ZERO-DIMENSIONAL COMPACT METRIZABLE SPACES AS ATTRACTORS OF GENERALIZED ITERATED FUNCTION SYSTEMS

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Abstract. R. Miculescu and A. Mihail in 2008 introduced the concept of a *generalized iterated function system* (GIFS in short), a particular extension of the classical IFS. The idea is that, instead of families of selfmaps of a metric space X, GIFSs consist of maps defined on a finite Cartesian m-th power X^m with values in X (in such a case we say that a GIFS is *of order* m). It turned out that a great part of the classical Hutchinson theory has natural counterpart in this GIFSs' framework. On the other hand, there are known only few examples of fractal sets which are generated by GIFSs, but which are not IFSs' attractors. In the paper we study 0-dimensional compact metrizable spaces from the perspective of GIFSs' theory. Such investigations for classical IFSs have been undertaken in the last several years, for example by T. Banakh, E. D'Aniello, M. Nowak, T.H. Steele and F. Strobin.
We prove that each such space X is homeomorphic to the attractor of some GIFS on the real line. Moreover, we prove that X can be embedded into the real line $\mathbb{R}$ as the attractor of some GIFS of order m and (in the same time) a nonattractor of any GIFS of order $m-1$, as well as it can be embedded as a nonattractor of any GIFS. Then we show that a relatively simple modifications of X deliver spaces whose each connected component is "big" and which are GIFS's attractors not homeomorphic with IFS's attractors. Finally, we use obtained results to show that a generic compact subset of a Hilbert space is not the attractor of any Banach GIFS.

1. Introduction

The classical Hutchinson theorem (proved by Hutchinson [17] and popularized by Barnsley [5]) from early 80s' states that if X is a complete metric space and $\mathcal{F}$ is a finite family of Banach contractions (that is, selfmaps with the Lipschitz constants $\mathrm{Lip}(f) < 1$), then there is a unique nonempty and compact $A \subset X$ such that

$$(1.1) \qquad\qquad A = \bigcup_{f \in \mathcal{F}} f(A).$$

In this setting, a finite family $\mathcal{F}$ of continuous selfmaps of X is called an *iterated function system* (IFS in short), and a nonempty and compact set $A_{\mathcal{F}}$ satisfying (1.1) is called an *attractor* or a *fractal generated by* $\mathcal{F}$. It is well known that the thesis of the Hutchinson theorem holds under weaker contractive assumptions on fumctions form $\mathcal{F}$ (like these due to Browder [7] or Matkowski [21], see for example [16]), which, in case of compact space X, reduce to the Edelstein contractivity (see [14]):

$$\forall\, x, y \in X,\ x \neq y \quad d(f(x), f(y)) < d(x, y).$$

We will call an IFS consisting of Banach contractions as a *Banach* IFS and consisting of weaker types of contractive maps – as a *weak* IFS. Also, for a given IFS $\mathcal{F}$, we denote $\mathrm{Lip}(\mathcal{F}) := \max\{\mathrm{Lip}(f) : f \in \mathcal{F}\}$.

By a *topological* IFS *fractal* or a *topological fractal* for short (see [18], [27]; in [18] it is called a topological self similar set) we will mean a compact Hausdorff space X such that for some IFS $\mathcal{F}$, $X = \bigcup_{f \in \mathcal{F}} f(X)$ and for every sequence $\{f_k\}_{k \in \mathbb{N}} \subset \mathcal{F}$, the set

$$\bigcap_{k \in \mathbb{N}} f_1 \circ \ldots \circ f_k(X)$$

(called sometimes a *fibre*), is singleton. As was proved in [2] and [25], X is a topological IFS fractal if and only if X is homeomorphic to the attractor of some weak IFS (in particular, it is metrizable). Finally, let us note that topological IFS fractals are attractors of so-called *topologically contracting* IFSs, studied in mentioned papers.

In the last years, there has been an effort to detect those sets (especially, subsets of Euclidean spaces) and topological spaces which are IFSs' fractals or topological fractals. In particular, in [28] Magdalena Nowak proved the following theorem (see Section 3.1 for the definition of the scattered height).

Theorem 1.1 ([28, Theorem 2, Corollary 1]). *Let X be a countable compact metrizable space.*

(1) *The following conditions are equivalent*:
    (a) *X is topological fractal*;

(b) *X can be embedded into the real line as the attractor of a Banach IFS;*

(c) *the scattered height of X is successor ordinal.*

(2) *If X is infinite, then it can be embedded into the real line as a nonattractor of any weak IFS.*

Recently, Banakh, Nowak and Strobin in [4] (see also [3] for a bit weaker result) proved that, additionally:

THEOREM 1.2. *Each uncountable 0-dimensional compact metrizable space can be embedded into the real line as the attractor of a Banach IFS that consists of two maps.*

A bit earlier, D'Aniello and Steele [10] proved an analogous result, but the IFSs they constructed consist of more than two maps.

An interesting generalization of the Hutchinson theory of fractals was introduced by Miculescu and Mihail in 2008 – instead of selfmaps of a metric space, they considered mappings defined on a finite Cartesian power of a given space with values in that space.

Let $m \in \mathbb{N}$ and (X, d) be a metric space. If not stated otherwise, on the Cartesian m-th power X^m we will consider the maximum metric d_m, i.e.

$$d_m((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \max\{d(x_1, y_1), \ldots, d(x_m, y_m)\}.$$

We say that a map $f \colon X^m \to X$ is *a generalized Banach contraction* if the Lipschitz constant $\mathrm{Lip}(f) < 1$, that is, there exists $\alpha < 1$ such that for any $x, y \in X^m$,

$$d(f(x), f(y)) \leq \alpha \cdot d_m(x, y).$$

Miculescu and Mihail in [23] and [26] proved that if X is complete, $m \in \mathbb{N}$ and $\mathcal{F}$ is a finite family of generalized Banach contractions $f \colon X^m \to X$, then there is a unique nonempty and compact $A \subset X$ such that

$$(1.2) \qquad\qquad A = \bigcup_{f \in \mathcal{F}} f(A \times \ldots \times A).$$

In this setting, a family $\mathcal{F}$ of continuous maps $f \colon X^m \to X$ is called a *generalized iterated function system of order m* (GIFS in short), and a unique nonempty and compact set $A_{\mathcal{F}}$ satisfying (1.2) is called the *attractor* or the *fractal generated by $\mathcal{F}$*.

After the papers of Miculescu and Mihail, other aspects of the theory of GIFSs and their fractals were considered, especially by them, Secelean and Strobin, see for example papers [23], [24], [29]–[32] and references therein. In particular, it was proved that GIFSs consisting of weaker contractive type mappings also generates a unique fractal sets (see [31]; also cf. [23]). Again, for

compact X, these conditions reduce to the following:

$$\forall\, x, y \in X^m, \ x \neq y \quad d(f(x), f(y)) < d_m(x, y).$$

If a GIFS consists of generalized Banach contractions, then we call it a *Banach GIFS*, and if it consists of weaker type of generalized contractions – we call it a *weak* GIFS. Similarly, for a GIFS $\mathcal{F}$, we set $\mathrm{Lip}(\mathcal{F}) := \max\{\mathrm{Lip}(f) : f \in \mathcal{F}\}$.

One of the problems considering GIFSs is the following:

> Is the class of GIFSs' attractors essentially wider than the class of IFSs' attractors?

and, related to it,

> Which sets/spaces are attractors of some GIFS?

Several interesting examples were given. In [24] it was observed that the Hilbert cube $I := \prod_{k=1}^{\infty} [0, 1/2^k]$ is generated by a Banach GIFS $\mathcal{F} = \{f, g\}$ of order 2 defined by

$$f((x_k), (y_k)) := \left(\frac{1}{2}\, x_1, \frac{1}{2}\, y_1, \frac{1}{2}\, y_2, \frac{1}{2}\, y_3, \dots \right)$$

and

$$g((x_k), (y_k)) := \left(\frac{1}{2}\, x_1 + \frac{1}{2}, \frac{1}{2}\, y_1, \frac{1}{2}\, y_2, \frac{1}{2}\, y_3, \dots \right).$$

On the other hand, it cannot be generated by any Banach IFS, as it has infinite dimension. However, to our best knowledge, it is not known whether I is the attractor of a weak IFS or a topological fractal.

In [30], for each $m \geq 2$, there is constructed a Cantor subset $C(m)$ of the plane, which is generated by some Banach GIFS on the plane of order m, but is not generated by any weak GIFS of order $m - 1$. Also, there is constructed a Cantor set C which is not the attractor of any weak GIFS. On the other hand, each such $C(m)$ and C are homeomorphic to the Cantor ternary set, hence they are homeomorphic to the attractor of a Banach IFS (in particular, they are topological fractals).

The aim of this paper is to study 0-dimensional compact metrizable spaces from the perspective of GIFSs' theory. It is organized as follows.

In the first part of the next section we show some simple, but useful observations concerning GIFSs and we also prove that certain quotients of topological fractals are also topological fractals.

In Section 3 we construct a wide class of metric spaces, (Λ, b)-*spaces*, which will be key in the proofs of main results. Also, we prove some basic properties of these spaces.

In Section 4 we prove that each compact 0-dimensional metrizable space X can be embedded into the real line as the attractor of a Banach GIFS of order 2.

In particular, if X is countable and its scattered height is limit ordinal, then we obtain the attractor of a Banach GIFS which is not a topological fractal.

In the main result of Section 5 we prove that for any $m \geq 2$, each infinite compact metrizable 0-dimensional space X can be embedded into the real line as the attractor of a Banach GIFS of order m and a nonattractor of any weak GIFS of order $m-1$, and also X can be embedded into the real line as a nonattractor of any weak GIFS. This extends both mentioned results from [30] and part (2) of Theorem 1.1.

In Section 6 we show that, replacing points in certain countable compact spaces by appropriately "big" sets, we obtain next examples of Banach GIFSs' fractals which are not topological fractals.

Finally, in Section 7 we use earlier machinery to prove that a generic compact subset of a Hilbert space is not the attractor of any Banach GIFS.

2. Basic definitions, observations and auxiliary constructions

2.1. Remarks on IFSs and GIFSs. Here we make some simple observations which we will use later. The first one is obvious and we skip the proof.

LEMMA 2.1. *Let X be a metric space, $m_1, \ldots, m_n \in \mathbb{N}$ and for $i = 1, \ldots, n$, $f_i \colon X^{m_i} \to X$ be a generalized Banach contraction. Take $m \geq \max\{m_1, \ldots, m_n\}$ and for each $i = 1, \ldots, n$, define $\widetilde{f_i} \colon X^m \to X$ by*

$$\widetilde{f_i}(x_1, \ldots, x_m) := f_i(x_1, \ldots, x_{m_i}),$$

and set $\widetilde{\mathcal{F}} := \{\widetilde{f_1}, \ldots, \widetilde{f_n}\}$. Then:

(a) *$\widetilde{\mathcal{F}}$ is a Banach GIFS of order m and $\mathrm{Lip}(\widetilde{\mathcal{F}}) = \max\{\mathrm{Lip}(f_1), \ldots, \mathrm{Lip}(f_n)\}$.*

(b) *If $A \subset X$ is nonempty and compact, and $A = \bigcup_{i=1}^{n} f_i(A^{m_i})$, then A is the attractor of $\widetilde{\mathcal{F}}$.*

The second lemma shows that in a certain cases, we can extend maps from a given GIFS to some wider space.

LEMMA 2.2. *Assume that H is a Hilbert space, $X \subset H$ is nonempty and compact, and $\mathcal{F}$ is a Banach GIFS on X of order m with $\mathrm{Lip}(\mathcal{F}) < 1/\sqrt{m}$ so that X is its attractor. Then, for every $f \in \mathcal{F}$, there is its extension $\widetilde{f} \colon H^m \to H$ such that $\widetilde{\mathcal{F}} := \{\widetilde{f} : f \in \mathcal{F}\}$ is a Banach GIFS with $\mathrm{Lip}(\mathcal{F}) \leq \sqrt{m}\,\mathrm{Lip}(\mathcal{F}) < 1$, whose attractor is X.*

PROOF. Take $f \in \mathcal{F}$, and denote by ρ_m the ℓ_2 metric on H^m, that is, $\rho_m((x_1, \ldots, x_m), (y_1, \ldots, y_m)) := \sqrt{d(x_1, y_1)^2 + \ldots + d(x_m, y_m)^2}$. Clearly,

(2.1) $$d_m \leq \rho_m \leq \sqrt{m}d_m$$

Hence $\mathrm{Lip}_{\rho_m}(f) \leq \mathrm{Lip}(f)$, and since (H^m, ρ_m) is a Hilbert space (more precisely, the metric ρ_m is generated by appropriate norm), using the Kirszbraun–Valentine

theorem (see [6, Theorem 1.12]), we can extend the map f to the map $\widetilde{f}\colon H^m \to$ H so that $\mathrm{Lip}_{\rho_m}(\widetilde{f}) = \mathrm{Lip}_{\rho_m}(f) \leq \mathrm{Lip}(f)$. Then, again by (2.1), $\mathrm{Lip}(\widetilde{f}) \leq$ $\sqrt{m}\,\mathrm{Lip}(f)$. Hence the GIFS $\widetilde{\mathcal{F}} := \{\widetilde{f} : f \in \mathcal{F}\}$ satisfies $\mathrm{Lip}(\widetilde{\mathcal{F}}) \leq \sqrt{m}\,\mathrm{Lip}(\mathcal{F})$. The second part of the thesis is obvious. $\qquad\square$

REMARK 2.3. Let us remark that similar result holds for ℓ^∞ space. If $K \subset$ ℓ^∞, then any map $f\colon K^m \to \ell^\infty$ with $\mathrm{Lip}(f) < \infty$ can be extended to a map $\widetilde{f}\colon (\ell^\infty)^m \to \ell^\infty$ with $\mathrm{Lip}(\widetilde{f}) = \mathrm{Lip}(f)$ (see [6, Lemma 1.1 (ii)]).

The next lemma shows that if two GIFS's fractals are appropriately separated, then their union is also a GIFS fractal.

LEMMA 2.4. *Assume that X is a compact metric space of the form $X = X_1 \cup \ldots \cup X_n$, such that*

(a) *$X_1, \ldots, X_n$ are attractors of some Banach GIFSs of order m, $\mathcal{F}_1, \ldots, \mathcal{F}_n$, respectively;*

(b) *there are projections $\pi_i\colon X \to X_i$, $i = 1, \ldots, n$, such that $\max\{\mathrm{Lip}(\pi_i) \cdot \mathrm{Lip}(\mathcal{F}_i) : i = 1, \ldots, n\} < 1$.*

Then X is the attractor of some Banach GIFS $\mathcal{H}$ of order m with $\mathrm{Lip}(\mathcal{H}) \leq \max\{\mathrm{Lip}(\pi_i) \cdot \mathrm{Lip}(\mathcal{F}_i) : i = 1, \ldots, n\}$, and which consists of certain extensions of maps from $\mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n$.

PROOF. Take $f \in \mathcal{F}_i$, and consider the map $\widetilde{f}\colon X^m \to X$ by $\widetilde{f}(x_1, \ldots, x_m) := f(\pi_i(x_1), \ldots, \pi_i(x_m))$. Then, obviously, $\widetilde{f}$ is an extension of f and $\mathrm{Lip}(\widetilde{f}) \leq \mathrm{Lip}(\pi_i) \cdot \mathrm{Lip}(f)$. Then it is enough to take $\mathcal{H} = \{\widetilde{f} : f \in \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_n\}$.

REMARK 2.5. Observe that condition (b) holds if

(c) $\lambda := \dfrac{\max\{\mathrm{diam}(X_i) : i = 1, \ldots, n\}}{\min\{\mathrm{dist}(X_i, X_j) : i, j = 1, \ldots, n, \ i \neq j\}}$
$$< \frac{1}{\max\{\mathrm{Lip}(\mathcal{F}_i) : i = 1, \ldots, n\}}.$$

Indeed, let the projection maps π_i be given by

$$\pi_i(x) := \begin{cases} x & \text{if } x \in X_i, \\ \widetilde{x} & \text{otherwise,} \end{cases}$$

where $\widetilde{x}$ is an initially chosen point of X_i. Then, for $x \in X_i$ and $y \in X_j$, where $i \neq j$, we have

$$d(\pi_i(x), \pi_i(y)) \leq \mathrm{diam}(X_i) \leq \lambda\,\mathrm{dist}(X_i, X_j) \leq \lambda d(x, y),$$

so $\mathrm{Lip}(\pi_i) \leq \max\{1, \lambda\} < 1/\max\{\mathrm{Lip}(\mathcal{F}_i) : i = 1, \ldots, n\}$ and (b) is satisfied.

The following lemma is an extension of [28, Lemma 1].

LEMMA 2.6. *Assume that $n \in \mathbb{N}$ and a metric space X is of the form $X = X_0 \cup \ldots \cup X_n$, where:*

(a) *each X_i is isometric copy of X_0;*
(b) $\operatorname{dist}(X_i, X_j) \geq \operatorname{diam}(X_0)$ *for distinct i, j.*

If X is the attractor of a weak GIFS *of order m, then X_0 is the attractor of a weak* GIFS *of order m.*

PROOF. For $i = 0, \ldots, n$, let $t_i \colon X_0 \to X_i$ be an isometry. Suppose that $\mathcal{F}$ is a GIFS of order m such that $X = \bigcup\limits_{f \in \mathcal{F}} f(X^m)$. For every $f \in \mathcal{F}$ and $\mathbf{i} = (i_1, \ldots, i_m) \in \{0, \ldots, n\}^m$, let $H_{f,\mathbf{i}} \colon X_0^m \to X_0$ be defined by

$$H_{f,\mathbf{i}}(x_1, \ldots, x_m) = \begin{cases} f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)) & \text{if } f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)) \in X_0, \\ z_0 & \text{if } f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)) \notin X_0, \end{cases}$$

where z_0 is an initially chosen point of X_0. Clearly,

$$\bigcup\limits_{f \in \mathcal{F},\ \mathbf{i} \in \{0, \ldots, n\}^m} H_{f,\mathbf{i}}(X_0^m) = X_0.$$

We will show that each $\operatorname{Lip}(H_{f,\mathbf{i}}) \leq \operatorname{Lip}(f)$. Take $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in X_0^m$ such that $H_{f,i}(x) \neq H_{f,i}(y)$.

If $f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), f(t_{i_1}(y_1), \ldots, t_{i_m}(y_m))$ belong to distinct X_i, X_j, then by assumption (b), we have

$$\begin{aligned} d(H_{f,\mathbf{i}}(x), H_{f,\mathbf{i}}(y)) &\leq \operatorname{diam}(X_0) \leq \operatorname{dist}(X_i, X_j) \\ &\leq d\big(f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), f(t_{i_1}(y_1), \ldots, t_{i_m}(y_m))\big) \\ &< d_m\big((t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), (t_{i_1}(y_1), \ldots, t_{i_m}(y_m))\big) = d_m(x, y). \end{aligned}$$

If $f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), f(t_{i_1}(y_1), \ldots, t_{i_m}(y_m)) \in X_0$ then

$$\begin{aligned} d(H_{f,\mathbf{i}}(x), H_{f,\mathbf{i}}(y)) &= d\big(f(t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), f(t_{i_1}(y_1), \ldots, t_{i_m}(y_m))\big) \\ &< d_m\big((t_{i_1}(x_1), \ldots, t_{i_m}(x_m)), (t_{i_1}(y_1), \ldots, t_{i_m}(y_m))\big) = d_m(x, y). \end{aligned}$$

In remaining cases we proceed in a trivial way. $\qquad \square$

2.2. Quotients of topological fractals. We first show that if X is a topological fractal, then also its certain quotient space is a topological fractal (in fact, this result was proved during preparations of the paper [2], but it was not published; let us also note that similar ideas appears in a recent paper of D'Aniello and Steele [12]). Let X be a topological space and R be an equivalence relation on X. By $[x]$ we denote the equivalence class containing x, by X/R, the set of all equivalence classes, and by $\pi \colon X \to X/R$, the map $\pi(x) := [x]$. If τ is the topology on X, then the quotient topology on X/R is defined by

$$\tau_R := \{V \subset X/R : \pi^{-1}(V) \in \tau\} = \left\{V \subset X/R : \bigcup V \in \tau\right\}$$

that is, τ_R is the richest topology so that π is continuous.

If $x \in X$, then by $[x]_c$ let us denote the connected component containing x, that is, the union of all connected subsets of X which contains x. It is well known (again, see [15]) that two different connected components are disjoint, hence the relation R_c defined by xR_cy if and only if x, y belong to the same connected component, is equivalence relation. Moreover, each connected component is connected and closed.

THEOREM 2.7. *If X is the topological fractal for an* IFS *$\mathcal{F}$, then X/R_c is the topological fractal for the* IFS *$\widetilde{\mathcal{F}} = \{\widetilde{f} : f \in \mathcal{F}\}$, where for each $f \in \mathcal{F}$ and $x \in X$, $\widetilde{f}([x]_c) := [f(x)]_c$.*

PROOF. As the image of connected set via a continuous function must be connected, and connected components are connected, the map $\widetilde{f} \colon X/R_c \to X/R_c$ defined in thesis is well defined. We will show that it is continuous. Let V be open in X/R_c. Then $\pi^{-1}(\widetilde{f}^{-1}(U)) = f^{-1}(\pi^{-1}(U))$ is open as f and π are continuous. Hence $\widetilde{f}^{-1}(U)$ is open in X/R_c and $\widetilde{f}$ is continuous. Now observe that

$$(2.2) \qquad \bigcup_{\widetilde{f} \in \widetilde{\mathcal{F}}} \widetilde{f}(X/R_c) = X/R_c.$$

Take $[y]_c \in X/R_c$. Then, for some $x \in X$ and $f \in \mathcal{F}$, $f(x) = y$, so $\widetilde{f}([x]_c) = [f(x)]_c = [y]_c$. We get (2.2).

Finally, take any sequence $(\widetilde{f}_n) \subset \widetilde{\mathcal{F}}$. We will show that $\bigcap_{n \in \mathbb{N}} \widetilde{f}_1 \circ \ldots \circ \widetilde{f}_n(X/R_c)$ is singleton. Let $[x]_c$ be any element of this intersection. Then, for every $n \in \mathbb{N}$, there exists $x_n \in X$ such that

$$[x]_c = \widetilde{f}_1 \circ \ldots \circ \widetilde{f}_n([x_n]_c) = \widetilde{f}_1 \circ \ldots \circ \widetilde{f}_{n-1}([f_n(x_n)]_c) = \ldots = [f_1 \circ \ldots \circ f_n(x_n)]_c.$$

In particular, $f_1 \circ \ldots \circ f_n(x_n) \in [x]_c$. In particular, $f_1 \circ \ldots \circ f_n(x_n) \in f_1 \circ \ldots \circ f_n(X)$. Now, for $n \in \mathbb{N}$, let $y_n := f_1 \circ \ldots \circ f_n(x_n)$. Since X is topological fractal, it is compact metrizable and by the first observation, (y_n) has a convergent subsequence whose limit $y \in [x]_c$ (as $[x]_c$ is closed). But by the second one, this limit must belong to $\bigcap_{n \in \mathbb{N}} f_1 \circ \ldots \circ f_n(X)$. Hence if $[x']_c$ is different from $[x]_c$ and contained in $\bigcap_{n \in \mathbb{N}} \widetilde{f}_1 \circ \ldots \circ \widetilde{f}_n(X/R_c)$, we would also find $y' \in [x']_c$ contained in $\bigcap_{n \in \mathbb{N}} f_1 \circ \ldots \circ f_n(X)$. As $y \neq y'$, this would be a contradiction. Finally, X/R_c is compact and Hausdorff, see for example [20, Chapter V, §46.Va]. □

3. (Λ, b)-spaces

If $\xi = (\xi_1, \ldots, \xi_k)$, then we define *the length of* ξ by $|\xi| := k$, if $\xi = (\xi_1, \xi_2, \ldots)$, then we set $|\xi| := \omega$, and also we put $|\emptyset| := 0$. If $i \leq |\xi|$, then we set $[\xi]_i := (\xi_1, \ldots, \xi_i)$ (and also $[\xi]_0 := \emptyset$). If ξ, η are sequences, then we write $\xi \preceq \eta$, if $[\eta]_k = \xi$ for some $k \leq |\eta|$. Finally, if ξ, η are sequences such that $|\xi| < \omega$, then by $\xi^\frown \eta$ we denote the concatenation of ξ and η.

If A is a set and $k \in \mathbb{N} \cup \{0\}$, then by A^ω, $A^{<\omega}$ and A^k we denote the families of all countable infinite, finite and of the length k sequences of elements of A, respectively.

Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\omega\} = \{\omega, 1, 2, \ldots\}$. We say that a nonempty set $\Lambda \subset \overline{\mathbb{N}}^{<\omega}$ is *a tree*, if for every $\xi \in \Lambda$ and $k \leq |\xi|$, also $[\xi]_k \in \Lambda$. If Λ is a tree, then the *boundary* of Λ is defined by

$$\widetilde{\Lambda} := \big\{ \xi \in \Lambda : \xi \hat{\ } k \notin \Lambda \text{ for all } k \in \overline{\mathbb{N}} \big\} \cup \big\{ \xi \in \overline{\mathbb{N}}^\omega : [\xi]_k \in \Lambda \text{ for all } k \in \mathbb{N} \big\}.$$

Observe that $\widetilde{\Lambda}$ can be identified with a family of all maximal $\preceq$-linearly ordered subsets of Λ. Moreover, $\widetilde{\Lambda} \cap \Lambda$ consists of all finite sequences from $\widetilde{\Lambda}$, and $\widetilde{\Lambda} \setminus \Lambda$, of all infinite sequences from $\widetilde{\Lambda}$.

We say that a tree Λ is *proper*, if

(p1) for every $\xi \in \Lambda$ which ends with ω, it holds $\xi \in \widetilde{\Lambda}$;

(p2) for every $\xi \in \Lambda \setminus \widetilde{\Lambda}$ and $k \in \overline{\mathbb{N}}$, $\xi \hat{\ } k \in \Lambda$.

By $\Lambda_{\max}$ we will mean the maximal proper tree, that is, the tree which consists of the empty sequence and all sequences $\eta = (\eta_1, \ldots, \eta_k)$ so that $\eta_i \in \mathbb{N}$ for $i = 1, \ldots, k-1$. In this case, the boundary $\widetilde{\Lambda}_{\max}$ consists of all finite sequences which ends with ω and all infinite sequences of elements of $\mathbb{N}$.

We say that a sequence $b = (b_i)_{0 \leq i \leq \omega}$ of positive reals is *good*, if $b_\omega = 0$ and

$$(3.1) \qquad M_b := \sup \left\{ \frac{b_k}{b_{k-1}} : k \in \mathbb{N} \right\} < \frac{1}{25}.$$

It is easy to see that in this case

$$(3.2) \qquad \lambda_b := 25 M_b < 1$$

and, for every $k \in \mathbb{N}$,

$$(3.3) \qquad b_k \leq \frac{1}{20} \lambda_b (b_{k-1} - 2b_k - b_{k+1}).$$

The above conditions looks artificial but, as we will see, they will be needed at some places later (even more technical assumptions will be made in Section 5). On the other hand, restricting to good sequences will not decrease the generality of our constructions.

Finally, for every $i \in \mathbb{N} \cup \{0\}$, define $b^i := (b_i, b_{i+1}, \ldots)$. Clearly, b^i is a good sequence, $M_{b^i} \leq M_b$ and $\lambda_{b^i} \leq \lambda_b$.

For a sequence $\eta \in \overline{\mathbb{N}}^{\leq \omega}$, we set $l(\eta)$ by (we assume $n + \omega = \omega$)

$$(3.4) \qquad l(\eta) := \begin{cases} \eta_1 + \ldots + \eta_k & \text{if } \eta = (\eta_1, \ldots, \eta_k), \\ \omega & \text{if } \eta \in \overline{\mathbb{N}}^\omega, \\ 0 & \text{if } \eta = \emptyset. \end{cases}$$

Now, relying on the above notations, we will define a certain class of metric spaces. All our next consideration will base on it.

DEFINITION 3.1. Let Λ be a proper tree and b be a good sequence. A compact metric space X will be called a (Λ, b)-space, if X is of the form

$$X = \bigcup_{\eta \in \Lambda \cup \widetilde{\Lambda}} X_\eta, \quad \text{where } X_\emptyset = X$$

and, for every $\eta \in \widetilde{\Lambda}$,

(i) X_η is nonempty, compact and contained in X_ξ for every $\xi \preceq \eta$;

for every $\eta \in \Lambda \setminus \widetilde{\Lambda}$,

    (ii) $X_\eta = \bigcup_{k \in \overline{\mathbb{N}}} X_{\eta^\smallfrown k}$;

    (iii) $\text{dist}\left(X_{\eta^\smallfrown k}, \bigcup_{k < j \leq \omega} X_{\eta^\smallfrown j}\right) \geq b_{l(\eta)+k-1} - 2b_{l(\eta)+k} - b_{l(\eta)+k+1}$ for every $k \in \mathbb{N}$;

    (iv) $\text{diam}\left(\bigcup_{k \leq j < \omega} X_{\eta^\smallfrown j}\right) \leq b_{l(\eta)+k-1}$;

    (v) there is $x_{\eta^\smallfrown \omega} \in X_{\eta^\smallfrown \omega}$ such that, for every $k \in \mathbb{N}$, $x_{\eta^\smallfrown \omega} \in \overline{\bigcup_{k \leq j < \omega} X_{\eta^\smallfrown j}}$.

If X is a (Λ, b)-space such that, for every $\eta \in \widetilde{\Lambda}$,

    (s) X_η is singleton,

then we call X a (Λ, b, s)-space. In such case, if $\eta \in \widetilde{\Lambda}$ then the unique element of X_η we denote by x_η (clearly, there is no collision with (e)). If X is a (Λ, b)-space such that for some compact metric space Z and every $\eta, \xi \in \Lambda \cap \widetilde{\Lambda}$,

    (Z) X_η is a similarity copy (that is, an image under similitude, i.e. isometry with a scale) of Z such that

       (Z1) if η does not end with ω, then $\text{diam}(X_\eta) = b_{l(\eta)} + b_{l(\eta)+1}$;

       (Z2) if η ends with ω, then $\text{diam}(X_\eta) = b_{l(\widetilde{\eta})+1}$, where $\widetilde{\eta}$ is such that $\eta = \widetilde{\eta}^\smallfrown \omega$;

       (Z3) if η and ξ end with ω, then there is a similitude $h \colon X_\eta \to X_\xi$ such that $h(x_\eta) = x_\xi$,

then we call X a (Λ, b, Z)-space.

Observe that conditions (i) and (iv) imply that X_η is singleton for $\eta \in \widetilde{\Lambda} \setminus \Lambda$. If X is a (Λ, b)-space, $\eta \in \Lambda \setminus \widetilde{\Lambda}$ and $k \in \mathbb{N}$, then we also set

$$\widetilde{X}_{\eta^\smallfrown k} := \bigcup_{k \leq j \leq \omega} X_{\eta^\smallfrown j}.$$

Observe that, if $\eta \in \Lambda \setminus \widetilde{\Lambda}$, then $X_\eta = \widetilde{X}_{\eta^\smallfrown 1}$.

We first list basic properties of (Λ, b)-spaces (by d we will denote the metric on a metric space X):

LEMMA 3.2. *Let X be a (Λ, b)-space.*

(a) *For every $\eta \in \Lambda$ which does not end with ω, X_η and $\widetilde{X}_\eta$ are compact and open.*

(b) *If $\eta \in \Lambda \setminus \widetilde{\Lambda}$ and $x \in X_{\eta^\frown\omega}$, $y \in X_\eta \setminus X_{\eta^\frown\omega}$, then $d(y, x_{\eta^\frown\omega}) \leq 2d(x, y)$ and $d(x, x_{\eta^\frown\omega}) \leq 3d(x, y)$.*

If X is a (Λ, b, s)-space which is not singleton, then

(c) *the family*

$$\{X_\eta : \eta \in \Lambda, \ \eta \text{ does not end with } \omega\} \cup \{\widetilde{X}_\eta : \eta \in \Lambda, \ \eta \text{ does not end with } \omega\}$$

is a basis of X consisting of clopen sets;

(d) *if $\eta \in \Lambda$, then $\Lambda_1[\eta] := \{\beta : \eta^\frown\beta \in \Lambda\}$ is proper tree and X_η is $(\Lambda_1[\eta], b^{l(\eta)}, s)$-space;*

(e) *if $\eta \in \Lambda$ and $k \in \mathbb{N}$ are such that $\eta^\frown k \in \Lambda$, then*

$$\Lambda_2[\eta^\frown k] := \{i^\frown\beta : i \in \mathbb{N}, \ \eta^\frown(i + k - 1)^\frown\beta \in \Lambda\} \cup \{\emptyset, \omega\}$$

is proper tree, and $\widetilde{X}_{\eta^\frown k}$ is $(\Lambda_2[\eta^\frown k], b^{l(\eta^\frown k)-1}, s)$-space;

(f) *if Y is a (Λ, b', s)-space for a good sequence b', then X and Y are homeomorphic.*

PROOF. We first prove (a). Let $\xi \in \Lambda$ does not end with ω. X_ξ and $\widetilde{X}_\xi$ are open as, by (ii), (iii) and (3.3),

$$\text{dist}(X_\xi, X \setminus X_\xi) \geq b_{l(\xi)-1} - 2b_{l(\xi)} - b_{l(\xi)+1} > 0,$$
$$\text{dist}(\widetilde{X}_\xi, X \setminus \widetilde{X}_\xi) \geq b_{l(\xi)-2} - 2b_{l(\xi)-1} - b_{l(\xi)} > 0.$$

Now we show that they are compact. We first prove that X_ξ is compact. Suppose $\xi \notin \widetilde{\Lambda}$, choose a sequence $(x_n) \subset X_\xi$ and consider the following cases: *Case 1.* Infinitely many elements x_n belong to $X_{\xi^\frown\eta}$ for some η with $\xi^\frown\eta \in \widetilde{\Lambda}$. Then (x_n) has convergent subsequence by (i).

Case 2. Some subsequence of (x_n) converges to some element of $X_{\xi^\frown\eta}$, where η ends with ω. Then we are done.

Case 3. The previous cases do not hold. Step by step, we will define a sequence $\eta = (\eta_1, \eta_2, \ldots)$ such that $\xi^\frown\eta \in \widetilde{\Lambda}$ and for every k, infinitely many elements x_n belong to $X_{\xi^\frown[\eta]_k}$. This will allow to choose a subsequence (x_{n_k}) such that $x_{n_k} \in X_{\xi^\frown[\eta]_k}$ for $k \in \mathbb{N}$. By (iv) and a fact that $\emptyset \neq X_{\xi^\frown\eta} \subset X_{\xi^\frown[\eta]_k}$ for every $k \in \mathbb{N}$, this will mean that (x_{n_k}) is convergent to the unique element of $X_{\xi^\frown\eta} \subset X_\xi$.

Now we show how to choose η. By our assumptions (iv) and (v), there is $k \in \mathbb{N}$ such that infinitely many elements x_n belong to $\bigcup_{j \leq k_1} X_{\xi^\frown j}$. Hence we can find $\eta_1 \in \{1, \ldots, k_1\}$ such that infinitely many elements of x_n belong to $X_{\xi^\frown\eta_1}$. Similarly, there is $k_2 \in \mathbb{N}$ so that infinitely many elements of x_n belong to $\bigcup_{j \leq k_2} X_{\xi^\frown\eta_1^\frown j}$, and hence we find $\eta_2 \in \{1, \ldots, k_2\}$ so that infinitely many elements x_n belong to $X_{\xi^\frown\eta_1^\frown\eta_2}$. We can continue this procedure as we assume that Cases 1 and 2 do not hold.

We proved that X_ξ is compact. Now let $\widetilde{\xi}$ be such that $\xi = \widetilde{\xi}^\smallfrown k$ for some $k \in \mathbb{N}$. Then $\widetilde{X}_\xi = X_{\widetilde{\xi}} \setminus \bigcup_{i<k} X_{\xi^\smallfrown i}$ and, by what we already proved, $\widetilde{X}_\xi$ is compact.

Now we prove (b). Assume that $x \in X_{\eta^\smallfrown \omega}$ and $y \in X_{\eta^\smallfrown k}$ for some $k \in \mathbb{N}$. By conditions (iii)–(v) and (3.3), we have

$$d(x_{\eta^\smallfrown \omega}, y) \leq \operatorname{diam}\left(\overline{\bigcup_{k \leq j < \omega} X_{\eta^\smallfrown j}}\right) = \operatorname{diam}\left(\bigcup_{k \leq j < \omega} X_{\eta^\smallfrown j}\right) \leq b_{l(\eta)+k-1}$$

and

$$d(x,y) \geq \operatorname{dist}\left(X_{\eta^\smallfrown k}, \bigcup_{k < j \leq \omega} X_{\eta^\smallfrown j}\right) \geq b_{l(\eta)+k-1} - 2b_{l(\eta)+k} - b_{l(\eta)+k+1}$$

$$\geq b_{l(\eta)+k-1} - \frac{5}{2}b_{l(\eta)+k} \geq b_{l(\eta)+k-1}\left(1 - \frac{5}{2}M_b\right) \geq \frac{1}{2}b_{l(\eta)+k-1} \geq \frac{1}{2}d(x_{\eta^\smallfrown \omega}, y).$$

Hence we get the first inequality. To see the second, use the first one:

$$d(x, x_{\eta^\smallfrown \omega}) \leq d(x,y) + d(y, x_{\eta^\smallfrown \omega}) \leq 3d(x,y).$$

Point (c) follows from (a), (ii) and facts that $\operatorname{diam}(X_\eta) \leq b_{l(\eta)}$ and $\operatorname{diam}(\widetilde{X}_{l(\eta)}) \leq b_{l(\eta)-1}$ (which are implied by (iii) and (iv)).

To see (d), it is routine to check that $\Lambda_1[\eta]$ is proper and the family $X'_\beta := X_{\eta^\smallfrown \beta}$, $\beta \in \Lambda_1[\eta]$, satisfies the required conditions.

Similarly, to get (e), it is easy to see that $\Lambda_2[\eta^\smallfrown k]$ is proper and that the family $X''_{i^\smallfrown \beta} := X_{\eta^\smallfrown(i+k-1)^\smallfrown \beta}$, $i^\smallfrown \beta \in \Lambda_2[\eta^\smallfrown k]$, satisfies the required conditions (in fact, (d) follows from (e) by our earlier observation).

To see (f), note that the map $X \ni x_\eta \to y_\eta \in Y$ is homeomorphism, which can be easily proved using (c). $\qquad\square$

Now we show that the above definition is non void, and we can embed (Λ, b) spaces into $\mathbb{R}$ or other Hilbert spaces. We should start with defining appropriate "skeleton" on the real line.

Let b be a good sequence. We say that a family $\mathcal{I} = \{I_\xi : \xi \in \Lambda_{\max}\}$ of closed intervals is a *b-family*, if for every $\eta \in \Lambda_{\max}$ which does not end with ω,

($\mathcal{I}1$) $\operatorname{diam}(I_\eta) = b_{l(\eta)} + b_{l(\eta)+1}$;
($\mathcal{I}2$) for every $k \in \overline{\mathbb{N}}$,

($\mathcal{I}2$a) if $k \in \mathbb{N}$, then $\max I_{\eta^\smallfrown k} = \min I_\eta + b_{l(\eta)+1} + b_{l(\eta)+k-1}$;
($\mathcal{I}2$b) if $k = \omega$, then $\min I_{\eta^\smallfrown \omega} = \min I_\eta$, and $\max I_{\eta^\smallfrown \omega} = \min I_\eta + b_{l(\eta)+1}$.

Observe that $I_{\eta^\smallfrown k} \subset I_\eta$, $\max I_\eta = \max I_{\eta^\smallfrown 1}$ for η which does not end with ω, and $\operatorname{dist}(I_{\eta^\smallfrown k}, I_{\eta^\smallfrown (k+1)}) = b_{l(\eta)-1} - 2b_{l(\eta)} - b_{l(\eta)+1}$. Additionally, for every infinite $\eta \in \widetilde{\Lambda}_{\max}$, set $I_\eta := \bigcap_{k \in \mathbb{N}} I_{[\eta]_k}$ (clearly, it is a singleton).

REMARK 3.3. It is worth to observe that we can look on the family $\{I_\eta : \eta \in \Lambda_{\max}, \eta$ does not end with $\omega\}$ as on standard ternary Cantor scheme on $\mathbb{R}$ (with

the length of each interval of the k-th step equal to b_k), but for our purposes we enumerate these intervals in a different way.

THEOREM 3.4. *Assume that Λ is a proper tree and b is a good sequence.*

(a) *There exists $X \subset \mathbb{R}$ which is a (Λ, b, s)-space.*
(b) *If Z is a nonempty, non singleton, compact subset of a normed space H, then there exists $X \subset H$ which is a (Λ, b, Z)-space.*
(c) *If Z is a nonempty compact metric space which is not a singleton, then there exists a metric space X which is a (Λ, b, Z)-space.*

PROOF. We first prove (a). Take a b-family $\mathcal{I}$ and $\eta \in \widetilde{\Lambda}$. If η ends with ω then set $x_\eta := \max I_\eta$, otherwise let x_η be any element of I_η. Then it is easy to see that $X := \{x_\eta : \eta \in \widetilde{\Lambda}\}$ is a (Λ, b, s)-space (for the family $X_\eta := \{x_\xi : \xi \in \widetilde{\Lambda}, \eta \preceq \xi\}$, $\eta \in \Lambda$).

Now we prove (b). We first state the following claim:

CLAIM. *For any distinct points x, y of a normed space H and any distinct points $x', y' \in \mathbb{R}(x - y) := \{t(x - y) : t \in \mathbb{R}\}$ there exists a bijective similitude $h \colon H \to H$ such that $h(x) = x'$ and $h(y) = y'$.*

The similitude h can be defined by

$$h(z) := x' + \frac{||y' - x'||}{||y - x||}(z - x) \quad \text{or} \quad h(z) := x' - \frac{||y' - x'||}{||y - x||}(z - x),$$

depending on the mutual relationship between x' and y'.

Since Z is compact, there are $x_0, y_0 \in Z$ such that $||x_0 - y_0|| = \operatorname{diam}(Z)$. Now choose any line $l \subset H$, and identify l with the real line $\mathbb{R}$, and find any b-family $\mathcal{I}$ on l. By the Claim, for every $\eta \in \Lambda \cap \widetilde{\Lambda}$, we can find a similitude $h_\eta \colon H \to H$ such that $h_\eta(x_0) = \min I_\eta$ and $h_\eta(y_0) = \max I_\eta$. Set $X_\eta := h_\eta(Z)$. Moreover, if $\eta \in \widetilde{\Lambda} \setminus \Lambda$, then let $X_\eta := I_\eta = \bigcap_{k \in \mathbb{N}} I_{[\eta]_k}$ (clearly, X_η is singleton). For every $\eta \in \Lambda$, let $X_\eta := \bigcup_{\xi \in \widetilde{\Lambda}, \eta \preceq \xi} X_\xi$, and finally set $X := X_\emptyset = \bigcup_{\eta \in \widetilde{\Lambda}} X_\eta$. We will prove that X is a (Λ, b, Z)-space.

Clearly, the condition (Z) is satisfied (as $\operatorname{diam}(h(Z)) = ||h(x_0) - h(y_0)||$ for every considered similitude and, if η ends with ω, then x_η is the image of y_0 via h). Condition (a) follows from the fact that Z is nonempty and compact, and (b) directly from definition.

Now we show that, for every $\eta \in \Lambda \cup \widetilde{\Lambda}$,

(3.5) $\qquad \min I_\eta, \max I_\eta \in X_\eta \quad \text{and} \quad \operatorname{diam}(X_\eta) = \operatorname{diam}(I_\eta).$

If $\eta \in \widetilde{\Lambda}$, then (3.5) clearly holds. Hence assume $\eta \in \Lambda \setminus \widetilde{\Lambda}$. By definition, $\min I_\eta = \min I_{\eta^\frown \omega} \in X_{\eta^\frown \omega} \subset X_\eta$. Now if $\eta^\frown(1, 1, 1, \dots) \in \widetilde{\Lambda}$, then $\max I_\eta \in$

$\bigcap_{k\in\mathbb{N}} I_{\eta^\frown[(1,1,\ldots)]_k} = X_{\eta^\frown(1,1,\ldots)} \subset X_\eta$, and if for some $k \in \mathbb{N}$, $\eta^\frown[(1,1,\ldots)]_k \in \widetilde{\Lambda}$,
then $\max I_\eta = \max I_{\eta^\frown[(1,1,\ldots)]_k} \in X_{\eta^\frown[(1,1,\ldots)]_k} \subset X_\eta$.

Now choose any $x, y \in X_\eta$ and let $\xi, \xi' \in \widetilde{\Lambda}$ be such that $\eta \preceq \xi$, $\eta \preceq \xi'$,
$x \in X_\xi$ and $y \in X_{\xi'}$. If $\xi = \xi'$, then $\|x - y\| \leq \mathrm{diam}(I_\xi) \leq \mathrm{diam}(I_\eta)$. Hence
assume $\xi \neq \xi'$ and, without loss of generality, that $\xi_k > \xi'_k$ for some $k > |\eta|$.
Then

$$\|x - y\| \leq \|x - \max I_\xi\| + \|\max I_\xi - \min I_{\xi'}\| + \|\min I_{\xi'} - y\|$$
$$\leq \|\min I_\xi - \max I_\xi\| + \|\max I_\xi - \min I_{\xi'}\| + \|\min I_{\xi'} - \max I_{\xi'}\|$$
$$= \|\min I_\xi - \max I_{\xi'}\| \leq \mathrm{diam}(I_\eta).$$

This ends the proof of (3.5).

Now we prove (iii), (iv) and (v). We will use (3.5) and definition of a b-family.
Take any $\eta \in \Lambda \setminus \widetilde{\Lambda}$ and $k < j \leq \omega$. Then for every $x \in X_{\eta^\frown k}$ and $y \in X_{\eta^\frown j}$, we
have by (3.5),

$$\|\max I_{\eta^\frown k} - \min I_{\eta^\frown k}\| + \|\min I_{\eta^\frown k} - \max I_{\eta^\frown j}\| + \|\max I_{\eta^\frown j} - \min I_{\eta^\frown j}\|$$
$$= \|\max I_{\eta^\frown k} - \min I_{\eta^\frown j}\| \leq \|\max I_{\eta^\frown k} - x\| + \|x - y\| + \|y - \min I_{\eta^\frown j}\|$$
$$\leq \|\max I_{\eta^\frown k} - \min I_{\eta^\frown k}\| + \|x - y\| + \|\max I_{\eta^\frown j} - \min I_{\eta^\frown j}\|$$

so

$$\|x - y\| \geq \|\min I_{\eta^\frown k} - \max I_{\eta^\frown j}\| \geq b_{l(\eta)+k-1} - 2b_{l(\eta)+k} - b_{l(\eta)+k+1}.$$

We proved (iii).

Now assume additionally $j < \infty$. Then

$$\|x - y\| \leq \|x - \min I_{\eta^\frown k}\| + \|\min I_{\eta^\frown k} - \max I_{\eta^\frown j}\| + \|\max I_{\eta^\frown j} - y\|$$
$$\leq \|\max I_{\eta^\frown k} - \min I_{\eta^\frown k}\|$$
$$+ \|\min I_{\eta^\frown k} - \max I_{\eta^\frown j}\| + \|\max I_{\eta^\frown j} - \min I_{\eta^\frown j}\|$$
$$\leq \mathrm{diam}\left(\bigcup_{k \leq i < \omega} I_{\eta^\frown i}\right) \leq b_{l(\eta)+k-1}$$

and we proved (iv).

Finally, (3.5) implies that $x_{\eta^\frown\omega} = \max I_{\eta^\frown\omega} \in \overline{\bigcup_{k\leq i<\omega} X_{\eta^\frown j}}$, hence (v) also
holds. This ends the proof of (b).

We just sketch the proof of (c). The idea is similar as in point (b). We choose
a b-family $\mathcal{I}$ and try to replace I_η by appropriate copy of Z, for $\eta \in \widetilde{\Lambda}$.

For every $\eta \in \widetilde{\Lambda} \cap \Lambda$, let X_η be a similarity copy of Z according to (Z1), (Z2)
and let d_η be initial metric on X_η. If $\eta \in \widetilde{\Lambda}$ ends with ω, then let $x_\eta \in X_\eta$ be
the image, via the similitude, of initially chosen point $y_0 \in Z$ such that for some
$x \in Z$, $\rho(x, y_0) = \mathrm{diam}(Z)$. Thanks to it, the condition (Z3) will be satisfied. If

$\eta \in \widetilde{\Lambda}$ is infinite, then let X_η be singleton. Moreover, for every $\eta \in \Lambda \setminus \widetilde{\Lambda}$, let X_η be defined like before in the proof of (b), and finally set $X := X_\emptyset$. Now we will define appropriate metric d on X.

For every $\xi \in \Lambda$, let

$$z_\xi := \begin{cases} \text{midpoint of } I_\xi & \text{if } \xi \text{ does not end with } \omega, \\ \max I_\xi & \text{if } \xi \text{ ends with } \omega. \end{cases}$$

If $\eta \in \widetilde{\Lambda} \cap \Lambda$, then let the metric d coincide with d_η on X_η. Now let $x, y \in X$ be distinct and assume that they do not belong to the same X_η for $\eta \in \widetilde{\Lambda}$. Without loss of generality, there exist $\eta \in \Lambda \setminus \widetilde{\Lambda}$ and $k < j \le \omega$, such that $x \in X_{\eta^\frown k}$, $y \in X_{\eta^\frown j}$. If $j \ne \omega$, then define

$$d(x, y) := |z_{\eta^\frown k} - z_{\eta^\frown j}|,$$

and if $j = \omega$, then define

$$d(x, y) := d(y, x_{\eta^\frown \omega}) + |z_{\eta^\frown \omega} - z_{\eta^\frown k}|.$$

It is routine to check that d is indeed a metric and that (X, d) is a (Λ, b, Z)-space. $\qquad\square$

We will also need to deal with certain subsets of (Λ, b, s)-spaces, hence we introduce the following notation: given a (Λ, b, s)-space X and its compact subset $M \subset X$, we set $\Lambda_M := \{\eta \in \Lambda : X_\eta \cap M \ne \emptyset\}$. We skip the proof of the following observation:

OBSERVATION 3.5. In the above frame:

(a) Λ_M is a tree (but need not be proper);
(b) $\widetilde{\Lambda}_M = \{\eta \in \widetilde{\Lambda} : x_\eta \in M\}$;
(c) $M = \{x_\eta : \eta \in \widetilde{\Lambda}_M\}$;
(d) If $N \ne M$ and N is also compact, then $\Lambda_N \ne \Lambda_M$.

We end this section with two general lemmas, which will be keys in later constructions of appropriate GIFSs.

LEMMA 3.6. *Assume that X is a (Λ, b)-space, $\emptyset \ne Z \subset X$ and for every $2 \le k < \omega$, there exists a map $h_k \colon Z \to X$ so that for every $2 \le k < j < \omega$,*

(a) *$\mathrm{Lip}(h_k) \le \lambda_b/4$;*
(b) *for all $x \in Z$, $d(h_k(x), h_j(x)) \le 2b_{k-1}$.*

Let $h_\omega(x) := \lim\limits_{k \to \infty} h_k(x), x \in X$, and for $k \in \overline{\mathbb{N}}$, let $Z_k := Z \cap X_k$. Then the map $F \colon Z \times Z \to X$ defined by (we set $\omega + 1 = \omega$)

$$F(x, y) := h_{k+1}(x), \quad \textit{if } y \in Z_k, \ k \in \overline{\mathbb{N}},$$

satisfies the following

$$\mathrm{Lip}(F) \leq \frac{1}{2}\lambda_b \quad and \quad F(Z \times Z) = \bigcup\{h_k(Z) : k \in \overline{\mathbb{N}}, \ k \geq 2, \ Z_{k-1} \neq \emptyset\}.$$

PROOF. Since X is compact and $(h_k(x))_{k\in\mathbb{N}}$ is Cauchy, the map h_ω is well defined. By definition, we have

$$F(Z \times Z) = F\left(Z \times \bigcup_{k \in \overline{\mathbb{N}}, \ Z_k \neq \emptyset} Z_k \right) = \bigcup\{h_k(Z) : k \in \overline{\mathbb{N}}, \ k \geq 2, \ Z_{k-1} \neq \emptyset\}.$$

We will show that $\mathrm{Lip}(F) \leq \lambda_b/2$. Take distinct $(x,y),(x',y') \in Z \times Z$ so that $F(x,y) \neq F(x',y')$ and consider cases.

Case 1. For some $k \in \overline{\overline{\mathbb{N}}}$, it holds $y, y' \in Z_k$. Then

$$d(F(x,y), F(x',y')) = d(h_{k+1}(x), h_{k+1}(x'))$$
$$\leq \mathrm{Lip}(h_{k+1})d(x,x') \leq \frac{1}{4}\lambda_b d_m((x,y),(x',y')).$$

Case 2. For some $k < j \leq \omega$, it holds $y \in Z_k$ and $y' \in Z_j$. Then by Case 1, (iii) and (3.3),

$$d(F(x,y), F(x',y')) \leq d(F(x,y), F(x,y')) + d(F(x,y'), F(x',y'))$$
$$\leq d(h_{k+1}(x), h_{j+1}(x)) + \frac{1}{4}\lambda_b d(x,x') \leq 2b_k + \frac{1}{4}\lambda_b d(x,x')$$
$$\leq \frac{1}{4}\lambda_b(b_{k-1} - 2b_k - b_{k+1}) + \frac{1}{4}\lambda_b d(x,x')$$
$$\overset{(*)}{\leq} \frac{1}{4}\lambda_b d(y,y') + \frac{1}{4}\lambda_b d(x,x') \leq \frac{1}{2}\lambda_b d_m((x,y),(x',y')),$$

where $(*)$ follows from

$$d(y,y') \geq \mathrm{dist}(Z_k, Z_j) \geq \mathrm{dist}(X_k, X_j) \geq b_{k-1} - 2b_k - b_{k+1}. \qquad \square$$

REMARK 3.7. Observe that assumption (b) of the above Lemma is satisfied if $h_k \colon Z \to X_k$ for all $2 \leq k < \omega$. Indeed, in this case we have by Definition 3.1 (iv),

$$d(h_k(x), h_j(x)) \leq \mathrm{diam}\left(\bigcup_{k \leq i < \omega} X_i \right) \leq b_{k-1}.$$

Then also $h_\omega(Z) = \{x_\omega\}$.

LEMMA 3.8. *Assume that* X, Y *are* (Λ, b, s) *and* (Λ', b', s)-*spaces such that* $\lambda := \sup\{b'_k/b_k : k \in \mathbb{N} \cup \{0\}\} < \infty$, *and let* M, M' *be their compact subsets such that* $\Lambda'_{M'} \subset \Lambda_M$. *Then there exists surjective map* $h \colon M \to M'$ *with* $\mathrm{Lip}(h) \leq 2\lambda$. *Moreover, if* $X = Y$ *(and* $\Lambda = \Lambda'$ *and* $b = b'$*) and* $M' \subset M$, *then additionally* $h(x) = x$ *for* $x \in M'$.

Proof. At first, fix a map $T\colon \Lambda'_{M'} \cup \widetilde{\Lambda}'_{M'} \to \widetilde{\Lambda}'_{M'}$ such that $\eta \preceq T(\eta)$ and if $\eta \notin \widetilde{\Lambda}'_{M'}$, then the value $T(\eta)_{|\eta|+1}$ is biggest possible.

Now, for every $\eta \in \widetilde{\Lambda}_M$, let $j := \sup\{k : [\eta]_k \in \Lambda'_{M'}\}$, and define $R(\eta) := T([\eta]_j)$, where we assume $[\eta]_\omega = \eta$. By assumptions and a fact that $T(\eta) = \eta$ for $\eta \in \widetilde{\Lambda}'_{M'}$, it is easy to see that:

(a) $R(\eta) = \eta$ for $\eta \in \widetilde{\Lambda}'_{M'} \cap \widetilde{\Lambda}_M$;

(b) $R(\widetilde{\Lambda}_M) = \widetilde{\Lambda}'_{M'}$.

Finally, set $h\colon M \to M'$ by $h(x_\eta) := y_{R(\eta)}$, $\eta \in \widetilde{\Lambda}_M$. By Observation 3.5, h is well defined, by (b), $h(M) = M'$, and by (a), if $X = Y$ and $M' \subset M$, we have that $h(x) = x$ for $x \in M'$. It remains to prove that $\mathrm{Lip}(h) \le 2\lambda$.

Choose any $\eta, \eta' \in \widetilde{\Lambda}_M$ so that $R(\eta) \ne R(\eta')$. We can assume that there exist a sequence ξ and $k < k' \le \omega$ such that $\xi\,\widehat{}\,k \preceq \eta$ and $\xi\,\widehat{}\,k' \preceq \eta'$. Observe that $\xi\,\widehat{}\,k$ or $\xi\,\widehat{}\,k'$ belongs to $\Lambda'_{M'}$. Indeed, otherwise

$$j := \max\{i : [\eta]_i \in \Lambda'_{M'}\} = \max\{i : [\eta']_i \in \Lambda'_{M'}\} \le |\xi|$$

and then $R(\eta) = T([\eta]_j) = T([\eta']_j) = R(\eta')$, which contradicts our assumption.

Assume first that $\xi\,\widehat{}\,k \in \Lambda'_{M'}$. Then $R(\eta) = T([\eta]_j) \succeq [\eta]_j \succeq \xi\,\widehat{}\,k$. Now, if $\xi\,\widehat{}\,k' \in \Lambda'_{M'}$, then also $R(\eta') \succeq \xi\,\widehat{}\,k'$.

If $\xi\,\widehat{}\,k \notin \Lambda'_{M'}$, then by definition of the map T and a fact that $\xi\,\widehat{}\,k \in \Lambda'_{M'}$, there is $k \le l \le \omega$ such that $R(\eta') = T(\xi) \succeq \xi\,\widehat{}\,l$. In both cases, we have $h(x_\eta), h(x_{\eta'}) \in \bigcup_{k \le j \le \omega} Y_{\xi\,\widehat{}\,j}$ and hence

$$d(h(x_\eta), h(x_{\eta'})) \le \mathrm{diam}\left(\bigcup_{k \le j \le \omega} Y_{\xi\,\widehat{}\,j}\right) \le b'_{(\xi)+k-1}.$$

On the other hand,

$$d(x_\eta, x_{\eta'}) \ge \mathrm{dist}(X_{\xi\,\widehat{}\,k}, X_{\xi\,\widehat{}\,k'}) \ge b_{l(\xi)+k-1} - 2b_{l(\xi)+k} - b_{l(\xi)+k+1} \ge \frac{1}{2}\,b_{l(\xi)+k-1}.$$

Hence

$$d(h(x_\eta), h(x_{\eta'})) \le 2\,\frac{1}{2}\,b_{l(\xi)+k-1}\frac{b'_{l(\xi)+k-1}}{b_{l(\xi)+k-1}} \le 2\lambda d(x_\eta, x_{\eta'}).$$

In the case when $\xi\,\widehat{}\,k' \in \Lambda'_{M'}$, we proceed in similar way. $\qquad\square$

3.1. Scattered spaces as (Λ, b, s)-spaces. A topological space X is called *scattered* if every its nonempty subspace Y has an isolated point in Y. It is well known that a compact metrizable space is scattered iff it is countable (it follows from the Cantor-Bendixson theorem and a fact that perfect sets are uncountable).

For a scattered space X, define

$$X' := \{x \in X : x \text{ is an accumulation point of } X\}.$$

Then X' is called *the Cantor–Bendixson derivative of X*. For each ordinal α, we can define the Cantor–Bendixson α-th derivative $X^{(\alpha)}$, according to the inductive procedure:

- $X^{(\alpha+1)} := (X^{(\alpha)})'$;
- $X^{(\alpha)} := \bigcap_{\beta < \alpha} X^{(\beta)}$ for a limit ordinal α.

(the definition is correct, as sets $X^{(\alpha)}$ are empty from some level). Then we can define *the scattered height* of a scattered space X, by

$$\mathrm{ht}(X) := \min \big\{ \alpha : X^{(\alpha)} \text{ is discrete} \big\}.$$

The classical Mazurkiewicz–Sierpiński theorem (see [22]), states that every countable compact scattered space X is homeomorphic to the space $\omega^\beta \cdot n + 1$ with the order topology, where $\beta = \mathrm{ht}(X)$ and $n = \mathrm{card}\, X^{(\beta)}$. In particular each countable compact scattered spaces with the same height and the same cardinality of elements with the "highest rank" are homeomorphic. We call a scattered space *unital*, if $X^{(\mathrm{ht}(X))}$ is a singleton.

Now we show that each compact metrizable scattered space is homeomorphic to certain (Λ, b, s)-space.

Let δ_0 be a countable limit ordinal, and $\{c_n^{\delta_0}(\beta) : n \in \mathbb{N}, \beta \le \delta_0,\ \beta \text{ is limit}\}$ be a monotone ladder system in δ_0, that is, a family of ordinals that satisfies

1. for each limit ordinal $\beta \le \delta_0$, the sequence $\{c_n^{\delta_0}(\beta)\}_{n \in \mathbb{N}}$ is strictly increasing and converges to β;
2. for every limit $\beta, \gamma \le \delta_0$, if $\beta \le \gamma$ then $c_n^{\delta_0}(\beta) \le c_n^{\delta_0}(\gamma)$ for every $n \in \mathbb{N}$.

(the proof of its existance is given in [28]).

Now, for every ordinal $\alpha \le \delta_0$, define the sequence (α_n) such that:

- if α is a limit ordinal, then $\alpha_n := c_n^{\delta_0}(\alpha)$;
- if α is a successor ordinal and $\alpha = \alpha' + 1$, then $\alpha_n := \min\{c_n^{\delta_0}(\alpha + \omega), \alpha'\}$.

Clearly,

(A) for every $\alpha \le \delta_0$, $(\alpha_n + 1)$ is nondecreasing and converges to α;

(B) for every $\alpha \le \beta \le \delta_0$ and every $n \in \mathbb{N}$, $\alpha_n \le \beta_n < \beta$.

Now we will define a family Λ^α, $\alpha \le \delta_0$, of proper trees. Definition will be recursive. At first, let $\Lambda^0 := \{\emptyset\}$. Assume that for some $\alpha \le \delta_0$, all sets Λ^β, $\beta < \alpha$, are defined. Then define

$$\Lambda^\alpha := \{\emptyset, \omega\} \cup \bigcup_{k \in \mathbb{N}} k^\smallfrown \Lambda^{\alpha_k} = \bigcup_{k \in \mathbb{N}} \{k^\smallfrown \eta : \eta \in \Lambda^{\alpha_k}\} \cup \{\emptyset, \omega\},$$

where $k^\smallfrown A := \{k^\smallfrown a : a \in A\}$.

The following proposition lists basic properties of sets Λ^α. It can be easily proved by induction and with a help of (A) and (B).

PROPOSITION 3.9. *If $\alpha \le \beta \le \delta_0$, then*

(a) $\widetilde{\Lambda}^{\beta} = \{\omega\} \cup \bigcup_{k \in \mathbb{N}} k^{\frown} \widetilde{\Lambda}^{\beta_k}$;

(b) $\widetilde{\Lambda}^{\beta}$ consists of finite sequences, that is, $\widetilde{\Lambda}^{\beta} \subset \Lambda^{\beta}$;

(c) Λ^{β} is proper;

(d) $\Lambda^{\alpha} \subset \Lambda^{\beta}$.

Finally, set $\Lambda^{\alpha,1} := \Lambda^{\alpha}$ and for $n \geq 2$, define $\Lambda^{\alpha,n}$ by

$$\Lambda^{\alpha,n} := \left(\bigcup_{1 \leq i \leq n-1} i^{\frown} \Lambda^{\alpha} \right) \cup \left(\bigcup_{i \geq n} i^{\frown} \Lambda^{\alpha_{i-n+1}} \right) \cup \{\emptyset, \omega\}.$$

Clearly, $\Lambda^{\alpha,n}$ is proper tree. As δ_0 was taken as arbitrary countable ordinal, the following result shows that all compact scattered spaces are certain (Λ, b, s)-spaces.

PROPOSITION 3.10. *If b is a good sequence and $\alpha \leq \delta_0$, then $(\Lambda^{\alpha,n}, b, s)$-space is homeomorphic to $\omega^{\alpha} \cdot n + 1$.*

PROOF. We first assume $n = 1$. Then the thesis holds by a simple inductive argument and the fact that if X is a (Λ^{α}, b, s)-space, then for every $k \in \mathbb{N}$, X_k is a $(\Lambda^{\alpha_k}, b^k, s)$-space (see Lemma 3.2 (d)). Hence take $(\Lambda^{\alpha,n}, b, s)$-space X. Then, from the definition, it can be easily seen (see again Lemma 3.2 (d)) that for $i = 1, \ldots, n-1$, the space X_i is a $(\Lambda^{\alpha}, b^i, s)$-space and that the space $\widetilde{X}_n := \bigcup_{n \leq i \leq \omega} X_i$ is a $(\Lambda^{\alpha}, b^{n-1}, s)$-space. Since $X_1, \ldots, X_{n-1}, \widetilde{X}_n$ are pairwise disjoint and open, the result follows. $\square$

REMARK 3.11. Nowak in [28] considered a particular case of (Λ^{α}, b, s)-spaces. Namely, take $r > 3$ and for each $n \in \mathbb{N}$, define the affine homeomorphism s_n by

$$s_n(x) := \frac{x}{r^n} + \frac{1}{r^n}.$$

Finally, construct the scattered compact sets $L_{\alpha} \subset [0, 1]$, $\alpha \leq \delta_0$, in the following recursive way:

(3.6) $$L_0 := \{0\}, \qquad L_{\alpha} := L_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n}).$$

Notice that L_{α} is a (Λ^{α}, b, s)-space for a sequence $b = (2/r^{k+1})_{k \in \mathbb{N} \cup \{0\}}$ with sufficiently large r. From the perspective of countable compact spaces and Section 4, Nowak's construction would be sufficient. However, in next sections we will need our, more flexible, setting.

3.2. Compact 0-dimensional uncountable spaces as (Λ, b, s)-spaces. In this section we show that each compact metrizable uncountable 0-dimensional space is homeomorphic to a certain subset of a $(\Lambda_{\max}, b, s)$-space. Note that

if Y is uncountable, compact, metrizable and 0-dimensional, by the Cantor–Bendixson theorem, there exists a maximal perfect subset $Y^{(\infty)}$ (called a perfect kernel) which is a Cantor space.

Define two subtrees of $\Lambda_{\max}$

$$(3.7) \qquad \begin{aligned} \Lambda_s &:= \{\eta \in \Lambda_{\max} : \eta \text{ does not contain } 1\}, \\ \Lambda_r &:= \{\eta \in \Lambda_{\max} : \eta \text{ contains at most one } 1\}. \end{aligned}$$

Observe that

$$(3.8) \qquad \begin{aligned} \widetilde{\Lambda}_s &:= \big\{\eta \in \widetilde{\Lambda}_{\max} : \eta \text{ does not contain } 1\big\}, \\ \widetilde{\Lambda}_r &:= \big\{\eta \in \widetilde{\Lambda}_{\max} : \eta \text{ contains at most one } 1\big\}. \end{aligned}$$

PROPOSITION 3.12. *Let b be a good sequence, Y be an uncountable 0-dimensional compact metrizable space and $y_0 \in Y^{(\infty)}$. Then, for every $(\Lambda_{\max}, b, s)$-space X, there is a compact set $M \subset X$ such that*

$$(3.9) \qquad \Lambda_s \subset \Lambda_M \subset \Lambda_r$$

and a homeomorphism $h\colon Y \to M$ such that $h(y_0) = x_\omega$.

Before we give the proof, let us observe that (3.9) implies

$$(3.10) \qquad \widetilde{\Lambda}_s \subset \widetilde{\Lambda}_M \subset \widetilde{\Lambda}_r.$$

PROOF. Let X be a $(\Lambda_{\max}, b, s)$-space and define

$$C_s := \big\{x_\eta : \eta \in \widetilde{\Lambda}_s\big\} \quad \text{and} \quad C_r := \big\{x_\eta : \eta \in \widetilde{\Lambda}_r\big\}.$$

Clearly, $\Lambda_{C_s} = \Lambda_s$ and $\Lambda_{C_r} = \Lambda_r$. Also, C_s, C_r are closed in X, as

$$C_s = X \setminus \bigcup\{X_\eta : \eta \in \Lambda_{\max},\ \eta \text{ ends with } 1\},$$
$$C_r = X \setminus \bigcup\{X_\eta : \eta \in \Lambda_{\max},\ \eta \text{ ends with } 1 \text{ and contains two } 1\text{'s}\},$$

and the sets which we remove are open. Moreover, it is easy to see that C_s, C_r are also perfect, hence they are Cantor spaces. Finally, $C_s \subset C_r$ and C_s has empty interior with respect to C_r. Indeed, take $x = x_\eta \in C_s$. If $\eta = \widetilde{\eta}^\frown \omega$, then for every $k \geq 2$, $x_k := x_{\widetilde{\eta}^\frown k ^\frown 1 ^\frown (2,2,\dots)} \in C_r \setminus C_s$ and $x_k \to x$, and if η is infinite, then for every $k \in \mathbb{N}$, $x_k := x_{[\eta]_k ^\frown 1 ^\frown (2,2,\dots)} \in C_r \setminus C_s$ and $x_k \to x$. Hence the result follows from the following Claim (whose version we also prove in [4]; we give a proof for the sake of completeness):

CLAIM. Assume that C_1, C_2 are Cantor spaces such that $C_1 \subset C_2$ and $\mathrm{Int}(C_1) = \emptyset$. Then there exists a homeomorphic embedding $h\colon Y \to C_2$ such that $h(Y^{(\infty)}) = C_1$ and $h(y_0) = x_0$, where $y_0 \in Y^{(\infty)}$ and $x_0 \in C_1$ are initially chosen.

Since $Y^{(\infty)}$ and C_1 are Cantor spaces, there is a homeomorphism $h \colon Y^{(\infty)} \to C_1$ such that $h(y_0) = x_0$. Consider a family

$$D := \{f \colon Y \to C_2 : f \text{ is continuous and} f_{|Y^{(\infty)}} = h\}$$

and consider it as a metric space with the supremum metric. Let us note that D is nonempty as the set $Y^{(\infty)}$ is a retract of zero dimensional space Y (see [19, Theorem 7.3]), so there exists a continuous retraction $r \colon Y \to Y^{(\infty)}$, and $h \circ r \in D$. Now let $z_1, z_2, \ldots$ be all elements of (clearly countable) set $Y \setminus Y^{(\infty)}$ (in the case when $Y \setminus Y^{(\infty)}$ is finite, we can proceed in similar, but simpler way), and for every $n, m \in \mathbb{N}$, let

$$U_n := \{f \in D : f(z_n) \notin C_1\}, \qquad U_{n,m} := \{f \in D : f(z_n) \neq f(z_m)\}.$$

Observe that any map $f \in \bigcap_{n,m \in \mathbb{N}} (U_n \cap U_{n,m})$ satisfies the thesis of the Claim. Hence, in view of completeness of D, it is enough to prove that all U_n and $U_{n,m}$ are open and dense. Openness of these sets is obvious. We will show that U_n is dense. Take $f \in D$ and $\varepsilon > 0$, and choose $y_0 \in C_2 \setminus C_1$ so that $d(f(z_n), y_0) < \varepsilon$ (where d is an original metric on C_2). Then take a clopen set $V \subset Y \setminus Y^{(\infty)}$ such that $z_n \in V$ and $f(V) \subset B(f(z_n), \varepsilon/2)$, and define

$$\widetilde{f}(x) := \begin{cases} f(x) & \text{if } x \notin V, \\ y_0 & \text{if } x \in V. \end{cases}$$

As V is clopen and $V \cap Y^{(\infty)} = \emptyset$, $\widetilde{f}$ is continuous and therefore $\widetilde{f} \in D$. Moreover, for every $x \in Y$,

- if $x \notin V$, then $d(f(x), \widetilde{f}(x)) = 0$;
- if $x \in V$, then

$$d(f(x), \widetilde{f}(x)) = d(f(x), y_0) \leq d(f(x), f(z_n)) + d(f(z_n), y_0) < \varepsilon.$$

Hence the supremum metric $d_s(f, \widetilde{f}) \leq \varepsilon$ and U_n is dense. Similarly we prove the density of $U_{n,m}$. Claim is proved. $\qquad\square$

4. 0-dimensional compact metrizable spaces as attractors of Banach GIFSs of order 2

The main result of this section is the following:

THEOREM 4.1. *Let X be a compact metrizable 0-dimensional space and $\lambda \in (0,1)$. Then X is homeomorphic to the attractor of a GIFS $\mathcal{F}$ of order 2 on the real line with $\mathrm{Lip}(\mathcal{F}) \leq \lambda$. Moreover, if X is scattered, then $\mathcal{F}$ can consist of two maps.*

The aforementioned theorem will follow automatically from Theorem 4.3, which give more "qualitative" result.

Remark 4.2. (a) As was mentioned in Theorem 1.1, Nowak in [28] showed that any compact scattered metrizable space of limit scattered height is not a topological fractal. Hence Theorem 4.1 shows that scattered spaces of limit scattered height distinguish the classes of GIFSs' fractals and IFSs' fractals.

(b) As was mentioned in Theorems 1.1 and 1.2, 0-dimensional compact metrizable spaces which are uncountable or scattered with successor height, are homeomorphic to attractors of IFSs on the real line consisting Banach contractions. In the constructions presented there, the number of required mappings in case of uncountable or unital case is two (but in a nonunital case it seems that we need at least three maps — see [4]). Hence our result, in the case of uncountable spaces, says nothing new. However, later we will need some particular properties that are guaranteed by our construction, so the proof of Theorem 4.1 for an uncountable case can be considered as a step in the proof of results presented in the next section.

(c) In [13], Strobin and his coauthors studied a topological version of a GIFS, and Theorem 4.1 shows that all compact scattered spaces are attractors of such kind of GIFSs.

The promised qualitative version of Theorem 4.1 is the following (then Theorem 4.1 follows from it and Propositions 3.10, 3.12, Lemma 2.2 and Theorem 3.4).

Theorem 4.3.

(a) *Let $\alpha \leq \delta_0$, $n \in \mathbb{N}$ and b be a good sequence. Then each $(\Lambda^{\alpha,n}, b, s)$-space is the attractor of a Banach GIFS $\mathcal{F}$ of order 2 which consists of two maps and such that $\mathrm{Lip}(\mathcal{F}) \leq \lambda_b$.*

(b) *Let X be a $(\Lambda_{\max}, b, s)$-space and $M \subset X$ be compact set which satisfies (3.9). Then M is the attractor of a Banach GIFS $\mathcal{F}$ of order 2 which consists of four maps and such that $\mathrm{Lip}(\mathcal{F}) \leq \lambda_b$.*

In the remaining part of this section we prove Theorem 4.3.

Proof. Assume now that X is a (Λ^{α}, b, s)-space and choose $k \in \mathbb{N}$. By Proposition 3.9 we see that $\Lambda^{\alpha_k} \subset \Lambda^{\alpha}$ and, as was already observed, the space X_k is $(\Lambda^{\alpha_k}, b^k, s)$-space. Moreover,

$$\lambda := \sup \left\{ \frac{b_n^k}{b_n} : n \in \mathbb{N} \right\} = \sup \left\{ \frac{b_{n+k}}{b_n} : n \in \mathbb{N} \right\} \leq M_b \leq \frac{1}{8} \lambda_b.$$

Hence, by Lemma 3.8, there exists a surjective map $h_k \colon X \to X_k$ with $\mathrm{Lip}(h_k) \leq \lambda_b/4$. Then, by Lemma 3.6 and Remark 3.7, there exists a map $F \colon X \times X \to X$ such that $\mathrm{Lip}(F) \leq \lambda_b/2$ and $F(X \times X) = X \setminus X_1$. Moreover, define $G \colon X \to X$ by $G(x) := h_1(x)$. Then $G(X) = X_1$ and $\mathrm{Lip}(G) \leq \lambda_b/4$. Hence, by Lemma 2.1, the proof of Theorem 4.3 (a) in the case when $n = 1$ is finished.

Now let X be a $(\Lambda^{\alpha,n}, b, s)$-space, where $n > 1$. As was observed, $X_1, \ldots, X_{n-1}$ are (Λ^α, b^i, s)-spaces, for $i = 1, \ldots, n-1$, respectively, and $\widetilde{X}_n := \bigcup_{n \leq k \leq \omega} X_k$ is a $(\Lambda^\alpha, b^{n-1}, s)$-space. By what we already proved, there is a surjection $F_1 \colon X_1 \times X_1 \to X_1 \setminus X_{(1,1)}$ with $\mathrm{Lip}(F_1) \leq \lambda_{b^1}/2 \leq \lambda_b/2$. Now (see Lemma 3.2 (d)) since $X \setminus X_1$ is a $(\Lambda^{\alpha,n-1}, b^1, s)$-space, $X_{(1,1)}$ is $(\Lambda^{\alpha_1}, b^2, s)$-space and $\Lambda^{\alpha_1} \subset \Lambda^{\alpha,n-1}$, by Lemma 3.8 there exists a surjection $h \colon X \setminus X_1 \to X_{(1,1)}$ with $\mathrm{Lip}(h) \leq 2M_b \leq \lambda_b$. Hence, by simple calculations (involving (3.3)) we can show that the map $P \colon X \times X \to X_1$ defined by

$$
P(x,y) := \begin{cases} F_1(x,y) & \text{if } (x,y) \in X_1 \times X_1, \\ h(x) & \text{if } x \notin X_1, \\ z & \text{if } x \in X_1, \ y \notin X_1, \end{cases}
$$

where z is an arbitrary point of X_1, is surjective and $\mathrm{Lip}(P) \leq \lambda_b$.

Now, if $n = 2$, then exactly the same reasoning, shows that there exists a surjection $Q \colon X \times X \to \widetilde{X}_2 = X \setminus X_1$ with $\mathrm{Lip}(Q) \leq \lambda_b$, and we are done. Hence let $n \geq 3$.

By Lemma 3.8, proceeding similarly as earlier, we can see that for $i = 1, \ldots, n-2$, there exists a surjective map $h_i \colon X_i \to X_{i+1}$ with $\mathrm{Lip}(h_i) \leq 2M_b \leq \lambda_b$. Moreover, for the same reason, there exists a surjection $h_{n-1} \colon X_1 \to \widetilde{X}_n$ with $\mathrm{Lip}(h_{n-1}) \leq 2M_b \leq \lambda_b$. Then the map $Q \colon X \times X \to X \setminus X_1$ defined by

$$
Q(x,y) := \begin{cases} h_i(x) & \text{if } x \in X_i, \ i = 1, \ldots, n-2, \ y \in X_1, \\ h_{n-1}(x) & \text{if } x \in X_1, \ y \notin X_1, \\ z & \text{otherwise}, \end{cases}
$$

where z is an arbitrary element of X_{n-1}, is surjective and $\mathrm{Lip}(Q) \leq \lambda_b$. All in all, the proof of Theorem 4.3 (a) is finished.

Now we switch to the case when M is a compact subset of a $(\Lambda_{\max}, b, s)$-space X which satisfies (3.9).

Let $Z := \{x_\eta : \eta \in \widetilde{\Lambda}_s\}$. By (3.10) and Observation 3.5, $Z \subset M$. We will define four maps: $G_i \colon Z \times Z \to M$, $i = 2, 3, 4$ and $F \colon Z \to M$, such that

$$
\mathrm{Lip}(G_i), \mathrm{Lip}(F) \leq \lambda_b/2 \quad \text{and} \quad \bigcup_{i=2}^{4} G_i(Z \times Z) \cup F(Z) = M.
$$

This will end the proof. Indeed, then we can define $\widetilde{G}_i \colon M \times M \to M$ and $\widetilde{F} \colon M \to M$ by $\widetilde{G}_i(x,y) = G_i(r(x), r(y))$ and $\widetilde{F}(x) = F(r(x))$, where $r \colon X \to Z$ is a projective map with $r(x) = x$ for $x \in Z$ and $\mathrm{Lip}(r) \leq 2$ (existence of r is guaranteed by Lemma 3.8). Then we obtain GIFS $\mathcal{F}$ with $\mathrm{Lip}(\mathcal{F}) \leq \lambda_b$ whose attractor is M.

Define $M_1, \ldots, M_4$ in the following way:

$$M_4 := M \setminus (X_1 \cup X_2 \cup X_3) \quad \text{and} \quad M_i := M \cap X_i, \quad i = 1, 2, 3.$$

Clearly, $M = \bigcup_{i=1}^{4} M_i$. We first define the map G_4. If $\eta \in \widetilde{\Lambda}_s$, then by η' we will mean the sequence so that $\eta = \eta_1{}^\frown\eta'$. Moreover, if $2 \le k < \omega$ and $|\eta| \ge k$, then by η^1, η^2 we will mean sequences so that $\eta = \eta_1{}^\frown\eta^1{}^\frown\eta_k{}^\frown\eta^2$ (if $k = 2$ and $|\eta| = 2$, then $\eta^1 = \eta^2 = \emptyset$). Let $s_k \colon Z \to X$ be the map defined by (we assume $\omega + 2 = \omega$ and $\omega - 1 = \omega$ and equate $x_\eta \in Z$ with $\eta \in \widetilde{\Lambda}_s$):

$$s_k(x_\eta) := \begin{cases} x_{(\eta_1+2)^\frown\eta'} & \text{if } |\eta| < k, \\ x_{(\eta_1+2)^\frown\eta^1{}^\frown(\eta_k-1)^\frown\eta^2} & \text{if } \eta = \eta_1{}^\frown\eta^1{}^\frown\eta_k{}^\frown\eta^2. \end{cases}$$

Observe that s_k is well defined as X is $(\Lambda_{\max}, b, s)$-space and η does not contain 1. Moreover, $\operatorname{Lip}(s_k) \le \lambda_b/8$. Indeed, take distinct $x = x_\eta$, $y = y_\xi \in Z$, and let j be such that $[\eta]_j = [\xi]_j$ and $x \in X_{[\eta]_j{}^\frown p}$ and $y \in X_{[\xi]_j{}^\frown q}$ for some $p < q \le \omega$. We consider two cases:

Case 1. $j > k - 1$. Then, setting $\beta := [\eta]_j$ and $\widetilde{\beta}$ — its modification due to definition of s_k (that is, $\widetilde{\beta} = (\eta_1 + 2)^\frown[\eta]_j^1{}^\frown(\eta_k - 1)^\frown[\eta]_j^2$) — we have by (3.3)

$$d(x, y) \ge \operatorname{dist}\left(X_{\beta^\frown p}, X_{\beta^\frown q}\right) \ge b_{l(\beta)+p-1} - 2b_{l(\beta)+p} - b_{l(\beta)+p+1}$$

and

$$d(s_k(x), s_k(y)) \le \operatorname{diam}\left(\bigcup_{p \le i \le \omega} X_{\widetilde{\beta}^\frown i}\right) \le b_{l(\widetilde{\beta})+p-1} = b_{l(\beta)+2-1+p-1}$$

$$= b_{l(\beta)+p} \le \frac{1}{8}\lambda_b(b_{l(\beta)+p-1} - 2b_{l(\beta)+p} - b_{l(\beta)+p+1}) \le \frac{1}{8}\lambda_b d(x, y).$$

Case 2. $j = k - 1$. Then, letting $\widetilde{\beta}$ be appropriate modification of β, we have that

$$d(s_k(x), s_k(y)) \le \operatorname{diam}\left(\bigcup_{p-1 \le i \le \omega} X_{\widetilde{\beta}^\frown i}\right) \le b_{l(\widetilde{\beta})+p-2} = b_{l(\beta)+2+p-2}$$

$$= b_{l(\beta)+p} \le \frac{1}{8}\lambda_b(b_{l(\beta)+p-1} - 2b_{l(\beta)+p} - b_{l(\beta)+p+1}) \le \frac{1}{8}\lambda_b d(x, y).$$

Case 3. $j < k - 1$. Then we obtain the desired inequality in a similar way.

Finally, we show that for $k < j < \omega$ and every $x \in Z$, $d(s_k(x), s_j(x)) \le b_{k+1}$. Let $x = x_\eta$. If $|\eta| < k$, then also $|\eta| < j$, and $s_k(x) = s_j(x)$. Hence assume that $|\eta| \ge k$. Then $s_k(x) \in X_{(\eta_1+2, \eta_2, \ldots, \eta_{k-1})}$ and $s_j(x) \in X_{(\eta_1+2, \eta_2, \ldots, \eta_{k-1})}$, so

$$d(s_k(x), s_j(x)) \le \operatorname{diam}\left(X_{(\eta_1+2, \eta_2, \ldots, \eta_{k-1})}\right) \le b_{\eta_1+2+\eta_2+\ldots+\eta_{k-1}} \le b_{k-1+2} = b_{k+1}.$$

Finally, let $r' \colon X \to M$ be the projective map with $r'(x) = x$ for $x \in M$ and $\operatorname{Lip}(r') \le 2$, and set $s_k' := r' \circ s_k$. Then $\operatorname{Lip}(s_k') \le \lambda_b/4$ and $d(s_k'(x), s_j'(x)) \le 2b_{k+1} \le b_k$. Hence the assumptions of Lemma 3.6 are satisfied for $h_k := s_{k-1}'$,

$k \geq 3$ and h_2 — any constant map with value in M_4. Hence there is a map $G_4 \colon Z \times Z \to X$ such that $\mathrm{Lip}(G_4) \leq \lambda_b/2$, and (since $Z \cap X_i \neq \emptyset$ for $i \geq 2$)

$$M \supset G_4(Z \times Z) = \bigcup \{h_k(Z) : k \in \overline{\mathbb{N}},\ k \geq 3\} \supset \bigcup \{s'_k(Z) : k \geq 2\} \supset M_4.$$

To see the last inclusion, take any $x_\eta \in M_4$. If $|\eta| \geq 2$, then we can choose $k \geq 2$ so that $\eta_k = \min\{\eta_i : i \geq 2\}$ and we put $\xi := (\eta_1 - 2)^\frown \eta^1 {}^\frown (\eta_k + 1)^\frown \eta^2$ and then $s'_k(x_\xi) = s_k(x_\xi) = x_\eta$ (since η can have just one 1, we have that $x_\xi \in Z$). If $|\eta| = 1$, then $\eta = \omega$ and $s'_2(x_\omega) = x_\omega$.

Now for $i = 2, 3$ and every $k \geq 2$, define $s^i_k \colon Z \to X$ by

$$s^i_k(x_\eta) := \begin{cases} x_{i^\frown \eta} & \text{if } |\eta| < k - 1, \\ x_{i^\frown \eta^1 {}^\frown (\eta_{k-1} - 1)^\frown \eta^2} & \text{if } \eta = \eta^1 {}^\frown \eta_{k-1} {}^\frown \eta^2. \end{cases}$$

Similarly as before we can show that $\mathrm{Lip}(s^i_k) \leq \lambda_b/2$ and $d(s^i_k(x), s^i_j(x)) \leq b_k$, and proceeding in an analogous way, we can define maps $G_i \colon Z \times Z \to X$ with $\mathrm{Lip}(G_i) \leq \lambda_b/2$ and such that for $i = 2, 3$, $M_i \subset G_i(Z \times Z) \subset M$.

Finally, define $F \colon Z \to M$ by $F(x_\eta) := r'(x_{1^\frown \eta})$. Then it is easy to see that $\mathrm{Lip}(F) \leq \lambda_b/2$ and $F(Z) \supset M_1$. The result follows. $\qquad \square$

REMARK 4.4. By Lemma 2.4 we can see that appropriately separated union of n many (Λ^α, b, s)-sets is the attractor of some Banach GIFS. However, presented construction shows that it is enough to take just two functions.

REMARK 4.5. It is worth underlining that GIFSs give much more possibilities than IFSs. Considering Cartesian power in the domain provides indispensable space when considering compact scattered spaces with countable limit height. Moreover, observe that the definition of function G (in the proof for (Λ^α, b, s)-spaces) does not make use of whole Cartesian power (it can be perceived as a selfmap).

5. 0-dimensional compact spaces as attractors of GIFSs of order m and nonattractors of GIFSs of order $m - 1$

In this section we prove the following theorem:

THEOREM 5.1. *Let X be an infinite compact 0-dimensional metrizable space.*

(a) *For every $\lambda \in (0, 1)$ and $m \geq 2$, there exists $Z \subset \mathbb{R}$ which is homeomorphic to X and such that*:
   (a1) *Z is the attractor of some GIFS $\mathcal{F}$ on the real line of order m with $\mathrm{Lip}(\mathcal{F}) \leq \lambda$;*
   (a2) *Z is not the attractor of any weak GIFS of order $m - 1$.*

(b) *There exists $Z \subset \mathbb{R}$ which is homeomorphic to X and which is not the attractor of any weak GIFS.*

Almost whole Theorem 5.1 will follow from more "qualitative" Theorem 5.7 and Lemma 2.2 (or [6, Lemma 1.1 (ii)]). However, the case when $X \setminus X^{(\infty)}$ is finite (where $X^{(\infty)}$ is the perfect kernel) need to be dealt separately. At first recall that in [30], for every $\lambda \in (0,1)$ and every $m \geq 2$, we constructed a Cantor subset of the plane which is the attractor of some Banach GIFS of order m which is not the attractor of any weak GIFS of order $m - 1$, and also we constructed a Cantor set which is not the attractor of any weak GIFS. As can be seen, the construction proved there can be modified so that we get appropriate Cantor subsets of the real line (in fact, in the first draft of that paper, exactly such sets were constructed, as they were inspired by the construction from [8]). Hence, Theorem 5.1 (a) for X with $X \setminus X^{(\infty)}$ finite, follows from Lemma 2.4, Remark 2.5 and the following:

LEMMA 5.2. *Assume that* $m \in \mathbb{N}$, $\emptyset \neq C \subset \mathbb{R}$, $\emptyset \neq P \subset \mathbb{R}$ *is finite and* $\mathrm{dist}(C, P) \geq \mathrm{diam}(C)$. *If* $C \cup P$ *is the attractor of a weak* GIFS *of order* m, *then* C *is the attractor of a weak* GIFS *of order* m.

PROOF. By our assumptions, we can define the projection $\pi \colon C \cup P \to C$ with $\mathrm{Lip}(\pi) = 1$. Let $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a weak GIFS of order m such that $C \cup P$ is its attractor. Now, for every $1 \leq k < m$, $1 \leq i_1 < \ldots < i_k \leq m$, $(z_{i_1}, \ldots, z_{i_k}) \in P^k$ and $j = 1, \ldots, n$, we define $f_j^{(i_1, \ldots, i_k),(z_{i_1}, \ldots, z_{i_k})} \colon C^{m-k} \to C$ in the following way: if $(x_1, \ldots, x_{m-k}) \in C^{m-k}$, then let $f_j^{(i_1, \ldots, i_k),(z_{i_1}, \ldots, z_{i_k})}(x_1, \ldots, x_{m-k})$ be equal to the value $\pi(f_j(y_1, \ldots, y_m))$, where $y_{i_1} = z_{i_1}, \ldots, y_{i_k} = z_{i_k}$ and remaining coordinates of $(y_1, \ldots, y_n)$ equals $x_1, \ldots, x_{m-k}$, consecutively. Define also $f_j^0 \colon C^m \to C$ as $f_j^0 := \pi \circ f_j$. As f_j are generalized weak contractions, so are f_j^0 and $f_j^{(i_1, \ldots, i_k),(z_{i_1}, \ldots, z_{i_k})}$. Finally, observe that

$$C \setminus \bigcup_{j=1,\ldots,n} f_j(P^m)$$
$$\subset \bigcup_{j=1,\ldots,n} \left(f_j^0(C^m) \cup \bigcup_{1 \leq k < m} \bigcup_{i_1 < \ldots < i_k} \bigcup_{(z_{i_1}, \ldots, z_{i_k})} f_j^{(i_1,\ldots,i_k),(z_{i_1},\ldots,z_{i_k})}(C^{m-k}) \right) \subset C.$$

Since $\bigcup\limits_{j=1,\ldots,n} f_j(P^m)$ is finite, we can define finitely many constant maps $g_1, \ldots, g_l \colon C^m \to C$ so that these mappings, together with maps f_j^0 and obvious extensions (like in Lemma 2.1) of all maps $f_j^{(i_1,\ldots,i_k),(z_{i_1},\ldots,z_{i_k})}$, will form a weak GIFS of order m whose attractor is C, and this would be a contradiction. $\qquad \square$

To prove Theorem 5.1 in the remaining case when $X \setminus X^{(\infty)}$ is infinite, we have to extend a bit the notion of (Λ, b, s)-space by adding finite set to each segment X_k. We say that a pair of sequences (b, p) is *a good pair*, if b is good

and p is a sequence of natural numbers such that $p_k \geq 2$ for $k \in \mathbb{N}$ and

$$(5.1)\ \sup\left\{8(1+4p_k)\frac{b_k}{b_{k-1}} : k \in \mathbb{N}\right\} < 1, \quad \sup\left\{\frac{b_k}{b_{k-1}}p_k : k \in \mathbb{N},\ k \geq 2\right\} \leq \lambda_b.$$

Definition 5.3. Let (b, p) be a good pair and Λ be a proper tree. We say that a metric space Z is a $(\Lambda, (b, p), s)$-space, if $Z = X \cup Y$, where

(a) X is a (Λ, b, s)-space;

(b) $Y = \bigcup\limits_{1 \leq k \leq \omega} Y_k$, where $Y_\omega = X_\omega$, and for every $k \in \mathbb{N}$,

(b1) $Y_k = \{a_1^k, \ldots, a_{p_k}^k\}$ for some $a_1^k, \ldots, a_{p_k}^k$ with $d(a_i^k, a_j^k) = 2(j - i)b_k$ for $i \leq j$;

(b2) $\mathrm{dist}(X_k, Y_k) = 2b_k p_k$.

Remark 5.4. It is easy to see that for every sequence $p = (p_k)$ of natural numbers with $p_k \geq 2$ for $k \in \mathbb{N}$, and $\lambda > 0$, there exists a good sequence b such that the pair (b, p) is good and $\lambda_b < \lambda$ (the important fact is that in the "middle condition" in (5.1) we consider only $k \geq 2$). It is also clear that for every good pair (b, p) and a proper tree Λ, there exists a set $X \subset \mathbb{R}$ which is a $(\Lambda, (p, b), s)$-space.

Later, when writing a $(\Lambda, (b, p), s)$-space as $X \cup Y$, we will automatically assume that X, Y have meaning as in the above definition. We start with making basic observations of the structure of $(\Lambda, (b, p), s)$-spaces. If $Z = X \cup Y$ is a $(\Lambda, (b, p), s)$-space, then for any $k \in \overline{\mathbb{N}}$, we put $Z_k := X_k \cup Y_k$.

Observation 5.5. Let $Z = X \cup Y$ be a $(\Lambda, (b, p), s)$-space with $p_n \geq 2$. Then, for every $1 \leq k < \omega$,

(a) $\mathrm{diam}(Z_k) \leq b_{k-1}/8$;

(b) $\mathrm{dist}\left(Z_k, \bigcup\limits_{k < j \leq \omega} Z_j\right) \geq 2b_{k-1}/3$;

(c) $\mathrm{diam}\left(\bigcup\limits_{k \leq j \leq \omega} Z_j\right) \leq 5b_{k-1}/4$;

(d) $\mathrm{diam}(X_k) \leq \mathrm{diam}(Y_k) \leq \mathrm{dist}(X_k, Y_k)$.

Proof. The properties follow from (5.1), Definition 3.1 and the fact that b is good. (a) follows from the fact that, for any $x, y \in Z_k$,

$$d(x, y) \leq \mathrm{diam}(X_k) + \mathrm{dist}(X_k, Y_k) + \mathrm{diam}(Y_k)$$

$$\leq b_k + 2b_k p_k + 2b_k(p_k - 1) \leq b_k(4p_k + 1) \leq \frac{1}{8}\,b_{k-1}.$$

To see (b), note that for every $k < j \leq \omega$,

$$\mathrm{dist}(Z_k, Z_j) \geq \mathrm{dist}(X_j, X_k) - \mathrm{diam}(Z_j) - \mathrm{diam}(Z_k)$$

$$\geq b_{k-1} - 2b_k - b_{k+1} - \frac{1}{8}\,b_{j-1} - \frac{1}{8}\,b_{k-1} \geq \frac{7}{8}\,b_{k-1} - 3b_k \geq \frac{2}{3}\,b_{k-1}.$$

To see (c), we calculate as follows:

$$\operatorname{diam}\left(\bigcup_{k \leq j \leq \omega} Z_k\right) \leq b_{k-1} + 2\sup\{4b_j p_j : j \geq k\} \leq \frac{5}{4}b_{k-1}.$$

Point (d) follows from definition. $\square$

PROPOSITION 5.6. *Let* (b,p) *be a good pair, and* K *be a compact, metrizable,* 0-*dimensional space such that* $K \setminus K^{(\infty)}$ *is infinite. There exists a proper tree* Λ *such that for every* $(\Lambda, (b,p), s)$-*space* $Z = X \cup Y$*, there exists a set* $M \subset X$ *such that*

(a) *M is the attractor of some Banach* GIFS *$\mathcal{F}$ of order 2 with* $\operatorname{Lip}(\mathcal{F}) \leq \lambda_b$;
(b) *$M \cup Y$ is homeomorphic to K;*
(c) *$M \cap X_k \neq \emptyset$ for $2 \leq k \leq \omega$.*

PROOF. We will consider three cases:

Case 1. $K^{(\infty)} = \emptyset$. This means that Z is countable and infinite. Then it is scattered so by the Mazurkiewicz–Sierpiński theorem and Proposition 3.10, there is a proper tree $\Lambda := \Lambda^{\alpha,n}$ such that K is homeomorphic to any (Λ, b, s)-space. Hence let $X \cup Y$ be a $(\Lambda, (b,p), s)$-space. By Theorem 4.3 (a), X is the attractor of some Banach GIFS $\mathcal{F}$ of order 2 with $\operatorname{Lip}(\mathcal{F}) \leq \lambda_b$. By Observation 5.5 (b), each Y_k is open, so we see that the first derivative $(X \cup Y)' = X'$. Using the Mazurkiewicz–Sierpiński theorem, we see that $X \cup Y$ is homeomorphic with K.

Case 2. $K^{(\infty)}$ is nonempty and not open in K. Then there is a sequence $(x_n) \subset K$ of isolated points which converges to some $y_0 \in K^{(\infty)}$. Indeed, by our assumption, there is a sequence $(x_n') \subset K \setminus K^{(\infty)}$ and $y_0 \in K^{(\infty)}$ such that $x_n' \to y_0$. However, for any $k \in \mathbb{N}$, there is a clopen set $V \ni x_k'$ which is disjoint with $K^{(\infty)}$ and has diameter $\leq 1/k$. In particular, V is countable compact metrizable space, hence scattered, and it must have an isolated point x_k, which is also isolated in K. Then $x_k \to y_0$. Now let $X \cup Y$ be a $(\Lambda_{\max}, (b,p), s)$-space. Since $K' := K \setminus \{x_k : k \in \mathbb{N}\}$ is compact, uncountable 0-dimensional and metrizable, by Proposition 3.12, we can find $M \subset X$ so that (3.9) is satisfied and a homeomorphism $h \colon K' \to M$ so that $h(y_0) = x_\omega$. Enumerate elements of $\bigcup_{k \in \mathbb{N}} Y_k$ by $\{y_k : k \in \mathbb{N}\}$, and extend h by adjusting $h(x_k) := y_k$ for $k \in \mathbb{N}$. It is easy to see that $h \colon K \to M \cup Y$ is homeomorphism. Also, by Theorem 4.3 (b), M is the attractor of some Banach GIFS of order 2 with $\operatorname{Lip}(\mathcal{F}) \leq \lambda_b$.

Case 3. $K^{(\infty)}$ is nonempty and open in K. Then $K' := K \setminus K^{(\infty)}$ is countable compact space, hence scattered, and we can find a proper tree Λ' as in Case 1. Now let

$$\Lambda := \{1^\frown \eta : \eta \in \Lambda_{\max}\} \cup \{i^\frown \eta : 2 \leq i \leq \omega, \ (i-1)^\frown \eta \in \Lambda'\}.$$

Clearly, Λ is a proper tree and (according to the notation from Lemma 3.2), $\Lambda_1[1] = \Lambda_{\max}$ and $\Lambda_2[2] = \Lambda'$. Now let $X \cup Y$ be a $(\Lambda, (b,p), s)$-space. Then by Lemma 3.2, X_1 is $(\Lambda_{\max}, b^1, s)$-space and $\widetilde{X}_1$ is (Λ', b^1, s)-space. In the same way as in Case 1, we can show that $\widetilde{X}_1 \cup Y$ is homeomorphic to $\widetilde{X}_1$ (and to K'). Also, as $K^{(\infty)}$ is a Cantor space, it is homeomorphic to a subset $N \subset X_1$ which satisfies (3.9) (see Proposition 3.12), and then $K = K' \cup K^{(\infty)}$ is homeomorphic to $(\widetilde{X}_1 \cup Y) \cup N$ (as the underlying sets are clopen). Finally, as $\widetilde{X}_1$ and N are attractors of some GIFSs $\mathcal{F}_1, \mathcal{F}_2$ of order 2 with $\mathrm{Lip}(\mathcal{F}_i) \le \lambda_b$ (by Theorem 4.3), $M := \widetilde{X}_1 \cup N$ is the attractor of a Banach GIFS $\mathcal{F}$ of order 2 with $\mathrm{Lip}(\mathcal{F}) \le \lambda$ (see Lemma 2.4). □

THEOREM 5.7. *Let $Z = X \cup Y$ be a $(\Lambda, (b,p), s)$-space and $M \subset X$ be the attractor of some Banach GIFS $\mathcal{F}$ of order 2 with $\mathrm{Lip}(\mathcal{F}) \le \lambda_b$ and such that $X_k \cap M \ne \emptyset$ for $2 \le k \le \omega$.*

   (a) *If $m \ge 2$, $p_1 \ge 2$, $p_{n+1} = p_n^m$ for $n \in \mathbb{N}$ and b is such that $\lambda_b < 1/2$, then*
      (a1) *$M \cup Y$ is the attractor of some Banach GIFS $\mathcal{G}$ of order m with $\mathrm{Lip}(\mathcal{G}) \le 2\lambda_b$,*
      (a2) *$M \cup Y$ is not the attractor of any weak GIFS of order $m - 1$.*
   (b) *If $p_1 \ge 2$ and $p_{n+1} := p_n^n$ for $n \in \mathbb{N}$, then $M \cup Y$ is not the attractor of any weak GIFS.*

Theorem 5.7 will follow from the next lemma:

LEMMA 5.8. *Let $Z = X \cup Y$ be a $(\Lambda, (b,p), s)$-space, $M \subset X$ be such that $M \cap X_k \ne \emptyset$ for $2 \le k \le \omega$ and $m \in \mathbb{N}$.*

   (a) *If $\lim_{n \to \infty} p_n^m / p_{n+1} = 0$, then $M \cup Y$ is not the attractor of any weak GIFS of order m.*
   (b) *If M is the attractor of some Banach GIFS $\mathcal{F}$ of order m with $\mathrm{Lip}(\mathcal{F}) < 1/2$ and for any $n \in \mathbb{N}$, $p_{n+1} \le p_n^m$, and b is such that $\lambda_b < 1/2$, then $M \cup Y$ is the attractor of some Banach GIFS $\mathcal{G}$ of order m such that $\mathrm{Lip}(\mathcal{G}) \le 2 \max\{\mathrm{Lip}(\mathcal{F}), \lambda_b\}$.*

We show that Theorem 5.7(1) follows from the above lemma. Observe that if $p_1 \ge 2$ and $p_{n+1} = p_n^m$, then

$$\lim_{n \to \infty} \frac{p_n^{m-1}}{p_{n+1}} = \lim_{n \to \infty} \frac{1}{p_n} = 0.$$

Hence (b1) follows from Lemma 5.8 (a). Clearly, (a1) follows from Lemma 5.8 (b). Finally, if $p_1 \ge 2$ and $p_{n+1} = p_n^n$, then for any $m \in \mathbb{N}$, we have

$$\lim_{n \to \infty} \frac{p_n^m}{p_n^n} = \lim_{n \to \infty} \frac{1}{p_n^{n-m}} = 0,$$

so (b) follows from Lemma 5.8 (a).

In the remaining part of this section we prove Lemma 5.8. For every $k \in \overline{\mathbb{N}}$, define $M_k = X_k \cap M$. If $M_1 = \emptyset$, then define $\pi_1 \colon M \cup Y \to M$ by

$$\pi_1(x) = \begin{cases} x & \text{if } x \in M, \\ z_{\max\{k,2\}} & \text{if } x \in Y_k, \ k \in \mathbb{N}, \end{cases}$$

where z_k is a fixed element of M_k. We will show that $\mathrm{Lip}(\pi_1) \leq 2$. Take $x, y \in M \cup Y$ and consider (most important) cases:

Case 1. $x \in M_k$ and $y \in Y_k$ for some $k \in \mathbb{N}$, $k \geq 2$. Then, by Observation 5.5 (d),

$$d(\pi_1(x), \pi_1(x)) \leq \mathrm{diam}(X_k) \leq \mathrm{dist}(X_k, Y_k) \leq \mathrm{dist}(M_k, Y_k) \leq d(x, y).$$

Case 2. $x \in Z_k$ and $y \in Z_j$ for $k < j \leq \omega$. By Observation 5.5 (b), (c), we have

$$d(\pi_1(y), \pi_1(x)) \leq \mathrm{diam}\left(\bigcup_{k \leq j \leq \omega} Z_j \right)$$
$$\leq \frac{5}{4}\, b_{k-1} \leq 2 \cdot \frac{2}{3}\, b_{k-1} \leq 2\, \mathrm{dist}(Z_k, Z_j) \leq 2d(x, y).$$

Case 3. Cases 1 and 2 do not hold. Then $d(\pi_1(x), \pi_1(y)) \leq d(x, y)$ clearly holds.

Similarly we can define a projective map $\pi_2 \colon M \cup Y \to Y$ with $\mathrm{Lip}(\pi_2) \leq 2$. If $M_1 \neq \emptyset$ we can also define appropriate projections π_1, π_2.

In view of Lemmas 2.1 and 2.4, to complete the proof of (b), it suffices to show that Y is the attractor of some GIFS $\mathcal{G}$ of order m with $\mathrm{Lip}(\mathcal{G}) \leq \lambda_b$. For every $k \in \overline{\mathbb{N}}$, let

$$G_k := \bigcup \big\{ Y_{i_1} \times \ldots \times Y_{i_m} : \max\{i_1, \ldots, i_m\} = k \big\}.$$

Then $Y^m = \bigcup_{k \in \overline{\mathbb{N}}} G_k$ and for $k \in \mathbb{N}$, the cardinality $|G_k| \geq p_k^m$. Hence by our assumptions, for every $2 \leq k \leq \omega$, there exists a surjection $h_k \colon G_{k-1} \to Y_k$ (we set $G_{\omega-1} := G_\omega$). Finally, if $x \in Y^m$, then we set

$$F(x) := h_k(x) \quad \text{if } x \in G_k.$$

Clearly, $F \colon Y^m \to (Y \setminus Y_1)$ is surjection. Now we show that

(5.2) $$\mathrm{Lip}(F) \leq \lambda_b.$$

Take distinct $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in Y^m$ and consider cases:

Case 1. For some $2 \leq k < \omega$ it holds $x, y \in G_{k-1}$. Then by Observation 5.5 (b),

$$d_m(x, y) \geq \min \{ \min\{\operatorname{dist}(Y_i, Y_j) : i < j \leq k-1\}, \min\{2b_j : j \leq k-1\}\}$$

$$\geq \min \left\{ \frac{2}{3} b_{k-2}, 2b_{k-1} \right\} = 2b_{k-1}$$

and, by (5.1),

$$d(F(x), F(y)) \leq 2b_k \leq \lambda_b p_k 2b_{k-1} \leq \lambda_b d_m(x, y).$$

Case 2. For some $2 \leq k < j \leq \omega$, it holds $x \in G_{k-1}$ and $y \in G_{j-1}$. Then there exists $i \in \{1, \ldots, m\}$ such that $y_i \in Y_{j-1}$. Then $x_i \in Y_n$ for some $n \leq k-1$, hence by (3.2), (5.1) and Observation 5.5 (b), (c),

$$d(F(x), F(y)) \leq \operatorname{diam}(Y_k \cup Y_j) \leq \frac{5}{4} b_{k-1} \leq \frac{5}{4} b_n \leq \lambda_b \frac{2}{3} b_{n-1}$$

$$\leq \lambda_b \operatorname{dist}(Y_n, Y_{j-1}) \leq \lambda_b d(x_i, y_i) \leq \lambda_b d_m(x, y).$$

Hence the proof of (5.2) is finished.

Finally, let $G_1, \ldots, G_{p_1} \colon Y \to Y_1$ be constant maps so that $\bigcup_{i=1}^{p_1} G_i(Y) = Y_1$. Then $\operatorname{Lip}(G_i) = 0 \leq \lambda_b$ and, in view of Lemma 2.1, Y is the attractor of a GIFS of order m with $\operatorname{Lip}(\mathcal{G}) \leq \lambda_b$. The proof of Lemma 5.8 (b) is finished.

Now we prove Lemma 5.8 (a). It is enough to prove that for every generalized weak contraction $F \colon (M \cup Y)^m \to Z$,

(5.3) $$\lim_{n \to \infty} \frac{|F((M \cup Y)^m) \cap Y_n|}{p_n} = 0.$$

We start with the additional "structural" observation concerning (Λ, b, s)-spaces.

OBSERVATION 5.9. Let X be a (Λ, b, s)-space. For every $k \in \mathbb{N}$ and $i \in \mathbb{N} \cup \{0\}$, there exist $X_{k,i}^1, \ldots, X_{k,i}^{2^i}$ such that $X_k = \bigcup_{j=1}^{2^i} X_{k,i}^j$ and $\operatorname{diam}(X_{k,i}^j) \leq b_{k+i}$.

PROOF. By Lemma 3.2(d) we see that X_k is a (Λ', b^k, s)-space of diameter $\leq b_k$, which can be divided into $(\Lambda'_1[(k, 1)], b^{k+1}, s)$-space and $(\Lambda'_2[(k, 2)], b^{k+1}, s)$-space of diameters $\leq b_{k+1}$. If both of these sets are not singletons, then we can divide each of them into next two, each of diameter $\leq b_{k+2}$. Proceeding in this way, for each i we can divide X_k into 2^i spaces, each of the diameter $\leq b_{k+i}$ (however, some of these sets can be empty, as during the procedure we can have singletons at some stages). $\square$

Fix $n \in \mathbb{N}$. We will estimate $|F((M \cup Y)^m) \cap Y_n|$. Let $A_1, \ldots, A_m$ be sets so that every A_l is of one of the following forms (see Observation 5.9 for the notation in the second option):

(a) A_l is a singleton;

(b) $A_l = X_{k,n-k}^j \cap M$ for some $1 \le k \le n$ and $1 \le j \le 2^{n-k}$;

(c) $A_l = \bigcup_{n+1 \le j \le \omega} ((X_j \cap M) \cup Y_j)$;

(d) $A_l = Y_n$.

We will observe that

(5.4) $$|F(A_1 \times \ldots \times A_m) \cap Y_n| \le 1.$$

Indeed, suppose that it is not true. Then we can find $x, y \in A_1 \times \ldots \times A_m$ such that $d(F(x), F(y)) \ge 2b_n$. Now consider two cases:

Case 1. All $A_1, \ldots, A_m$ are of the form (a), (b) or (c). Since F is a weak contraction and by earlier calculations (see Observation 5.5 (vii?)) we get a contradiction since:

$$2b_n \le d(F(x), F(y)) < d_m(x, y) \le \operatorname{diam}(A_1 \times \ldots \times A_m)$$
$$= \max\{\operatorname{diam}(A_l) : l = 1, \ldots, m\} \le \max\{b_n, 2b_n\} = 2b_n.$$

Case 2. $A_{i_1} = \ldots = A_{i_k} = Y_n$ for some $i_1, \ldots, i_k$. Assume, without loss of generality, that $i_1 = 1, \ldots, i_k = k$. Then, setting $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, we can assume that

$$\max\{d(x_i, y_i) : i = 1, \ldots, k\} = j2b_n, \quad \text{for some } 2 \le j \le p_n - 1.$$

Supposing otherwise we would obtain a contradiction with contractivity of F similarly as in Case 1.

Moreover, we assume that x, y are chosen so that j is minimal in this sense, i.e. there are no $x', y' \in A_1 \times \ldots \times A_m$ with $F(x'), F(y') \in Y_n$, $F(x') \ne F(y')$ and $\max\{d(x_i', y_i') : i = 1, \ldots, k\} < j2b_n$. Now let $i = 1, \ldots, k$. If $d(x_i, y_i) \le 2b_n$, then set $z_i := x_i$, and if $d(x_i, y_i) > 2b_n$, then let z_i be an element of Y_n so that $d(x_i, z_i) < d(x_i, y_i)$ and $d(z_i, y_i) < d(x_i, y_i)$ (the existence of z_i is guaranteed by Definition 5.3 (a2)). For $i = k+1, \ldots, m$ set $z_i := x_i$. Then $z := (z_1, \ldots, z_m) \in A_1 \times \ldots \times A_m$, $\max\{d(x_i, z_i) : i = 1, \ldots, k\} < j2b_n$ and $\max\{d(y_i, z_i) : i = 1, \ldots, k\} < j2b_n$. The following conditions can hold: *Case 2a.* $F(z) \in Y_n$. Then by the minimality of j, $F(z) = F(x)$ and $F(z) = F(y)$ — it is a contradiction with $F(x) \ne F(y)$.

Case 2b. $F(z) \in Z_i$ for some $i \ne n$, then by Observation 5.5 (b) and (5.1), we arrive to a contradiction:

$$d(F(z), F(x)) \ge \operatorname{dist}(Z_n, Z_i) \ge \frac{1}{2}\min\{b_0, \ldots, b_{n-1}\}$$
$$= \frac{1}{2}b_{n-1} \ge 2p_n b_n \ge 2jb_n > d_m(z, x).$$

Case 2c. $F(z) \in X_n$, then we get a contradiction since

$$d(F(z), F(x)) \ge \operatorname{dist}(X_n, Y_n) = 2p_n b_n > d_m(z, x).$$

All in all, we proved (5.4).

Notice that (see Observation 5.9)

$$M \cup Y = \left(\bigcup_{1 \le i \le n-1} Y_i \right) \cup Y_n$$

$$\cup \left(\bigcup_{1 \le k \le n} \bigcup_{1 \le j \le 2^{n-k}} (X_{k,n-k}^j \cap M) \right) \cup \left(\bigcup_{n+1 \le j \le \omega} (X_j \cap M) \cup Y_j \right).$$

Hence, for $n \in \mathbb{N}$,

$$|F((M \cup Y)^m) \cap Y_n| \le ((p_1 + \ldots + p_{n-1}) + 1 + (1 + 2 + \ldots + 2^{n-1}) + 1)^m$$
$$\le (1 + p_1 + \ldots + p_{n-1} + n2^n)^m.$$

In order to prove (5.3), in view of the assumption $\lim_{n \to \infty} p_n^m / p_{n+1} = 0$, it is enough to show that for some $M > 0$ and all $n \in \mathbb{N}$,

$$(5.5) \qquad \frac{1 + p_1 + \ldots + p_n + (n+1)2^{n+1}}{p_n} < M.$$

By our assumptions, there exists n_0 such that, for $n \ge n_0$,

$$\frac{p_n}{p_{n+1}} \le \frac{p_n^m}{p_{n+1}} \le \frac{1}{4}.$$

Observe that

$$(5.6) \qquad \frac{p_{n_0+1} + \ldots + p_{n_0+k}}{p_{n_0+k}} \le 2.$$

Indeed, for $k = 1$ it clearly holds, and if it holds for $k \ge 1$, then

$$\frac{p_{n_0+1} + \ldots + p_{n_0+k+1}}{p_{n_0+k+1}} = \frac{p_{n_0+1} + \ldots + p_{n_0+k}}{p_{n_0+k}} \cdot \frac{p_{n_0+k}}{p_{n_0+k+1}} + 1 \le 2\frac{1}{2} + 1 \le 2.$$

Similarly we can show that for some N and every $n \in \mathbb{N}$, $(n+1)2^{n+1}/p_n \le N$ (we can take $N = 8/p_1$). Hence, for every $k \in \mathbb{N}$,

$$\frac{1 + p_1 + \ldots + p_{n_0+k} + (n_0 + k + 1)2^{n_0+k+1}}{p_{n_0+k}} \le \frac{1 + p_1 + \ldots + p_{n_0}}{p_{n_0}} + 2 + N.$$

We proved (5.5), and proof of Lemma 5.8 (a) is finished.

6. (Λ^ω, b, Z)-spaces as attractors of GIFSs

The main result of this section is the following:

THEOREM 6.1. *Let $\lambda \in (0,1)$ and Z be a compact, connected metric space which is the attractor of some Banach GIFS $\mathcal{F}$ of order 2 with $\mathrm{Lip}(\mathcal{F}) \le \lambda/3$. Then there exists a compact metric space X such that:*

    (a) *each connected component of X is a similarity copy of Z,*

    (b) *X is the attractor of some Banach GIFS $\mathcal{F}$ of order 2 with $\mathrm{Lip}(\mathcal{F}) \le \lambda$,*

    (c) *X is not a topological fractal.*

Additionally, if Z is a subset of a normed space H, then X can be taken so that

   (d) $X \subset H$.

Theorem 6.1 will follow from its qualitative version Theorem 6.2 together with Theorem 3.4

Recall that in Subsection 3.1, for each countable limit ordinal δ_0 and $\alpha \leq \delta_0$, we defined a certain sequence (α_n). Consider the case $\delta_0 = \omega$. Then we can assume that sequences (α_n) are of the following forms:

   - if $\alpha = k \in \mathbb{N}$, then $(\alpha_n) = (0, 1, \ldots, k-1, k-1, k-1, \ldots)$ i.e. $k_n = \min\{n-1, k-1\}$;
   - if $\alpha = \omega$, then $(\omega) := (\alpha_n) = (0, 1, \ldots)$, i.e. $\omega_n = n-1$.

It is easy to see that (A) and (B) from Section 3.1 are satisfied.

Observe that if X is a (Λ^ω, b, Z) space and Z is connected, then each connected component of X is a similarity copy of Z. Hence Theorem 6.1 follows from the following:

THEOREM 6.2. *In the above framework, let X be a (Λ^ω, b, Z)-space, where Z is connected.*

   (a) *If Z is the attractor of a Banach GIFS $\mathcal{F}$ of order 2, then X is the attractor of a Banach GIFS $\mathcal{G}$ of order 2 with $\mathrm{Lip}(\mathcal{G}) \leq \max\{3\,\mathrm{Lip}(\mathcal{F}), \lambda_b\}$.*
   (b) *X is not a topological fractal.*

In the remaining part of this section we prove Theorem 6.2. Directly by definition of families Λ^α for $\alpha \leq \omega$ and the above assumption, we can see that for every $k \geq 2$ (recall Proposition 3.9),

$$(6.1) \qquad \widetilde{\Lambda}^{k-1} = \{\omega\} \cup \left(\bigcup_{i \leq k-1} i^\frown \widetilde{\Lambda}^{i-1} \right) \cup \left(\bigcup_{\omega > i \geq k} i^\frown \widetilde{\Lambda}^{k-2} \right);$$

$$(6.2) \quad \widetilde{\Lambda}^\omega = \{\omega\} \cup \left(\bigcup_{i \leq k-1} i^\frown \widetilde{\Lambda}^{i-1} \right) \cup \left(\bigcup_{\omega > i \geq k} i^\frown (k-1)^\frown \widetilde{\Lambda}^{k-2} \right)$$
$$\cup \left(\bigcup_{\omega > i \geq k} \left(\bigcup_{j \neq k-1} i^\frown j^\frown \widetilde{\Lambda}^{(i-1)_j} \cup \{(i, \omega)\} \right) \right).$$

Now define $R_k \colon \widetilde{\Lambda}^\omega \to k^\frown \widetilde{\Lambda}^{k-1}$ by

$$R_k(\xi) := \begin{cases} k^\frown i^\frown \eta & \text{if } \xi = i^\frown \eta, \ i \leq k-1, \\ k^\frown i^\frown \eta & \text{if } \xi = i^\frown (k-1)^\frown \eta, \ k \leq i < \omega, \\ k^\frown i^\frown \omega & \text{if } \xi = i^\frown \eta, \ k \leq i < \omega, \ \eta_1 \neq k-1, \\ k^\frown \omega & \text{if } \xi = \omega. \end{cases}$$

By (6.1) and (6.2) we see that the map R_k is well defined and $R_k(\widetilde{\Lambda}^\omega) = k^\frown \widetilde{\Lambda}^{k-1}$. Now define the map $g_k \colon X \to X_k$ in the following way: if $\xi \in \widetilde{\Lambda}^\omega$ is of the form

$\xi = i^\frown\eta$ for $i \leq k-1$ or $\xi = i^\frown(k-1)^\frown\eta$ for $\omega > i \geq k$ or $\xi = \omega$, then let $g_k|_{X_\xi}$ be a similarity transformation of X_ξ onto $X_{R_k(\xi)}$ so that, if ξ ends with ω, then $g_k(x_\xi) = x_{R(\xi)}$ (condition (Z3) guarantees that we can choose such similitude). If $\xi \in \widetilde{\Lambda}^\omega$ is of the form $\xi = i^\frown\eta$ for $i \geq k$ and $\eta_1 \neq k-1$, then let $g_k|_{X_\xi}$ be a constant map from X_ξ to $X_{R_k(\xi)}$ such that $g_k(X_\xi) = \{x_{R(\xi)}\}$.

Clearly, $g_k(X) = X_k$. We will show that $\mathrm{Lip}(g_k) \leq \lambda_b/4$. If $\xi \in \widetilde{\Lambda}^\omega$ is of the form $\xi = i^\frown\eta$ for $i \leq k-1$ or $\xi = i^\frown(k-1)^\frown\eta$ for $i \geq k$ or $\xi = \omega$, then

- if ξ does not end with ω, then

$$\mathrm{diam}(X_\xi) = b_{l(\xi)} + b_{l(\xi)+1} \quad\text{and}\quad \mathrm{diam}(g_k(X_\xi))$$
$$= b_{l(R_k(\xi))} + b_{l(R_k(\xi))+1} \leq b_{l(\xi)+1} + b_{l(\xi)+2}.$$

- if ξ ends with ω, then letting $\widetilde{\xi}$ be such that $\xi = \widetilde{\xi}^\frown\omega$, we get

$$\mathrm{diam}(X_\xi) = b_{l(\widetilde{\xi})+1}, \quad\text{and}\quad \mathrm{diam}(g_k(X_\xi)) = b_{l(\widetilde{R_k(\xi)})+1} \leq b_{l(\widetilde{\xi})+2}.$$

If $\xi \in \widetilde{\Lambda}^\omega$ is of different form than those above, then $\mathrm{diam}(g_k(X_\xi)) = 0$. Hence $\mathrm{Lip}(g_k|X_\xi) \leq M_b \leq \lambda_b/4$ for all $\xi \in \widetilde{\Lambda}^\omega$.

Now let $x, y \in X$ be such that for some $\beta \in \Lambda^\omega$ and $1 \leq p < q \leq \omega$, $x \in X_{\beta^\frown p}$ and $y \in X_{\beta^\frown q}$, but if $q = \omega$, then we assume $y = x_{\beta^\frown\omega}$, and such that $g_k(x) \neq g_k(y)$. Then, by (3.3),

$$\frac{1}{20}\lambda_b d(x,y) \geq \frac{1}{20}\lambda_b\,\mathrm{dist}(X_{\beta^\frown p}, X_{\beta^\frown q})$$
$$\geq \frac{1}{20}\lambda_b(b_{l(\beta)+p-1} - 2b_{l(\beta)+p} - b_{l(\beta)+p+1}) \geq b_{l(\beta)+p}.$$

Now consider cases: *Case 1.* $\beta = i^\frown\eta$ for $i \leq k-1$. Then

$$d(g_k(x), g_k(y)) \leq \mathrm{diam}\left(\overline{\bigcup_{p \leq j < \omega} X_{k^\frown i^\frown\eta^\frown j}}\right)$$
$$\leq b_{k+l(\beta)+p-1} \leq b_{l(\beta)+p} \leq \frac{1}{20}\lambda_b d(x,y).$$

Case 2. $\beta = i^\frown(k-1)^\frown\eta$ for some $k \leq i < \omega$. Then

$$d(g_k(x), g_k(y)) \leq \mathrm{diam}\left(\overline{\bigcup_{p \leq j < \omega} X_{k^\frown i^\frown\eta^\frown j}}\right)$$
$$\leq b_{k+i+l(\eta)+p-1} = b_{l(\beta)+p} \leq \frac{1}{20}\lambda_b d(x,y).$$

Case 3. $\beta = i$ for $k \leq i < \omega$ and $p = k-1$ or $q = k-1$. Then

$$d(g_k(x), g_k(y)) \leq \mathrm{diam}\left(\overline{\bigcup_{1 \leq j < \omega} X_{(k,i,j)}}\right) \leq b_{k+i} \leq b_{l(\beta)+p} \leq \frac{1}{20}\lambda_b d(x,y).$$

Case 4. $\beta = \emptyset$. Then

$$d(g_k(x), g_k(y)) \leq \operatorname{diam}\left(\overline{\bigcup_{p \leq j < \omega} X_{(k,j)}}\right) \leq b_{k+p-1} \leq b_{l(\beta)+p} \leq \frac{1}{20}\lambda_b d(x, y).$$

Finally, assume that $y \in X_{\beta \frown \omega}$. By Definition 3.1 (v), Lemma 3.2 (b) and what we already proved, we have

$$d(g_k(x), g_k(y)) \leq d(g_k(x), g_k(x_{\beta \frown \omega})) + d(g_k(x_{\beta \frown \omega}), g_k(y))$$

$$\leq \frac{1}{20}\lambda_b d(x, x_{\beta \frown \omega}) + \frac{1}{20}\lambda_b d(x_{\beta \frown \omega}, y)$$

$$\leq \frac{2}{20}\lambda_b d(x, y) + \frac{3}{20}\lambda_b d(x, y) = \frac{1}{4}\lambda_b d(x, y).$$

Now, by Lemma 3.6 and Remark 3.7, there is a map $F \colon X \times X \to X$ such that $\operatorname{Lip}(F) \leq \lambda_b/2$ and

$$F(X \times X) = \{x_\omega\} \cup \bigcup_{1 \leq k < \omega} g_k(X) = \{x_\omega\} \cup \bigcup_{2 \leq k < \omega} X_k.$$

Since Z is the attractor of a Banach GIFS $\mathcal{F}$ of order 2 and X_1 and X_ω are similarity copy of Z, there are GIFSs $\mathcal{F}_1 = \{f_1, \ldots, f_n\}$ and $\mathcal{F}_\omega = \{g_1, \ldots, g_n\}$ on X_1 and X_ω, respectively, of order 2, with $\operatorname{Lip}(\mathcal{F}_1), \operatorname{Lip}(\mathcal{F}_\omega) = \operatorname{Lip}(\mathcal{F})$. Observe that the maps $\pi_1 \colon X \to X_1$ and $\pi_\omega \colon X \to X_\omega$ given by

$$\pi_1(x) := \begin{cases} x & \text{if } x \in X_1, \\ x_{(1,\omega)} & \text{if } x \notin X_1, \end{cases} \qquad \pi_\omega(x) := \begin{cases} x & \text{if } x \in X_\omega, \\ x_\omega & \text{if } x \notin X_\omega. \end{cases}$$

It is routine to check that $\operatorname{Lip}(\pi_1), \operatorname{Lip}(\pi_\omega) \leq 3$ (we use Lemma 3.2 (b)). Finally, define

$$f'_k(x, y) := f_k(\pi_1(x), \pi_1(y)) \quad \text{and} \quad g'_k(x, y) := g_k(\pi_\omega(x), \pi_\omega(y))$$

and observe that $\mathcal{H} = \{f'_1, \ldots, f'_k, g'_1, \ldots, g'_k, F\}$ is a Banach GIFS with $\operatorname{Lip}(\mathcal{H}) \leq \max\{3\operatorname{Lip}(\mathcal{F}), \lambda_b/2\}$ and X is its attractor. This ends the proof of (a).

Now we prove (b). It is easy to see that X/R_c is a scattered space with the height ω. By the result of Nowak and Theorem 2.7, X is not a topological fractal. $\qquad\square$

EXAMPLE 6.3. Theorem 6.1 gives us a way of constructing many mutually nonhomeomorphic GIFSs fractals which are not topological fractals. Indeed, if Z_1, Z_2 are not homeomorphic, then spaces which have all components homeomorphic with Z_1 and Z_2, respectively, are not homeomorphic.

For example, for Z we can take any cube $I_n = [0, 1]^n$. Thanks to Lemma 2.2, we obtain in this way fractals of GIFSs defined on the whole Euclidean spaces $\mathbb{R}^n$.

A slight modification of the Hilbert cube leads to an example in ℓ^∞ space. Indeed, let

$$Z := [0,1] \times \left[0, \frac{1}{4}\right] \times \left[0, \frac{1}{4^2}\right] \times \ldots$$

Then it is easy to see that Z is the attractor of the GIFS $\mathcal{F} = \{f_i : i = 0,1,2,3\}$, where

$$f_i((x_k),(y_k)) := \left(\frac{1}{4}\, x_1 + \frac{i}{4}, \frac{1}{4}\, y_1, \frac{1}{4}\, y_2, \ldots\right),$$

and $\mathrm{Lip}(\mathcal{F}) = 1/4$. Hence by Theorem 6.1 and Remark 2.3, there is $X \subset \ell_\infty$ whose all connected components are similarity copy of Z, which is not a topological fractal and which is the attractor of some GIFS $\mathcal{G}$ on ℓ^∞ with $\mathrm{Lip}(\mathcal{G}) \leq 3/4$. Similarly, starting with

$$Z := [0,1] \times \left[0, \frac{1}{8}\right] \times \left[0, \frac{1}{8^2}\right] \times \ldots$$

and using Theorem 6.1 and Lemma 2.2, we obtain appropriate example in the Hilbert space ℓ^2.

D'Aniello in [9] showed that for any $n \in \mathbb{N}$ and $0 < s \leq n$, there is a set $A \subset \mathbb{R}^n$ whose Hausdorff dimension $\dim_H(A) = s$ and which is not the fractal generated by any weak IFS on $\mathbb{R}^n$. However, the sets she constructed were certain Cantor sets, hence topological fractals. Theorem 6.2 implies the following:

COROLLARY 6.4. *If* $1 \leq s \leq n$, *then there is a set* $A \subset \mathbb{R}^n$ *such that* $\dim_H(A) = s$ *and which is not a topological fractal but is the attractor of some Banach GIFS on* $\mathbb{R}^n$ *of order* 2.

PROOF. It is enough to take as Z a connected IFS fractal of Hausdorff dimension equal to s.   □

REMARK 6.5. It is easy to see that justification of Theorem 6.2 (b) is more general — if X is a (Λ, b, Z)-space and Z is connected, then the quotient space X/R_c is homeomorphic with (Λ, b, s)-space. Hence, for any connected space Z and any countable α with limit height, the space (Λ, b, Z)-space is not a topological fractal.

In a mentioned paper [12], the authors independently obtained similar conclusion (only for subsets of the real line).

7. Topological properties of class of GIFSs' attractors

It is well known (see for example [1] and [11]) that (in reasonable metric spaces X) the class of attractors of weak IFSs is meager in $\mathcal{K}(X)$. In this section we extend this result in some sense — we show that the class of all attractors of *Banach* GIFSs on a Hilbert space H is meager.

Given a Hilbert space H and $m \in \mathbb{N}$, let

$$\mathcal{A}^m(H) := \{K \in \mathcal{K}(H) : K \text{ is the attractor of a Banach GIFS of order } m\},$$

$$\mathcal{A}^m_w(H) := \{K \in \mathcal{K}(H) : K \text{ is the attractor of a weak GIFS of order } m\},$$

where $\mathcal{K}(H)$ is the family of all nonempty and compact subsets of H. We consider it as a metric space with the Hausdorff metric h.

Finally, define

$$\mathcal{A}(H) := \bigcup_{m \in \mathbb{N}} \mathcal{A}^m(H).$$

That is, $\mathcal{A}^m(H)$ and $\mathcal{A}^m_w(H)$ are classes of all attractors of Banach and weak GIFSs of order m, respectively, and $\mathcal{A}(H)$ is the class of all attractors of Banach GIFSs .

THEOREM 7.1. *In the above frame*:

(a) *The set $\mathcal{A}(H)$ is meager F_σ in $\mathcal{K}(H)$ and, in particular, typical compact subset of H is not the attrator of any Banach* GIFS.

(b) *For every $m \in \mathbb{N}$, the set $\mathcal{A}^{m+1}(H) \setminus \mathcal{A}^m_w(H)$ is dense in $\mathcal{K}(H)$.*

PROOF. We first prove (b). Let $K \in \mathcal{K}(H)$ and fix $\varepsilon > 0$. Then choose a finite set $K' = \{x_1, \ldots, x_k\}$ so that $h(K, K') < \varepsilon/2$, where h is the Hausdorff metric on $\mathcal{K}(X)$. Identifying a fixed line in H with $\mathbb{R}$, by Theorem 5.1, we can find a set $Y \in \mathcal{A}^{m+1}(H) \setminus \mathcal{A}^m_w(H)$. Then we replace each point x_i in K' by appropriately small copy Y_i of Y so that $\mathrm{diam}(Y_i) < \varepsilon/2$ and $\mathrm{dist}(Y_i, Y_j) \geq \mathrm{diam}(Y_l)$ for $i \neq j$ and l. Then, denoting by $K'' := \bigcup_{i=1}^{k} Y_i$, we have

$$h(K, K'') \leq h(K, K') + h(K', K'') < \varepsilon,$$

and by Lemmas 2.4, 2.6 and Remark 2.5, $K'' \in \mathcal{A}^{m+1}(H) \setminus \mathcal{A}^m_w(H)$.

Now we prove (a). For every $n, m \in \mathbb{N}$ and $\alpha < 1$, denote by $\mathcal{A}^m_{n,\alpha}(H)$ the family of all attractors of Banach GIFSs $\mathcal{F}$ of order m consisting of $\leq n$ maps and such that $\mathrm{Lip}(\mathcal{F}) \leq \alpha$. Clearly,

$$\mathcal{A}(H) = \bigcup_{n,m,i \in \mathbb{N}} \mathcal{A}^m_{n,(i-1)/i}(H).$$

By (b), sets $\mathcal{A}^m_{n,\alpha}(H)$ have empty interior. Thus it is enough to prove that each $\mathcal{A}^m_{n,\alpha}(X)$ is closed. We will mimic the proof of [11, Proposition 3.6]. Choose a sequence $(K_k) \subset \mathcal{A}^m_{n,\alpha}(X)$ which converges to some compact set K. Calculating, if needed, the same functions more than once, we may assume that for every $k \in \mathbb{N}$, there are maps $f_1^k, \ldots, f_n^k \colon (K_k)^m \to K_k$ so that

$$\mathrm{Lip}(f_i^k) \leq \alpha \quad \text{and} \quad K_k = \bigcup_{i=1}^{n} f_i^n((K_k)^m).$$

Now let C be the closed convex hull of the compact set $K \cup \bigcup_{k \in \mathbb{N}} K_k$. Then C is compact and convex subset of a Hilbert space, so there exists a retraction $r \colon X \to C$ with $\mathrm{Lip}(r) = 1$ ($r(x)$ can be the nearest point to x in C). Using the Kirszbraun–Valentine theorem, extend each f_i^k to a map $\overline{f}_i^k \colon H^m \to H$, with the condition $\mathrm{Lip}(\overline{f}_i^k) \leq \sqrt{m}$ (see the proof of Lemma 2.2). Finally, set $\widetilde{f}_i^k := \left(r \circ \overline{f}_i^k\right)\big|_{C^m}$. In particular, $\widetilde{f}_i^k \colon C^m \to C$. Now since C is compact and $\mathrm{Lip}\left(\widetilde{f}_i^k\right) \leq \sqrt{m}$, for every $i = 1, \ldots, n$, the sequence $\left(\widetilde{f}_i^k\right)_{k \in \mathbb{N}}$ satisfies the assumptions of the Arzelá–Ascoli theorem. Hence there is a subsequence (k_j) (which can be appropriate for all i) and a map $f_i \colon C^m \to H$ such that $\left(\widetilde{f}_i^{k_j}\right)$ converges uniformly to f_i. For simplicity of notation, we assume that $k_j = j$.

To end the proof, it is enough to show that $\mathrm{Lip}(f_i|_K) \leq \alpha$ and $K = \bigcup_{i=1}^{n} f_i(K^m)$. To see the first assertion, take $x, y \in K^m$. As $(K_j)^m \to K^m$, we can find sequences $(x_j), (y_j)$ such that $x_j, y_j \in (K_j)^m$, $x_j \to x$ and $y_j \to y$. As $f_i^j(x_j) = \widetilde{f}_i^j(x_j) \to f_i(x)$ and $f_i^j(y_j) \to f_i(y)$, we have

$$d(f_i(x), f_i(y)) = \lim_{j \to \infty} d\left(f_i^j(x_j), f_i^j(y_j)\right) \leq \lim_{j \to \infty} \alpha d_m(x_j, y_j) = \alpha d_m(x, y).$$

Hence $\mathrm{Lip}(f_i) \leq \alpha$. In a similar way we can show that $f_i^j((K_j)^m) \to f_i(K^m)$, so

$$K_j = \bigcup_{i=1}^{n} f_i^j((K_j)^m) \to \bigcup_{i=1}^{n} f_i(K^m).$$

On the other hand, $K_j \to K$, and hence $\bigcup_{i=1}^{n} f_i(K^m) = K$. $\qquad\square$

In view of mentioned results from [1] and [11], it is natural to ask is the set of all attractors of weak GIFSs is meager in $\mathcal{K}(X)$. We leave it as an open problem:

PROBLEM 7.2. Let $\mathcal{A}_w(X)$ be the family of all sets $K \in \mathcal{K}(X)$ which are attractors of some (possibly weak) GIFSs. Is $\mathcal{A}_w(X)$ meager in $\mathcal{K}(X)$?

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