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POSITIVE GROUND STATES FOR A SUBCRITICAL AND CRITICAL COUPLED SYSTEM INVOLVING KIRCHHOFF–SCHRÖDINGER EQUATIONS

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Abstract. In this paper we prove the existence of positive ground state solution for a class of linearly coupled systems involving Kirchhoff-Schrödinger equations. We study the subcritical and critical case. Our approach is variational and based on minimization technique over the Nehari manifold. We also obtain a nonexistence result using a Pohozaev identity type.

1. Introduction

In this article we study the following class of nonlocal linearly coupled systems

$$(\mathrm{S}_\mu)\qquad \begin{cases} \big(a_1+\alpha'(\|u\|^2_{E_1})\big)\big(-\Delta u+V_1(x)u\big)=\mu|u|^{p-2}u+\lambda(x)v & \text{for } x\in\mathbb{R}^3,\\ \big(a_2+\beta'(\|v\|^2_{E_2})\big)\big(-\Delta v+V_2(x)v\big)=|v|^{q-2}v+\lambda(x)u & \text{for } x\in\mathbb{R}^3, \end{cases}$$

where $a_1,a_2>0$, $\alpha,\beta\in C^2(\mathbb{R}_+,\mathbb{R}_+)$ and for each $i=1,2$ we consider the following weighted Sobolev space

$$E_i:=\bigg\{w\in H^1(\mathbb{R}^3):\int_{\mathbb{R}^3}V_i(x)w^2\,dx<\infty\bigg\},$$

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endowed with the norm

$$\|w\|_{E_i}^2 = \int_{\mathbb{R}^3} |\nabla w|^2\,dx + \int_{\mathbb{R}^3} V_i(x) w^2\,dx.$$

Moreover, we assume that the coupling term $\lambda(x)$ is related with the potentials by $|\lambda(x)| \le \delta\sqrt{V_1(x)V_2(x)}$, for some suitable $\delta > 0$. Our main contribution here is to prove the existence of positive ground states for the subcritical case, that is, when $4 < p \le q < 2^* = 6$ and for the critical case when $4 < p < q = 6$. In the critical case, the existence of ground states will be related with the parameter μ introduced in the first equation. In fact, we obtain the existence result when $\mu > 0$ is large enough. For the critical case when $p = q = 6$, we make use of the Pohozaev identity type to prove that system (S_μ) does not admit positive solution.

In order to motivate our results we begin by giving a brief survey on Kirchhoff problems. First, we mention that system (S_μ) is called *nonlocal* due the dependence of the norms $\|\cdot\|_{E_1}$ and $\|\cdot\|_{E_2}$. This type of equations take care of the behavior of the solution in the whole space, which implies that the equations in (S_μ) are no longer a pointwise identity. We point out that nonlocal problems have been applied in many different contexts, for instance, biological systems, where can be used to describe the growth and movement of a particular species. Moreover, we cite also conservation laws, applications on population density, etc. The Kirchhoff–Schrödinger equations introduced in system (S_μ) are also motivated by some physical models. The first study in this direction was proposed by Kirchhoff [9] in the study of the following hyperbolic equation

$$\rho\,\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right| dx\right)\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.1}$$

Equation (1.1) is a generalization of the classical d'Alembert's wave equation, by considering the vibrations of the strings.

When $\lambda \equiv 0$, the uncoupled Kirchhoff–Schrödinger equation from system (S_μ) is related with the following stationary type equation

$$\begin{cases} u_{tt} - M\bigg(\displaystyle\int_\Omega |\nabla u|^2\,dx\bigg)\Delta u = f(x,u), & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. In 1878, J.L. Lions [12] introduced a functional analysis approach to study problem (1.2). Motivated by the physical interest and impulsed by [12], Kirchhoff problems has been extensively studied by many authors in the last years. Concerning the scalar case related with (1.2), there are several works with respect to the following Kirchhoff–Schrödinger equation

$$(1.3)\qquad \begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $a,b>0$ and $\Omega\subset\mathbb{R}^3$ is a smooth bounded domain. For existence and multiplicity of solutions for related problems to (1.3), we refer the readers to [1], [3], [6], [15] and references therein.

Here we are concerned to study the existence of ground states for a class of nonlocal linearly coupled systems defined in the whole space $\mathbb{R}^3$. We are motivated by recent works which obtain existence of solutions for nonlocal systems by using a variational approach. In this direction, D. Lü and J. Xiao [7] studied the following class of coupled systems involving Kirchhoff equations

$$\begin{cases} -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)\Delta u+\lambda V(x)u=\dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x\in\mathbb{R}^3,\\ -\left(a+b\displaystyle\int_{\mathbb{R}^3}|\nabla v|^2\,dx\right)\Delta v+\lambda W(x)v=\dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\mathbb{R}^3,\\ u(x)\to 0,\quad v(x)\to 0,\quad \text{as } |x|\to\infty,\end{cases}$$

where $\alpha,\beta>2$ satisfying $\alpha+\beta<2^*=6$. The authors obtained existence and multiplicity of solutions when the parameter $\lambda>0$ is large. For more related results to coupled Kirchhoff-type systems, we refer the readers to [8], [14].

Motivated by the above discussion, our purpose is to study the class of coupled systems (S_μ), by considering a more general class of Kirchhoff–Schrödinger equations. The class of systems (S_μ) imposes several difficulties. The first one is that the nonlocal terms here are introduced by functions $\alpha,\beta\in C^2(\mathbb{R}_+,\mathbb{R}_+)$ which generalize the standard Kirchhoff–Schrödinger equations (see Remark 1.4). Moreover, we deal with the "lack of compactness" due the fact that the problem is defined in the whole space $\mathbb{R}^3$. Another obstacle is the fact that system (S_μ) involves strongly coupled Kirchhoff–Schrödinger equations because the linear terms in the right hand side. The work is divided into three parts: The first one consists to study system (S_μ) in the subcritical case, that is, when $4<p\le q<2^*=6$. By using a minimization method over the Nehari manifold we obtain the existence of at least one positive ground state solution for system (S_μ), for any parameter $\mu>0$. After that, we study the critical case, when $4<p<q=6$. In this case, we use the parameter $\mu>0$ to control the range of the ground state energy level associated to system (S_μ). Finally, we make use of a Pohozaev identity type to conclude that system (S_μ) does not admit positive solution when $p=q=6$.

1.1. Assumptions and main results. In order to establish a variational approach to study system (S_μ), we introduce some suitable assumptions on the Kirchhoff functions and on the potentials. Throughout the paper, we assume that $a_1, a_2 > 0$ and $\alpha, \beta \in C^2(\mathbb{R}_+, \mathbb{R}_+)$. In addition, we suppose that α and β satisfy the following assumptions:

(M_1) $\alpha'(s)$ and $\beta'(t)$ are increasing on $s, t > 0$.

(M_2) $s \mapsto \dfrac{\alpha'(s)}{s}$ and $t \mapsto \dfrac{\beta'(t)}{t}$ are non-increasing on $s, t > 0$.

(M_3) $\alpha'(s) \le b_1 s$, $\beta'(t) \le b_2 t$ and

$$\frac{1}{2}\,\alpha'(s)s + \frac{1}{2}\,\beta'(t)t \le \alpha(s) + \beta(t) \le \alpha'(s)s + \beta'(t)t, \quad \text{for all } s, t \ge 0.$$

(M_4) $\alpha''(s)s \le \alpha'(s)$ and $\beta''(t)t \le \beta'(t)$, for all $s, t \ge 0$.

Due the presence of the potentials V_1 and V_2, we have introduced above suitable spaces E_1 and E_2. For each $i = 1, 2$, we assume the following hypotheses:

(V_1) $V_i, \lambda \in C(\mathbb{R}^3, \mathbb{R})$ are $\mathbb{Z}^3$-periodic.

(V_2) $V_i(x) \ge 0$ for all $x \in \mathbb{R}^3$ and

$$\inf_{u \in E_i} \bigg\{ \int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} V_i(x)u^2\,dx : \int_{\mathbb{R}^3} u^2\,dx = 1 \bigg\} > 0.$$

(V_3) $|\lambda(x)| \le \delta\sqrt{V_1(x)V_2(x)}$, for some $\delta \in (0, \min\{a_1, a_2\})$, for all $x \in \mathbb{R}^3$.

(V_3') Assumption (V_3) holds and $\lambda(x) > 0$, for all $x \in \mathbb{R}^3$.

In view of [13, Lemma 2.1], E_i is a Hilbert space continuously embedded into $L^r(\mathbb{R}^3)$ for $r \in [2, 6]$ and $i = 1, 2$. We set the product space $E = E_1 \times E_2$. We have that E is a Hilbert space when endowed with the inner product

$$((u,v),(z,w))_E = \int_{\mathbb{R}^3} (\nabla u \nabla z + V_1(x)uz + \nabla v \nabla w + V_2(x)vw)\,dx,$$

to which corresponds the induced norm

$$\|(u,v)\|_E^2 = ((u,v),(u,v))_E = \|u\|_{E_1}^2 + \|v\|_{E_2}^2.$$

Associated to system (S_μ) we have the energy functional $I \in C^1(E, \mathbb{R})$ defined by

$$I(u,v) = \frac{1}{2}\big(a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2\big) + \frac{1}{2}\big(\alpha\big(\|u\|_{E_1}^2\big) + \beta\big(\|v\|_{E_2}^2\big)\big) - \frac{\mu}{p}\,\|u\|_p^p - \frac{1}{q}\,\|v\|_q^q - \int_{\mathbb{R}^3} \lambda(x)uv\,dx.$$

By standard arguments it can be checked that critical points of I correspond to weak solutions of (S_μ) and conversely. We say that a weak solution $(u_0, v_0) \in E$ for system (S_μ) is a *ground state solution* (or least energy solution) if $I(u_0, v_0) \le I(u, v)$ for any other weak solution $(u, v) \in E \setminus \{(0, 0)\}$. We say that (u_0, v_0) is nonnegative (nonpositive) if $u_0, v_0 \ge 0$ ($u_0, v_0 \le 0$) and positive (negative) if $u_0, v_0 > 0$ ($u_0, v_0 < 0$), respectively.

Now we are able to state our main results.

THEOREM 1.1. *Assume that* (M_1)–(M_4) *and* (V_1)–(V_3) *hold. If* $4 < p \leq q < 6$, *then there exists a nonnegative ground state solution for system* (S_μ), *for all* $\mu \geq 0$. *If* (V_3') *holds, then the ground state is positive.*

THEOREM 1.2. *Assume that* (M_1)–(M_4) *and* (V_1)–(V_3) *hold. If* $4 < p < q = 6$, *then there exists* $\mu_0 > 0$ *such that system* (S_μ) *possesses a nonnegative ground state solution* $(u_0, v_0) \in E$, *for all* $\mu \geq \mu_0$. *If* (V_3') *holds, then the ground state is positive.*

THEOREM 1.3. *Let* $p = q = 6$. *In addition, for* $i = 1, 2$, *we consider the following assumptions*:

(V_4) $V_i \in C^1(\mathbb{R}^3)$ *and* $0 \leq \langle \nabla V_i(x), x\rangle \leq CV_i(x)$, *for all* $x \in \mathbb{R}^3$.

(V_5) $\lambda \in C^1(\mathbb{R}^3)$, $|\langle \nabla \lambda(x), x\rangle| \leq C|\lambda(x)|$ *and* $\langle \nabla \lambda(x), x\rangle \leq 0$, *for all* $x \in \mathbb{R}^3$.

Then, system (S_μ) *has no positive classical solution for all* $\mu \geq 0$.

REMARK 1.4. A typical example of function verifying assumptions (M_1)–(M_4) is given by

$$\alpha(s) = b_1 \frac{s^2}{2} \quad \text{and} \quad \beta(t) = b_2 \frac{t^2}{2},$$

where $b_1, b_2 > 0$. This is the example which was considered in [9] for the scalar case. More generally, the functions

$$\alpha(s) = b_1 \frac{s^2}{2} + \sum_{i=1}^{k} a_i s^{\gamma_i} \quad \text{and} \quad \beta(t) = b_2 \frac{t^2}{2} + \sum_{i=1}^{k} b_i t^{\gamma_i},$$

with $a_i, b_i > 0$ and $\gamma_i \in (0,1)$ for all $i \in \{1, \ldots, k\}$ verify hypotheses (M_1)–(M_4). Another example is given by

$$\alpha(s) = \int_0^s \ln(1+r)\,dr \quad \text{and} \quad \beta(t) = \int_0^t \ln(1+r)\,dr.$$

In the present work we introduce a class of Kirchhoff–Schrödinger coupled systems by considering different types of functions α and β.

1.2. Notation. Let us introduce the following notation:

- C, $\widetilde{C}$, C_1, C_2, … denote positive constants (possibly different).
- $o_n(1)$ denotes a sequence which converges to 0 as $n \to \infty$.
- The norm in $L^p(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$, will be denoted respectively by $\|\cdot\|_p$ and $\|\cdot\|_\infty$.
- The norm in $L^p(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ is given by $\|(u,v)\|_p = \left(\|u\|_p^p + \|v\|_p^p\right)^{1/p}$.
- We write $\int u$ instead of $\int_{\mathbb{R}^3} u\,dx$.
- We denote by S the sharp constant of the embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$

$$S\left(\int |u|^6\right)^{1/3} \leq \int |\nabla u|^2, \tag{1.4}$$

where $D^{1,2}(\mathbb{R}^3) := \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$.

1.3. Outline. In the forthcoming section we introduce the Nehari manifold associated to system (S_μ). In Section 3 we study the existence of ground states for system (S_μ) in the subcritical case. Section 4 is devoted to the critical case. In Section 5 we make use of a Pohozaev identity type to prove the nonexistence result.

2. The Nehari manifold

The main goal of the paper is to prove the existence of ground state solutions. For this purpose, we use a minimization technique over the Nehari manifold. In order to obtain some properties for the Nehari manifold, we have the following technical lemma:

Lemma 2.1. *If* (V_3) *holds, then we have*

$$a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 - 2\int \lambda(x)uv \geq (\min\{a_1,a_2\} - \delta)\|(u,v)\|_E^2. \tag{2.1}$$

Proof. Note that

$$2\sqrt{V_1(x)V_2(x)}|u||v| \leq V_1(x)u^2 + V_2(x)v^2.$$

Thus, using (V_3) we can deduce that

$$-2\int \lambda(x)uv \geq -\delta\int \big(V_1(x)u^2 + V_2(x)v^2\big) \geq -\delta\|(u,v)\|_E^2,$$

which easily implies (2.1). □

The Nehari manifold associated to System (S_μ) is given by

$$\mathcal{N} = \{(u,v) \in E \setminus \{(0,0)\} : I'(u,v)(u,v) = 0\}.$$

Notice that, if $(u,v) \in \mathcal{N}$, then

$$a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 + \alpha'\big(\|u\|_{E_1}^2\big)\|u\|_{E_1}^2 + \beta'\big(\|v\|_{E_2}^2\big)\|v\|_{E_2}^2 = \mu\|u\|_p^p + \|v\|_q^q + 2\int \lambda(x)\,uv. \tag{2.2}$$

Lemma 2.2. *Suppose that* (V_2), (V_3), (M_3) *and* (M_4) *hold. Then, there exists* $\alpha > 0$ *such that*

$$\|(u,v)\|_E \geq \alpha, \quad \text{for all } (u,v) \in \mathcal{N}. \tag{2.3}$$

Moreover, $\mathcal{N}$ *is a* C^1*-manifold.*

PROOF. If $(u,v)\in\mathcal{N}$, then using Lemma 2.1, (2.2) and Sobolev embedding we deduce that

$$\begin{aligned}(\min\{a_1,a_2\}-\delta)\|(u,v)\|_E^2 &\le a_1\|u\|_{E_1}^2+a_2\|v\|_{E_2}^2-2\int\lambda(x)\,uv\\ &\le \mu\|u\|_p^p+\|v\|_q^q\le C\big(\|(u,v)\|_E^p+\|(u,v)\|_E^q\big).\end{aligned}$$

Hence, we have that

$$0<\frac{\min\{a_1,a_2\}-\delta}{C}\le\|(u,v)\|_E^{p-2}+\|(u,v)\|_E^{q-2},$$

which implies (2.3). In order to prove that $\mathcal{N}$ is a C^1-manifold, we define the C^1-functional $J\colon E\setminus\{(0,0)\}\to\mathbb{R}$ given by $J(u,v)=I'(u,v)(u,v)$. Notice that $\mathcal{N}=J^{-1}(0)$ and

$$\begin{aligned}J'(u,v)(u,v)=&\,2\bigg(a_1\|u\|_{E_1}^2+a_2\|v\|_{E_2}^2-2\int\lambda(x)\,uv\bigg)\\ &+2\alpha''\big(\|u\|_{E_1}^2\big)\|u\|_{E_1}^2+2\alpha'\big(\|u\|_{E_1}^2\big)\|u\|_{E_1}^2\\ &+2\beta''\big(\|v\|_{E_2}^2\big)\|v\|_{E_2}^2+2\beta'\big(\|v\|_{E_2}^2\big)\|v\|_{E_2}^2-\mu p\|u\|_p^p-q\|v\|_q^q,\end{aligned}$$

which together with assumption (M_4) implies that

$$\begin{aligned}J'(u,v)(u,v)\le&\,2\bigg(a_1\|u\|_{E_1}^2+a_2\|v\|_{E_2}^2-2\int\lambda(x)\,uv\bigg)\\ &+4\alpha'(\|u\|_{E_1}^2)\|u\|_{E_1}^2+4\beta'(\|v\|_{E_2}^2)\|v\|_{E_2}^2-\mu p\|u\|_p^p-q\|v\|_q^q.\end{aligned}$$

Since $(u,v)\in\mathcal{N}$, we can conclude that

$$J'(u,v)(u,v)\le-2(\min\{a_1,a_2\}-\delta)\|(u,v)\|_E^2+\mu(4-p)\|u\|_p^p+(4-q)\|v\|_q^q<0,$$

where we have used Lemma 2.1 and the fact that $4<p\le q$. Therefore, 0 is a regular value of J and $\mathcal{N}$ is a C^1-manifold. □

REMARK 2.3. If $(u_0,v_0)\in\mathcal{N}$ is a critical point of the constrained functional $I|_{\mathcal{N}}$, then $I'(u_0,v_0)=0$. In fact, notice that $I'(u_0,v_0)=\eta J'(u_0,v_0)$, where $\eta\in\mathbb{R}$ is the corresponding Lagrange multiplier. By taking the scalar product with (u_0,v_0) we conclude that $\eta=0$.

LEMMA 2.4. *Assume* (V_2), (V_3), (M_2) *and* (M_3) *hold. Then, for any* $(u,v)\in E\setminus\{(0,0)\}$ *there exists a unique* $t_0>0$, *depending only on* (u,v), *such that*

$$(t_0u,t_0v)\in\mathcal{N}\quad\textit{and}\quad I(t_0u,t_0v)=\max_{t\ge0}I(tu,tv).$$

Proof. Let $(u,v)\in E\backslash\{(0,0)\}$ be fixed and consider the fiber map $g\colon [0,\infty)\to\mathbb{R}$ defined by $g(t)=I(tu,tv)$. Notice that $\langle I'(tu,tv)(tu,tv)\rangle = tg'(t)$. Therefore, t_0 is a positive critical point of g if and only if $(t_0u,t_0v)\in\mathcal{N}$. Note that

$$\begin{aligned} g(t) = {} & \frac{t^2}{2}\bigg(a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 - 2\int\lambda(x)\,uv\bigg) \\ & + \frac{1}{2}\big(\alpha\big(\|tu\|_{E_1}^2\big)+\beta\big(\|tv\|_{E_2}^2\big)\big) - \mu\frac{t^p}{p}\,\|u\|_p^p - \frac{t^q}{q}\,\|v\|_q^q. \end{aligned}$$

By using Lemma 2.1 and Sobolev embeddings, we have that

$$g(t)\geq (\min\{a_1,a_2\}-\delta)\,\frac{t^2}{2}\,\|(u,v)\|_E^2 - C_1\mu\,\frac{t^p}{p}\,\|(u,v)\|_E^p - C_2\,\frac{t^q}{q}\,\|(u,v)\|_E^q\ >0,$$

provided $t>0$ is sufficiently small. On the other hand, by using (M_3) we deduce that

$$\begin{aligned} g(t) \leq {} & \frac{t^2}{2}\bigg(a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 - 2\int\lambda(x)\,uv\bigg) \\ & + b_1\,\frac{t^4}{2}\,\|u\|_{E_1}^2 + b_2\,\frac{t^4}{2}\,\|v\|_{E_2}^2 - \mu\frac{t^p}{p}\|u\|_p^p - \frac{t^q}{q}\|v\|_q^q. \end{aligned}$$

Since $4<p\leq q$ we conclude that $g(t)<0$ for $t>0$ sufficiently large. Thus g has maximum points in $(0,\infty)$. It remains to prove that the critical point is unique. In fact, notice that if $g'(\bar t)=0$ then

$$\begin{aligned} & \frac{1}{\bar t^2}\bigg(a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 - 2\int\lambda(x)\,uv\bigg) \\ & \quad + \frac{\alpha'(\|\bar t u\|_{E_1}^2)}{\|\bar t u\|_{E_1}^2}\|u\|_{E_1}^4 + \frac{\beta'(\|\bar t v\|_{E_2}^2)}{\|\bar t v\|_{E_2}^2}\|v\|_{E_2}^4 = \mu\bar t^{p-4}\|u\|_p^p + \bar t^{\,q-4}\|v\|_q^q. \end{aligned}$$

It follows from (M_2) that the left-hand side is decreasing on $\bar t>0$. Since the right-hand side is increasing on $\bar t>0$, the maximum point of g is unique. □

Remark 2.5. Let us define the following energy levels associated to system (S_μ):

$$c_{\mathcal{N}} = \inf_{(u,v)\in\mathcal{N}} I(u,v),\quad c^* = \inf_{(u,v)\in E\backslash\{(0,0)\}}\ \max_{t\geq 0} I(tu,tv),\quad c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=(0,0),\ I(\gamma(1))<0\}$. By a similar argument used in [16, Theorem 4.2] we can deduce that $0<c_{\mathcal{N}}=c^*=c$.

3. The subcritical case

In this section we are concerned to prove existence of ground states for the subcritical System (S_μ). For this purpose, we follow some ideas from [2, Theorem 2.5]. Let $(u_n,v_n)\subset\mathcal{N}$ be a minimizing sequence for $c_{\mathcal{N}}$, that is

$$I(u_n,v_n)\to c_{\mathcal{N}}\quad\text{and}\quad I'(u_n,v_n)\to 0. \tag{3.1}$$

Proposition 3.1. *If* (V_2), (V_3) *and* (M_3) *hold, then the minimizing sequence* (u_n, v_n) *is bounded in* E.

Proof. In fact, recalling that we are assuming $4 < p \le q$, it follows from (2.2) that

$$\begin{aligned} I(u_n, v_n) \ge \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big(a_1\|u\|_{E_1}^2 + a_2\|v\|_{E_2}^2 - 2\int \lambda(x)\, uv\Big) \\ + \Big(\frac{1}{p} - \frac{1}{q}\Big)\|v_n\|_q^q + \frac{1}{2}\big(\alpha\big(\|u_n\|_{E_1}^2\big) + \beta(\|v_n\|_{E_2}^2)\big) \\ + \frac{1}{4}\big(\alpha'\big(\|u_n\|_{E_1}^2\big)\|u_n\|_{E_1}^2 + \beta'\big(\|v_n\|_{E_2}^2\big)\|v_n\|_{E_2}^2\big), \end{aligned}$$

which together with assumption (M_3) and Lemma 2.1 implies that

$$I(u_n, v_n) \ge \Big(\frac{1}{2} - \frac{1}{p}\Big)(\min\{a_1, a_2\} - \delta)\|(u_n, v_n)\|_E^2.$$

Since $I(u_n, v_n)$ is bounded, we conclude that (u_n, v_n) is bounded in E. □

By the preceding proposition we may assume that, up to a subsequence, we have

$$\begin{cases} (u_n, v_n) \rightharpoonup (u_0, v_0) & \text{weakly in } E, \\ \|u_n\|_{E_1} \to \varrho_0 \text{ and } \|v_n\|_{E_2} \to \varrho_1 & \text{strongly in } \mathbb{R}, \\ (u_n, v_n) \to (u_0, v_0) & \text{strongly in } L^t_{\mathrm{loc}}(\mathbb{R}^3) \times L^t_{\mathrm{loc}}(\mathbb{R}^3) \\ & \text{for all } t \in (2, 6), \\ (u_n(x), v_n(x)) \to (u_0(x), v_0(x)) & \text{almost everywhere in } \mathbb{R}^3. \end{cases} \tag{3.2}$$

Without loss of generality, we can assume that $(u_0, v_0) \ne (0, 0)$. In fact, by using a standard argument we can use the result due to Lions [16, Lemma 1.21] (see also [11]) to prove that there exist $\eta > 0$ and $(y_n) \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} (u_n^2 + v_n^2) \ge \eta > 0. \tag{3.3}$$

A direct computation shows that we can assume $(y_n) \subset \mathbb{Z}^3$. Let us define the shift sequence

$$(\widetilde{u}_n(x), \widetilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n)).$$

In view of assumption (V_1) one can see that $(\widetilde{u}_n, \widetilde{v}_n)$ is a Palais–Smale sequence of I at level $c_{\mathcal{N}}$, that is, also satisfies (3.1). Moreover, we have that $(\widetilde{u}_n, \widetilde{v}_n)$ is also bounded in E and its weak limit denoted by $(\widetilde{u}_0, \widetilde{v}_0)$ is nontrivial, because (3.3) and the local convergence imply that

$$\int_{B_R(0)} (\widetilde{u}_0^2 + \widetilde{v}_0^2) \ge \eta > 0.$$

Therefore, we can assume $(u_0, v_0) \ne (0, 0)$.

Proposition 3.2. *Suppose that* (V_2), (V_3), (M_1)–(M_4) *hold. Then, the weak limit* (u_0, v_0) *is a critical point of* I.

Proof. By using (3.2) and the fact that α' and β' are continuous, we have the convergences

$$\alpha'\big(\|u_n\|_{E_1}^2\big) \to \alpha'(\varrho_0^2) \quad \text{and} \quad \beta'(\|v_n\|_{E_2}^2) \to \beta'(\varrho_1^2).$$

Since $I'(u_n, v_n) = o_n(1)$, we conclude that (u_0, v_0) is a nontrivial solution of the following system

$$\begin{cases} \big(a_1 + \alpha'\big(\varrho_0^2\big)\big)(-\Delta u_0 + V_1(x)u_0) = \mu|u_0|^{p-2}u_0 + \lambda(x)v_0 & \text{for } x \in \mathbb{R}^3, \\ \big(a_2 + \beta'\big(\varrho_1^2\big)\big)(-\Delta v_0 + V_2(x)v_0) = |v_0|^{q-2}v_0 + \lambda(x)u_0, & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{3.4}$$

We claim that $\alpha'(\|u_0\|_{E_1}^2) = \alpha'(\varrho_0^2)$ and $\beta'(\|v_0\|_{E_2}^2) = \beta'(\varrho_1^2)$. In fact, by the lower semicontinuity of the norm we have

$$\liminf_{n\to\infty} \|u_n\|_{E_1} \geq \|u_0\|_{E_1}.$$

Consequently, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|u_n\|_{E_1} \geq \|u_0\|_{E_1} - \varepsilon, \quad \text{for all } n \geq n_0.$$

Arguing by the same argument we get

$$\|v_n\|_{E_2} \geq \|v_0\|_{E_2} - \varepsilon, \quad \text{for all } n \geq n_0.$$

Thus, for $n \geq n_0$, it follows from (M_1) that

$$\alpha'\big(\|u_n\|_{E_1}^2\big) \geq \alpha'\big(\|u_0\|_{E_1}^2 - \varepsilon\big) \quad \text{and} \quad \beta'\big(\|v_n\|_{E_2}^2\big) \geq \beta'\big(\|v_0\|_{E_2}^2 - \varepsilon\big).$$

Letting $n \to \infty$, and after $\varepsilon \to 0$, we obtain

$$\alpha'\big(\varrho_0^2\big) \geq \alpha'\big(\|u_0\|_{E_1}^2\big) \quad \text{and} \quad \beta'\big(\varrho_1^2\big) \geq \beta'\big(\|v_0\|_{E_2}^2\big).$$

Let us suppose by contradiction that at least one of the preceding estimates is strictly. Without loss of generality, let us assume that $\alpha'(\varrho_0^2) > \alpha'(\|u_0\|_{E_1}^2)$. Let $\mathcal{H}\colon (0, +\infty) \to \mathbb{R}$ be defined by

$$\mathcal{H}(t) := \frac{1}{2}\big[\alpha\big(\|tu_0\|_{E_1}^2\big) + \beta\big(\|tv_0\|_{E_2}^2\big)\big] - \frac{1}{4}\big[\alpha'\big(\|tu_0\|_{E_1}^2\big)\|tu_0\|_{E_1}^2 + \beta'\big(\|tv_0\|_{E_2}^2\big)\|tv_0\|_{E_2}^2\big].$$

By using (M_4) one can see that $\mathcal{H}(t)$ is increasing on $t > 0$. Since (u_0, v_0) is a weak solution for the problem (3.4) we can deduce that $I'(u_0, v_0)(u_0, v_0) < 0$.

Moreover, we have that $I'(t_0u_0,t_0v_0)(t_0u_0,t_0v_0)>0$ for some $t\in(0,1)$. Therefore, there exists $\bar t\in(0,1)$ such that $\bar t(u_0,v_0)\in\mathcal{N}$. Thus, it follows that

$$\begin{aligned}c_{\mathcal N}&\le I(\bar t u_0,\bar t v_0)-\frac14\,I'(\bar t u_0,\bar t v_0)(\bar t u_0,\bar t v_0)\\&=\frac{\bar t^2}{4}\left(a_1\|u_0\|_{E_1}^2+a_2\|v_0\|_{E_2}^2-2\int\lambda(x)u_0v_0\right)\\&\quad+\bar t^p\left(\frac14-\frac1p\right)\mu\|u_0\|_p^p+\bar t^q\left(\frac14-\frac1q\right)\|v_0\|_q^q+\mathcal H(\bar t).\end{aligned}$$

Since the right-hand side is strictly increasing on $\bar t>0$ one has

$$\begin{aligned}c_{\mathcal N}<\frac14&\left(a_1\|u_0\|_{E_1}^2+a_2\|v_0\|_{E_2}^2-2\int\lambda(x)u_0v_0\right)\\&+\left(\frac14-\frac1p\right)\mu\|u_0\|_p^p+\left(\frac14-\frac1q\right)\|v_0\|_q^q+\mathcal H(1).\end{aligned}$$

By using Fatou's Lemma we conclude that

$$c_{\mathcal N}<\liminf_{n\to\infty}\left\{I(u_n,v_n)-\frac14\,I'(u_n,v_n)(u_n,v_n)\right\}=c_{\mathcal N},$$

which is not possible. Therefore, $I'(u_0,v_0)=0$. □

PROOF OF THEOREM 1.1. In the preceding proposition we have obtained a nontrivial weak solution $(u_0,v_0)\in E$ for system (S_μ). Thus, $(u_0,v_0)\in\mathcal N$ and consequently $c_{\mathcal N}\le I(u_0,v_0)$. On the other hand, by using (M_3), Lemma 2.1, the semicontinuity of norm and Fatou's Lemma, we can deduce that

$$\begin{aligned}c_{\mathcal N}+o_n(1)&=I(u_n,v_n)-\frac14\,I'(u_n,v_n)(u_n,v_n)\\&\ge I(u_0,v_0)-\frac14\,I'(u_0,v_0)(u_0,v_0)+o_n(1)=I(u_0,v_0)+o_n(1),\end{aligned}$$

which implies that $c_{\mathcal N}\ge I(u_0,v_0)$. Therefore, $I(u_0,v_0)=c_{\mathcal N}$. In order to get a nonnegative ground state solution, in view of Lemma 2.4, there exists $t_0>0$ such that $(t_0|u_0|,t_0|v_0|)\in\mathcal N$. Thus, we can deduce that

$$I(t_0|u_0|,t_0|v_0|)\le I(t_0u_0,t_0v_0)\le I(u_0,v_0)=c_{\mathcal N},$$

which implies that $(t_0|u_0|,t_0|v_0|)$ is also a minimizer of I on $\mathcal N$. Therefore, $(t_0|u_0|,t_0|v_0|)$ is a nonnegative ground state solution for system (S_μ). Now, let us suppose that (V_3') holds and let us denote $(\overline u_0,\overline v_0)=(t_0|u_0|,t_0|v_0|)$. Since $(\overline u_0,\overline v_0)\ne(0,0)$ we may assume without loss of generality that $\overline u_0\ne0$. We claim that $\overline v_0\ne0$. In fact, arguing by contradiction we suppose that $\overline v_0=0$. Thus, since $(\overline u_0,\overline v_0)$ is a critical point of I, we deduce that

$$0=\langle I'(\overline u_0,\overline v_0),(0,\phi)\rangle=-\int\lambda(x)\,\overline u_0\phi,\quad\text{for all }\phi\in C_0^\infty(\mathbb R^3).$$

Since $\lambda(x)$ is positive, we have that $\overline{u}_0 = 0$ which is a contradiction. Therefore, $\overline{v}_0 \neq 0$. By using strong maximum principle (see [5]) in each equation of (S_μ), we conclude that $(\overline{u}_0, \overline{v}_0)$ is positive, which finishes the proof of Theorem 1.1. □

4. The critical case

In the preceding section we have obtained a positive ground state solution to the subcritical system (S_μ), for any parameter $\mu > 0$. In this section, we are concerned with the critical case, that is, when $2 < p < q = 6$. Analogously to the subcritical case, we have a minimizing sequence $(u_n, v_n) \subset \mathcal{N}$ satisfying (3.1). Moreover, the sequence is bounded and $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in E.

We may assume that, up to a subsequence, $\|v_n\|_6^6 \to A_\mu \in [0, +\infty)$. If $A_\mu = 0$, then the proof follows by the same arguments from the subcritical case, using [16, Lemma 1.21] to obtain (3.3) and a nontrivial critical point for the energy functional I. Let us assume $A_\mu > 0$. In this case, we do not obtain (3.3) directly, since $q = 2^*$ and we are not able to use strong convergence in the argument. In order to overcome this difficulty, we choose $\mu > 0$ large enough such that the level $c_{\mathcal{N}}$ be in the suitable range where we can recover the compactness.

Proposition 4.1. *Suppose that* (V_2), (V_3), (M_2) *and* (M_3) *hold. There exists* $\mu_0 > 0$ *such that*

$$c_{\mathcal{N}} < \left(\frac{1}{4} - \frac{1}{p}\right)[(\min\{a_1, a_2\} - \delta)S]^{3/2}, \quad \text{for all } \mu \geq \mu_0. \tag{4.1}$$

Proof. Let $(u, v) \in E$ be nonnegative and $u, v \not\equiv 0$. We define $u_\mu = \mu u$ and $v_\mu = \mu v$. It follows from Lemma 2.4 that for any $\mu > 0$, there exists a unique $t_\mu > 0$ such that $(t_\mu u_\mu, t_\mu v_\mu) \in \mathcal{N}$. Hence, we have

$$\begin{aligned}(t_\mu \mu)^2 \bigg(a_1 \|u\|_{E_1}^2 + a_2 \|v\|_{E_2}^2 - 2 \int \lambda(x) uv\bigg) \\ + (t_\mu \mu)^2 \alpha'(\|t_\mu u_\mu\|_{E_1}^2) \|u\|_{E_1}^2 + (t_\mu \mu)^2 \beta'(\|t_\mu v_\mu\|_{E_2}^2) \|v\|_{E_2}^2 \\ = \mu (t_\mu \mu)^p \|u\|_p^p + (t_\mu \mu)^6 \|v\|_6^6,\end{aligned} \tag{4.2}$$

which together with (M_3) implies that

$$a_1 \|u\|_{E_1}^2 + a_2 \|v\|_{E_2}^2 - 2 \int \lambda(x) uv + (t_\mu \mu)^2 \big(b_1 \|u\|_{E_1}^4 + b_2 \|v\|_{E_2}^4\big) \geq (t_\mu \mu)^4 \|v\|_6^6.$$

Therefore, $(t_\mu \mu)_\mu$ is a real bounded family. Hence, up to a subsequence, $t_\mu \mu \to t_0 \geq 0$, as $\mu \to +\infty$. We claim that $t_0 = 0$. Indeed, let us suppose by contradiction that $t_0 > 0$. Thus, it follows from (4.2) that

$$a_1 \|u\|_{E_1}^2 + a_2 \|v\|_{E_2}^2 - 2 \int \lambda(x) uv + (t_\mu \mu)^2 \big(b_1 \|u\|_{E_1}^4 + b_2 \|v\|_{E_2}^4\big) \geq \mu (t_\mu \mu)^{p-2} \|u\|_p^p,$$

which is not possible since the right-hand side goes to infinity as $\mu \to +\infty$. Therefore, $t_\mu \mu \to 0$ as $\mu \to +\infty$.

By using (M_3) and the fact that $(t_\mu u_\mu, t_\mu v_\mu) \in \mathcal{N}$, we have

$$c_{\mathcal{N}} \leq (t_\mu \mu)^2 \bigg(a_1 \|u\|_{E_1}^2 + a_2 \|v\|_{E_2}^2 - 2 \int \lambda(x) uv \bigg) + (t_\mu \mu)^4 (b_1 \|u\|_{E_1}^4 + b_2 \|v\|_{E_2}^4).$$

Therefore, there exists $\mu_0 > 0$ such that (4.1) holds, for all $\mu \geq \mu_0$. □

Corollary 4.2. *Assume that* (V_2), (V_3), (M_2) *and* (M_3) *hold. Let* (u_n, v_n) *be the minimizing sequence satisfying* (3.1). *If* $\mu \geq \mu_0$, *then there exists* $(y_n) \subset \mathbb{R}^3$ *and constants* $R, \eta > 0$ *such that*

$$\liminf_{n \to \infty} \int_{B_R(y_n)} \big(u_n^2 + v_n^2 \big) \geq \eta > 0. \tag{4.3}$$

Proof. Arguing by contradiction, we suppose that (4.3) does not hold. Hence, for any $R > 0$ we have

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} \big(u_n^2 + v_n^2 \big) = 0.$$

In light of [16, Lemma 1.21] (see also [11]) we have that $u_n \to 0$ and $v_n \to 0$ strongly in $L^p(\mathbb{R}^3)$, for $2 < p < 2^*$. Thus, it follows from (2.2) that

$$\begin{aligned} a_1 \|u_n\|_{E_1}^2 + a_2 \|v_n\|_{E_2}^2 - 2 \int \lambda(x) u_n v_n + \alpha'(\|u_n\|_{E_1}^2) \|u_n\|_{E_1}^2 \\ + \beta'(\|v_n\|_{E_2}^2) \|v_n\|_{E_2}^2 = A_\mu + o_n(1). \end{aligned} \tag{4.4}$$

By using (1.4), Lemma 2.1 and (4.4) we deduce that

$$\begin{aligned} &(\min\{a_1, a_2\} - \delta) S A_\mu^{1/3} + o_n(1) \\ &\qquad = (\min\{a_1, a_2\} - \delta) S \|v_n\|_6^2 \leq (\min\{a_1, a_2\} - \delta) \|(u_n, v_n)\|_E^2 \\ &\qquad \leq a_1 \|u_n\|_{E_1}^2 + a_2 \|v_n\|_{E_2}^2 - 2 \int \lambda(x) u_n v_n \leq A_\mu + o_n(1), \end{aligned}$$

which implies that

$$A_\mu \geq [(\min\{a_1, a_2\} - \delta) S]^{3/2}. \tag{4.5}$$

By using (M_3), (3.1), (4.4) and (4.5) we can deduce that

$$\begin{aligned} c_{\mathcal{N}} + o_n(1) &= I(u_n, v_n) - \frac{1}{4} I'(u_n, v_n)(u_n, v_n) \\ &\geq \bigg(\frac{1}{4} - \frac{1}{p} \bigg) A_\mu + o_n(1) \geq \bigg(\frac{1}{4} - \frac{1}{p} \bigg) [(\min\{a_1, a_2\} - \delta) S]^{3/2}, \end{aligned}$$

which contradicts Proposition 4.1. □

Proof of Theorem 1.2. Since (4.3) holds, we can introduce the shift sequence $(\widetilde{u}_n(x), \widetilde{v}_n(x)) = (u_n(x + y_n), v_n(x + y_n))$ and we are able to repeat the same arguments used in the proof of Theorem 1.1 to finish the proof of Theorem 1.2. □

5. The nonexistence result

In this section we are concerned to prove that does not exist positive classical solution for System (S_μ) when $p = q = 6$. Let us denote

$$f(x,u,v) = -\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)V_1(x)u + \mu|u|^4u + \lambda(x)v,$$
$$g(x,u,v) = -\big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)V_2(x)v + |v|^4v + \lambda(x)u.$$

Thus, we write system (S_μ) in the following form

$$\begin{cases} -\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)\Delta u = f(x,u,v) & \text{for } x \in \mathbb{R}^3, \\ -\big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\Delta v = g(x,u,v) & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{5.1}$$

In order to obtain a nonexistence result we introduce the following Pohozaev type identity:

Lemma 5.1. *If $(u,v) \in E$ is a classical solution of system* (5.1)*, then satisfies the following Pohozaev identity*

$$\begin{aligned}\big(a_1 &+ \alpha'\big(\|u\|^2_{E_1}\big)\big)\int \big(|\nabla u|^2 + 3V_1(x)u^2\big) \\ &+ \big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\int \big(|\nabla v|^2 + 3V_2(x)v^2\big) \\ = &-\int \big(\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)\langle\nabla V_1(x), x\rangle u^2 + \big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\langle\nabla V_2(x), x\rangle v^2\big) \\ &+ 2\int\langle\nabla\lambda(x), x\rangle uv + \int\big(\mu u^6 + v^6 + 6\lambda(x)\,uv\big).\end{aligned}$$

Proof. The proof is quite similar to [4], [16], [10] but for the sake of convenience we give a sketch here. Let $(u,v) \in E$ be a classical solution of system (5.1). We introduce the cut-off function $\psi \in C_0^\infty(\mathbb{R})$ given by $\psi(t) = 1$ if $|t| \le 1$, $\psi(t) = 0$ if $|t| \ge 2$ and $|\psi'(t)| \le C$, for some $C > 0$. Moreover, we define $\psi_n(x) = \psi(|x|^2/n^2)$ and we note that $\nabla\psi_n(x) = 2\psi'(|x|^2/n^2)x/n^2$. Multiplying the first equation in (5.1) by the factor $\langle\nabla u, x\rangle\psi_n$, the second equation by the factor $\langle\nabla v, x\rangle\psi_n$, summing and integrating we get

$$-\int\big[\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)\Delta u\langle\nabla u, x\rangle + \big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\Delta v\langle\nabla v, x\rangle\big]\psi_n = \int[f(x,u,v)\langle\nabla u, x\rangle + g(x,u,v)\langle\nabla v, x\rangle]\psi_n. \tag{5.2}$$

The idea is to take the limit as $n \to +\infty$ in (5.2). Similarly to [4], [16], we deduce that

$$-\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)\lim_{n\to\infty}\int\Delta u\langle\nabla u, x\rangle = -\big(a_1 + \alpha'\big(\|u\|^2_{E_1}\big)\big)\frac{1}{2}\int|\nabla u|^2,$$
$$-\big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\lim_{n\to\infty}\int\Delta v\langle\nabla v, x\rangle = -\big(a_2 + \beta'\big(\|v\|^2_{E_2}\big)\big)\frac{1}{2}\int|\nabla v|^2.$$

In order to study the limit in the right-hand side of (5.2), we note that

$$\operatorname{div}(\psi_n F(x,u,v)x) = \psi_n \langle \nabla F(x,u,v), x\rangle + F(x,u,v)\langle \nabla \psi_n, x\rangle + 3\psi_n F(x,u,v),$$

where $F(x,u,v) = -\big(a_1 + \alpha'\big(\|u\|_{E_1}^2\big)\big)V_1(x)u^2/2 + (\mu/6)u^6 + \lambda(x)uv$. Notice that

$$\begin{aligned}\langle \nabla F(x,u,v), x\rangle = -\big(a_1 + \alpha'\big(\|u\|_{E_1}^2\big)\big)\frac{1}{2}\langle \nabla V_1(x), x\rangle u^2 \\ + f(x,u,v)\langle \nabla u, x\rangle + \langle \nabla \lambda(x), x\rangle uv + \lambda(x)\langle u\nabla v, x\rangle.\end{aligned}$$

Thus, we have that

$$\begin{aligned}\int f(x,u,v)\langle \nabla u, x\rangle \psi_n &= \int (\operatorname{div}(\psi_n F(x,u,v)x) - F(x,u,v)\langle \nabla \psi_n, x\rangle) \\ &+ \int \Big[(a_1 + \alpha'(\|u\|_{E_1}^2))\frac{1}{2}\langle \nabla V_1(x), u\rangle u^2 - 3F(x,u,v)\Big]\psi_n \\ &- \int \left[\langle \nabla \lambda(x), x\rangle uv + \lambda(x)\langle u\nabla v, x\rangle\right]\psi_n.\end{aligned}$$

Analogously, denoting

$$G(x,u,v) = -\big(a_2 + \beta'\big(\|v\|_{E_2}^2\big)\big)\frac{1}{2}\, V_2(x)v^2 + \frac{1}{6}\, v^6 + \lambda(x)\, uv$$

we deduce

$$\begin{aligned}\int g(x,u,v)\langle \nabla v, x\rangle \psi_n &= \int (\operatorname{div}(\psi_n G(x,u,v)x) - G(x,u,v)\langle \nabla \psi_n, x\rangle \psi_n) \\ &+ \int \Big[(a_2 + \beta'(\|v\|_{E_2}^2))\frac{1}{2}\langle \nabla V_2(x), v\rangle v^2 - 3G(x,u,v)\Big]\psi_n \\ &- \int [\langle \nabla \lambda(x), x\rangle uv + \lambda(x)\langle v\nabla u, x\rangle]\psi_n.\end{aligned}$$

By using integration by parts we conclude that

$$-\lim_{n\to\infty}\int \lambda(x)\langle u\nabla v + v\nabla u, x\rangle \psi_n = \int \langle \nabla \lambda(x), x\rangle uv + 3\int \lambda(x)\, uv.$$

Therefore, using integration by parts and Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}\lim_{n\to\infty}\int (f(x,u,v)\langle \nabla u, x\rangle + g(x,u,v)\langle \nabla v, x\rangle)\psi_n = -3\int (F(x,u,v) + G(x,u,v)) \\ + \frac{1}{2}\int \big(\big(a_1 + \alpha'\big(\|u\|_{E_1}^2\big)\big)\langle \nabla V_1(x), x\rangle u^2 + \big(a_2 + \beta'\big(\|v\|_{E_2}^2\big)\big)\langle \nabla V_2(x), x\rangle v^2\big) \\ - \int \langle \nabla \lambda(x), x\rangle uv + 3\int \lambda(x)\, uv.\end{aligned}$$

Replacing $F(x,u,v)$ and $G(x,u,v)$ in the equation above, we get the right-hand side of (5.2) which finishes the proof. □

Now, we are able to prove that system (5.1) does not admit positive classical solution.

Proof of Theorem 1.3. Let $(u,v)\in E$ be a positive classical solution of system (5.1). By the definition of weak solution we have

$$(5.3)\quad \big(a_1+\alpha'\big(\|u\|^2_{E_1}\big)\big)\|u\|^2_{E_1}+\big(a_2+\beta'\big(\|v\|^2_{E_2}\big)\big)\|v\|^2_{E_2} = \mu\int u^6+\int v^6+2\int \lambda(x)\,uv.$$

Combining Lemma 5.1 and (5.3) we deduce that

$$\begin{aligned}&\int \big[\big(a_1+\alpha'\big(\|u\|^2_{E_1}\big)\big)V_1(x)u^2+\big(a_2+\beta'\big(\|v\|^2_{E_2}\big)\big)V_2(x)v^2-2\lambda(x)\,uv\big]\\ &= -\frac{1}{2}\int \big[\big(a_1+\alpha'\big(\|u\|^2_{E_1}\big)\big)\langle\nabla V_1(x),x\rangle u^2+\big(a_2+\beta'\big(\|v\|^2_{E_2}\big)\big)\langle\nabla V_2(x),x\rangle v^2\big]\\ &\qquad+\int\langle\nabla\lambda(x),x\rangle uv,\end{aligned}$$

which together with assumptions (V_4) and (V_5) implies that

$$(5.4)\quad \int \big[\big(a_1+\alpha'\big(\|u\|^2_{E_1}\big)\big)V_1(x)u^2+\big(a_2+\beta'\big(\|v\|^2_{E_2}\big)\big)V_2(x)v^2-2\lambda(x)\,uv\big]\le 0.$$

On the other hand, by using assumption (V_3) and (5.4) we have

$$\begin{aligned}0&\le C\int \big(a_1V_1(x)u^2+a_2V_2(x)v^2-2\min\{a_1,a_2\}\sqrt{V_1(x)V_2(x)}\,uv\big)\\ &\le C\int\bigg(a_1V_1(x)u^2+a_2V_2(x)v^2-2\,\frac{\min\{a_1,a_2\}}{\delta}\,\lambda(x)\,uv\bigg)\\ &< C\int \big[\big(a_1+\alpha'\big(\|u\|^2_{E_1}\big)\big)V_1(x)u^2+\big(a_2+\beta'\big(\|v\|^2_{E_2}\big)\big)V_2(x)v^2-2\lambda(x)\,uv\big]\le 0,\end{aligned}$$

which is a contradiction and finishes the proof. □

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