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ON GROUND STATE SOLUTIONS FOR THE NONLINEAR KIRCHHOFF TYPE PROBLEMS WITH A GENERAL CRITICAL NONLINEARITY

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Abstract. In this paper, we are concerned with the following Kirchhoff type problem with critical growth:

$$-\bigg(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)\Delta u+V(x)u=f(u)+|u|^4u,\quad u\in H^1(\mathbb{R}^3),$$

where $a,b>0$ are constants. Under certain assumptions on V and f, we prove that the above problem has a ground state solution of Nehari–Pohozaev type and a least energy solution via variational methods. Furthermore, we also show that the mountain pass value gives the least energy level for the above problem. Our results improve and extend some recent ones in the literature.

1. Introduction and statement of results

In this paper, we study the existence of ground state solutions for the following Kirchhoff type problem with a critical nonlinearity:

$$-\bigg(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)\Delta u+V(x)u=f(u)+|u|^4u,\quad x\in\mathbb{R}^3. \tag{1.1}$$

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Here, $a, b > 0$ are constants, on the potential V, we make the following assumptions:

- (V_1) $V \in \mathcal{C}(\mathbb{R}^3, [0,\infty))$;
- (V_2) For almost every $x \in \mathbb{R}^3$, $V(x) \leq \liminf_{|y|\to\infty} V(y) := V_\infty$ and the inequality is strict in a set of positive Lebesgue measure;
- (V_3) $V(x)$ is weakly differentiable and there exists $\theta \in [0,1)$ such that

$$(\nabla V(x), x) \leq \frac{\theta a}{2|x|^2}, \quad \text{a.e. } x \in \mathbb{R}^3 \setminus \{0\};$$

- (V_4) $V(x)$ is weakly differentiable and there exists $\theta \in [0,1)$ such that

$$4t^4[V(x) - V(tx)] - (1-t^4)(\nabla V(x), x) \geq -\frac{\theta a(1-t^2)^2}{2|x|^2},$$

  for all $t \geq 0$, $x \in \mathbb{R}^3 \setminus \{0\}$;

and we assume that f satisfies the following conditions:

- (F_1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(t) = o(t)$ as $t \to 0$;
- (F_2) f has a "quasicritical" growth, namely, $\lim_{|t|\to\infty} f(t)/t^5 = 0$;
- (F_3) there exist $D > 0$ and $2 < q < 6$ such that $f(t) \geq D|t|^{q-2}t$ for $t \in \mathbb{R}$;
- (F_4) $[f(t)t + 6F(t)]/|t|t$ is nondecreasing on $(-\infty, 0) \cup (0,\infty)$.

It is well known that under the above hypotheses, weak solutions for (1.1) correspond to critical points of the energy functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V(x)u^2\big) + \frac{b}{4}\bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6. \tag{1.2}$$

Problem (1.1) is often referred to be nonlocal because of the appearance of the term $\int_{\mathbb{R}^3} |\nabla u|^2$, which indicates that (1.1) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties and makes the study of such a problem particularly interesting. Indeed, if $\mathbb{R}^3$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^3$, then problem (1.1) describes the stationary state of the Kirchhoff type quasilinear hyperbolic equation of the following form:

$$u_{tt} - \bigg(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)\Delta u = f(t, x, u). \tag{1.3}$$

In [13], (1.3) was regarded as an extension of the classical d'Alembert's wave equation by sufficiently considering the effects of the changes in the length of the string during the vibrations. For more mathematical and physical background on Kirchhoff type problems, we refer the readers to [1], [8].

Recently, the Kirchhoff type problem

$$-\bigg(a + b\int_{\mathbb{R}^3} |\nabla u|^2\,dx\bigg)\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3 \tag{1.4}$$

has been well studied in a general dimension by various authors only after Lions [16] introduced an abstract functional analysis framework to such problems. See, for example, [2], [6], [7], [10], [14], [15], [17], [25], [27], [31], [33]–[37]. Let us briefly recall some known results on (1.4). For the case (1.4) with pure power nonlinearities, i.e.

$$(1.5)\qquad -\bigg(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)\Delta u+V(x)u=|u|^{p-2}u,\quad x\in\mathbb{R}^3,$$

when $2<p\le 4$, it is very difficult to verify the mountain pass geometry and the boundedness of (PS) sequences for the corresponding energy functional to (1.5). However, following the procedure of [23] which considered Schrödinger–Poisson system, Li and Ye [14] obtained the existence of a ground state solution of (1.5) for $3<p\le 4$. Afterwards, He and Li [12] proved that the existence and concentration of positive solutions of (1.1) with $f(u)=|u|^{p-2}u$ if $2<p\le 4$.

Very recently, Tang and Chen [30] extended the results obtained in [29] for Schrödinger–Poisson system to the Kirchhoff problem (1.4). Motivated by [9], they took the minimization on a new Nehari–Pohozaev manifold $\mathcal{M}_0=\{u\in H^1(\mathbb{R}^3)\setminus\{0\}:\langle I_0'(u),u\rangle/2+P_0(u)=0\}$ different from [12], [14]. Here, I_0 and P_0 are the corresponding energy functional and Pohozaev functional to (1.4), respectively. Then, under (V_1)–(V_4), (F_1), (F_4) and some additional hypotheses on f, the authors proved that a minimizer of I_0 on $\mathcal{M}_0$ exists (it will be called a ground state solution of Nehari–Pohozaev type), which improvements and generalizes the results in [9], [14].

In [18], Liu and Guo also generalized problem (1.5) in [14] to (1.4). However, different from [30], the authors assumed that V fulfills (V_2) and some suitable conditions. In addition, f satisfies (F_1), (F_2) and the following assumptions:

- (F_4') there exists $\mu>2$ such that $f(t)t\ge\mu F(t)>0$ for all $t\in\mathbb{R}\setminus\{0\}$;
- (F_5) there exists $\zeta>0$ such that

$$\inf_{x\in\mathbb{R}^3}G(x,\zeta):=\int_0^\zeta(f(s)-V(x)s)\,ds>0.$$

Afterwards, Liu and Guo [19] considered the following autonomous case of (1.1):

$$(1.6)\qquad -\bigg(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\bigg)\Delta u+u=f(u)+|u|^4u,\quad x\in\mathbb{R}^3,$$

to which the corresponding functional is defined in $H^1(\mathbb{R}^3)$ by

$$(1.7)\quad \overline{I}(u)=\frac12\int_{\mathbb{R}^3}\big(a|\nabla u|^2+u^2\big)+\frac b4\bigg(\int_{\mathbb{R}^3}|\nabla u|^2\bigg)^2-\int_{\mathbb{R}^3}F(u)-\frac16\int_{\mathbb{R}^3}|u|^6.$$

If $-u+f(u)$ satisfies (F_1)–(F_3) and f is odd. Then the authors established the existence of a least energy solution of (1.6) for $q\in(2,4]$ with D sufficiently large or $q\in(4,6)$.

Recently, Liu and Luo [20] extended the results in [18] to the critical case. Precisely, if V verifies the same assumptions as in [18] and f fulfills (F_1)–(F_3) and (F_4'). Then they proved that (1.1) admits a positive ground state solution for $q \in (2,4]$ with D sufficiently large or $q \in (4,6)$. To this end, the authors applied the Jeanjean monotonicity trick [11] and established a global compactness lemma, which extends the subcritical compactness result in [14] to critical case. Very recently, the similar arguments have been used in [21] in study of the fractional Kirchhoff type problem.

Motivated by all results mentioned previously, it is very natural for us to pose a series of interesting questions, in particular, such as:

(1) In [19], the least energy solution was obtained in the radially symmetric space $H_r^1(\mathbb{R}^3)$ due to the compact embedding of $H_r^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ with $p \in (2,6)$. If we use the standard space $H^1(\mathbb{R}^3)$ to take place of $H_r^1(\mathbb{R}^3)$, will (1.6) also admit a least energy solution in $H^1(\mathbb{R}^3)$?

(2) As we can see, the condition (F_4') appears necessary in the study [20], as well as in [18]. Can one establish the same results as described in [20] by replacing (F_4') with a weaker condition or other suitable condition?

(3) Since the critical case was not dealt with in [30], we would much like to know whether (1.1) and (1.6) respectively possess ground state solutions of Nehari–Pohozaev type, like that in [30]. Can we show it?

In this paper, we restrict our attention to the ground state solutions of (1.1) and (1.6) and are most interested in seeking definite answers to questions (1)–(3). To the best of our knowledge, little is known about the existence of ground state solutions of Nehari–Pohozaev type for (1.1) or (1.6). It is worth mentioning that (F_1)–(F_2) is firstly introduced by [3] in the study of ground state solutions to the nonlinear elliptic equations. Another aim of the paper is to extend Berestycki–Lions theorem to critical and non-radial case on Kirchhoff problem.

To answer question (3), motivated by [30], we set the manifolds

$$\mathcal{M} := \big\{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J(u) = 0\big\},$$
$$\overline{\mathcal{M}} := \big\{u \in H^1(\mathbb{R}^3) \setminus \{0\} : \overline{J}(u) = 0\big\},$$

where

$$\begin{aligned} J(u) = {} & a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4V(x) + (\nabla V(x), x)]u^2 \\ & + b\|\nabla u\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3}[f(u)u + 6F(u)] - \|u\|_6^6, \end{aligned} \tag{1.8}$$

$$\overline{J}(u) = a\|\nabla u\|_2^2 + 2\|u\|_2^2 + b\|\nabla u\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3}[f(u)u + 6F(u)] - \|u\|_6^6. \tag{1.9}$$

Then, it is easy to see that $\mathcal{M}$ and $\overline{\mathcal{M}}$ are the Nehari–Pohozaev manifold for (1.1) and (1.6), respectively.

Now we state our main results.

THEOREM 1.1. *Under the assumptions* (F_1)–(F_4), *assume that either* $q \in (2,4]$ *with* D *sufficiently large or* $q \in (4,6)$. *Then problem* (1.6) *has a solution* $u_0 \in H^1(\mathbb{R}^3)$ *such that* $\overline{I}(u_0) = \inf_{\overline{\mathcal{M}}} \overline{I} > 0$.

THEOREM 1.2. *Under the assumptions* (F_1)–(F_4), (V_1), (V_2) *and* (V_4), *assume that either* $q \in (2,4]$ *with* D *sufficiently large or* $q \in (4,6)$. *Then problem* (1.1) *has a solution* $u_0 \in H^1(\mathbb{R}^3)$ *such that* $I(u_0) = \inf_{\mathcal{M}} I > 0$.

THEOREM 1.3. *Under the assumptions* (F_1)–(F_4) *and* (V_1)–(V_3), *assume that either* $q \in (2,4]$ *with* D *sufficiently large or* $q \in (4,6)$. *Then problem* (1.1) *admits a least energy solution.*

THEOREM 1.4. *Under the assumptions* (F_1)–(F_4), *assume that either* $q \in (2,4]$ *with* D *sufficiently large or* $q \in (4,6)$. *Then problem* (1.6) *admits a least energy solution.*

REMARK 1.5. There are many functions satisfying (V_1)–(V_4). For instance, $V(x) = V_\infty - A/(|x|^2+1)$, where $V_\infty > 1$ and $0 < A < a/8$ are two constants. In addition, the function

$$f(s) = 2s \ln\left(1+s^2\right) + \frac{2s^3}{1+s^2}$$

satisfies (F_4) but not (F_4'). Hence, Theorem 1.3 gives an answer to question (2). Furthermore, Theorem 1.4 and Theorems 1.1–1.2 give an answer to questions (1) and (3), respectively.

The remainder of this paper is organized as follows. In Section 2 we give some preliminaries and the proof of Theorem 1.1. The proof of Theorem 1.2 will be given in Section 3. Section 4 is devoted to dealing with the proof of Theorems 1.3–1.4.

Notation. Throughout the article, we let $u_t(x) := t^{1/2}u(t^{-1}x)$ for $t > 0$ and denote by C, C_k, $k = 1,2,\ldots$ various positive constants whose exact value is inessential. For $r > 0$ and $y \in \mathbb{R}^3$, let $B_r(y)$ be the open ball in $\mathbb{R}^3$ with center y and radius r. We denote by $\to$ ($\rightharpoonup$) the strong (weak) convergence. We consider the Hilbert space $H^1(\mathbb{R}^3)$ with the norm

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right).$$

Denote the norm of $D^{1,2}(\mathbb{R}^3)$ by

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2$$

and the usual L^s-norm by $\|u\|_s$ for $s \geq 2$. Let S be the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ given by

$$S = \inf_{v \in D^{1,2}(\mathbb{R}^3),\, \|v\|_6 = 1} \|\nabla v\|_2^2.$$

Recall that I satisfies the PS condition at level c ($(\mathrm{PS})_c$ for short) if any sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ satisfying $I(u_n) \to c$ and $I'(u_n) \to 0$ in $H^{-1}(\mathbb{R}^3)$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$.

2. Ground state solutions for the case $V(x) \equiv 1$

In this section we will prove that a ground state solution of Nehari–Pohozaev type for problem (1.6) can be obtained and it is the minimizer of $\overline{I}$ on the manifold $\overline{\mathcal{M}}$. In addition, another aim of this section is to formulate the existence of a ground state solution of Nehari-Pohozaev type for the associated "limited problem" of (1.1)

$$-\bigg(a + b \int_{\mathbb{R}^3} |\nabla u|^2\, dx\bigg) \Delta u + V_\infty u = f(u) + |u|^4 u, \quad x \in \mathbb{R}^3. \tag{2.1}$$

Its functional is given in $H^1(\mathbb{R}^3)$ by

$$I^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^3} \big(a|\nabla u|^2 + V_\infty u^2\big) + \frac{b}{4} \bigg(\int_{\mathbb{R}^3} |\nabla u|^2\bigg)^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6. \tag{2.2}$$

Set $\mathcal{M}^\infty := \big\{u \in H^1(\mathbb{R}^3) \setminus \{0\} : J^\infty(u) = 0\big\}$, where

$$J^\infty(u) = a\|\nabla u\|_2^2 + 2V_\infty \|u\|_2^2 + b\|\nabla u\|_2^4 - \frac{1}{2} \int_{\mathbb{R}^3} [f(u)u + 6F(u)] - \|u\|_6^6. \tag{2.3}$$

To show some properties of $\overline{\mathcal{M}}$, we first give the following lemma.

Lemma 2.1. *Assume that* (F_1), (F_2) *and* (F_4) *hold. Then, for any* $t > 0$ *and* $u \in H^1(\mathbb{R}^3)$,

$$\overline{I}(u) \geq \overline{I}(u_t) + \frac{1 - t^4}{4} \overline{J}(u) + \frac{a(1 - t^2)^2}{4} \|\nabla u\|_2^2 + \frac{2t^6 - 3t^4 + 1}{12} \|u\|_6^6. \tag{2.4}$$

In particular,

$$I^\infty(u) \geq I^\infty(u_t) + \frac{1 - t^4}{4} J^\infty(u) + \frac{a(1 - t^2)^2}{4} \|\nabla u\|_2^2 + \frac{2t^6 - 3t^4 + 1}{12} \|u\|_6^6. \tag{2.5}$$

Proof. For any $t > 0$, $\tau \in \mathbb{R}$, (F_4) yields

$$\begin{aligned} &\frac{1 - t^4}{8} f(\tau)\tau - \frac{1 + 3t^4}{4} F(\tau) + t^3 F(t^{1/2}\tau) \\ &\quad = \int_t^1 \frac{1}{2} s^3 \tau^2 \bigg[\frac{f(\tau)\tau + 6F(\tau)}{\tau^2} - \frac{f(s^{1/2}\tau)s^{1/2}\tau + 6F(s^{1/2}\tau)}{s\tau^2}\bigg]\, ds \geq 0. \end{aligned} \tag{2.6}$$

Note that

$$\overline{I}(u_t) = \frac{at^2}{2} \|\nabla u\|_2^2 + \frac{t^4}{2} \|u\|_2^2 + \frac{bt^4}{4} \|\nabla u\|_2^4 - t^3 \int_{\mathbb{R}^3} F(t^{1/2}u) - \frac{t^6}{6} \|u\|_6^6. \tag{2.7}$$

Hence, by (1.7), (1.9), (2.6) and (2.7), we get

$$\begin{aligned}\overline{I}(u)-\overline{I}(u_t) &= \frac{1}{2}\int_{\mathbb{R}^3}\left[a(1-t^2)|\nabla u|^2+(1-t^4)u^2\right]+\frac{b(1-t^4)}{4}\|\nabla u\|_2^4\\ &\quad+\int_{\mathbb{R}^3}\left(t^3F(t^{1/2}u)-F(u)\right)+\frac{t^6-1}{6}\int_{\mathbb{R}^3}u^6\\ &=\frac{(1-t^4)}{4}\,\overline{J}(u)+\frac{2t^6-3t^4+1}{12}\,\|u\|_6^6+\frac{a(1-t^2)^2}{4}\,\|\nabla u\|_2^2\\ &\quad+\int_{\mathbb{R}^3}\left[\frac{1-t^4}{8}\,f(u)u-\frac{1+3t^4}{4}\,F(u)+t^3F(t^{1/2}u)\right]\\ &\geq\frac{(1-t^4)}{4}\,\overline{J}(u)+\frac{2t^6-3t^4+1}{12}\,\|u\|_6^6+\frac{a(1-t^2)^2}{4}\,\|\nabla u\|_2^2,\end{aligned}$$

which implies that (2.4) holds. Similarly, one gets (2.5). □

LEMMA 2.2. *Assume that* (F_1), (F_2) *and* (F_4) *hold. Then, for* $u \in H^1(\mathbb{R}^3)\setminus\{0\}$, *there exists a unique* $t(u)>0$ *such that* $u_{t(u)} \in \overline{\mathcal{M}}$. *Moreover,*

$$\overline{I}(u_{t(u)})=\max_{t\geq 0}\overline{I}(u_t).$$

PROOF. Fix $u \in H^1(\mathbb{R}^3)\setminus\{0\}$ and consider a function $\xi(t) := I(u_t)$ on $[0,\infty)$. By (F_1), (F_2) and (2.7), it is easy to check that $\xi(0)=0$, $\xi(t)>0$ for $t>0$ small and $\xi(t)<0$ for t large. Hence, $\max_{t\geq 0}\xi(t)$ is achieved at $t_0=t(u)>0$ and then $\xi'(t_0)=0$, that is,

$$at_0^2\|\nabla u\|_2^2+2t_0^4\|u\|_2^2+bt_0^4\|\nabla u\|_2^4-\frac{t_0^3}{2}\int_{\mathbb{R}^3}\left[f\left(t_0^{1/2}u\right)t_0^{1/2}u+6F\left(t_0^{1/2}u\right)\right]-t_0^6\|u\|_6^6=0.$$

This shows that $\overline{J}(u_{t_0})=0$ and $u_{t_0}\in\overline{\mathcal{M}}$. Next, we prove that $t(u)$ is unique for any $u\in H^1(\mathbb{R}^3)\setminus\{0\}$. Suppose arguing by contradiction that there exist $t_1,t_2>0$ and $t_2=st_1,\ s>1$ such that for given $u\in H^1(\mathbb{R}^3)\setminus\{0\}$, $u_{t_1},u_{t_2}\in\overline{\mathcal{M}}$. Then $\overline{J}(u_{t_1})=\overline{J}(u_{t_2})=0$. Together with (2.4), one has

$$\begin{aligned}\overline{I}(u_{t_1}) &\geq \overline{I}(u_{st_1})+\frac{a(1-s^2)^2}{4}\,\|\nabla u_{t_1}\|_2^2+\frac{2s^6-3s^4+1}{12}\,\|u_{t_1}\|_6^6\\ &\geq \overline{I}(u_{t_2})+\frac{a(1-s^2)^2}{4}\,t_1^2\|\nabla u\|_2^2+\frac{2s^6-3s^4+1}{12}\,t_1^6\|u\|_6^6,\end{aligned}$$

Similarly,

$$\overline{I}(u_{t_2})\geq\overline{I}(u_{t_1})+\frac{a(1-s^{-2})^2}{4}\,t_2^2\|\nabla u\|_2^2+\frac{2s^{-6}-3s^{-4}+1}{12}\,t_2^6\|u\|_6^6,$$

Then we have

$$\begin{aligned}\frac{a}{4}\left[\left(1-s^2\right)^2t_1^2+\left(1-s^{-2}\right)^2t_2^2\right]\|\nabla u\|_2^2 &\\ +\frac{1}{12}\left[\left(2s^6-3s^4+1\right)t_1^6+\left(2s^{-6}-3s^{-4}+1\right)t_2^6\right]\|u\|_6^6 &\leq 0,\end{aligned}\tag{2.8}$$

i.e.

$$\big[\big(2s^6 - 3s^4 + 1\big)t_1^6 + \big(2s^{-6} - 3s^4 + 1\big)t_2^6\big] \le 0. \tag{2.9}$$

By a simple calculation, we know that if $t \neq 1$, then

$$g(t) = 2t^6 - 3t^4 + 1 > g(1) = 0, \quad \text{for all } t \ge 0. \tag{2.10}$$

This contradicts with (2.9). Hence, $t(u)$ is unique for any $u \in H^1(\mathbb{R}^3) \setminus \{0\}$. Finally, it follows from Lemma 2.1 and (2.10) that, for $u \in \overline{\mathcal{M}}$, $\overline{I}(u) = \max_{t \ge 0} \overline{I}(u_t)$.□

LEMMA 2.3. *Assume that* (F_1), (F_2) *and* (F_4) *hold. Then* $\overline{I}$ *possesses the mountain pass geometry.*

PROOF. From (F_1)–(F_2), there exists $C_1 > 0$ such that

$$F(t) \le \frac{1}{2}\min\{a, 1\}t^2 + C_1 t^6, \ \forall t \in \mathbb{R}. \tag{2.11}$$

By (1.7) and (2.11), we see that there exist $\rho, \alpha > 0$ such that

$$\overline{I}(u) \ge \frac{1}{4}\min\{a, 1\}\|u\|^2 - C_2\|u\|^6 \ge \alpha > 0$$

for $\|u\| = \rho > 0$ small. Fix $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, by (2.7), one has $I(u_t) < 0$ for $t > 0$ large, then there exists $t_0 > 0$, set $v_0 := u_{t_0}$, $I(v_0) < 0$. □

As in [24], set the mountain pass level of $\overline{I}$:

$$c_0 = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \overline{I}(\gamma(t)),$$

where $\Gamma := \big\{\gamma \in \mathcal{C}([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } \overline{I}(\gamma(1)) < 0\big\}$. Then the above lemma implies that $c_0 \ge \alpha$ and $\overline{I}$ possesses a (PS) sequence $\{u_n\}$ for c_0. To describe the property of the sequence $\{u_n\}$, we introduce the general minimax principle as follows:

LEMMA 2.4 ([32, Theorem 2.8]). *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \in \mathcal{C}(M_0, X)$. Define*

$$\Gamma := \{\gamma \in \mathcal{C}(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in \mathcal{C}(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u))$$

then, for every $\varepsilon \in (0, (c - a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \le c + \varepsilon$, there exists $u \in X$ such that

(a) $c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon$,
(b) $\mathrm{dist}(u, \gamma(M)) \le 2\delta$,
(c) $\|\varphi'(u)\| \le 8\varepsilon/\delta$.

Motivated by [12], we show the following lemma.

LEMMA 2.5. *Assume that* (F_1), (F_2) *and* (F_4) *hold. Then there exists sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *such that, as* $n \to \infty$,

$$\overline{I}(u_n) \to c_0, \qquad \overline{I}'(u_n) \to 0, \qquad \overline{J}(u_n) \to 0. \tag{2.12}$$

PROOF. Define a map $\Phi\colon \mathbb{R} \times H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ for $\theta \in \mathbb{R}$, $v \in H^1(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$ by $\Phi(\theta, v) = e^{\theta/2} v(e^{-\theta} x)$. For any $\theta \in \mathbb{R}$, $v \in H^1(\mathbb{R}^3)$, the functional $\overline{I} \circ \Phi$ is computed as

$$\begin{aligned} \overline{I} \circ \Phi(\theta, v) = \frac{a}{2} e^{2\theta} \|\nabla v\|_2^2 + \frac{1}{2} e^{4\theta} \int_{\mathbb{R}^3} v^2 \\ + \frac{b}{4} e^{4\theta} \|\nabla v\|_2^4 - e^{3\theta} \int_{\mathbb{R}^3} F(e^{\theta/2} v) - \frac{1}{6} e^{6\theta} \int_{\mathbb{R}^3} v^6. \end{aligned}$$

Similar to Lemma 2.3, we can easily verify that $\overline{I} \circ \Phi(\theta, v) > 0$ for all (θ, v) with θ, $\|v\|$ small and $\overline{I} \circ \Phi(0, v_0) < 0$, i.e. $\overline{I} \circ \Phi$ has the mountain pass geometry in $\mathbb{R} \times H^1(\mathbb{R}^3)$. Hence, set

$$\widetilde{c_0} = \inf_{\widetilde{\gamma} \in \widetilde{\Gamma}} \sup_{t \in [0,1]} \overline{I} \circ \Phi(\widetilde{\gamma}(t)),$$

where $\widetilde{\Gamma} := \{\widetilde{\gamma} \in \mathcal{C}([0,1], \mathbb{R} \times H^1(\mathbb{R}^3)) : \widetilde{\gamma}(0) = (0,0) \text{ and } \overline{I} \circ \Phi(\widetilde{\gamma}(1)) < 0\}$. As $\Gamma = \{\Phi \circ \widetilde{\gamma} : \widetilde{\gamma} \in \widetilde{\Gamma}\}$, the mountain pass level of $\overline{I}$ and $\overline{I} \circ \Phi$ coincide, i.e. $c_0 = \widetilde{c_0}$. By Lemma 2.4, we see that there exists a sequence $\{(\theta_n, v_n)\} \subset \mathbb{R} \times H^1(\mathbb{R}^3)$ such that as $n \to \infty$,

$$\overline{I} \circ \Phi(\theta_n, v_n) \to c_0, \qquad (\overline{I} \circ \Phi)'(\theta_n, v_n) \to 0. \tag{2.13}$$

Then we claim $\theta_n \to 0$ as $n \to \infty$. Indeed, set $\varepsilon = \varepsilon_n := 1/n^2$, $\delta = \delta_n := 1/n$ in Lemma 2.4. For $\varepsilon = \varepsilon_n := 1/n^2$, there exists $\gamma_n \in \Gamma$ such that

$$\sup_{t \in [0,1]} \overline{I}(\gamma_n(t)) \le c_0 + \frac{1}{n^2}.$$

Set $\widetilde{\gamma}_n(t) = (0, \gamma_n(t))$, then

$$\sup_{t \in [0,1]} \overline{I} \circ \Phi(\widetilde{\gamma}_n(t)) = \sup_{t \in [0,1]} \overline{I}(\gamma_n(t)) \le c_0 + \frac{1}{n^2}.$$

By (b) of Lemma 2.4, there exists $(\theta_n, v_n) \in \mathbb{R} \times H^1(\mathbb{R}^3)$ such that

$$\operatorname{dist}((\theta_n, v_n), (0, \gamma_n(t))) \le \frac{2}{n},$$

then $\theta_n \to 0$ as $n \to \infty$.

Next we show that, for any $(h, w) \in \mathbb{R} \times H^1(\mathbb{R}^3)$,

$$\langle (\overline{I} \circ \Phi)'(\theta_n, v_n), (h, w) \rangle = \langle \overline{I}'(\Phi(\theta_n, v_n)), \Phi(\theta_n, w) \rangle + \overline{J}(\Phi(\theta_n, v_n)) h. \tag{2.14}$$

Indeed,

$$\begin{aligned}(2.15)\quad &\big\langle(\overline{I}\circ\Phi)'(\theta_n,v_n),(h,w)\big\rangle\\ &\qquad= \lim_{t\to 0}\frac{1}{t}\big[\big(\overline{I}\circ\Phi\big)'(\theta_n+th,v_n+tw)-\big(\overline{I}\circ\Phi\big)(\theta_n,v_n)\big]=\sum_{i=1}^{5}\overline{I}_i,\end{aligned}$$

where

$$\begin{aligned}
\overline{I}_1&=\lim_{t\to 0}\frac{a}{2t}\big[e^{2(\theta_n+th)}\|\nabla(v_n+tw)\|_2^2-e^{2\theta_n}\|\nabla v_n\|_2^2\big],\\
\overline{I}_2&=\lim_{t\to 0}\frac{1}{2t}\bigg[e^{4(\theta_n+th)}\int_{\mathbb{R}^3}|v_n+tw|^2-e^{4\theta_n}\int_{\mathbb{R}^3}|v_n|^2\bigg],\\
\overline{I}_3&=\lim_{t\to 0}\frac{b}{4t}\big[e^{4(\theta_n+th)}\|\nabla(v_n+tw)\|_2^4-e^{4\theta_n}\|\nabla v_n\|_2^4\big],\\
\overline{I}_4&=-\lim_{t\to 0}\frac{1}{t}\bigg[e^{3(\theta_n+th)}\int_{\mathbb{R}^3}F(e^{(\theta_n+th)/2}(v_n+tw))-e^{3\theta_n}\int_{\mathbb{R}^3}F(e^{\theta_n/2}v_n)\bigg],\\
\overline{I}_5&=-\lim_{t\to 0}\frac{1}{6t}\bigg[e^{6(\theta_n+th)}\int_{\mathbb{R}^3}|v_n+tw|^6-e^{6\theta_n}\int_{\mathbb{R}^3}v_n^6\bigg].
\end{aligned}$$

By Mean Value Theorem, one has

$$(2.16)\qquad\begin{aligned}
\overline{I}_1=&\,ahe^{2\theta_n}\int_{\mathbb{R}^3}|\nabla v_n|^2+ae^{2\theta_n}\int_{\mathbb{R}^3}\nabla v_n\nabla w,\\
\overline{I}_2=&\,2he^{4\theta_n}\int_{\mathbb{R}^3}|v_n|^2+e^{4\theta_n}\int_{\mathbb{R}^3}v_nw,\\
\overline{I}_3=&\,bhe^{4\theta_n}\bigg(\int_{\mathbb{R}^3}|\nabla v_n|^2\bigg)^2+be^{4\theta_n}\int_{\mathbb{R}^3}|\nabla v_n|^2\int_{\mathbb{R}^3}\nabla v_n\nabla w,\\
\overline{I}_4=&-\frac{h}{2}e^{3\theta_n}\int_{\mathbb{R}^3}\big[f(e^{\theta_n/2}v_n)e^{\theta_2}v_n+3F(e^{\theta_n}/2v_n)\big]\\
&-e^{3\theta_n}\int_{\mathbb{R}^3}f(e^{\theta_n/2}v_n)e^{\theta_n/2}w,\\
\overline{I}_5=&-he^{6\theta_n}\int_{\mathbb{R}^3}|v_n|^6-e^{6\theta_n}\int_{\mathbb{R}^3}|v_n|^4v_nw.
\end{aligned}$$

This means (2.14) holds. Taking $h=1$, $w=0$ in (2.14), we have

$$\overline{J}(\Phi(\theta_n,v_n))\to 0,\quad \text{as } n\to\infty.$$

Denote $u_n:=\Phi(\theta_n,v_n)$, we get $\overline{J}(u_n)\to 0$, as $n\to\infty$. For any $v\in H^1(\mathbb{R}^3)$, set $w(x)=e^{-\theta/2}v(e^{\theta}x)$, $h=0$ in (2.14), we get

$$o(1)\|w\|=\big\langle\overline{I}'(u_n),v\big\rangle=o(1)\|v\|$$

for $\theta_n\to 0$ as $n\to\infty$, i.e. $\overline{I}'(u_n)\to 0$ in $(H^1(\mathbb{R}^3))^{-1}$ as $n\to\infty$. Hence, we have got a sequence $\{u_n\}\subset H^1(\mathbb{R}^3)$ satisfying (2.12). □

Moreover, using the same arguments as in [22], we also have the following equivalent characterization of c_0:

$$c_0 = \inf_{u\in H^1(\mathbb{R}^3)\setminus\{0\}} \max_{t\geq 0} \overline{I}(u_t) = \inf_{u\in\overline{\mathcal{M}}} \overline{I}(u) > 0. \tag{2.17}$$

In the following lemma, we devote to estimating the mountain pass level c_0.

LEMMA 2.6. *Assume that* (F_1)–(F_4) *hold and either* $q \in (2,4]$ *with* D *sufficiently large or* $q \in (4,6)$. *Then*

$$c_0 < \Lambda := \frac{1}{4}\, abS^3 + \frac{1}{24}\, b^3S^6 + \frac{1}{24}\left(b^2S^4 + 4aS\right)^{3/2}.$$

PROOF. Let $\eta(x) \in C_0^\infty(\mathbb{R}^3,[0,1])$ is such that $\eta(x) = 1$ for $|x| \leq R$ and $\eta(x) = 0$ for $|x| \geq 2R$ for some $R > 0$. Given $\varepsilon > 0$, set $v_\varepsilon := \eta w_\varepsilon$, where

$$w_\varepsilon := \frac{3^{1/4}\varepsilon^{1/4}}{(\varepsilon+|x|^2)^{1/2}}$$

is a family of functions on which S is attained. Then

$$\int_{R^3} |\nabla w_\varepsilon|^2 = \int_{R^3} |w_\varepsilon|^6 = S^{3/2}.$$

It is well known that the following asymptotic estimates hold for ε small enough (see [5]):

$$\|\nabla v_\varepsilon\|_2^2 = \int_{\mathbb{R}^3} \frac{|x|^2}{(1+|x|^2)^3} + O(\varepsilon^{1/2}) := A_1 + O\big(\varepsilon^{1/2}\big), \tag{2.18}$$

$$\|v_\varepsilon\|_6^6 = \int_{\mathbb{R}^3} \frac{1}{(1+|x|^2)^3} + O(\varepsilon^{3/2}) := A_2 + O\big(\varepsilon^{3/2}\big), \tag{2.19}$$

$$\|v_\varepsilon\|_s^s = \begin{cases} O\big(\varepsilon^{s/4}\big) & \text{if } s\in[2,3),\\ O\big(\varepsilon^{s/4}|\ln\varepsilon|\big) & \text{if } s=3,\\ O\big(\varepsilon^{(6-s)/4}\big) & \text{if } s\in(3,6), \end{cases} \tag{2.20}$$

where A_1 and A_2 are positive constants and $S = A_1/A_2^{1/3}$.

In view of Lemma 2.2 and (2.17), we infer that there exists $t_\varepsilon > 0$ such that $\overline{I}((v_\varepsilon)_{t_\varepsilon}) = \max_{t\geq 0} \overline{I}((v_\varepsilon)_t)$ and then $c_0 \leq \overline{I}((v_\varepsilon)_{t_\varepsilon})$. We just need to verify $\overline{I}((v_\varepsilon)_{t_\varepsilon}) < \Lambda$. We first claim that that for $\varepsilon > 0$ small enough, there exist constants t_1 and t_2 independent of ε such that $0 < t_1 \leq t_{\varepsilon,z} \leq t_2 < \infty$. In fact, for the mountain pass level c_0, it follows from Lemma 2.3 that $\overline{I}((v_\varepsilon)_{t_\varepsilon}) \geq c_0 \geq \alpha > 0$. Then from the continuity of $\overline{I}$, we can assume that $t_\varepsilon \geq t_1 > 0$. On the other hand, since $(v_\varepsilon)_{t_\varepsilon} \in \overline{\mathcal{M}}$, we have $\overline{J}((v_\varepsilon)_{t_\varepsilon}) = 0$. Noting that $F(t) \geq 0$ for $t \in \mathbb{R}$ and (1.9), one gets

$$\big(at_\varepsilon^2 + 2t_\varepsilon^4\big)\|v_\varepsilon\|^2 + bt_\varepsilon^4\|v_\varepsilon\|^4 \geq t_\varepsilon^6\|v_\varepsilon\|_6^6.$$

Joint with (2.18)–(2.20), we have

$$t_\varepsilon^6\big(A_2 + O\big(\varepsilon^{3/2}\big)\big) \leq \big(at_\varepsilon^2 + 2t_\varepsilon^4\big)\|v_\varepsilon\|^2 + bt_\varepsilon^4\|v_\varepsilon\|^4.$$

Then there exists $t_2 > 0$ such that $t_\varepsilon < t_2$ since $\|v_\varepsilon\|$ is bounded for ε small enough.

Now we estimate $\overline{I}((v_\varepsilon)_{t_\varepsilon})$. Define function

$$h(t) := \frac{at^2}{2}\,\|\nabla v_\varepsilon\|_2^2 + \frac{bt^4}{4}\,\|\nabla v_\varepsilon\|_2^4 - \frac{t^6}{6}\,\|v_\varepsilon\|_6^6.$$

It is clear that $h(t)$ attains its maximum at

$$t_h = \bigg(\frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^2\|v_\varepsilon\|_6^6}}{2\|v_\varepsilon\|_6^6}\bigg)^{1/2}.$$

Using (2.18)–(2.20), we have

$$h(t_h) = \frac{ab\|\nabla v_\varepsilon\|_2^6}{4\|v_\varepsilon\|_6^6} + \frac{1}{24}\bigg(\frac{\|\nabla v_\varepsilon\|_2^8}{\|v_\varepsilon\|_6^8} + \frac{\|\nabla v_\varepsilon\|_2^2}{\|v_\varepsilon\|_6^2}\bigg) + \frac{b^3\|\nabla v_\varepsilon\|_2^{12}}{24\|v_\varepsilon\|_6^{12}} = \Lambda + O(\varepsilon^{1/2})$$

and hence

$$\begin{aligned} \overline{I}((v_\varepsilon)_{t_\varepsilon}) &\le \Lambda + O(\varepsilon^{1/2}) + \frac{t_2^4}{2}\int_{\mathbb{R}^3} v_\varepsilon^2 - \frac{D}{q}t_1^{(q+6)/2}\int_{\mathbb{R}^3} v_\varepsilon^q \\ &\le \Lambda + O(\varepsilon^{1/2}) - C_3 D\int_{\mathbb{R}^3} v_\varepsilon^q. \end{aligned} \tag{2.21}$$

By a standard argument, one can obtain $c_0 < \Lambda$. □

Proof of Theorem 1.1. Let c_0 be the mountain pass value for $\overline{I}$ and $\{u_n\}$ satisfy (2.12). From (2.12), (2.7) and (2.4) with $t \to 0$, we get

$$c_0 + o(1) = \overline{I}(u_n) \ge \frac{a}{4}\,\|\nabla u_n\|_2^2 + \frac{1}{12}\,\|u_n\|_6^6, \tag{2.22}$$

which means that $\{\|\nabla u_n\|_2\}$ is bounded. Then by (F_1), (F_2), (1.9) and Sobolev embedding theorem,

$$\begin{aligned} \min\{a,2\}\|u_n\|^2 &\le a\|\nabla u_n\|_2^2 + 2\|u_n\|_2^2 + b\|\nabla u_n\|_2^4 \\ &= \frac{1}{2}\int_{\mathbb{R}^3}[f(u_n)u_n + 6F(u_n)] + \|u_n\|_6^6 \\ &\le \frac{1}{2}\min\{a,2\}\|u_n\|^2 + C_4\|u_n\|_6^6 \\ &\le \frac{1}{2}\min\{a,2\}\|u_n\|^2 + C_4S^{-3}\|\nabla u_n\|_2^6. \end{aligned}$$

This shows that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. We claim that there exist $r,\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\limsup_{n\to\infty}\int_{B_r(y_n)} |u_n|^2 \ge \delta. \tag{2.23}$$

Arguing by contradiction, suppose $\{u_n\}$ is vanishing. Then Lions' Vanishing Lemma [32, Lemma 1.21] implies that $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Note that (F_1)–(F_3) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(u)| \le \varepsilon\big(|u| + |u|^5\big) + C_\varepsilon|u|^{s-1}.$$

Joining with (2.12), we have

$$(2.24)\qquad \begin{aligned} c_0 + o(1) &= \frac{a}{2}\,\|\nabla u_n\|_2^2 + \frac{1}{2}\,\|u_n\|_2^2 + \frac{b}{4}\,\|\nabla u_n\|_2^4 - \frac{1}{6}\,\|u_n\|_6^6, \\ o(1) &= a\|\nabla u_n\|_2^2 + \|u_n\|_2^2 + b\|\nabla u_n\|_2^4 - \|u_n\|_6^6. \end{aligned}$$

Then, up to subsequence, we have

$$(2.25)\qquad \|\nabla u_n\|_2^2 \to l_1 \geq 0 \quad \text{and} \quad \|u_n\|_6^6 \to l_2, \quad \text{as } n \to \infty.$$

Suppose $l_1 = 0$, then it is easy to obtain that $c_0 = 0$. This is impossible, since c_0 is mountain pass value of $\overline{I}$. Thus $l_1 > 0$. Then we deduce from (2.24) and $S(l_2)^{1/3} \leq l_1$ that

$$a l_1 + b l_1^2 \leq l_2 \quad \text{and} \quad \frac{bS^2 + \sqrt{b^2S^4 + 4aS}}{2} \leq l_2^{1/3}.$$

Then

$$\begin{aligned} c_0 + o(1) &= \overline{I}(u_n) - \frac{1}{4}\langle \overline{I}'(u_n), u_n\rangle \\ &= \frac{a}{4}\,\|\nabla u_n\|_2^2 + \frac{1}{4}\,\|u_n\|_2^2 + \frac{1}{12}\,\|u_n\|_6^6 \geq \frac{a}{4}\,l_1 + \frac{1}{12} l_2 \geq \frac{aS}{4}(l_2)^{1/3} + \frac{1}{12}\,l_2 \\ &\geq \frac{1}{4}\,aS\,\frac{bS^2 + \sqrt{b^2S^4 + 4aS}}{2} + \frac{1}{12}\left(\frac{bS^2 + \sqrt{b^2S^4 + 4aS}}{2}\right)^3 \\ &= \frac{1}{4}\,abS^3 + \frac{1}{24}\,b^3S^6 + \frac{1}{24}(b^2S^4 + 4aS)^{3/2} = \Lambda, \end{aligned}$$

which contradicts with Lemma 2.6. Thus (2.23) holds.

Set $v_n(x) = u_n(x + y_n)$. Then we have $\|v_n\| = \|u_n\|$ and $\{v_n\}$ satisfies $\int_{B_r(0)} |v_n|^2 > \delta$ and

$$(2.26)\qquad \overline{I}(v_n) \to c_0, \quad \overline{I}'(v_n) \to 0, \quad \overline{J}(v_n) \to 0.$$

Hence, there exists $v \in H^1(\mathbb{R}^3)\backslash\{0\}$ such that, taking a subsequence if necessary, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Then $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $s \in [1,6)$ and $v_n \to v$ for almost every x in $\mathbb{R}^3$. It is easy to see that v satisfies

$$(2.27)\qquad -\big(a + bl^2\big)\Delta v + v = f(v) + |v|^4 v,$$

where $l^2 := \lim\limits_{n\to\infty} \|\nabla v_n\|_2^2$ and $\|\nabla v\|_2^2 \leq l^2$. The corresponding functional to (2.27) is defined by

$$E(u) = \frac{a + bl^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} u^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6.$$

As in Lemma 2.2 in [20], since $E'(v) = 0$, we have the Pohozaev identity applying to (2.27)

$$(2.28)\qquad P_E(v) = \frac{a + bl^2}{2}\,\|\nabla v\|_2^2 + \frac{3}{2}\,\|v\|_2^2 - 3\int_{\mathbb{R}^3} F(v) - \frac{1}{2}\,\|v\|_6^6 = 0.$$

It follows from $\|\nabla v\|_2^2 \le l^2$ and (1.9) that

$$\begin{aligned} \overline{J}(v) &= a\|\nabla v\|_2^2 + 2\|v\|_2^2 + b\|\nabla v\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3}[f(v)v + 6F(v)] - \|v\|_6^6 \\ &\le \big(a + bl^2\big)\|\nabla v\|_2^2 + 2\|v\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^3}[f(v)v + 6F(v)] - \|v\|_6^6 \\ &= \frac{1}{2}\big\langle E'(v), v\big\rangle + P_E(v) := J_E(v) = 0. \end{aligned} \tag{2.29}$$

Since $v \in H^1(\mathbb{R}^3) \setminus \{0\}$, in view of Lemma 2.2, there exists $t_v > 0$ such that $v_{t_v} \in \overline{\mathcal{M}}$. By (1.7), (1.9), (2.4), (2.26), (2.29) and Fatou's lemma, we infer that

$$\begin{aligned} E(v) &= E(v) - \frac{1}{4}\,J_E(v) \\ &= \frac{a + bl^2}{4}\,\|\nabla v\|_2^2 + \frac{1}{8}\int_{\mathbb{R}^3}[f(v)v - 2F(v)] + \frac{1}{12}\,\|v\|_6^6 \\ &= \frac{bl^2}{4}\,\|\nabla v\|_2^2 + \overline{I}(v) - \frac{1}{4}\,\overline{J}(v) \\ &\ge \frac{bl^2}{4}\,\|\nabla v\|_2^2 + \overline{I}(v_{t_v}) - \frac{t_v^4}{4}\,\overline{J}(v) + \frac{2t_v^6 - 3t_v^4 + 1}{12}\,\|v\|_6^6 \\ &\ge \frac{bl^2}{4}\|\nabla v\|_2^2 + c_0 = \frac{bl^2}{4}\,\|\nabla v\|_2^2 + \lim_{n\to\infty}\bigg(\overline{I}(v_n) - \frac{1}{4}\,\overline{J}(v_n)\bigg) \\ &= \frac{bl^2}{4}\|\nabla v\|_2^2 + \lim_{n\to\infty}\bigg(\frac{a}{4}\,\|\nabla v_n\|_2^2 \\ &\qquad + \frac{1}{8}\int_{\mathbb{R}^3}[f(v_n)v_n - 2F(v_n)] + \frac{1}{12}\int_{\mathbb{R}^3}|v_n|^6\bigg) \\ &\ge \frac{a + bl^2}{4}\,\|\nabla v\|_2^2 + \frac{1}{8}\int_{\mathbb{R}^3}[f(v)v - 2F(v)] + \frac{1}{12}\int_{\mathbb{R}^3}|v|^6 = E(v). \end{aligned} \tag{2.30}$$

Thus $\overline{J}(v) = 0$, $t_v = 1$ and $\overline{I}(v) = c_0$. □

Remark 2.7. Similarly, under the assumptions (F_1)–(F_4), the "limited problem" (2.1) admits a solution $v \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $I^\infty(v) = \inf_{\mathcal{M}^\infty} I^\infty > 0$. Furthermore, let $\widetilde{f}(t) = 0$ for $t < 0$ and $\widetilde{f}(t) = f(t)$ for $t \ge 0$. It is easy to see that $\widetilde{f}$ fulfills (F_1)–(F_4). Using $\widetilde{f}$ instead of f in (2.1), we also can prove that $v > 0$ by the standard elliptic estimate and strong maximum principle.

3. Ground state solutions for (1.1)

In this section we will prove that a ground state solution of Nehari–Pohozaev type for problem (1.1) can be obtained. From now on we assume that (F_1)–(F_4), (V_1), (V_2) and (V_4) hold. By Lemma 2.5 in [30], we have the norm

$$\|u\|_0 := \bigg(a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4V(x) + (\nabla V(x), x)]u^2\bigg)^{1/2}$$

is equivalent to $\|\cdot\|$ in $H^1(\mathbb{R}^3)$.

Similar to Lemma 2.1, we have

LEMMA 3.1. *For all* $u \in H^1(\mathbb{R}^3)$ *and* $t > 0$

$$I(u) \geq I(u_t) + \frac{1-t^4}{4} J(u) + \frac{a(1-\theta)(1-t^2)^2}{4} \|\nabla u\|_2^2 + \frac{2t^6 - 3t^4 + 1}{12} \|u\|_6^6. \tag{3.1}$$

PROOF. Joint with Hardy inequality

$$\int_{\mathbb{R}^3} |\nabla u|^2 \geq \frac{1}{4} \int_{\mathbb{R}^3} \frac{u^2}{|x|^2}, \quad \text{for all } u \in H^1(\mathbb{R}^3), \tag{3.2}$$

the proof is analogous to Lemma 2.1. So we omit it here. □

Using the arguments in Lemma 2.2, we can prove the following lemma with the help of Lemma 3.1.

LEMMA 3.2. *For* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, *there exists a unique* $t(u) > 0$ *such that* $u_{t(u)} \in \mathcal{M}$. *Moreover,* $I(u_{t(u)}) = \max_{t \geq 0} I(u_t)$.

It is easy to see that I possesses the mountain pass geometry. Then define the mountain pass value of I:

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} \overline{I}(\gamma(t)),$$

where $\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0 \text{ and } \overline{I}(\gamma(1)) < 0\}$. Analogous to (2.17), we have

$$c = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \max_{t \geq 0} I(u_t) = \inf_{u \in \mathcal{M}} I(u) > 0. \tag{3.3}$$

As in Lemma 2.5, the $(\mathrm{PS})_c$ sequence of I has also the following property.

LEMMA 3.3. *There exists a sequence* $\{u_n\} \subset H^1(\mathbb{R}^3)$ *such that, as* $n \to \infty$,

$$I(u_n) \to c, \qquad I'(u_n) \to 0, \qquad J(u_n) \to 0. \tag{3.4}$$

PROOF. We consider

$$\begin{aligned} I_2' &= \lim_{t \to 0} \frac{1}{2t} \left[e^{4(\theta_n + th)} \int_{\mathbb{R}^3} V\big(e^{\theta_n + th} x\big) |v_n + tw|^2 - e^{4\theta_n} \int_{\mathbb{R}^3} V\big(e^{\theta_n} x\big) |v_n|^2 \right] \\ &= \frac{h}{2} e^{4\theta_n} \int_{\mathbb{R}^3} \big[4V\big(e^{\theta_n} x\big) + \big(\nabla V\big(e^{\theta_n} x\big), e^{\theta_n} x\big)\big] |v_n|^2 + e^{4\theta_n} \int_{\mathbb{R}^3} V\big(e^{\theta_n} x\big) v_n w. \end{aligned}$$

Similar to the proof of Lemma 2.5, we can prove the lemma by using I, J and I_2' instead of $\overline{I}$, $\overline{J}$ and $\overline{I}_2$, respectively. □

LEMMA 3.4. *Assume that either* $q \in (2, 4]$ *with* D *sufficiently large or* $q \in (4, 6)$. *Then* $c < \Lambda$, *which* Λ *is given by Lemma* 2.6.

Proof. We first claim that

$$c < c^\infty. \tag{3.5}$$

Indeed, from Remark 2.1, I^∞ has a minimizer $u^\infty > 0$ on $\mathcal{M}^\infty$ and $c^\infty := I^\infty(u^\infty)$. By Lemma 3.2, there exists $t_u > 0$ such that $(u^\infty)_{t_u} \in \mathcal{M}$. Thus, by (V_2), (1.2), (2.2) and (2.5), we have

$$c \le I((u^\infty)_{t_u}) < I^\infty((u^\infty)_{t_u}) \le I^\infty(u^\infty) = c^\infty.$$

Using the arguments in Lemma 2.6, we can obtain $c^\infty < \Lambda$, which together with (3.5) means $c < \Lambda$. □

Proof of Theorem 1.2. Let c be the mountain pass value for I and $\{u_n\}$ satisfy (3.4). From (3.4) and (3.1) with $t \to 0$, we get

$$c + o(1) = I(u_n) \ge \frac{a(1-\theta)}{4} \|\nabla u_n\|_2^2 + \frac{1}{12} \|u_n\|_6^6, \tag{3.6}$$

which means that $\{\|\nabla u_n\|_2\}$ is bounded. Then, by (F_1), (F_2), (1.8) and Sobolev embedding theorem,

$$\begin{aligned}\gamma_1\|u_n\|^2 &\le a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + (\nabla V(x), x)]u^2 + b\|\nabla u\|_2^4 \\ &= \frac{1}{2}\int_{\mathbb{R}^3} [f(u)u + 6F(u)] + \|u\|_6^6 \le \frac{\gamma_1}{2}\|u_n\|^2 + C_5\|u_n\|_6^6 \\ &\le \frac{\gamma_1}{2}\|u_n\|^2 + C_5 S^{-3}\|\nabla u_n\|_2^6.\end{aligned}$$

This shows $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Taking a subsequence if necessary, we have $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^3)$. Next we show $u_0 \ne 0$. Arguing by contradiction, suppose that $u_0 = 0$, i.e. $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $s \in [1, 6)$ and $u_n \to 0$ for almost every x in $\mathbb{R}^3$. Let $t = 0$ and $t \to \infty$ in (V_4), respectively, and using (V_2), one has

$$-\frac{\theta a}{2|x|^2} + 4V_\infty \le 4V(x) + (\nabla V(x), x) \le \frac{\theta a}{2|x|^2} + 4V_\infty, \quad \text{for all } x \in \mathbb{R}^3\{0\}.$$

Together with (V_2), it is easy to see that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} [V_\infty - V(x)]u_n^2 = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} (\nabla V(x), x)u_n^2 = 0. \tag{3.7}$$

From (1.2), (1.8), (2.2), (2.3), (3.4) and (3.7), we get

$$I^\infty(u_n) \to c, \quad (I^\infty)'(u_n) \to 0 \quad \text{and} \quad J^\infty(u_n) \to 0. \tag{3.8}$$

Using the same arguments in Theorem 1.1, one also gets (2.23). Set $v_n(x) = u_n(x + y_n)$. Then we have $\|v_n\| = \|u_n\|$ and $\{v_n\}$ satisfies $\int_{B_r(0)} |v_n|^2 > \delta$ and

$$I^\infty(v_n) \to c, \quad (I^\infty)'(v_n) \to 0 \quad \text{and} \quad J^\infty(v_n) \to 0. \tag{3.9}$$

Hence, there exists $v \in H^1(\mathbb{R}^3)\setminus\{0\}$ such that, taking a subsequence if necessary, $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Similar to the proof of Theorem 1.1, we have $J^\infty(v) = 0$ and

$I^\infty(v) = c$, which contradicts with (3.5). So $u_0 \neq 0$. It is easy to see that v satisfies

$$-(a+bl^2)\Delta u_0 + V(x)u_0 = f(u_0) + |u_0|^4 u_0 \tag{3.10}$$

where $l^2 := \lim_{n\to\infty} \|\nabla u_n\|_2^2$ and $\|\nabla u_0\|_2^2 \leq l^2$. The corresponding functional to (3.10) is defined by

$$E(u) = \frac{a+bl^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6.$$

Since $E'(u_0) = 0$, we have the Pohozaev identity applying to (3.10)

$$\begin{aligned} P_V(u_0) = \frac{a+bl^2}{2}\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [3V(x) + (\nabla V(x), x)]u_0^2 \\ - 3\int_{\mathbb{R}^3} F(u_0) - \frac{1}{2}\|u_0\|_6^6 = 0. \end{aligned} \tag{3.11}$$

Let $t = 0$ in (V_4), together with (3.2), one has

$$\frac{1}{2}\int_{\mathbb{R}^3} (\nabla V(x), x)u_0^2 \leq \frac{a}{4}\int_{\mathbb{R}^3} \frac{u^2}{|x|^2} \leq a\int_{\mathbb{R}^3} |\nabla u_0|^2. \tag{3.12}$$

It follows from (1.8) and $\|\nabla v\|_2^2 \leq l^2$ that

$$\begin{aligned} J(u_0) &= a\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + (\nabla V(x), x)]u_0^2 \\ &\quad + b\|\nabla u_0\|_2^4 - \frac{1}{2}\int_{\mathbb{R}^3} [f(u_0)u_0 + 6F(u_0)] - \|u_0\|_6^6 \\ &\leq (a+bl^2)\|\nabla u_0\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} [4V(x) + (\nabla V(x), x)]u_0^2 \\ &\quad - \frac{1}{2}\int_{\mathbb{R}^3} [f(u_0)u_0 + 6F(u_0)] - \|u_0\|_6^6 \\ &= \frac{1}{2}\langle E'(u_0), u_0\rangle + P_V(u_0) = 0. \end{aligned} \tag{3.13}$$

In view of Lemma 3.2 and $u_0 \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists $t_0 > 0$ such that $(u_0)_{t_0} \in \mathcal{M}$. Let $t = 0$ in (2.6), we have

$$f(\tau)\tau - 2F(\tau) \geq 0, \quad \text{for all } \tau \in \mathbb{R}. \tag{3.14}$$

Then by (1.2), (1.8), (3.14), (3.1), (3.4), (3.12), (3.13) and Fatou's lemma, we infer that

$$\begin{aligned} E(u_0) &= E(u_0) - \frac{1}{4}\left[\frac{1}{2}\langle E'(u_0), u_0\rangle + P_V(u_0)\right] \\ &= \frac{bl^2}{4}\|\nabla u_0\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left[a|\nabla u_0|^2 - \frac{1}{2}(\nabla V(x), x)u_0^2\right] \\ &\quad + \frac{1}{8}\int_{\mathbb{R}^3} [f(u_0)u_0 - 2F(u_0)] + \frac{1}{12}\|u_0\|_6^6 \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&= \frac{bl^2}{4}\|\nabla u_0\|_2^2 + I(u_0) - \frac{1}{4}J(u_0)\\
&\geq \frac{bl^2}{4}\|\nabla u_0\|_2^2 + I((u_0)_{t_0}) - \frac{t_0^4}{4}J(u_0) + \frac{2t_0^6 - 3t_0^4 + 1}{12}\|u_0\|_6^6\\
&\geq \frac{bl^2}{4}\|\nabla u_0\|_2^2 + c = \frac{bl^2}{4}\|\nabla u_0\|_2^2 + \lim_{n\to\infty}\Big(I(u_n) - \frac{1}{4}J(u_n)\Big)\\
&= \frac{bl^2}{4}\|\nabla u_0\|_2^2 + \lim_{n\to\infty}\frac{1}{4}\int_{\mathbb{R}^3}\Big[a|\nabla u_n|^2 - \frac{1}{2}(\nabla V(x), x)u_n^2\Big]\\
&\quad + \lim_{n\to\infty}\frac{1}{8}\int_{\mathbb{R}^3}[f(u_n)u_n - 2F(u_n)] + \frac{1}{12}\lim_{n\to\infty}\|u_n\|_6^6\\
&\geq \frac{a + bl^2}{4}\|\nabla u_0\|_2^2 - \frac{1}{8}\int_{\mathbb{R}^3}(\nabla V(x), x)u_0^2\\
&\quad + \frac{1}{8}\int_{\mathbb{R}^3}[f(u_0)u_0 - 2F(u_0)] + \frac{1}{12}\|u_0\|_6^6 = E(u_0).
\end{aligned}
$$

So $J(u_0) = 0$, $t_0 = 1$ and $I(u_0) = c$. □

4. The least energy solutions

In this section we will prove Theorems 1.3 and 1.4. From now on we assume that (F_1)–(F_4) hold. From (V_1) and (V_2), it follows that the norm

$$
\|u\|_V := \bigg(a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2\bigg)^{1/2}
$$

is equivalent to $\|\cdot\|$ in $H^1(\mathbb{R}^3)$.

PROPOSITION 4.1 ([11]). *Let $(E, \|\cdot\|)$ be a real Banach space and $J \subset \mathbb{R}^+$ be an interval. Consider the family of $\mathcal{C}^1$–functionals on E of the form*

$$
I_\lambda = A(u) - \lambda B(u), \quad \textit{for all } \lambda \in J,
$$

with B nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$. we assume that there are two points v_1, v_2 in E such that, for any $\lambda \in J$,

$$
c_\lambda = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}, \tag{4.1}
$$

where $\Gamma := \{\gamma \in \mathcal{C}([0,1], E) : \gamma(0) = v_1$ and $\gamma(1) = v_2\}$. Then, for almost every $\lambda \in J$, there is a bounded $(\mathrm{PS})_{c_\lambda}$ sequences for I_λ, that is, $\{u_n\}$ is bounded satisfying $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in E^*. Moreover, the map $\lambda \mapsto c_\lambda$ is left-continuous.

To apply Proposition 5.1, we denote $E = H^1(\mathbb{R}^3)$ and define $\lambda \in [1/2, 1]$ and two families of functional defined by

$$
I_\lambda(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)u^2 + \frac{b}{4}\|\nabla u\|_2^4 - \lambda\int_{\mathbb{R}^3} F(u) - \frac{\lambda}{6}\int_{\mathbb{R}^3}|u|^6 \tag{4.2}
$$

and

$$I_\lambda^\infty(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{V_\infty}{2}\|u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \lambda\int_{\mathbb{R}^3} F(u) - \frac{\lambda}{6}\int_{\mathbb{R}^3}|u|^6. \tag{4.3}$$

Similar to the definitions of J, J^∞, $\mathcal{M}^\infty$ and c^∞, we set

$$\begin{aligned} J_\lambda(u) = {} & a\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3}[4V(x) + (\nabla V(x), x)]u^2 \\ & + b\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}[f(u)u + 6F(u)] - \lambda\|u\|_6^6, \end{aligned} \tag{4.4}$$

$$\begin{aligned} J_\lambda^\infty(u) = {} & a\|\nabla u\|_2^2 + 2V_\infty\|u\|_2^2 \\ & + b\|\nabla u\|_2^4 - \frac{\lambda}{2}\int_{\mathbb{R}^3}[f(u)u + 6F(u)] - \lambda\|u\|_6^6, \end{aligned} \tag{4.5}$$

$$\mathcal{M}_\lambda^\infty := \{u \in H^1(\mathbb{R}^3)\setminus\{0\} : J_\lambda^\infty(u) = 0\} \quad \text{and} \quad m_\lambda^\infty = \inf_{u\in\mathcal{M}_\lambda^\infty} I_\lambda^\infty(u)$$

for $\lambda \in [1/2, 1]$. Using the arguments in Remark 2.1 and Lemma 2.6, we give the following lemma.

LEMMA 4.2. *Assume that either $q \in (2, 4]$ with D sufficiently large or $q \in (4, 6)$. Then I_λ^∞ has a minimizer $u_\lambda^\infty > 0$ on $\mathcal{M}_\lambda^\infty$ for any $\lambda \in [1/2, 1]$. Moreover, $(I_\lambda^\infty)'(u_\lambda^\infty) = 0$ and*

$$m_\lambda^\infty < \Lambda_\lambda := \frac{1}{4\lambda}abS^3 + \frac{1}{24\lambda^2}b^3S^6 + \frac{1}{24\lambda^2}(b^2S^4 + 4\lambda aS)^{3/2}.$$

Analogous to (2.5), we obtain that, for any $u \in H^1(\mathbb{R}^3)$, $t \geq 0$ and $\lambda \geq 0$,

$$\begin{aligned} I_\lambda^\infty(u) \geq {} & I_\lambda^\infty(u_t) + \frac{1-t^4}{4}J_\lambda^\infty(u) \\ & + \frac{a(1-t^2)^2}{4}\|\nabla u\|_2^2 + \lambda\frac{2t^6 - 3t^4 + 1}{12}\|u\|_6^6. \end{aligned} \tag{4.6}$$

Moreover, the following lemma shows that I_λ has the mountain pass geometry and the corresponding mountain pass level denoted by c_λ.

LEMMA 4.3. *Assume that (V_1) and (V_2) hold. Then*

(a) *there exists a $v \in E\setminus\{0\}$ such that $I_\lambda(v) \leq 0$ for all $\lambda \in [1/2, 1]$;*
(b) *there exists a positive constant δ_0 independent of λ such that for all $\lambda \in [1/2, 1]$,*

$$c_\lambda := \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t)) \geq \delta_0 > \max\{I_\lambda(0), I_\lambda(v)\}, \tag{4.7}$$

where $\Gamma := \{\gamma \in \mathcal{C}([0, 1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = v\}$.

The proof is similar to Lemma 2.3, so we omit it.

LEMMA 4.4 ([30, Lemma 4.5]). *Assume that (V_1)–(V_3) hold. Then there exists $\overline{\lambda} \in [1/2, 1)$ such that $c_\lambda < m_\lambda^\infty$ for all $\lambda \in [\overline{\lambda}, 1]$.*

The following lemma can also be seen in [20], which shows the decomposition for the bounded $(\mathrm{PS})_{c_\lambda}$ and extends Lemma 3.4 in [14] to a critical growing nonlinearity. However, the proof is different from the one in [20] since we use (F_4) to take place of the condition (F_4').

Lemma 4.5. *Assume that* (V_1)–(V_3) *hold. Let* $\{u_n\}$ *be a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence for* I_λ, *for every* $\lambda \in [1/2, 1]$ *and* $0 < c_\lambda < \Lambda_\lambda$. *Then there exist* u_0 *and* $A \in \mathbb{R}$ *such that* $E_\lambda'(u_0) = 0$, *where*

$$E_\lambda(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 - \lambda \int_{\mathbb{R}^3} F(u) - \frac{\lambda}{6} \int_{\mathbb{R}^3} |u|^6, \tag{4.8}$$

and an integer $k \in \mathbb{N} \cup \{0\}$, *nontrivial solutions* $w^1, \ldots, w^k$ *of the following problem*

$$-(a + bA^2)\Delta u + V_\infty u = \lambda f(u) + \lambda |u|^4 u$$

and k *sequences of points* $\{y_n^j\} \subset \mathbb{R}^3$, $1 \le j \le k$, *such that*

(a) $|y_n^j| \to \infty$ *and* $|y_n^j - y_n^i| \to \infty$ *for* $i \ne j$, $n \to \infty$;

(b) $w^j \ne 0$ *and* $(E_\lambda^\infty)'(w^j) = 0$ *for* $1 \le j \le k$;

(c) $u_n - u_0 - \sum_{j=1}^{k} w^j(\,\cdot - y_n^j) \to 0$ *in* E *as* $n \to \infty$;

(d) $c_\lambda + \dfrac{bA^4}{4} = E_\lambda(u_0) + \sum_{j=1}^{k} E_\lambda^\infty(w^j)$ *as* $n \to \infty$, *where*

$$E_\lambda^\infty(u) = \frac{a + bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V_\infty u^2 - \lambda \int_{\mathbb{R}^3} F(u) - \frac{\lambda}{6} \int_{\mathbb{R}^3} |u|^6; \tag{4.9}$$

(e) $A^2 = \|\nabla u_0\|_2^2 + \sum_{j=1}^{k} \|\nabla w^j\|_2^2$.

Moreover, in the case $k = 0$ *the above conclusions hold without* w^j *and* $\{y_n^j\}$.

Proof. Note that $\{u_n\}$ is bounded in E, then there exist $u_0 \in E$ and $A \in \mathbb{R}$ such that $u_n \rightharpoonup u_0$ and $A^2 := \lim_{n\to\infty} \|\nabla u_n\|_2^2$ after extracting a subsequence. It follows from $E_\lambda'(u_n) \to 0$ that $E_\lambda'(u_0) = 0$. Since it is easy to see that

$$E_\lambda(u_n) = I_\lambda(u_n) + \frac{bA^4}{4} + o(1) \quad \text{and} \quad \langle E_\lambda'(u_n), \phi\rangle = \langle I_\lambda'(u_n), \phi\rangle + o(1)$$

for any $\phi \in E$, we conclude that $E_\lambda(u_n) \to c_\lambda + bA^4/4$ and $E_\lambda'(u_n) \to 0$ in E^*. Moreover, taking a subsequence if necessary, $u_n \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$ for $s \in [1, 6)$ and $u_n \to u_0$ for almost every x in $\mathbb{R}^3$.

Set $v_n^1 = u_n - u_0$, then one has $v_n \rightharpoonup 0$ in E. From the Brezis–Lieb Lemma [4], we have

$$\begin{aligned} \|\nabla v_n^1\|_2^2 &= \|\nabla u_n\|_2^2 - \|\nabla u_0\|_2^2 + o(1), \\ \|v_n^1\|_s^s &= \|u_n\|_s^s - \|u_0\|_s^s + o(1) \quad \text{for } s \in [2, 6]. \end{aligned} \tag{4.10}$$

As in Lemma 3.2 of [18], we have

$$\begin{aligned}\int_{\mathbb{R}^3} F(v_n^1) &= \int_{\mathbb{R}^3} F(u_n) - \int_{\mathbb{R}^3} F(u_0) + o(1),\\ \int_{\mathbb{R}^3} f(v_n^1)v_n^1 &= \int_{\mathbb{R}^3} f(u_n)u_n - \int_{\mathbb{R}^3} f(u_0)u_0 + o(1).\end{aligned} \tag{4.11}$$

It follows from (V_2), (4.10) and (4.11) that

$$\begin{aligned}E_\lambda(v_n^1) &= c_\lambda + \frac{bA^4}{4} - E_\lambda(u_0) + o(1),\\ \langle E'_\lambda(v_n^1), v_n^1\rangle &= \langle E'_\lambda(u_n)u_n\rangle - \langle E'_\lambda(u_0), u_0\rangle + o(1) = o(1).\end{aligned} \tag{4.12}$$

Similar to (3.11), since $E'_\lambda(u_0) = 0$, we have

$$\begin{aligned}P_\lambda(u_0) = {}& \frac{a+bA^2}{2}\|\nabla u_0\|_2^2\\ &+ \frac{1}{2}\int_{\mathbb{R}^3} [3V(x) + (\nabla V(x), x)]u_0^2 - 3\lambda \int_{\mathbb{R}^3} F(u_0) - \frac{\lambda}{2}\|u_0\|_6^6.\end{aligned} \tag{4.13}$$

It follows from Hardy inequality (3.2) and (V_3) imply (3.12). By (3.14), (3.12), (4.8) and (4.13), we have

$$\begin{aligned}E_\lambda(u_0) &= E_\lambda(u_0) - \frac{1}{4}\left[\frac{1}{2}\langle E'_\lambda(u_0), u_0\rangle + P_\lambda(u_0)\right]\\ &= \frac{bA^2}{4}\|\nabla u_0\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\left[a|\nabla u_0|^2 - \frac{1}{2}(\nabla V(x), x)u_0^2\right]\\ &\quad + \frac{\lambda}{8}\int_{\mathbb{R}^3} [f(u_0)u_0 - 2F(u_0)] + \frac{\lambda}{12}\|u_0\|_6^6\\ &\geq \frac{bA^2}{4}\|\nabla u_0\|_2^2.\end{aligned} \tag{4.14}$$

We claim that one of the following conclusions holds for v_n^1:

(v1) $v_n^1 \to 0$ in E;

(v2) there exist r', $m > 0$ and a sequence $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_{r'}(y_n^1)} |v_n^1|^2 = \sigma^1 > 0. \tag{4.15}$$

Indeed, suppose that (v2) does not occur. Then for any $r > 0$, we have

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |v_n^1|^2 = 0. \tag{4.16}$$

Using the arguments in Theorem 1.1, we see that Lions' Vanishing Lemma implies that $v_n^1 \to 0$ in $L^s(\mathbb{R}^3)$ for $2 < s < 6$. Then we deduce from (4.10)–(4.12) that

$$
\begin{aligned}
&c_\lambda + \frac{bA^4}{4} - E_\lambda(u_0) + o(1) \\
(4.17)\qquad &\qquad = \frac{a+bA^2}{2}\,\|\nabla v_n^1\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(x)|v_n^1|^2 - \frac{\lambda}{6}\int_{\mathbb{R}^3}|v_n^1|^6, \\
&o(1) = (a+bA^2)\|\nabla v_n^1\|_2^2 + \int_{\mathbb{R}^3} V(x)|v_n^1|^2 - \lambda\int_{\mathbb{R}^3}|v_n^1|^6.
\end{aligned}
$$

Then, up to subsequence, we have

$$
(4.18)\qquad \|\nabla v_n^1\|_2^2 \to l_1 \geq 0 \quad \text{and} \quad \lambda\|v_n^1\|_6^6 \to l_2, \quad \text{as } n\to\infty.
$$

Suppose $l_1 > 0$, then we deduce from (4.10), (4.17) and $S(\lambda^{-1}l_2)^{1/3} \leq l_1$ that

$$
(4.19)\qquad al_1 + bl_1^2 \leq l_2 \quad \text{and} \quad \frac{bS^2 + \sqrt{b^2S^4 + 4\lambda aS}}{2\lambda^{2/3}} \leq l_2^{1/3}.
$$

Then, it follows from (4.10), (4.14) and (4.17)–(4.19) that

$$
\begin{aligned}
c_\lambda + \frac{bA^4}{4} &= E_\lambda(u_0) + \frac{a+bA^2}{4}\,l_1 + \frac{1}{4}\,l_2 - \frac{1}{6}l_2 \\
&\geq E_\lambda(u_0) + \frac{bA^2}{4}\,l_1 + \frac{a}{4}\,l_1 + \frac{1}{12}\,l_2 \\
&\geq \frac{bA^2}{4}(\|\nabla u_0\|_2^2 + l_1) + \frac{aS}{4\lambda^{1/3}}(l_2)^{1/3} + \frac{1}{12}\,l_2 \\
&\geq \frac{bA^4}{4} + \frac{1}{8\lambda}\,aS\big(bS^2 + \sqrt{b^2S^4 + 4aS}\big) \\
&\quad + \frac{1}{96\lambda^2}\big(bS^2 + \sqrt{b^2S^4 + 4\lambda aS}\big)^3 \\
&= \frac{bA^4}{4} + \frac{1}{4\lambda}\,abS^3 + \frac{1}{24\lambda^2}\,b^3S^6 + \frac{1}{24\lambda^2}(b^2S^4 + 4aS)^{3/2},
\end{aligned}
$$

which contradicts with Lemma 4.2. Thus $l_1 = 0$ and it is easy to obtain that $u_n \to u_0$ in E as $n \to \infty$ and the the proof is completed.

If (v1) hold for $\{v_n^1\}$, then Lemma 4.5 holds with $k = 0$. Otherwise, suppose (v2) holds; that is (4.15) holds. Let $w_n^1 := v_n^1(\,\cdot\, + y_n^1)$. Then $\{w_n^1\}$ is bounded in E and we may assume that $w_n^1 \rightharpoonup w^1$ in E. Hence $(E_\lambda^\infty)'(w_1) = 0$. Since

$$
\int_{B_{r'}(0)} |v_n^1(x + y_n^1)|^2 \geq \frac{\sigma^1}{2} > 0
$$

for n large. By a standard argument, we have $|y_n^1| \to \infty$ and $w^1 \neq 0$. The rest of proof is similar to Steps 3 and 4 in Lemma 3.3 of [20]. □

In what follows, for simplicity, let $V_\infty = 1$ and denote $\overline{I}_\lambda := I_\lambda^\infty$.

Lemma 4.6. *If $\{u_n(\lambda)\}$ is a bounded $(\mathrm{PS})_{\overline{c}_\lambda}$ sequence for $\overline{I}_\lambda$. Then, for every $\lambda \in [1/2, 1]$ and $0 < \overline{c}_\lambda < \Lambda_\lambda$, there exist an integer $k \in \mathbb{N}\cup\{0\}$, $u_0 \in E$, $A \in \mathbb{R}$, nonzero critical points $w^1, \ldots, w^k$ of E_λ^∞ given by (4.9) and k sequences $\{y_n^j\} \subset \mathbb{R}^3$, $1 \leq j \leq k$, such that*

(a) $u_n \rightharpoonup u_0$ *in* E *with* $(E_\lambda^\infty)'(u_0) = 0$;

(b) $|y_n^j| \to \infty$ *and* $|y_n^j - y_n^i| \to \infty$ *for* $i \neq j$, $n \to \infty$;

(c) $u_n - u_0 - \sum_{j=1}^{k} w^j(\,\cdot\, - y_n^j) \to 0$ *in* E *as* $n \to \infty$;

(d) $\bar{c}_\lambda + \dfrac{bA^4}{4} = E_\lambda^\infty(u_0) + \sum_{j=1}^{k} E_\lambda^\infty(w^j)$ *as* $n \to \infty$;

(e) $A^2 = \|\nabla u_0\|_2^2 + \sum_{j=1}^{k} \|\nabla w^j\|_2^2$.

PROOF. For term (a), note that $\{u_n\}$ is bounded in E, then there exist $u_0 \in E$ and $A \in \mathbb{R}$ such that $u_n \rightharpoonup u_0$ in E and $A^2 := \lim_{n\to\infty} \|\nabla u_n\|_2^2$ after extracting a subsequence. It follows from $(E_\lambda^\infty)'(u_n) \to 0$ that $(E_\lambda^\infty)'(u_0) = 0$. Then the remaining proof is similar to Lemma 4.5. So we omit it. □

By Lemma 4.5, we can prove that I_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition, which together with Proposition 4.1 means the following result.

LEMMA 4.7. *Assume that* (V_1)–(V_3) *hold. For almost all* $\lambda \in [\overline{\lambda}, 1]$, *let* $\{u_n\}$ *be a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence for* I_λ, *then there exists* $u_\lambda \in E$ *such that* $u_n \to u_\lambda$.

PROOF. Note that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. Using Lemmas 4.2 and 4.4, we get $0 < c_\lambda < m_\lambda^\infty < \Lambda_\lambda$. Then, by Lemma 4.5, there exist a subsequence $\{u_n\}$, still denoted by $\{u_n\}$, $A \in \mathbb{R}$ and $u_\lambda \in E$ such that

$$u_n \rightharpoonup u_\lambda, \quad A^2 := \lim_{n\to\infty} \|\nabla u_n\|_2^2 \quad \text{and} \quad E'_\lambda(u_\lambda) = 0.$$

If $k = 0$, then the conclusion follows. Otherwise, we deal with the case $k \in \mathbb{N}$. Analogous to (2.28), for each nontrivial critical point $w^j = 0$ $(1 \le j \le k)$ of E_λ^∞, we have the following Pohozaev identity

$$P_\lambda^\infty(w^j) = \frac{a + bA^2}{2} \|\nabla w^j\|_2^2 + \frac{3}{2} \|w^j\|_2^2 - 3\lambda \int_{\mathbb{R}^3} F(w^j) - \frac{\lambda}{2} \|w^j\|_6^6 = 0. \tag{4.20}$$

Using the argument of (3.13), joint with Lemma 4.5 (e), we also have $J_\lambda^\infty(w^j) \le 0$. In view of Lemma 2.2 and $w^j \neq 0$, there exists $t_j > 0$ such that $(w^j)_{t_j} \in \mathcal{M}_\lambda^\infty$. By (4.3), (4.5), (4.6), (4.9) and (4.20), we infer that

$$\begin{aligned} E_\lambda^\infty(w^j) &= E_\lambda^\infty(w^j) - \frac{1}{4}\left[\frac{1}{2}\big\langle (E_\lambda^\infty)'(w^j), w^j \big\rangle + P_\lambda^\infty(w^j)\right] \\ &= \frac{a + bA^2}{4} \|\nabla w^j\|_2^2 + \frac{1}{8} \int_{\mathbb{R}^3} \big[f(w^j)w^j - 2F(w^j)\big] + \frac{1}{12} \|w^j\|_6^6 \\ &= \frac{bA^2}{4} \|\nabla w^j\|_2^2 + I_\lambda^\infty(w^j) - \frac{1}{4} J_\lambda^\infty(w^j) \end{aligned} \tag{4.21}$$

$$\geq \frac{bA^2}{4}\left\|\nabla w^j\right\|_2^2 + I((w^j)_{t_j}) - \frac{t_j^4}{4} J_\lambda^\infty\left(w^j\right) + \frac{2t_j^6 - 3t_j^4 + 1}{12}\left\|w^j\right\|_6^6$$
$$\geq \frac{bA^2}{4}\left\|\nabla w^j\right\|_2^2 + m_\lambda^\infty.$$

Then, from (4.14), (4.21) and Lemma 4.5, we deduce that

$$c_\lambda + \frac{bA^4}{4} = E_\lambda(u_\lambda) + \sum_{j=1}^{k} E_\lambda^\infty\left(w^j\right)$$
$$\geq k m_\lambda^\infty + \frac{bA^2}{4}\left[\|\nabla u_\lambda\|_2^2 + \sum_{j=1}^{k}\left\|\nabla w^j\right\|_2^2\right] \geq m_\lambda^\infty + \frac{bA^4}{4},$$

which contradicts with Lemma 4.4. Thus we have $u_n \to u_\lambda$ in E. □

Similarly, we have the following result for the functional $\overline{I}_\lambda$.

Lemma 4.8. *For almost all $\lambda \in [1/2, 1]$ holds: if $\{u_n\}$ is a bounded* $(\mathrm{PS})_{\overline{c}_\lambda}$ *sequence for $\overline{I}_\lambda$, then there exists $u_\lambda \in E$ such that, after translating the sequence suitably and passing to a subsequence, $u_n \to u_\lambda$.*

Proof. By the similar argument to Proposition 4.1 and Lemma 4.3, we get that for almost every $\lambda \in [1/2, 1]$, that there exists a bounded sequence $\{u_n(\lambda)\} \subset E$, denoted by $\{u_n\}$ for simplicity, such that $\overline{I}_\lambda(u_n) \to \overline{c}_\lambda$ and $\overline{I}'_\lambda(u_n) \to 0$ in E^*, where $\overline{c}_\lambda$ is defined as in (4.7) using $\overline{I}_\lambda$ instead of I_λ. It is readily checked that

$$\delta_0 \leq \overline{c}_\lambda = m_\lambda^\infty < \Lambda_\lambda. \tag{4.22}$$

Repeating the argument in the proof of Theorem 1.1, one has that the sequence $\{u_n\}$ satisfies (2.23). So we may assume that, up to translations, a subsequence of $\{u_n\}$ converges weakly to $u_\lambda \in E \setminus \{0\}$. Then we claim that $u_n \to u_\lambda$ in E. In fact, it follows from Lemma 4.6 (a) that

$$u_n \rightharpoonup u_\lambda, \quad A^2 := \lim_{n\to\infty}\|\nabla u_n\|_2^2 \quad \text{and} \quad (E_\lambda^\infty)'(u_\lambda) = 0.$$

If $k = 0$, the proof is finished. Otherwise, similar to the proof of Lemma 4.7, we can also prove that (4.21) holds. In particular,

$$E_\lambda^\infty(u_\lambda) \geq \frac{bA^2}{4}\|\nabla u_\lambda\|_2^2 + m_\lambda^\infty. \tag{4.23}$$

Consequently, we deduce from (4.23) and Lemma 4.6 (d) that

$$\overline{c}_\lambda + \frac{bA^4}{4} = E_\lambda^\infty(u_\lambda) + \sum_{j=0}^{k} E_\lambda^\infty\left(w^j\right) \geq 2m_\lambda^\infty + \frac{bA^4}{4},$$

which contradicts with (4.22). Thus, we have $u_n \to u_\lambda$ in E. □

PROOF OF THEOREM 1.3. It follows from Proposition 4.1 and Lemma 4.3 that, for almost every $\lambda \in [1/2, 1]$, there exists a bounded sequence $\{u_n(\lambda)\} \subset E$, denoted by $\{u_n\}$ for simplicity, such that $I_\lambda(u_n) \to c_\lambda$ and $I'_\lambda(u_n) \to 0$ in E^*. In view of Lemma 4.7, there exist two sequences of $\{\lambda_n\} \subset [\overline{\lambda}, 1]$ and $\{u_{\lambda_n}\}$, denoted by $\{u_n\}$, such that

$$\lambda_n \to 1, \qquad I'_{\lambda_n}(u_n) = 0, \qquad I_{\lambda_n}(u_n) = c_{\lambda_n}. \tag{4.24}$$

Using (V_3), (3.14), (4.2) and (4.3), we refer that

$$\begin{aligned} c_{1/2} \geq c_{\lambda_n} &= I_{\lambda_n}(u_n) - \frac{1}{4} J_{\lambda_n}(u_n) \\ &\geq \frac{a}{4} \|\nabla u_n\|_2^2 - \frac{1}{8}(\nabla V(x), x)u_n^2 + \frac{\lambda_n}{8} \int_{\mathbb{R}^3} [f(u_n)u_n - 2F(u_n)] + \frac{\lambda_n}{12}\|u_n\|_6^6 \\ &\geq \frac{(1-\theta)a}{4}\|\nabla u_n\|_2^2. \end{aligned}$$

Thus $\{\|\nabla u_n\|_2\}$ is bounded. Then from (F_1), (F_2) and the Sobolev embedding inequality, we can easily deduce that that $\{\|u_n\|\}$ is bounded. The rest of the proof is the same as the one in [20], so we omit it. □

PROOF OF THEOREM 1.4. By virtue of Lemma 4.8, there exist two sequence of $\{\lambda_n\} \subset [1/2, 1]$ and $\{v_{\lambda_n}\}$, denoted by $\{v_n\}$, such that

$$\lambda_n \to 1, \qquad \overline{I}'_{\lambda_n}(v_n) = 0, \qquad \overline{I}_{\lambda_n}(v_n) = \overline{c}_{\lambda_n}. \tag{4.25}$$

Let us show that $\{v_n\}$ is bounded in E. It is easy to check that $\|\nabla v_n\|_2 \leq C_1$. Then, it follows from (F_1), (F_2), (4.5) and Sobolev embedding inequality that for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|v_n\|_2^2 \leq C_2 + \varepsilon\|v_n\|_2^2 + C_\varepsilon\|v_n\|_6^6 \leq C_2 + \varepsilon\|v_n\|_2^2 + C_\varepsilon S^{-3}\|\nabla v_n\|_2^6.$$

Therefore, $\{\|v_n\|\}$ is bounded. By Proposition 4.1

$$\lim_{n\to\infty} \overline{I}(v_n) = \lim_{n\to\infty} \bigg(\overline{I}_{\lambda_n}(v_n) + (\lambda_n - 1)\int_{\mathbb{R}^3} F(v_n)\bigg) = \lim_{n\to\infty} \overline{c}_{\lambda_n} = \overline{c}_1$$

and, for any $\varphi \in E$,

$$\lim_{n\to\infty} \langle \overline{I}(v_n), \varphi\rangle = \lim_{n\to\infty} \bigg(\overline{I}_{\lambda_n}(v_n), \varphi\rangle + (\lambda_n - 1)\int_{\mathbb{R}^3} f(v_n)\varphi\bigg) = 0.$$

That is to say, $\{v_n\}$ is a bounded (PS) sequence for $\overline{I}$ at level $\overline{c}_1$. As in the proof of Lemma 4.8, we may assume that $v_n \to v \neq 0$ in E with $\overline{I}(v) = \overline{c}_1$. Set

$$\nu = \big\{\overline{I}(u) : u \in E \setminus \{0\},\ \overline{I}'(u) = 0\big\}.$$

It is easy to see that $0 < \nu \leq \overline{c}_1 < \Lambda$. By the definition of ν, there exists a sequence $\{w_n\}$ such that $\overline{I}'(w_n) \to 0$ and $\overline{I}(w_n) \to \nu$ as $n \to \infty$. Then it is readily seen that $\{w_n\}$ is bounded in E. By the preceding arguments, there exists a nontrivial $w \in E$ such that, up to translations and a subsequence, we have $w_n \to w$ in E as $n \to \infty$, i.e. $\overline{I}(w) = \overline{c}_1$ and $\overline{I}'(w) = 0$. Therefore, we see that w is a least energy solution of problem (1.6). □

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