varepsilon \rightharpoonup 0$ weakly in $D^{1,2}(\mathbb{R}^N)$ and so $v_\varepsilon^2 \rightharpoonup 0$ weakly in $L^{N/(N-2)}(\mathbb{R}^N)$ as $\varepsilon \to +\infty$. Combining these with

$$\bigg(\frac{1}{|x|^4} * u_1^2\bigg) \in L^{N/2}(\mathbb{R}^N),$$

we know

$$\lim_{\varepsilon \to +\infty} \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_1^2\bigg)v_\varepsilon^2 = 0.$$

Therefore, for $\varepsilon > 0$ sufficiently large, the equations

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} |\nabla u_1|^2 = \mu_1 \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_1^2\bigg)u_1^2 \\ \displaystyle\qquad\qquad = t_{1,\varepsilon}\mu_1 \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_1^2\bigg)u_1^2 + t_{2,\varepsilon}\beta \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_1^2\bigg)v_\varepsilon^2, \\ \displaystyle\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 = \mu_1 \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * v_\varepsilon^2\bigg)v_\varepsilon^2 \\ \displaystyle\qquad\qquad = t_{2,\varepsilon}\mu_2 \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * v_\varepsilon^2\bigg)v_\varepsilon^2 + t_{1,\varepsilon}\beta \int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_1^2\bigg)v_\varepsilon^2, \end{cases}$$

have a solution $(t_{1,\varepsilon}, t_{2,\varepsilon})$ with

$$\lim_{\varepsilon\to+\infty}(|t_{1,\varepsilon}-1|+|t_{2,\varepsilon}-1|)=0.$$

Noting that $\big(\sqrt{t_{1,\varepsilon}}u_1, \sqrt{t_{2,\varepsilon}}v_\varepsilon\big) \in \mathcal{M}$ for ε sufficiently large, by (2.4), we have

$$\begin{aligned} A_0 \le I\big(\sqrt{t_{1,\varepsilon}}u_1, \sqrt{t_{2,\varepsilon}}v_\varepsilon\big) &= \frac{1}{4}\bigg(t_{1,\varepsilon}\int_{\mathbb{R}^N}|\nabla u_1|^2 + t_{2,\varepsilon}\int_{\mathbb{R}^N}|\nabla v_\varepsilon|^2\bigg) \\ &= \frac{1}{4}\big(t_{1,\varepsilon}\mu_1^{-1}+t_{2,\varepsilon}\mu_2^{-1}\big)S_{H,L}^2. \end{aligned}$$

Letting $\varepsilon\to+\infty$, we know that $A_0 \le (\mu_1^{-1}+\mu_2^{-1})S_{H,L}^2/4$.

On the other hand, for any $(u,v)\in\mathcal{M}$, we see from (2.1) and $\beta<0$ that

$$\int_{\mathbb{R}^N}|\nabla u|^2 \le \mu_1\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^4}*u^2\bigg)u^2 \le \mu_1 S_{H,L}^{-2}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\bigg)^2,$$

and so

$$\int_{\mathbb{R}^N}|\nabla u|^2 \ge \mu_1^{-1}S_{H,L}^2.$$

Similarly,

$$\int_{\mathbb{R}^N}|\nabla v|^2 \ge \mu_2^{-1}S_{H,L}^2.$$

Combining these with (2.7), we get that $A_0 \ge (\mu_1^{-1}+\mu_2^{-1})S_{H,L}^2/4$. Therefore,

$$A_0 = \frac{1}{4}\big(\mu_1^{-1}+\mu_2^{-1}\big)S_{H,L}^2. \tag{2.9}$$

Now, if we assume that A_0 is attained by some $(u,v)\in\mathcal{M}$, then $(|u|,|v|)\in\mathcal{M}$ and $I(|u|,|v|)=A_0$. By Proposition 2.3, we have $(|u|,|v|)$ is a nontrivial solution of (2.5). By the maximum principle, $u>0$, $v>0$ and so

$$\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^4}*u^2\bigg)v^2>0.$$

Then

$$\int_{\mathbb{R}^N}|\nabla u|^2 < \mu_1\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^4}*u^2\bigg)u^2 \le \mu_1 S_{H,L}^{-2}\bigg(\int_{\mathbb{R}^N}|\nabla u|^2\bigg)^2.$$

Therefore, it is easy to see that

$$A_0 = I(u,v) = \frac{1}{4}\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2 > \frac{1}{4}\big(\mu_1^{-1}+\mu_2^{-1}\big)S_{H,L}^2,$$

which is a contradiction. □

PROOF OF (b) IN THEOREM 2.2. Since $0<\beta<\min\{\mu_1,\mu_2\}$ or $\beta>\max\{\mu_1,\mu_2\}$, then equation (1.14) has a solution (π_1,π_2) satisfying $\pi_1>0$, $\pi_2>0$. Apparently, we see that $\big(\sqrt{\pi_1}\widetilde{U}, \sqrt{\pi_2}\widetilde{U}\big)$ is a nontrivial solution of (2.5) and

$$A_0 \le I\big(\sqrt{\pi_1}\widetilde{U}, \sqrt{\pi_2}\widetilde{U}\big) = \frac{1}{4}(\pi_1+\pi_2)S_{H,L}^2.$$

Let $(u_n, v_n) \subset \mathcal{M}$ be a minimizing sequence for A_0, that is, $I(u_n, v_n) \to A_0$. We set

$$a_n = \bigg(\int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * u_n^2\bigg) u_n^2\bigg)^{1/2}, \qquad b_n = \bigg(\int_{\mathbb{R}^N} \bigg(\frac{1}{|x|^4} * v_n^2\bigg) v_n^2\bigg)^{1/2}.$$

Then, by Lemma 1.2 and (2.1) we have

$$\begin{aligned} S_{H,L} a_n &\le \int_{\mathbb{R}^N} |\nabla u_n|^2 \\ &= \int_{\mathbb{R}^N} \mu_1 \bigg(\frac{1}{|x|^4} * u_n^2\bigg) u_n^2 + \beta \bigg(\frac{1}{|x|^4} * u_n^2\bigg) v_n^2 \le \mu_1 a_n^2 + \beta a_n b_n, \\ S_{H,L} b_n &\le \int_{\mathbb{R}^N} |\nabla v_n|^2 \\ &= \int_{\mathbb{R}^N} \mu_2 \bigg(\frac{1}{|x|^4} * v_n^2\bigg) v_n^2 + \beta \bigg(\frac{1}{|x|^4} * v_n^2\bigg) u_n^2 \le \mu_2 b_n^2 + \beta a_n b_n. \end{aligned}$$

Since

$$I(u_n, v_n) = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\nabla v_n|^2,$$

we have

$$\begin{gathered} S_{H,L}(a_n + b_n) \le 4I(u_n, v_n) = 4A_0 + o(1) \le (\pi_1 + \pi_2) S_{H,L}^2 + o(1), \\ \mu_1 a_n + \beta b_n \ge S_{H,L}, \qquad \mu_2 b_n + \beta a_n \ge S_{H,L}. \end{gathered}$$

By (1.14) the above three inequalities are equivalent to

$$\begin{aligned} (a_n - \pi_1 S_{H,L}) + (b_n - \pi_2 S_{H,L}) &\le o(1), \\ \mu_1 (a_n - \pi_1 S_{H,L}) + \beta (b_n - \pi_2 S_{H,L}) &\ge 0, \\ \beta (a_n - \pi_1 S_{H,L}) + \mu_2 (b_n - \pi_2 S_{H,L}) &\ge 0. \end{aligned}$$

So $a_n \to \pi_1 S_{H,L}$ and $b_n \to \pi_2 S_{H,L}$ as $n \to \infty$. Then

$$4A_0 = \lim_{n\to\infty} 4I(u_n, v_n) \ge \lim_{n\to\infty} S_{H,L}(a_n + b_n) = (\pi_1 + \pi_2) S_{H,L}^2.$$

This yields

$$A_0 = \frac{1}{4}(\pi_1 + \pi_2) S_{H,L}^2 = I\big(\sqrt{\pi_1}\widetilde{U}, \sqrt{\pi_2}\widetilde{U}\big), \tag{2.10}$$

and so $\big(\sqrt{\pi_1}\widetilde{U}, \sqrt{\pi_2}\widetilde{U}\big)$ is a positive least energy solution of (2.5). □

3. Proof of Theorem 1.5

In this section, we assume that $-\lambda_1(\Omega) < \nu_2 \le \nu_1 < 0$. Multiply the equation for u in (1.5) by v, the equation for v by u, and integrate over Ω, which yields

$$\int_\Omega uv \bigg[(\nu_1 - \nu_2) + (\beta - \mu_1)\bigg(\frac{1}{|x|^4} * u^2\bigg) + (\mu_2 - \beta)\bigg(\frac{1}{|x|^4} * v^2\bigg)\bigg] = 0.$$

This implies that (d) of Theorem 1.5 holds. Note that

$$\int_\Omega (|\nabla u|^2 + \nu_i u^2) \geq \left(\frac{\lambda_1(\Omega) + \nu_i}{\lambda_1(\Omega)}\right) \int_\Omega |\nabla u|^2, \quad i = 1, 2. \tag{3.1}$$

It is easy to see that $A > 0$ just as (2.8). As has been pointed out in Section 1, by [12] the problem

$$-\Delta u + \nu_i u = \mu_i \left(\frac{1}{|x|^4} * u^2\right) u, \quad u \geq 0, \ u \in H_0^1(\Omega), \tag{3.2}$$

has a positive least energy solution $\omega_i \in C^2(\Omega) \cap C(\overline{\Omega})$ (see Appendix) with energy

$$\frac{1}{4}\left(\frac{\lambda_1(\Omega) + \nu_i}{\lambda_1(\Omega)}\right)^2 \mu_i^{-1} S_{H,L}^2 \leq m_i < \frac{1}{4}\,\mu_i^{-1} S_{H,L}^2, \quad i = 1, 2, \tag{3.3}$$

where

$$m_i := \frac{1}{2} \int_\Omega (|\nabla \omega_i|^2 + \nu_i \omega_i^2) - \frac{1}{4} \int_\Omega \mu_i \left(\frac{1}{|x|^4} * \omega_i^2\right) \omega_i^2, \quad i = 1, 2. \tag{3.4}$$

Moreover,

$$\int_\Omega (|\nabla u|^2 + \nu_i u^2) \geq 2\sqrt{m_i} \left(\int_\Omega \mu_i \left(\frac{1}{|x|^4} * u^2\right) u^2\right)^{1/2}, \quad \text{for all } u \in H_0^1(\Omega). \tag{3.5}$$

The following lemma is very important. In the proof, we need the assumption $\nu_1, \nu_2 < 0$, and define

$$\beta_0 := \min\left\{\mu_2, \sqrt{\frac{\mu_1\mu_2 m_1}{m_2}}, \sqrt{\frac{\mu_1\mu_2 m_2}{m_1}}, \mu_1 \frac{\nu_2 + \lambda_1(\Omega)}{\nu_1 + \lambda_1(\Omega)}, \frac{(\nu_1 + \lambda_1(\Omega))(\nu_2 + \lambda_1(\Omega))}{\lambda_1^2(\Omega)(\mu_1^{-1} + \mu_2^{-1})}\right\}, \tag{3.6}$$

$$\overline{\beta} := \min\left\{\frac{|\nu_2|}{2\widetilde{C}\left(2 + \sqrt{4 + S_{H,L}^2/(\mu_2 m_1)}\right)|\omega_1|_{L^\infty}^2}, \frac{|\nu_1|}{2\widetilde{C}\left(2 + \sqrt{4 + S_{H,L}^2/(\mu_1 m_2)}\right)|\omega_2|_{L^\infty}^2}\right\}, \tag{3.7}$$

where $\widetilde{C}$ is defined in (3.25).

Lemma 3.1. *Assume that $-\lambda_1(\Omega) < \nu_2 \leq \nu_1 < 0$ and $N \geq 5$.*

(a) *If $\beta \in (0, \beta_0)$, then $\beta_0 \leq \min\{\mu_1, \mu_2\}$ and $A < \min\{m_1 + m_2, A_0\}$.*

(b) *If $\beta \in (-\overline{\beta}, 0)$, then*

$$A < \min\left\{m_1 + \frac{1}{4}\,\mu_2^{-1} S_{H,L}^2, m_2 + \frac{1}{4}\,\mu_1^{-1} S_{H,L}^2, A_0\right\}.$$

Proof. First, we prove (a) of Lemma 3.1. Define

$$G(u,v) := \begin{pmatrix} \mu_1 \int_\Omega \left(\frac{1}{|x|^4} * u^2\right) u^2 & \beta \int_\Omega \left(\frac{1}{|x|^4} * u^2\right) v^2 \\ \beta \int_\Omega \left(\frac{1}{|x|^4} * u^2\right) v^2 & \mu_2 \int_\Omega \left(\frac{1}{|x|^4} * v^2\right) v^2 \end{pmatrix}, \tag{3.8}$$

when $|G(u,v)| := \det G(u,v) > 0$, the inverse matrix of $G(u,v)$ is

$$G^{-1}(u,v) := \frac{1}{\det G(u,v)} \begin{pmatrix} \mu_2 D(v^2,v^2)) & -\beta D(u^2,v^2) \\ -\beta D(u^2,v^2) & \mu_1 D(u^2,u^2) \end{pmatrix}, \tag{3.9}$$

where $D(\,\cdot\,,\,\cdot\,)$ is defined in (1.8). Since $\nu_2 \le \nu_1$, then $\beta_0 \le \min\{\mu_1,\mu_2\}$. It follows from (1.9) that $|G(\omega_1,\omega_2)| > 0$. Note that $(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) \in \mathcal{N}$ for some $t_1, t_2 > 0$ is equivalent to

$$\begin{aligned} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} &:= G^{-1}(\omega_1,\omega_2) \begin{pmatrix} \mu_1 D(\omega_1^2,\omega_1^2) \\ \mu_2 D(\omega_2^2,\omega_2^2) \end{pmatrix} \\ &= \frac{1}{|G(\omega_1,\omega_2)|} \begin{pmatrix} \mu_2 D(\omega_2^2,\omega_2^2)(\mu_1 D(\omega_1^2,\omega_1^2) - \beta D(\omega_1^2,\omega_2^2)) \\ \mu_1 D(\omega_1^2,\omega_1^2)(\mu_2 D(\omega_2^2,\omega_2^2) - \beta D(\omega_1^2,\omega_2^2)) \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Meanwhile, we deduce from (1.9) and $0 < \beta < \beta_0 \le \sqrt{\mu_1\mu_2}$ that

$$\beta D(\omega_1^2,\omega_2^2) < \sqrt{\mu_1\mu_2 \frac{m_1}{m_2}}\, D^{1/2}(\omega_1^2,\omega_1^2) D^{1/2}(\omega_2^2,\omega_2^2) = \mu_1 D(\omega_1^2,\omega_1^2).$$

Similarly, when $\beta \in (0,\beta_0)$ we have $\beta D(\omega_1^2,\omega_2^2) < \mu_2 D(\omega_2^2,\omega_2^2)$. Then (3.10) holds and $(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) \in \mathcal{N}$. Therefore,

$$\begin{aligned} A \le E(\sqrt{t_1}\omega_1, \sqrt{t_2}\omega_2) &= \frac{t_1}{4} \int_\Omega |\nabla \omega_1|^2 + \nu_1 \omega_1^2 + \frac{t_2}{4} \int_\Omega |\nabla \omega_2|^2 + \nu_2 \omega_2^2 \\ &= \frac{t_1}{4} \int_\Omega \mu_1 \left(\frac{1}{|x|^4} * \omega_1^2\right) \omega_1^2 + \frac{t_2}{4} \int_\Omega \mu_2 \left(\frac{1}{|x|^4} * \omega_2^2\right) \omega_2^2 \\ &< \frac{t_1}{4} \int_\Omega \mu_1 \left(\frac{1}{|x|^4} * \omega_1^2\right) \omega_1^2 + \beta \left(\frac{1}{|x|^4} * \omega_1^2\right) \omega_2^2 \\ &\quad + \frac{t_2}{4} \int_\Omega \mu_2 \left(\frac{1}{|x|^4} * \omega_2^2\right) \omega_2^2 + \beta \left(\frac{1}{|x|^4} * \omega_1^2\right) \omega_2^2 \\ &= \frac{1}{4} \int_\Omega |\nabla \omega_1|^2 + \nu_1 \omega_1^2 + \frac{1}{4} \int_\Omega |\nabla \omega_2|^2 + \nu_2 \omega_2^2 = m_1 + m_2. \end{aligned}$$

Hence, $A < m_1 + m_2$. It remains to prove $A < A_0$. Without loss of generality, we may assume that $0 \in \Omega$. Then there exists $\delta > 0$ such that $\{x : |x| \le \delta\} \subset \Omega$. Let $\xi \in C_0^1(\Omega)$ be a nonnegative function with $\xi \equiv 1$ for $|x| \le \delta$. Recall that $N \ge 5$ and $\widetilde{U}$ in Proposition 2.1. For $\varepsilon > 0$, we define

$$\widetilde{U}_\varepsilon(x) := \varepsilon^{(2-N)/2}\, \widetilde{U}\left(\frac{x}{\varepsilon}\right).$$

It is easy to see that

$$\int_{\mathbb{R}^N} |\nabla \widetilde{U}_\varepsilon|^2 = S_{H,L}^2, \qquad \int_{\mathbb{R}^N} \left(\frac{1}{|x|^4} * \widetilde{U}_\varepsilon^2\right)\widetilde{U}_\varepsilon^2 = S_{H,L}^2.$$

Define $\overline{U}_\varepsilon := \xi \widetilde{U}_\varepsilon$. First we claim that there holds

$$\int_\Omega |\nabla \overline{U}_\varepsilon| = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big), \tag{3.11}$$

$$\int_\Omega |\overline{U}_\varepsilon|^2 \ge C\varepsilon^2 + O\big(\varepsilon^{N-2}\big), \tag{3.12}$$

$$\int_\Omega \left(\frac{1}{|x|^4} * \overline{U}_\varepsilon^2\right)\overline{U}_\varepsilon^2 = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big), \tag{3.13}$$

where C is a positive constant.

By [36], it is standard to see that (3.11) and (3.12) hold. It remains to prove (3.13). Let $0 < \varepsilon \ll \delta$. On one hand, by (2.1) we have

$$S_{H,L}\bigg(\int_\Omega \left(\frac{1}{|x|^4} * \overline{U}_\varepsilon^2\right)\overline{U}_\varepsilon^2\bigg)^{1/2} \le \int_\Omega |\nabla \overline{U}_\varepsilon|^2 = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big),$$

that is

$$\int_\Omega \left(\frac{1}{|x|^4} * \overline{U}_\varepsilon^2\right)\overline{U}_\varepsilon^2 \le S_{H,L}^2 + O\big(\varepsilon^{N-2}\big). \tag{3.14}$$

On the other hand,

$$\begin{aligned}\int_\Omega \left(\frac{1}{|x|^4} * \overline{U}_\varepsilon^2\right)\overline{U}_\varepsilon^2 &\ge \int_{B_\delta}\int_{B_\delta} \frac{\overline{U}_\varepsilon^2(x)\overline{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\widetilde{U}_\varepsilon^2(x)\,\widetilde{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy - 2\int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{\widetilde{U}_\varepsilon^2(x)\,\widetilde{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy \\ &\qquad - \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{\widetilde{U}_\varepsilon^2(x)\,\widetilde{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy := S_{H,L}^2 - 2I_1 - I_2,\end{aligned} \tag{3.15}$$

where

$$I_1 = \int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{\widetilde{U}_\varepsilon^2(x)\,\widetilde{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy, \quad I_2 = \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{\widetilde{U}_\varepsilon^2(x)\,\widetilde{U}_\varepsilon^2(y)}{|x-y|^4}\,dx\,dy.$$

We are going to estimate I_1 and I_2. By a direct computation, we have

$$\begin{aligned} I_1 &= \varepsilon^{4-2N} \int_{\mathbb{R}^N\setminus B_\delta}\int_{B_\delta} \frac{S^{4-N}C^{-2}(N,4)[N(N-2)]^{N-2}}{(1+|x/\varepsilon|^2)^{N-2}|x-y|^4(1+|y/\varepsilon|^2)^{N-2}}\,dx\,dy \\ &\le O\big(\varepsilon^{2N-4}\big)\bigg(\int_{\mathbb{R}^N\setminus B_\delta} \frac{1}{(\varepsilon^2+|x|^2)^N}\,dx\bigg)^{(N-2)/N} \\ &\quad \cdot \bigg(\int_{B_\delta} \frac{1}{(\varepsilon^2+|y|^2)^N}\,dy\bigg)^{(N-2)/N} \\ &\le O\big(\varepsilon^{N-2}\big)\bigg(\int_0^{+\infty} \frac{r^{N-1}}{(1+r^2)^N}\,dr\bigg)^{(N-2)/N} = O\big(\varepsilon^{N-2}\big),\end{aligned} \tag{3.16}$$

and

$$(3.17)\quad I_2 = \varepsilon^{4-2N} \int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta} \frac{S^{4-N}C^{-2}(N,4)[N(N-2)]^{N-2}}{(1+|x/\varepsilon|^2)^{N-2}|x-y|^4(1+|y/\varepsilon|^2)^{N-2}}\,dx\,dy$$
$$\leq O\big(\varepsilon^{2N-4}\big)\int_{\mathbb{R}^N\setminus B_\delta}\int_{\mathbb{R}^N\setminus B_\delta}\frac{1}{|x|^{2N-4}|x-y|^4|y|^{2N-4}}\,dx\,dy = O(\varepsilon^{2N-4}).$$

It follows from (3.14)–(3.17) that (3.13) holds. Recalling π_1 and π_2 in (1.14), we set

$$(3.18)\qquad (u_\varepsilon, v_\varepsilon) := \big(\sqrt{\pi_1}\,\overline{U}_\varepsilon, \sqrt{\pi_2}\,\overline{U}_\varepsilon\big).$$

Combining this with (3.11)–(3.13) and recalling that $\nu_1,\nu_2<0$, $N\geq 5$, we have

$$(3.19)\qquad \begin{aligned} E(\sqrt{t}u_\varepsilon,\sqrt{s}v_\varepsilon) &= \frac{1}{2}t\int_\Omega |\nabla u_\varepsilon|^2+\nu_1 u_\varepsilon^2 \\ &\quad + \frac{1}{2}s\int_\Omega |\nabla v_\varepsilon|^2+\nu_2 v_\varepsilon^2 - \frac{1}{4}\int_\Omega t^2\mu_1\bigg(\frac{1}{|x|^4}*u_\varepsilon^2\bigg)u_\varepsilon^2 \\ &\quad + 2ts\beta\bigg(\frac{1}{|x|^4}*u_\varepsilon^2\bigg)v_\varepsilon^2 + s^2\mu_2\bigg(\frac{1}{|x|^4}*v_\varepsilon^2\bigg)v_\varepsilon^2 \\ &\leq \frac{1}{2}(\pi_1 t+\pi_2 s)\big(S_{H,L}^2 - C\varepsilon^2 + O\big(\varepsilon^{N-2}\big)\big) \\ &\quad - \frac{1}{4}\big(\mu_1\pi_1^2t^2+2\beta\pi_1\pi_2 ts+\mu_2\pi_2^2s^2\big)\big(S_{H,L}^2+O\big(\varepsilon^{N-2}\big)\big). \end{aligned}$$

Denote

$$(3.20)\qquad B_\varepsilon = S_{H,L}^2 - C\varepsilon^2 + O\big(\varepsilon^{N-2}\big), \qquad D_\varepsilon = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big).$$

Obviously, $0<B_\varepsilon<D_\varepsilon$ and $B_\varepsilon<S_{H,L}^2$ for $\varepsilon>0$ small enough. Consider

$$h(t,s) := \frac{1}{2}\,B_\varepsilon(\pi_1 t+\pi_2 s) - \frac{1}{4}\,D_\varepsilon\big(\mu_1\pi_1^2t^2+2\beta\pi_1\pi_2 ts+\mu_2\pi_2^2 s^2\big),$$

then it is easy to see that there exists $t_\varepsilon, s_\varepsilon>0$ such that

$$h(t_\varepsilon,s_\varepsilon) = \max_{t,s>0} h(t,s) \quad\text{and}\quad \frac{\partial}{\partial t}h(t,s)|_{(t_\varepsilon,s_\varepsilon)} = \frac{\partial}{\partial s}h(t,s)|_{(t_\varepsilon,s_\varepsilon)} = 0.$$

Combining these with (1.14), we get that $t_\varepsilon = s_\varepsilon = B_\varepsilon/D_\varepsilon$. Therefore, it follows from (1.14), (2.10) and (3.19) that

$$(3.21)\qquad \begin{aligned} \max_{t,s>0} E(\sqrt{t}u_\varepsilon,\sqrt{s}v_\varepsilon) &\leq \max_{t,s>0} h(t,s) \\ &= \frac{1}{2}(\pi_1+\pi_2)\frac{B_\varepsilon^2}{D_\varepsilon} - \frac{1}{4}\big(\mu_1\pi_1^2+2\beta\pi_1\pi_2+\mu_2\pi_2^2\big)\frac{B_\varepsilon^2}{D_\varepsilon} \\ &< \frac{1}{4}(\pi_1+\pi_2)S_{H,L}^2 = A_0 \end{aligned}$$

for ε small enough. As above, we have $|G(u_\varepsilon,v_\varepsilon)|>0$. Note that $\beta\in(0,\beta_0)$, by a similar argument as that of Lemma 5.1 in [8], we know that there exists

$t_\varepsilon, s_\varepsilon > 0$ such that $\left(\sqrt{\widetilde{t_\varepsilon}}u_\varepsilon, \sqrt{\widetilde{s_\varepsilon}}v_\varepsilon\right) \in \mathcal{N}$. Therefore,

$$A \le E\left(\sqrt{\widetilde{t_\varepsilon}}u_\varepsilon, \sqrt{\widetilde{s_\varepsilon}}v_\varepsilon\right) \le \max_{t,s>0} E\big(\sqrt{t}u_\varepsilon, \sqrt{s}v_\varepsilon\big) < A_0.$$

We complete the proof of (a) of Lemma 3.1.

It suffices to prove (b) of Lemma 3.1. As (a) of Lemma 3.1, we define $v_\varepsilon := \xi \widetilde{U}_\varepsilon$. Similarly, we have the following inequalities

$$\int_\Omega |\nabla v_\varepsilon|^2 = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big), \quad \int_\Omega \left(\frac{1}{|x|^4} * v_\varepsilon^2\right) v_\varepsilon^2 = S_{H,L}^2 + O\big(\varepsilon^{N-2}\big), \tag{3.22}$$

$$C_1\varepsilon^2 + O\big(\varepsilon^{N-2}\big) \le \int_\Omega v_\varepsilon^2 \le C_2\varepsilon^2 + O\big(\varepsilon^{N-2}\big). \tag{3.23}$$

It follows from (3.22) and (3.23) that, for ε small enough,

$$\int_\Omega v_\varepsilon^2 \le 2\int_\Omega \left(\frac{1}{|x|^4} * v_\varepsilon^2\right) v_\varepsilon^2. \tag{3.24}$$

Note that $N \ge 5$, then we know that

$$\left|\frac{1}{|x|^4} * \omega_1^2\right|_{L^\infty(\Omega)} \le \widetilde{C}|\omega_1|_{L^\infty(\Omega)}^2, \quad \text{where } \widetilde{C} = \int_{B(0,\mathrm{diam}(\Omega))} \frac{1}{|y|^4}\, dy. \tag{3.25}$$

Combining (3.25) with (3.23) and (3.24) we have for $t, s > 0$ that

$$\begin{aligned} 2|\beta|ts\int_\Omega \left(\frac{1}{|x|^4} * \omega_1^2\right) v_\varepsilon^2 &\le 2\widetilde{C}|\beta|ts|\omega_1|_{L^\infty(\Omega)}^2 \int_\Omega v_\varepsilon^2 \\ &\le 4\widetilde{C}^2|\beta|^2|\omega_1|_{L^\infty(\Omega)}^4 \mu_2^{-1} t^2 \int_\Omega v_\varepsilon^2 + \frac{1}{4}\, s^2\mu_2 \int_\Omega v_\varepsilon^2 \\ &\le \frac{1}{2}\, t^2 \int_\Omega \mu_1 \left(\frac{1}{|x|^4} * \omega_1^2\right)\omega_1^2 + \frac{1}{2} s^2 \int_\Omega \mu_2 \left(\frac{1}{|x|^4} * v_\varepsilon^2\right) v_\varepsilon^2, \end{aligned} \tag{3.26}$$

for ε small enough. Therefore, by (3.26) we have

$$\begin{aligned} E\big(\sqrt{t}\omega_1, \sqrt{s}v_\varepsilon\big) &\le \frac{1}{2}\, t \int_\Omega |\nabla \omega_1|^2 + \nu_1\omega_1^2 - \frac{1}{8}\, t^2 \int_\Omega \mu_1 \left(\frac{1}{|x|^4} * \omega_1^2\right)\omega_1^2 \\ &\quad + \frac{1}{2}\, s\int_\Omega |\nabla v_\varepsilon|^2 + \nu_2 v_\varepsilon^2 - \frac{1}{8}\, s^2 \int_\Omega \mu_2 \left(\frac{1}{|x|^4} * v_\varepsilon^2\right) v_\varepsilon^2 = f_1(t) + g_1(s). \end{aligned} \tag{3.27}$$

By (3.22) and (3.23), it is standard to check, for ε small enough, that

$$\max_{s>0} g_1(s) < \frac{1}{2}\, \mu_2^{-1} S_{H,L}^2. \tag{3.28}$$

Let t_0 be the larger root of

$$\mu_2 m_1 t^2 - 4\mu_2 m_1 t - S_{H,L}^2 = 0, \tag{3.29}$$

and

$$t_0 = 2 + \sqrt{4 + S_{H,L}^2/\mu_2 m_1}.$$

Then we have

$$2m_1 t - \frac{1}{2}\, m_1 t^2 < -\frac{1}{2\mu_2}\, S_{H,L}^2, \quad \text{for all } t > t_0. \tag{3.30}$$

Note that

$$f_1(t) = 2m_1 t - \frac{1}{2}\, m_1 t^2.$$

Combining this with (3.30), we have

$$f_1(t) + g_1(s) < 0, \quad \text{for all } t > t_0,\ s > 0,$$

and so it follows from (3.27) that

$$\max_{t,s>0} E\big(\sqrt{t}\omega_1, \sqrt{s} v_\varepsilon\big) = \max_{0<t\le t_0, s>0} E\big(\sqrt{t}\omega_1, \sqrt{s} v_\varepsilon\big). \tag{3.31}$$

For $0 < t \le t_0$, $s > 0$, we see from (3.7) that

$$|\beta| ts \int_\Omega \bigg(\frac{1}{|x|^4} * \omega_1^2\bigg) v_\varepsilon^2 \le \widetilde{C} |\beta| t_0 s |\omega_1|_{L^\infty(\Omega)}^2 \int_\Omega v_\varepsilon^2 \le -\frac{\nu_2}{2} s \int_\Omega v_\varepsilon^2.$$

Therefore,

$$\begin{aligned} E\big(\sqrt{t}\omega_1, \sqrt{s} v_\varepsilon\big) &\le \frac{1}{2}\, t \int_\Omega |\nabla \omega_1|^2 + \nu_1 \omega_1^2 - \frac{1}{4}\, t^2 \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * \omega_1^2\bigg) \omega_1^2 \\ &\quad + \frac{1}{2}\, s \int_\Omega |\nabla v_\varepsilon|^2 + \frac{\nu_2}{2}\, v_\varepsilon^2 - \frac{1}{4}\, s^2 \int_\Omega \mu_2 \bigg(\frac{1}{|x|^4} * v_\varepsilon^2\bigg) v_\varepsilon^2 = f_2(t) + g_2(s). \end{aligned} \tag{3.32}$$

Note that $\max\limits_{t>0} f_2(t) = f_2(1) = m_1$. Similarly to (3.28), we know

$$\max_{s>0} g_2(s) < \frac{1}{4}\, \mu_2^{-1} S_{H,L}^2 \quad \text{for } \varepsilon \text{ small enough.}$$

It follows from (3.31) and (3.32) that

$$\begin{aligned} \max_{t,s>0} E\big(\sqrt{t}\omega_1, \sqrt{s} v_\varepsilon\big) &= \max_{0<t\le t_0,\, s>0} E\big(\sqrt{t}\omega_1, \sqrt{s} v_\varepsilon\big) \\ &\le \max_{t>0} f_2(t) + \max_{s>0} g_2(s) < m_1 + \frac{1}{4}\, \mu_2^{-1} S_{H,L}^2 \end{aligned}$$

for ε small enough. Similarly to (3.26), we get that

$$\begin{aligned} \bigg(\int_\Omega \beta \bigg(\frac{1}{|x|^4} * \omega_1^2\bigg) v_\varepsilon^2\bigg)^2 &\le \widetilde{C}^2 |\beta|^2 |\omega_1|_{L^\infty(\Omega)}^4 \bigg(\int_\Omega v_\varepsilon^2\bigg)^2 \\ &< \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * \omega_1^2\bigg) \omega_1^2 \int_\Omega \mu_2 \bigg(\frac{1}{|x|^4} * v_\varepsilon^2\bigg) v_\varepsilon^2 \end{aligned}$$

for ε small enough, that is $\det G(\omega_1, v_\varepsilon) > 0$. Similarly, $\Big(\sqrt{\widehat{t_\varepsilon}}\omega_1, \sqrt{\widehat{s_\varepsilon}} v_\varepsilon\Big) \in \mathcal{N}$ for some $\widehat{t_\varepsilon}, \widehat{s_\varepsilon} > 0$ is equivalent to

$$\begin{pmatrix} \widehat{t_\varepsilon} \\ \widehat{s_\varepsilon} \end{pmatrix} := G^{-1}(\omega_1, v_\varepsilon) \begin{pmatrix} \displaystyle\int_\Omega |\nabla \omega_1|^2 + \nu_1 \omega_1^2 \\ \displaystyle\int_\Omega |\nabla v_\varepsilon|^2 + \nu_2 v_\varepsilon^2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.33}$$

Since $\beta < 0$, we see from (3.9) that every element of $G^{-1}(\omega_1, v_\varepsilon)$ is positive. Hence, (3.33) holds and $\left(\sqrt{\widehat{t_\varepsilon}}\omega_1, \sqrt{\widehat{s_\varepsilon}}v_\varepsilon\right) \in \mathcal{N}$. Therefore,

$$A \leq E\left(\sqrt{\widehat{t_\varepsilon}}\omega_1, \sqrt{\widehat{s_\varepsilon}}v_\varepsilon\right) \leq \max_{t,s>0} E\left(\sqrt{t}\omega_1, \sqrt{s}v_\varepsilon\right) < m_1 + \frac{1}{4}\,\mu_2^{-1} S_{H,L}^2.$$

By a similar argument, we can also prove that $A < m_2 + \mu_1^{-1} S_{H,L}^2/4$. We see from (2.9) and (3.3) that

$$A_0 > \max\left\{m_1 + \frac{1}{4}\,\mu_2^{-1} S_{H,L}^2,\ m_2 + \frac{1}{4}\,\mu_1^{-1} S_{H,L}^2\right\}. \qquad \square$$

Lemma 3.2. *Assume that $\beta \in (-\overline{\beta}, \beta_0)$, where β_0, $\overline{\beta}$ are defined in* (3.6) *and* (3.7), *respectively. Then there exists $C_2 > C_1 > 0$ such that, for any $(u,v) \in \mathcal{N}$ with $E(u,v) \leq (\mu_1^{-1} + \mu_2^{-1}) S_{H,L}^2/4$, there holds*

$$C_1 \leq \int_\Omega \left(\frac{1}{|x|^4} * u^2\right) u^2, \qquad \int_\Omega \left(\frac{1}{|x|^4} * v^2\right) v^2 \leq C_2.$$

Proof. For any $(u,v) \in \mathcal{N}$ with $E(u,v) \leq (\mu_1^{-1} + \mu_2^{-1}) S_{H,L}^2/4$. Denote

$$D_1 = \left(\int_\Omega \left(\frac{1}{|x|^4} * u^2\right) u^2\right)^{1/2}, \qquad D_2 = \left(\int_\Omega \left(\frac{1}{|x|^4} * v^2\right) v^2\right)^{1/2}.$$

It follows from (2.1) and (3.1) that

$$\begin{aligned}
\frac{\lambda_1(\Omega) + \nu_1}{\lambda_1(\Omega)} S_{H,L} D_1 &\leq \int_\Omega |\nabla u|^2 + \nu_1 u^2 \\
&= \int_\Omega \mu_1 \left(\frac{1}{|x|^4} * u^2\right) u^2 + \beta\left(\frac{1}{|x|^4} * u^2\right) v^2 \\
&\leq \mu_1 D_1^2 + \beta_+ D_1 D_2, \\
\frac{\lambda_1(\Omega) + \nu_2}{\lambda_1(\Omega)} S_{H,L} D_2 &\leq \int_\Omega |\nabla v|^2 + \nu_2 v^2 \\
&= \int_\Omega \mu_2 \left(\frac{1}{|x|^4} * v^2\right) v^2 + \beta\left(\frac{1}{|x|^4} * u^2\right) v^2 \\
&\leq \mu_2 D_2^2 + \beta_+ D_1 D_2,
\end{aligned}$$

where $\beta_+ = \max\{\beta, 0\}$. Since

$$\int_\Omega |\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2 \leq \left(\mu_1^{-1} + \mu_2^{-1}\right) S_{H,L}^2,$$

then there exists C_2 such that $D_1, D_2 \leq C_2$. Obviously, $D_1, D_2 \geq C$ for some C if $\beta \leq 0$. It suffices to consider the case $\beta > 0$. Note that

$$\mu_1 D_1 + \beta D_2 \geq \frac{\lambda_1(\Omega) + \nu_1}{\lambda_1(\Omega)} S_{H,L}, \tag{3.34}$$

$$\beta D_1 + \mu_2 D_2 \geq \frac{\lambda_1(\Omega) + \nu_2}{\lambda_1(\Omega)} S_{H,L}, \tag{3.35}$$

$$\frac{\lambda_1(\Omega)+\nu_1}{\lambda_1(\Omega)}\,D_1+\frac{\lambda_1(\Omega)+\nu_2}{\lambda_1(\Omega)}\,D_2\le\big(\mu_1^{-1}+\mu_2^{-1}\big)S_{H,L}. \tag{3.36}$$

It follows from (3.6), (3.34) and (3.36) that

$$D_1\ge\frac{S_{H,L}\big[(\lambda_1(\Omega)+\nu_1)(\lambda_1(\Omega)+\nu_2)-\beta\lambda_1^2(\Omega)\big(\mu_1^{-1}+\mu_2^{-1}\big)\big]}{\lambda_1(\Omega)[\mu_1(\lambda_1(\Omega)+\nu_2)-\beta(\lambda_1(\Omega)+\nu_1)]}>0.$$

Similarly, we have

$$D_2\ge\frac{S_{H,L}\big[(\lambda_1(\Omega)+\nu_1)(\lambda_1(\Omega)+\nu_2)-\beta\lambda_1^2(\Omega)\big(\mu_1^{-1}+\mu_2^{-1}\big)\big]}{\lambda_1(\Omega)[\mu_2(\lambda_1(\Omega)+\nu_1)-\beta(\lambda_1(\Omega)+\nu_2)]}>0. \qquad \square$$

Lemma 3.3. *Assume that $-\overline{\beta}<\beta<\beta_0$ and $\beta\neq 0$, where β_0, $\overline{\beta}$ are defined in* (3.6) *and* (3.7)*, respectively. Then there exists a sequence $\{(u_n,v_n)\}\subset\mathcal{N}$ satisfying*

$$\lim_{n\to\infty}E(u_n,v_n)=A,\qquad \lim_{n\to\infty}E'(u_n,v_n)=0. \tag{3.37}$$

Proof. Note that E is coercive and bounded from below on $\mathcal{N}$. Then, by the Ekeland variational principle (see [36]), there exists a minimizing sequence $\{(u_n,v_n)\}\subset\mathcal{N}$ satisfying

$$E(u_n,v_n)\le\min\bigg\{A+\frac{1}{n},\frac{1}{4}\big(\mu_1^{-1}+\mu_2^{-1}\big)S_{H,L}^2\bigg\}, \tag{3.38}$$

$$E(u,v)\ge E(u_n,v_n)-\frac{1}{n}\,\|(u_n,v_n)-(u,v)\|,\quad \text{for all } (u,v)\in\mathcal{N}. \tag{3.39}$$

Then $\{(u_n,v_n)\}$ is bounded in H. For any $(\varphi,\phi)\in H$ with $\|\varphi\|,\|\phi\|\le 1$ and each $n\in\mathbb{N}$, we define h_n and $g_n\colon\mathbb{R}^3\to\mathbb{R}$ by

$$\begin{aligned} h_n(t,s,l)={}&\int_\Omega\Big|\nabla\Big(u_n+t\varphi+\frac{s}{2}\,u_n\Big)\Big|^2+\nu_1\int_\Omega\Big|u_n+t\varphi+\frac{s}{2}\,u_n\Big|^2\\ &-\mu_1\int_\Omega\bigg(\frac{1}{|x|^4}*\Big|u_n+t\varphi+\frac{s}{2}\,u_n\Big|^2\bigg)\Big|u_n+t\varphi+\frac{s}{2}\,u_n\Big|^2\\ &-\beta\int_\Omega\bigg(\frac{1}{|x|^4}\Big|u_n+t\varphi+\frac{s}{2}\ u_n\Big|^2\bigg)\Big|v_n+t\phi+\frac{l}{2}\,v_n\Big|^2 \end{aligned}$$

and

$$\begin{aligned} g_n(t,s,l)={}&\int_\Omega\Big|\nabla\Big(v_n+t\phi+\frac{l}{2}\,v_n\Big)\Big|^2+\nu_2\int_\Omega\Big|v_n+t\phi+\frac{l}{2}\,v_n\Big|^2\\ &-\mu_2\int_\Omega\bigg(\frac{1}{|x|^4}*\Big|v_n+t\phi+\frac{l}{2}v_n\Big|^2\bigg)\Big|v_n+t\phi+\frac{l}{2}\,v_n\Big|^2\\ &-\beta\int_\Omega\bigg(\frac{1}{|x|^4}*\Big|u_n+t\varphi+\frac{s}{2}\,u_n\Big|^2\bigg)\Big|v_n+t\phi+\frac{l}{2}\,v_n\Big|^2. \end{aligned}$$

let $\mathbf{0}=(0,0,0)$. Then $h_n,g_n\in C^1(\mathbb{R}^3,\mathbb{R})$ and $h_n(\mathbf{0})=g_n(\mathbf{0})=0$. Moreover,

$$\frac{\partial h_n}{\partial s}(\mathbf{0})=-\mu_1\int_\Omega\bigg(\frac{1}{|x|^4}*u_n^2\bigg)u_n^2,\qquad \frac{\partial h_n}{\partial l}(\mathbf{0})=-\beta\int_\Omega\bigg(\frac{1}{|x|^4}*u_n^2\bigg)v_n^2,$$

$$\frac{\partial g_n}{\partial s}(\mathbf{0}) = -\beta \int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right) v_n^2, \qquad \frac{\partial g_n}{\partial l}(\mathbf{0}) = -\mu_2 \int_\Omega \left(\frac{1}{|x|^4} * v_n^2\right) v_n^2.$$

Define the matrix

$$G_n := \begin{bmatrix} \dfrac{\partial h_n}{\partial s}(\mathbf{0}) & \dfrac{\partial h_n}{\partial l}(\mathbf{0}) \\ \dfrac{\partial g_n}{\partial s}(\mathbf{0}) & \dfrac{\partial g_n}{\partial l}(\mathbf{0}) \end{bmatrix}.$$

Then, for $0 < \beta < \beta_0$, we see from Lemma 3.2 and (1.9) that

$$\begin{aligned} \det(G_n) &= \mu_1\mu_2 D\big(u_n^2, u_n^2\big) D\big(v_n^2, v_n^2\big) - \beta^2 D^2\big(u_n^2, v_n^2\big) \\ &\geq \big(\mu_1\mu_2 - \beta^2\big) a_n^2 b_n^2 \geq C > 0, \end{aligned} \tag{3.40}$$

where C is independent of n and $D(\,\cdot\,,\,\cdot\,)$ is defined in (1.8). For $\beta < 0$, by Lemma 3.2 and $(u_n, v_n) \in \mathcal{N}$ we have

$$\begin{aligned} \det(G_n) &= \left(|\beta| \int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right) v_n^2 + \int_\Omega \big(|\nabla u_n|^2 + \nu_1 u_n^2\big)\right) \\ &\quad \times \left(|\beta| \int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right) v_n^2 + \int_\Omega \big(|\nabla v_n|^2 + \nu_2 v_n^2\big)\right) \\ &\quad - \beta^2 \left(\int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right) v_n^2\right)^2 \\ &\geq \int_\Omega \big(|\nabla u_n|^2 + \nu_1 u_n^2\big) \int_\Omega \big(|\nabla v_n|^2 + \nu_2 v_n^2\big) \\ &\geq \frac{(\lambda_1(\Omega) + \nu_1)(\lambda_1(\Omega) + \nu_2)}{\lambda_1(\Omega)^2} \\ &\quad \times S_{H,L}^2 \left(\int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right) u_n^2\right)^{1/2} \left(\int_\Omega \left(\frac{1}{|x|^4} * v_n^2\right) v_n^2\right)^{1/2} \geq C. \end{aligned}$$

Therefore, $\det(G_n) \geq C > 0$ holds for all $-\overline{\beta} < \beta < \beta_0$. By the implicit function theorem, functions s_n and l_n are well defined and class C^1 on some interval $(-\delta_n, \delta_n)$ for $\delta_n > 0$. Moreover, $s_n(0) = l_n(0) = 0$ and

$$h_n(t, s_n(t), l_n(t)) \equiv 0, \quad g_n(t, s_n(t), l_n(t)) \equiv 0, \quad t \in (-\delta_n, \delta_n).$$

This implies that

$$\begin{cases} s_n'(0) = \dfrac{1}{\det(G_n)} \left(\dfrac{\partial g_n}{\partial t}(\mathbf{0}) \dfrac{\partial h_n}{\partial l}(\mathbf{0}) - \dfrac{\partial g_n}{\partial l}(\mathbf{0}) \dfrac{\partial h_n}{\partial t}(\mathbf{0})\right), \\ l_n'(0) = \dfrac{1}{\det(G_n)} \left(\dfrac{\partial g_n}{\partial s}(\mathbf{0}) \dfrac{\partial h_n}{\partial t}(\mathbf{0}) - \dfrac{\partial g_n}{\partial t}(\mathbf{0}) \dfrac{\partial h_n}{\partial s}(\mathbf{0})\right). \end{cases}$$

While, since (u_n, v_n) is bounded in H, we have

$$\left|\frac{\partial h_n}{\partial t}(\mathbf{0})\right| = 2\left|\int_\Omega \left(\nabla u_n \nabla\varphi + \nu_1 u_n\varphi - \mu_1\left(\frac{1}{|x|^4} * u_n^2\right)u_n\varphi - \mu_1\left(\frac{1}{|x|^4} * (u_n\varphi)\right)u_n^2 \right.\right.$$
$$\left.\left. - \beta\left(\frac{1}{|x|^4} * u_n^2\right)v_n\phi - \beta\left(\frac{1}{|x|^4} * (u_n\varphi)\right)v_n^2\right)\right| \leq C,$$

where C is independent of n. Similarly, $\left|\frac{\partial g_n}{\partial t}(\mathbf{0})\right| \leq C$. From Lemma 3.2 we also have

$$\left|\frac{\partial h_n}{\partial s}(\mathbf{0})\right|, \ \left|\frac{\partial h_n}{\partial l}(\mathbf{0})\right|, \ \left|\frac{\partial g_n}{\partial s}(\mathbf{0})\right|, \ \left|\frac{\partial g_n}{\partial l}(\mathbf{0})\right| \leq C.$$

Hence, combining these with (3.40), we have

$$|s_n'(0)|, |l_n'(0)| \leq C, \tag{3.41}$$

where C is independent of n.

With these, it is standard to prove that (see [8, Theorem 1.3 (1)])

$$\lim_{n\to\infty} E'(u_n, v_n) = 0. \tag{3.42}$$

Proof of (a)–(b) in Theorem 1.5. By Lemma 3.3, we know that there exists a sequence $\{(u_n, v_n)\} \subset \mathcal{N}$ satisfying (3.37). Then $\{(u_n, v_n)\}$ is bounded in H, passing to a subsequence, we may assume that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in H, and so

$$\begin{aligned} &u_n \rightharpoonup u, \quad &&v_n \rightharpoonup v, \quad &&\text{weakly in } L^{2N/(N-2)}(\Omega),\\ &u_n^2 \rightharpoonup u^2, \quad &&v_n^2 \rightharpoonup v^2, \quad &&\text{weakly in } L^{N/(N-2)}(\Omega),\\ &u_n \to u, \quad &&v_n \to v, \quad &&\text{strongly in } L^2(\Omega). \end{aligned}$$

It follows from (3.37) that $E'(u, v) = 0$. Denote $\omega_n = u_n - u$ and $\sigma_n = v_n - v$. Note that

$$\begin{aligned} \omega_n &\rightharpoonup 0, & \sigma_n &\rightharpoonup 0, && \text{weakly in } H_0^1(\Omega),\\ |\omega_n|^{N/(N-2)} &\rightharpoonup 0, & |\sigma_n|^{N/(N-2)} &\rightharpoonup 0, && \text{weakly in } L^2(\Omega),\\ \int_\Omega |\omega_n u|^{N/(N-2)} &\to 0, & \int_\Omega |\sigma_n v|^{N/(N-2)} &\to 0, && \text{as } n \to \infty,\\ |\omega_n|^2 &\rightharpoonup 0, & |\sigma_n|^2 &\rightharpoonup 0, && \text{weakly in } L^{N/(N-2)}(\Omega). \end{aligned} \tag{3.43}$$

Thus

$$\left|\int_\Omega \left(\frac{1}{|x|^4} * (\omega_n u)\right)v_n^2\right| \leq C|\omega_n u|_{N/(N-2)}|v_n|_{2^*}^2 = o(1), \tag{3.44}$$

$$\left|\int_\Omega \left(\frac{1}{|x|^4} * u^2\right)(\sigma_n v)\right| \leq C|u|_{2^*}^2|\sigma_n v|_{N/(N-2)} = o(1), \tag{3.45}$$

$$\left|\int_\Omega \left(\frac{1}{|x|^4} * \omega_n^2\right)(\sigma_n v)\right| \leq C|\omega_n|_{2^*}^2|\sigma_n v|_{N/(N-2)} = o(1). \tag{3.46}$$

By the Hardy–Littlewood–Sobolev inequality, we have

$$\left(\frac{1}{|x|^4} * u^2\right) \in L^{N/2}(\Omega), \qquad \left(\frac{1}{|x|^4} * v^2\right) \in L^{N/2}(\Omega).$$

Combining these with (3.43) we get that

$$\int_\Omega \left(\frac{1}{|x|^4} * u^2\right)\sigma_n^2 = o(1), \qquad \int_\Omega \left(\frac{1}{|x|^4} * v^2\right)\omega_n^2 = o(1). \tag{3.47}$$

It follows from (3.44)–(3.47) that

$$\begin{aligned}\int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right)v_n^2 &= \int_\Omega \left(\frac{1}{|x|^4} * \omega_n^2\right)\sigma_n^2 \\ &\quad + \int_\Omega \left(\frac{1}{|x|^4} * u^2\right)v^2 + 2\int_\Omega \left(\frac{1}{|x|^4} * (\omega_n u)\right)v_n^2 \\ &\quad + \int_\Omega \left(\frac{1}{|x|^4} * u^2\right)\sigma_n(2v+\sigma_n) + \int_\Omega \left(\frac{1}{|x|^4} * \omega_n^2\right)v(v+2\sigma_n) \\ &= \int_\Omega \left(\frac{1}{|x|^4} * \omega_n^2\right)\sigma_n^2 + \int_\Omega \left(\frac{1}{|x|^4} * u^2\right)v^2 + o(1).\end{aligned} \tag{3.48}$$

Similarly, we have

$$\int_\Omega \left(\frac{1}{|x|^4} * u_n^2\right)u_n^2 = \int_\Omega \left(\frac{1}{|x|^4} * \omega_n^2\right)\omega_n^2 + \int_\Omega \left(\frac{1}{|x|^4} * u^2\right)u^2 + o(1), \tag{3.49}$$

$$\int_\Omega \left(\frac{1}{|x|^4} * v_n^2\right)v_n^2 = \int_\Omega \left(\frac{1}{|x|^4} * \sigma_n^2\right)\sigma_n^2 + \int_\Omega \left(\frac{1}{|x|^4} * v^2\right)v^2 + o(1). \tag{3.50}$$

Note that $(u_n, v_n) \in \mathcal{N}$ and $E'(u,v) = 0$. Combining these with (3.48)–(3.50) we have

$$\int_\Omega |\nabla\omega_n|^2 - \int_\Omega \mu_1\left(\frac{1}{|x|^4} * \omega_n^2\right)\omega_n^2 + \beta\left(\frac{1}{|x|^4} * \omega_n^2\right)\sigma_n^2 = o(1), \tag{3.51}$$

$$\int_\Omega |\nabla\sigma_n|^2 - \int_\Omega \mu_2\left(\frac{1}{|x|^4} * \sigma_n^2\right)\sigma_n^2 + \beta\left(\frac{1}{|x|^4} * \omega_n^2\right)\sigma_n^2 = o(1), \tag{3.52}$$

$$E(u_n, v_n) = E(u,v) + I(\omega_n, \sigma_n) + o(1). \tag{3.53}$$

Passing to a subsequence, we may assume that

$$\lim_{n\to+\infty}\int_\Omega |\nabla\omega_n|^2 = d_1, \qquad \lim_{n\to+\infty}\int_\Omega |\nabla\sigma_n|^2 = d_2.$$

Then by (3.51) and (3.52) we have $I(\omega_n, \sigma_n) = (d_1 + d_2)/4 + o(1)$. Letting $n \to +\infty$ in (3.53), we see that

$$0 \le E(u,v) \le E(u,v) + \frac{1}{4}(d_1 + d_2) = \lim_{n\to+\infty} E(u_n, v_n) = A. \tag{3.54}$$

Case 1. $u \equiv 0, v \equiv 0$. It follows from Lemma 3.2 that $d_1 > 0$ and $d_2 > 0$, and so we may assume that both $\omega_n \not\equiv 0$ and $\sigma_n \not\equiv 0$ for n large. It follows from

(3.51) and (3.52) that there exists $t_n, s_n > 0$ such that $\big(\sqrt{t_n}\omega_n, \sqrt{s_n}\sigma_n\big) \in \mathcal{M}$ and

$$\lim_{n\to+\infty}(|t_n - 1| + |s_n - 1|) = 0.$$

Therefore

$$A = \frac{1}{4}(d_1 + d_2) = \lim_{n\to+\infty} I(\omega_n, \sigma_n) = \lim_{n\to+\infty} I\big(\sqrt{t_n}\omega_n, \sqrt{s_n}\sigma_n\big) \geq A_0,$$

which implies $A \geq A_0$, a contradiction with Lemma 3.1. Hence, Case 1 is impossible.

Case 2. $u \not\equiv 0, v \equiv 0$ or $u \equiv 0, v \not\equiv 0$. Without loss of generality, we may assume that $u \not\equiv 0, v \equiv 0$. Then $d_2 > 0$. By Case 1 we may assume that $d_1 = 0$. Then $\omega_n \to 0$ strongly in $H_0^1(\Omega)$. Hence, by (1.9) and (1.10) we know that

$$\lim_{n\to+\infty} \int_\Omega \bigg(\frac{1}{|x|^4} * \omega_n^2\bigg)\sigma_n^2 = 0.$$

Combining this with (3.52) we have

$$\int_\Omega |\nabla\sigma_n|^2 = \int_\Omega \mu_2\bigg(\frac{1}{|x|^4} * \sigma_n^2\bigg)\sigma_n^2 + o(1) \leq \mu_2 S_{H,L}^{-2}\bigg(\int_\Omega |\nabla\sigma_n|^2\bigg)^2 + o(1).$$

This implies that $d_2 \geq \mu_2^{-1} S_{H,L}^2$. Note that u is a nontrivial solution of $-\Delta u + \nu_1 u = \mu_1\big(1/|x|^4 * u^2\big)u$, then $E(u,0) \geq m_1$. By (3.54) we get that

$$A \geq m_1 + \frac{1}{4}\,d_2 \geq m_1 + \frac{1}{4}\,\mu_2^{-1} S_{H,L}^2 > m_1 + m_2,$$

a contradiction with Lemma 3.1. Hence, Case 2 is impossible.

Since Cases 1 and 2 are both impossible, then we get that $u \not\equiv 0$ and $v \not\equiv 0$. Therefore, $(u,v) \in \mathcal{N}$. It follows from (3.54) that $E(u,v) = A$. Then we have $(|u|,|v|) \in \mathcal{N}$ and $E(|u|,|v|) = A$. By Proposition 2.3, $(|u|,|v|)$ is a least energy solution of (1.5). Then, using the maximum principle, we know $|u|,|v| > 0$ in Ω. Therefore, $(|u|,|v|)$ is a positive least energy solution of (1.5). $\square$

It suffices to prove (c) of Theorem 1.5. Assume that $\beta > \max\{\mu_1, \mu_2\}$. Define

$$\mathcal{A} := \inf_{h\in\Gamma} \max_{t\in[0,1]} E(h(t)), \tag{3.55}$$

where $\Gamma = \{h \in C([0,1], H) : h(0) = (0,0),\ E(h(1)) < 0\}$. By (1.12), we know that, for any $(u,v) \in H$ and $(u,v) \neq (0,0)$,

$$\begin{aligned} \max_{t>0} E\big(\sqrt{t}u, \sqrt{t}v\big) &= E\big(\sqrt{t_{u,v}}u, \sqrt{t_{u,v}}v\big) \\ &= \frac{1}{4}\,t_{u,v} \int_\Omega \big(|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2\big), \end{aligned} \tag{3.56}$$

where $t_{u,v} > 0$ satisfies

$$t_{u,v} = \frac{\displaystyle\int_\Omega \big(|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2\big)}{\displaystyle\int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * u^2\bigg) u^2 + 2\beta \bigg(\frac{1}{|x|^4} * u^2\bigg) v^2 + \mu_2 \bigg(\frac{1}{|x|^4} * v^2\bigg) v^2}.$$

Note that $\big(\sqrt{t_{u,v}}u, \sqrt{t_{u,v}}v\big) \in \mathcal{N}'$, where

$$\begin{aligned}(3.57)\quad \mathcal{N}' := \bigg\{&(u,v) \in H \setminus \{(0,0)\}, \\ &F(u,v) := \int_\Omega \big(|\nabla u|^2 + \nu_1 u^2 + |\nabla v|^2 + \nu_2 v^2\big) - \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * u^2\bigg) u^2 \\ &\qquad + 2\beta \bigg(\frac{1}{|x|^4} * u^2\bigg) v^2 + \mu_2 \bigg(\frac{1}{|x|^4} * v^2\bigg) v^2 = 0\bigg\},\end{aligned}$$

it is standard to see that

$$\mathcal{A} = \inf_{H \ni (u,v) \neq (0,0)} \max_{t>0} E(tu, tv) = \inf_{(u,v)\in\mathcal{N}'} E(u,v). \tag{3.58}$$

Note that $\mathcal{N} \subseteq \mathcal{N}'$, one has that $\mathcal{A} \leq A$. Similarly as (2.8), we have $\mathcal{A} > 0$. Denote

$$\gamma := \min\bigg\{\mu_i^{-1}\bigg(1 + \frac{\nu_i}{\lambda_1(\Omega)}\bigg)^2,\ i = 1,2\bigg\}. \tag{3.59}$$

Then we have the following lemma.

Lemma 3.4. *Assume that $-\lambda_1(\Omega) < \nu_2 \leq \nu_1 < 0$. Let β_1 be the larger root of the equation*

$$\beta^2 - \frac{2}{\gamma}\beta + \frac{\mu_1 + \mu_2}{\gamma} - \mu_1\mu_2 = 0,$$

where γ is defined in (3.59). *Then $\beta_1 \geq \max\{\mu_1, \mu_2\}$, and for any $\beta > \beta_1$ there holds $\mathcal{A} < \min\{m_1, m_2, A_0\}$.*

Proof. First, we prove that $\mathcal{A} < A_0$. Without loss of generality, we assume that $\mu_2 \geq \mu_1$. Define

$$f(\beta) := \beta^2 - \frac{2}{\gamma}\beta + \frac{\mu_1 + \mu_2}{\gamma} - \mu_1\mu_2,$$

by (3.59) we have $\gamma < \mu_2^{-1}$ and so $f(\mu_2) \leq 0$, that is, $\beta_1 \geq \mu_2 = \max\{\mu_1, \mu_2\}$. Then $\pi_1, \pi_2 > 0$, where (π_1, π_2) is defined in (1.14). Recall (3.18) and (3.20), similarly to (3.19) we know that

$$\begin{aligned}E\big(\sqrt{t}u_\varepsilon, \sqrt{t}v_\varepsilon\big) &= \frac{1}{2}\,t \int_\Omega |\nabla u_\varepsilon|^2 + \nu_1 u_\varepsilon^2 + |\nabla v_\varepsilon|^2 + \nu_2 v_\varepsilon^2 \\ &\quad - \frac{1}{4}\,t^2 \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * u_\varepsilon^2\bigg) u_\varepsilon^2 + 2\beta \bigg(\frac{1}{|x|^4} * u_\varepsilon^2\bigg) v_\varepsilon^2 + \mu_2 \bigg(\frac{1}{|x|^4} * v_\varepsilon^2\bigg) v_\varepsilon^2 \\ &\leq \frac{1}{2}\,B_\varepsilon(\pi_1 + \pi_2)t - \frac{1}{4}\,D_\varepsilon\big(\mu_1\pi_1^2 + 2\beta\pi_1\pi_2 + \mu_2\pi_2^2\big)t^2.\end{aligned}$$

Similarly to (3.21) we get that

$$0 < \mathcal{A} \le \max_{t>0} E\big(\sqrt{t}u_\varepsilon, \sqrt{t}v_\varepsilon\big) < \frac{1}{4}(\pi_1+\pi_2)S_{H,L}^2 = A_0. \tag{3.60}$$

It remains to prove $\mathcal{A} < \min\{m_1, m_2\}$. Note that $\beta > \beta_1$, then $f(\beta) > 0$ and so

$$\gamma > \frac{2\beta - \mu_1 - \mu_2}{\beta^2 - \mu_1\mu_2} = \pi_1 + \pi_2.$$

Combining this with (3.3) and (3.60) we have

$$\min\{m_1, m_2\} \ge \frac{1}{4}\gamma S_{H,L}^2 > \frac{1}{4}(\pi_1+\pi_2)S_{H,L}^2 = A_0 > \mathcal{A}. \qquad \square$$

With these, the following proof is similar to that of (3) in Theorem 1.3 in [8], and so we omit it.

4. Proof of Theorem 1.9

This section is devoted to the proof of Theorem 1.9. Recall the definitions of E, $\mathcal{N}$, A, they all depend on β, and we use notations E_β, $\mathcal{N}_\beta$, A_β in this section.

Proof of Theorem 1.9. Let $0 < |\beta_n| < \min\{\overline{\beta}, \beta_0\}, n \in \mathbb{N}$, satisfy $\beta_n \to 0$ as $n \to +\infty$, and (u_n, v_n) be the positive least energy solutions of (1.5) with $\beta = \beta_n$. By Lemma 3.1, we know that $E_{\beta_n}(u_n, v_n)$ is uniformly bounded, and so (u_n, v_n) is uniformly bounded in H. Passing to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup \widehat{u}, \quad v_n \rightharpoonup \widehat{v} \quad \text{weakly in } H_0^1(\Omega),\\ &u_n \to \widehat{u}, \quad v_n \to \widehat{v} \quad \text{strongly in } L^2(\Omega),\\ &u_n \to \widehat{u}, \quad v_n \to \widehat{v} \quad \text{almost everywhere } x \in \Omega. \end{aligned}$$

Hence, $\widehat{u}(x), \widehat{v}(x) \ge 0$ for almost everywhere $x \in \Omega$.

First, we claim that for $|\beta_n| < \min\{\overline{\beta}, \beta_0\}$ sufficiently small, there holds

$$\lim_{n\to\infty} A_{\beta_n} \le m_1 + m_2. \tag{4.1}$$

For $0 < \beta_n < \beta_0$, by Lemma 3.1 we have

$$\lim_{n\to\infty} A_{\beta_n} \le m_1 + m_2. \tag{4.2}$$

On the other hand, for $-\overline{\beta} < \beta_n < 0$ sufficiently small, by a similar argument as that of Lemma 3.1, we know that there exists $t_{1,n}, s_{1,n} > 0$ such that $\big(\sqrt{t_{1,n}}\omega_1, \sqrt{s_{1,n}}\omega_2\big) \in \mathcal{N}_{\beta_n}$ and

$$\lim_{n\to\infty} (|t_{1,n} - 1| + |s_{1,n} - 1|) = 0. \tag{4.3}$$

It follows from (3.4) that for $-\overline{\beta} < \beta_n < 0$ we have

$$\begin{aligned}\lim_{n\to\infty} A_{\beta_n} &\le \frac{1}{4}\lim_{n\to\infty} t_{1,n}\int_\Omega |\nabla\omega_1|^2 + \nu_1\omega_1^2 + \frac{1}{4}\lim_{n\to\infty} s_{1,n}\int_\Omega |\nabla\omega_2|^2 + \nu_2\omega_2^2 \\ &= E_0(\omega_1, 0) + E_0(0, \omega_2) = m_1 + m_2.\end{aligned}$$

Combining this with (4.2) we get that (4.1) holds.

Case 1. $\widehat{u} \equiv 0$, $\widehat{v} \equiv 0$. Note that

$$\int_\Omega u_n^2 \to 0 \quad \text{and} \quad \int_\Omega \beta_n\bigg(\frac{1}{|x|^4} * u_n^2\bigg)v_n^2 \to 0.$$

It follows from $(u_n, v_n) \in \mathcal{N}_{\beta_n}$ that

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 + \nu_1\int_\Omega u_n^2 &= \int_\Omega \mu_1\bigg(\frac{1}{|x|^4} * u_n^2\bigg)u_n^2 + \beta_n\bigg(\frac{1}{|x|^4} * v_n^2\bigg)u_n^2 \\ &\le \mu_1 S_{H,L}^{-2}\bigg(\int_\Omega |\nabla u_n|^2\bigg)^2 + o(1),\end{aligned}$$

and so

$$d_1 := \lim_{n\to\infty}\int_\Omega |\nabla u_n|^2 \ge \mu_1^{-1} S_{H,L}^2.$$

Similarly, we have

$$d_2 := \lim_{n\to\infty}\int_\Omega |\nabla v_n|^2 \ge \mu_2^{-1} S_{H,L}^2. \tag{4.4}$$

We see from (4.1) and (3.3) that

$$m_1 + m_2 \ge \lim_{n\to\infty} A_{\beta_n} = \frac{1}{4}(d_1 + d_2) > m_1 + m_2,$$

a contradiction. Therefore, Case 1 is impossible.

Case 2. $\widehat{u} \not\equiv 0$, $\widehat{v} \equiv 0$ or $\widehat{u} \equiv 0$, $\widehat{v} \not\equiv 0$. Without loss of generality, we assume that $\widehat{u} \not\equiv 0, \widehat{v} \equiv 0$. Then (4.4) holds. Multiply the equation for u in (1.5) by $\widehat{u}$ and integrate over Ω, which implies

$$\int_\Omega \nabla u_n\nabla\widehat{u} + \nu_1 u_n\widehat{u} = \int_\Omega \mu_1\bigg(\frac{1}{|x|^4} * u_n^2\bigg)u_n\widehat{u} + \beta_n\bigg(\frac{1}{|x|^4} * v_n^2\bigg)u_n\widehat{u}. \tag{4.5}$$

We claim that

$$\int_\Omega \big(|x|^{-4} * u_n^2\big)u_n\widehat{u} \to \int_\Omega \big(|x|^{-4} * \widehat{u}^2\big)\widehat{u}^2, \quad \text{as } n\to\infty. \tag{4.6}$$

Note that

$$u_n^2 \rightharpoonup \widehat{u}^2 \quad \text{weakly in } L^{N/(N-2)}(\Omega), \text{ as } n\to\infty.$$

By the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{N/(N-2)}(\Omega)$ to $L^{N/2}(\Omega)$, hence

$$|x|^{-4} * u_n^2 \rightharpoonup |x|^{-4} * \widehat{u}^2 \quad \text{weakly in } L^{N/2}(\Omega), \text{ as } n\to\infty.$$

Combining this with the fact that

$$u_n \rightharpoonup \widehat{u} \quad \text{weakly in } L^{2N/(N-2)}(\Omega), \text{ as } n\to\infty,$$

we have

$$\big(|x|^{-4} * u_n^2\big)u_n \rightharpoonup \big(|x|^{-4} * \widehat{u}^2\big)\widehat{u} \quad \text{weakly in } L^{2N/(N+2)}(\Omega), \text{ as } n \to \infty.$$

Therefore, (4.6) holds. It follows from (3.5), (4.5) and (4.6) that

$$\text{(4.7)} \quad \int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big) \le \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * \widehat{u}^2\bigg)\widehat{u}^2 \le (4m_1)^{-1} \bigg(\int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big)\bigg)^2,$$

and so

$$\int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big) \ge 4m_1.$$

Combining this with (4.1) and (3.3) we know that

$$\begin{aligned} \text{(4.8)} \qquad m_1 + m_2 &\ge \lim_{n\to\infty} A_{\beta_n} = \frac{1}{4} \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 + \nu_1 u_n^2 + |\nabla v_n|^2 + \nu_2 v_n^2 \\ &= \frac{1}{4} \int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big) + \frac{1}{4} d_2 + \lim_{n\to\infty} \frac{1}{4} \int_\Omega |\nabla (u_n - \widehat{u})|^2 \\ &\ge m_1 + \frac{1}{4} \mu_2^{-1} S_{H,L}^2 + \lim_{n\to\infty} \frac{1}{4} \int_\Omega |\nabla (u_n - \widehat{u})|^2 > m_1 + m_2, \end{aligned}$$

a contradiction. Therefore, Case 2 is impossible.

Since Cases 1 and 2 are both impossible, then we get that $\widehat{u} \not\equiv 0$ and $\widehat{v} \not\equiv 0$. Similarly as the Case 2, we can get that (4.7) holds and

$$\int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big) \ge 4m_1, \qquad \int_\Omega \big(|\nabla \widehat{v}|^2 + \nu_2 \widehat{v}\big) \ge 4m_2.$$

Combining these with (4.1) we know that

$$\begin{aligned} \text{(4.9)} \qquad m_1 + m_2 &\ge \lim_{n\to\infty} A_{\beta_n} \\ &= \frac{1}{4} \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 + \nu_1 u_n^2 + |\nabla v_n|^2 + \nu_2 v_n^2 \\ &= \frac{1}{4} \int_\Omega |\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2 + \frac{1}{4} \int_\Omega |\nabla \widehat{v}|^2 + \nu_2 \widehat{v}^2 \\ &\quad + \frac{1}{4} \lim_{n\to\infty} \int_\Omega \big(|\nabla (u_n - \widehat{u})|^2 + |\nabla (v_n - \widehat{v})|^2\big) \\ &\ge m_1 + m_2 + \frac{1}{4} \lim_{n\to\infty} \int_\Omega \big(|\nabla (u_n - \widehat{u})|^2 + |\nabla (v_n - \widehat{v})|^2\big) \\ &\ge m_1 + m_2. \end{aligned}$$

This means that

$$\lim_{n\to\infty} \int_\Omega \big(|\nabla (u_n - \widehat{u})|^2 + |\nabla (v_n - \widehat{v})|^2\big) = 0,$$

and so $(u_n, v_n) \to (\widehat{u}, \widehat{v})$ strongly in $H_0^1(\Omega) \times H_0^1(\Omega)$ as $n \to +\infty$. Moreover,

$$\int_\Omega \big(|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2\big) = 4m_1,$$

and so we see from (4.7) that

$$\int_\Omega (|\nabla \widehat{u}|^2 + \nu_1 \widehat{u}^2) = \int_\Omega \mu_1 \bigg(\frac{1}{|x|^4} * \widehat{u}^2\bigg)\widehat{u}^2 = 4m_1.$$

Therefore, $\widehat{u}$ is a positive least energy solution of

$$-\Delta u + \nu_1 u = \mu_1 \bigg(\frac{1}{|x|^4} * u^2\bigg)u, \quad u \in H_0^1(\Omega).$$

Similarly, we know that $\widehat{v}$ is a positive least energy solution of

$$-\Delta v + \nu_2 v = \mu_2 \bigg(\frac{1}{|x|^4} * v^2\bigg)v, \quad v \in H_0^1(\Omega). \qquad \square$$

Appendix A

Proposition A.1. *Assume that Ω is a smooth bounded domain. Let $u \in H_0^1(\Omega)$. If*

$$-\Delta u + \nu u = \mu \bigg(\frac{1}{|x|^4} * u^2\bigg)u,$$

where $\nu < 0$, $\mu > 0$ are constants. Then $u \in C^2(\overline{\Omega})$.

Proof. By the Hardy–Littlewood–Sobolev inequality, we have

$$\bigg(\frac{1}{|x|^4} * u^2\bigg) \in L^{N/2}(\Omega).$$

Denote

$$a(x) = \mu \bigg(\frac{1}{|x|^4} * u^2\bigg) - \nu,$$

then $a(x) \in L^{N/2}(\Omega)$. From Lemma B.3 of [31], we have $u \in L^q(\Omega)$ for any $q < \infty$. Therefore $((1/|x|^4) * u^2) \in L^\infty(\Omega)$, and so

$$\bigg| - \nu u + \mu \bigg(\frac{1}{|x|^4} * u^2\bigg)u \bigg| \leq C\big(1 + |u|^{2^*-1}\big).$$

By Theorem 1.16 of [4], we know that $u \in C^2(\overline{\Omega})$. $\square$

Proposition A.2. *Assume that Ω is a smooth bounded domain. Let $u, v \in H_0^1(\Omega)$. If*

$$\begin{cases} -\Delta u + \nu_1 u = \mu_1 \bigg(\dfrac{1}{|x|^4} * u^2\bigg)u + \beta\bigg(\dfrac{1}{|x|^4} * v^2\bigg)u, & x \in \Omega, \\ -\Delta v + \nu_2 v = \mu_2 \bigg(\dfrac{1}{|x|^4} * v^2\bigg)v + \beta\bigg(\dfrac{1}{|x|^4} * u^2\bigg)v, & x \in \Omega, \\ u, v \geq 0 \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then $u, v \in C^2(\overline{\Omega})$.

The proof is similar to that of Proposition A.1, and so we omit it.

Proposition A.3. *Let $u,v\in W^{1,2}(\mathbb{R}^N)\cap L^{2Np/(2N-\mu)}(\mathbb{R}^N)$. If*

$$\text{(A.1)}\qquad \begin{cases} -\Delta u+\nu_1 u=\mu_1\bigg(\dfrac{1}{|x|^\mu}*|u|^p\bigg)|u|^{p-2}u \\ \qquad\qquad +\beta\bigg(\dfrac{1}{|x|^\mu}*|v|^p\bigg)|u|^{p-2}u, \quad x\in\mathbb{R}^N, \\ -\Delta v+\nu_2 v=\mu_2\bigg(\dfrac{1}{|x|^\mu}*|v|^p\bigg)|v|^{p-2}v \\ \qquad\qquad +\beta\bigg(\dfrac{1}{|x|^\mu}*|u|^p\bigg)|v|^{p-2}v, \quad x\in\mathbb{R}^N, \end{cases}$$

and $u,v\in W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N)\cap W^{1,2Np/(2N-\mu)}(\mathbb{R}^N)$, then

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2+\frac{N}{2}\int_{\mathbb{R}^N}\nu_1u^2+\nu_2v^2 \\ =\frac{2N-\mu}{2p}\int_{\mathbb{R}^N}\mu_1\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|u|^p+\frac{2N-\mu}{2p}\int_{\mathbb{R}^N}2\beta\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|v|^p \\ +\mu_2\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|v|^p. \end{aligned}$$

Proof. The idea of the following proof comes from [27], where Moroz and Van Schaftingen establish the Pohožaev type identity for single Choquard equation. We take $\phi\in C^1_c(\mathbb{R}^N)$ such that $\phi=1$ on B_1. Denote

$$u_\lambda(x)=\phi(\lambda x)x\cdot\nabla u(x),$$

where $\lambda\in(0,\infty)$ and $x\in\mathbb{R}^N$. We multiply the first equation in (A.1) by u_λ, which yields

$$\int_{\mathbb{R}^N}\nabla u\cdot\nabla u_\lambda+\nu_1uu_\lambda=\int_{\mathbb{R}^N}\mu_1\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|u|^{p-2}uu_\lambda+\beta\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|u|^{p-2}uu_\lambda.$$

We compute for every $\lambda>0$,

$$\begin{aligned} \int_{\mathbb{R}^N}\nabla u\cdot\nabla u_\lambda&=\int_{\mathbb{R}^N}(\lambda\nabla u\cdot\nabla\phi(\lambda x)x\cdot\nabla u+\phi(\lambda x)\nabla u\cdot\nabla(x\nabla u))\,dx \\ &=\int_{\mathbb{R}^N}\lambda\nabla u\cdot\nabla\phi(\lambda x)x\cdot\nabla u\,dx \\ &\quad-\int_{\mathbb{R}^N}((N-2)\phi(\lambda x)+\lambda x\cdot\nabla\phi(\lambda x))\frac{|\nabla u|^2}{2}\,dx. \end{aligned}$$

Thus

$$\lim_{\lambda\to0}\int_{\mathbb{R}^N}\nabla u\nabla u_\lambda=-\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2.$$

Note that

$$\int_{\mathbb{R}^N}uu_\lambda=-\int_{\mathbb{R}^N}(N\phi(\lambda x)+\lambda x\cdot\nabla\phi(\lambda x))\frac{|u|^2}{2}\,dx.$$

Hence

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N} uu_\lambda = -\frac{N}{2}\int_{\mathbb{R}^N}|u|^2.$$

Next

$$\begin{aligned}
&\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|u|^{p-2}uu_\lambda \\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)|^p\frac{1}{|x-y|^\mu}\,\phi(\lambda x)x\cdot\nabla\bigg(\frac{|u|^p}{p}\bigg)(x)\,dx\,dy \\
&\quad=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^\mu}\bigg(|u(y)|^p\phi(\lambda x)x\cdot\nabla\bigg(\frac{|u|^p}{p}\bigg)(x) \\
&\qquad+|u(x)|^p\phi(\lambda y)y\cdot\nabla\bigg(\frac{|u|^p}{p}\bigg)(y)\bigg)\,dx\,dy \\
&\quad=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)|^p\frac{1}{|x-y|^\mu}(N\phi(\lambda x)+\lambda x\cdot\nabla\phi(\lambda x))\frac{|u(x)|^p}{p}\,dx\,dy \\
&\qquad+\frac{\mu}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(y)|^p\frac{1}{|x-y|^\mu} \\
&\qquad\times\frac{(x-y)\cdot(x\phi(\lambda x)-y\phi(\lambda y))}{|x-y|^2}\,|u(x)|^p\,dx\,dy.
\end{aligned}$$

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|u|^{p-2}uu_\lambda=-\frac{2N-\mu}{2p}\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|u|^p.$$

Finally,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|u|^{p-2}uu_\lambda \\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|v(y)|^p\frac{1}{|x-y|^\mu}\phi(\lambda x)x\cdot\nabla\bigg(\frac{|u|^p}{p}\bigg)(x)\,dx\,dy \\
&\quad=-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(y)|^p}{|x-y|^\mu}(N\phi(\lambda x)+\lambda x\cdot\nabla\phi(\lambda x))\,\frac{|u(x)|^p}{p})\,dx\,dy \\
&\qquad+\frac{\mu}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(y)|^p}{|x-y|^\mu}\cdot\frac{(x-y)\cdot x\phi(\lambda x)}{|x-y|^2}\,|u(x)|^p\,dx\,dy.
\end{aligned}\tag{A.2}$$

Define $v_\lambda(x)=\phi(\lambda x)x\cdot\nabla v(x)$, where $\lambda\in(0,\infty)$ and $x\in\mathbb{R}^N$. Similarly, we get that

$$\int_{\mathbb{R}^N}\nabla v\cdot\nabla v_\lambda+\nu_2 vv_\lambda=\int_{\mathbb{R}^N}\mu_2\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|v|^{p-2}vv_\lambda+\beta\bigg(\frac{1}{|x|^\mu}*|u|^p\bigg)|v|^{p-2}vv_\lambda,$$

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\nabla v\cdot\nabla v_\lambda=-\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla v|^2,\qquad \lim_{\lambda\to 0}\int_{\mathbb{R}^N}vv_\lambda=-\frac{N}{2}\int_{\mathbb{R}^N}|v|^2,$$

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|v|^{p-2}vv_\lambda=-\frac{2N-\mu}{2p}\int_{\mathbb{R}^N}\bigg(\frac{1}{|x|^\mu}*|v|^p\bigg)|v|^p,$$

$$\begin{aligned}\tag{A.3} &\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*|u|^p\right)|v|^{p-2}vv_\lambda \\ &\quad= -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p}{|x-y|^\mu}\left(N\phi(\lambda y)+\lambda y\cdot\nabla\phi(\lambda y)\right)\frac{|v(y)|^p}{p}\,dx\,dy \\ &\qquad+\frac{\mu}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)|^p\frac{1}{|x-y|^\mu}\frac{(y-x)\cdot y\phi(\lambda y)}{|x-y|^2}|v(y)|^p\,dx\,dy.\end{aligned}$$

It follows from (A.2) and (A.3) that

$$\begin{aligned}\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*|v|^p\right)|u|^{p-2}uu_\lambda+\left(\frac{1}{|x|^\mu}*|u|^p\right)|v|^{p-2}vv_\lambda \\ =-\frac{2N-\mu}{p}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu}*|u|^p\right)|v|^p.\end{aligned}$$

From the above, we get that

$$\begin{aligned}&\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2+|\nabla v|^2+\frac{N}{2}\int_{\mathbb{R}^N}\nu_1u^2+\nu_2v^2 \\ &=\frac{2N-\mu}{2p}\int_{\mathbb{R}^N}\mu_1\left(\frac{1}{|x|^\mu}*|u|^p\right)|u|^p+2\beta\left(\frac{1}{|x|^\mu}*|u|^p\right)|v|^p+\mu_2\left(\frac{1}{|x|^\mu}*|v|^p\right)|v|^p.\ \square\end{aligned}$$

Proposition A.4. *Assume that Ω is a smooth bounded domain and $0\in\Omega$. Let $u,v\in H_0^1(\Omega)\cap L^{2Np/(2N-\mu)}(\Omega)$. If*

$$\tag{A.4}\begin{cases}-\Delta u+\nu_1u=\mu_1\left(\dfrac{1}{|x|^\mu}*|u|^p\right)|u|^{p-2}u \\ \qquad\qquad +\beta\left(\dfrac{1}{|x|^\mu}*|v|^p\right)|u|^{p-2}u, \quad x\in\Omega, \\ -\Delta v+\nu_2v=\mu_2\left(\dfrac{1}{|x|^\mu}*|v|^p\right)|v|^{p-2}v \\ \qquad\qquad +\beta\left(\dfrac{1}{|x|^\mu}*|u|^p\right)|v|^{p-2}v, \quad x\in\Omega, \\ u=v=0 \quad \text{on } \partial\Omega,\end{cases}$$

and $u,v\in W^{2,2}(\Omega)\cap W^{1,2Np/(2N-\mu)}(\Omega)$. Then

$$\begin{aligned}&\frac{N-2}{2}\int_\Omega|\nabla u|^2+|\nabla v|^2+\frac{1}{2}\int_{\partial\Omega}(|\nabla u|^2+|\nabla v|^2)(x\cdot\mathbf{n})\,d\sigma+\frac{N}{2}\int_\Omega\nu_1u^2+\nu_2v^2 \\ &=\frac{2N-\mu}{2p}\int_\Omega\mu_1\left(\frac{1}{|x|^\mu}*|u|^p\right)|u|^p+2\beta\left(\frac{1}{|x|^\mu}*|u|^p\right)|v|^p+\mu_2\left(\frac{1}{|x|^\mu}*|v|^p\right)|v|^p,\end{aligned}$$

where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$.

The proof is similar to that of Proposition A.3, and so we omit it.

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SONG YOU (corresponding author) AND PEIHAO ZHAO
School of Mathematics and Statistics
Lanzhou University
Lanzhou, Gansu 730000, P.R. CHINA

E-mail address: yous16@lzu.edu.cn, zhaoph@lzu.edu.cn

QINGXUN WANG
Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang 321004, P.R. CHINA

E-mail address: wangqx@zjnu.edu.cn