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# MULTIPLICITY OF POSITIVE SOLUTIONS FOR FRACTIONAL LAPLACIAN EQUATIONS INVOLVING CRITICAL NONLINEARITY

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Abstract. In this paper, we consider the following problem involving fractional Laplacian operator

$$(-\Delta)^s u = \lambda f(x)|u|^{q-2}u + |u|^{2_s^*-2}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $0 < s < 1$, $2_s^* = 2N/(N-2s)$, and $(-\Delta)^s$ is the fractional Laplacian. We will prove that there exists $\lambda_* > 0$ such that the problem has at least two positive solutions for each $\lambda \in (0, \lambda_*)$. In addition, the concentration behavior of the solutions are investigated.

## 1. Introduction

In this paper, we consider the following problem with the fractional Laplacian:

$$\begin{cases} (-\Delta)^s u = \lambda f(x)|u|^{q-2}u + |u|^{2_s^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N > 2s$, $0 < s < 1$, $1 < q < 2$, $\lambda > 0$, $2_s^* := 2N/(N-2s)$ is the critical exponent in fractional Sobolev inequalities, and $f\colon \Omega \to \mathbb{R}$ is a continuous function with $f^+(x) = \max\{f(x), 0\} \neq 0$ on $\Omega$, and $f \in L^{2_s^*/(2_s^*-q)}(\Omega)$. From the assumptions on $f$ and $q$, we know that

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the problem (1.1) involving the concave-convex nonlinearities and sign-changing weight function.

In a bounded domain $\Omega \subset \mathbb{R}^N$, we define the operator $(-\Delta)^s$ as follows. Let $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ be the eigenvalues and eigenfunctions of the Laplacian operator $-\Delta$ in $\Omega$ with zero Dirichlet boundary values on $\partial\Omega$ normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega; \qquad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

For any $u \in L^2(\Omega)$, we may write

$$u = \sum_{k=1}^{\infty} u_k\varphi_k, \quad \text{where } u_k = \int_{\Omega} u\varphi_k\, dx.$$

We define the space

$$\mathbb{H}^s(\Omega) = \bigg\{u = \sum_{k=1}^{\infty} u_k\varphi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} u_k^2\lambda_k^s < \infty\bigg\}, \tag{1.2}$$

which is equipped with the norm

$$\|u\|_{\mathbb{H}^s(\Omega)} = \bigg(\sum_{k=1}^{\infty} u_k^2\lambda_k^s\bigg)^{1/2}.$$

For any $u \in \mathbb{H}^s(\Omega)$, the spectral fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u = \sum_{k=1}^{\infty} \lambda_k^s u_k\varphi_k. \tag{1.3}$$

We wish to point out that a different notion of fractional Laplacian, available in the literature, is given by

$$(-\Delta)^s u(x) = c_{N,s}\text{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy, \quad \text{for all } x \in \Omega, \tag{1.4}$$

where $\Omega$ is bounded, $c_{N,s}$ is a normalization constant and P.V. stands for the principle value. This is also called the integral fractional Laplacian. This definition, in bounded domains, is really different from the spectral one. In the case of the integral notion, due to the strong nonlocal character of the operator, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$. We point out that we adopt in the paper the spectral definition of the fractional Laplacian in a bounded domain based upon a Caffarelli–Silvestre type extension (see [3], [8] and [7]), and not the integral definition. We shall refer to [21] for a nice comparison between these two different notions.

With this definition (1.3), we see that problem (1.1) with $f(x) = 1$ and $q = 2$ is the Brézis–Nirenberg type problem with the fractional Laplacian. Such a problem involves the fractional critical Sobolev exponent $2_s^* = 2N/(N-2s)$ for $N > 2s$, and it is well known that the Sobolev embedding $\mathbb{H}^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$ is not compact even if $\Omega$ is bounded. Hence, the associated functional of problem (1.1) does not satisfy the Palais–Smale ((PS) for short) condition, and critical point theory cannot be applied directly to find solutions of this problem. However, for the classical Laplace problem, $s = 1$, Brézis–Nirenberg in [5] proved that the functional satisfies the local $(\mathrm{PS})_c$ condition for $c \in (0, S^{N/2}/N)$, where $S$ is the best Sobolev constant and $S^{N/2}/N$ is the least energy level at which the (PS)-condition fails. So a solution can be found if the mountain pass value is strictly less than $S^{N/2}/N$ . Using this methods, Brändle, Colorado, de Pablo and Sánchez [3], [4] considered (1.1) with $f(x) = 1$, and the existence of nontrivial solution was proved. In this paper, we will investigate the existence and multiplicity of positive solutions for problem (1.1) with the concave-convex nonlinearities and sign-changing weight function.

Recently, Caffarelli and Silvestre [8] developed a local interpretation of the fractional Laplacian given in $\mathbb{R}^N$ by considering a Dirichlet to Neumann type operator in the domain $\{(x, y) \in \mathbb{R}^{N+1} : y > 0\}$. A similar extension, in a bounded domain with zero Dirichlet boundary condition, was established by Cabré ans Tan [7], Tan [24], Brändle, Colorado, de Pablo and Sánchez [3], [4]. For any $u \in \mathbb{H}^s(\Omega)$, the solution $U \in H^1_{0,L}(\mathcal{C}_\Omega)$ of

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega = \Omega \times (0,\infty), \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ U = u & \text{on } \Omega \times \{0\}, \end{cases} \tag{1.5}$$

is called the $s$-harmonic extension $U = E_s(u)$, and it belongs to the space

$$H^1_{0,L}(\mathcal{C}_\Omega) = \left\{ U \in L^2(\mathcal{C}_\Omega) : U = 0 \text{ on } \partial_L \mathcal{C}_\Omega \text{ and } \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla U|^2 \,dx\,dy < \infty \right\}.$$

It is proved [4] that

$$-k_s \lim_{y\to 0^+} y^{1-2s} \frac{\partial U}{\partial y}(x, y) = (-\Delta)^s u(x),$$

where $k_s = 2^{1-2s}\Gamma(1 - s)/\Gamma(s)$ is a normalization constant. Here $H^1_{0,L}(\mathcal{C}_\Omega)$ is a Hilbert space endowed with the norm

$$\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)} = \left( k_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla U|^2 \,dx\,dy \right)^{1/2}.$$

With this extension, the nonlocal problem (1.1) can be reformulated to the following local problem:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ k_s y^{1-2s} \dfrac{\partial U}{\partial \nu} = \lambda f(x) |U(x,0)|^{q-2} U(x,0) + |U(x,0)|^{2^*_s - 2} U(x,0) & \\ & \text{on } \Omega, \end{cases} \tag{1.6}$$

here the outward normal derivative should be understood as

$$y^{1-2s}\frac{\partial U}{\partial \nu} = -\lim_{y\to 0^+} y^{1-2s}\frac{\partial U}{\partial y}.$$

The energy functional $I\colon H^1_{0,L}(\mathcal{C}_\Omega)\to\mathbb{R}$ associated to problem (1.6) is defined by

$$I(U) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{\lambda}{q}\int_\Omega f(x)|U(x,0)|^q\,dx - \frac{1}{2^*_s}\int_\Omega |U(x,0)|^{2^*_s}\,dx.$$

In view of the hypotheses on $f$, $I$ is well-defined and $I\in C^1(H^1_{0,L}(\mathcal{C}_\Omega),\mathbb{R})$. Its derivative is given by

$$\begin{aligned}\langle I'(U),V\rangle = k_s&\int_{\mathcal{C}_\Omega} y^{1-2s}\nabla U\cdot\nabla V\,dx\,dy \\ &- \lambda\int_\Omega f(x)|U(x,0)|^{q-2}U(x,0)V(x,0)\,dx \\ &\qquad - \int_\Omega |U(x,0)|^{2^*_s-2}U(x,0)V(x,0)\,dx,\end{aligned}$$

for all $U,V\in H^1_{0,L}(\mathcal{C}_\Omega)$. Hence, the solutions of problem (1.6) are the critical points of the energy functional $I$. By the argument as above, if $U\in H^1_{0,L}(\mathcal{C}_\Omega)$ is a weak solution of problem (1.6), then $u=U(x,0)$, defined in the sense of traces, belong to the space $\mathbb{H}^s(\Omega)$ and it is a weak solution of original problem (1.1). The converse is also right.

The main purpose of this paper is to generalize the partial results of [3] to the problem involving sign-changing weight function. Using the variational methods and the Nehari manifold decomposition, we first prove that the problem (1.6) has at least two positive solutions for $\lambda$ sufficiently small.

The following existence result will be obtained.

Theorem 1.1. *There exists $\lambda_*>0$ such that for all $\lambda\in(0,\lambda_*)$, the problem* (1.6) *has at least two positive solutions.*

For any function $W$ defined on $\mathbb{R}^{N+1}_+$, $x\in\mathbb{R}^N$, $\sigma>0$, we define

$$\rho_{x,\sigma}(W) = \sigma^{(N-2s)/2}W(\sigma(\,\cdot\,-(x,0))).$$

As for the asymptotic behavior of the solutions obtained in Theorem 1.1 as $\lambda\to 0$, we have the following result.

Theorem 1.2. *Assume that a sequence $\{\lambda_n\}$ satisfies $\lambda_n>0$ and*

$$\lambda_n\to 0 \quad \text{as } n\to\infty.$$

*Then there exist a subsequence $\{\lambda_n\}$ and two sequences $\{U_{i,n}\}\subset H^1_{0,L}(\mathcal{C}_\Omega)$ ($i=1,2$) of positive solutions of problem* (1.6) *such that*

(a) $\|U_{1,n}\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\to 0$ *as* $n\to\infty$;

(b) *There exist two sequence* $\{x_n\} \subset \Omega$, $\{\sigma_n\} \subset (0,\infty)$ *and a positive solution* $W \in H^1_{0,L}(\mathbb{R}^{N+1}_+)$ *of critical problem*

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ k_s y^{1-2s}\dfrac{\partial U}{\partial y} = |U(x,0)|^{2^*_s-2}U(x,0) & \text{on } \mathbb{R}^N, \end{cases} \tag{1.7}$$

*such that* $\sigma_n \to +\infty$ *as* $n \to +\infty$ *and*

$$\|U_{2,n} - \rho_{x_n,\sigma_n}(W)\|_{H^1_{0,L}(\mathbb{R}^{N+1}_+)} \to 0 \quad \text{as } n \to \infty.$$

Before concluding this introduction, we would like to mention some related important results to fractional Laplace problem, such as in [3], [4], [6]–[8], [10], [11], [19], [20], [22], [24], [28] and the references therein. Caffarelli and Silvestre [8] gave a new formulation of the fractional Laplacian through Dirichlet–Neumann maps. This is commonly used in the recent literature since it allows us to write nonlocal problems in a local way and this permits us to use the variational methods for those kinds of problems. In [7], Cabré and Tan defined the operator of the square root of Laplacian through the spectral decomposition of the Laplacian operator on $\Omega$ with zero Dirichlet boundary conditions. With classical local techniques, they established existence of positive solutions for problems with subcritical nonlinearities, regularity and $L^\infty$-estimate of Brezis–Kato type for weak solutions.

Chi, Kim and Lee [10] studied the asymptotic behavior of least energy solutions and the existence of multiple bubbling solutions of nonlinear elliptic equations involving the fractional Laplacian and the critical exponents. Zhang and Liu [28] have investigated the existence and multiplicity of solutions to the fractional laplacian elliptic problem involving critical and supercritical Sobolev exponent. In [11], the authors took into account the singularly perturbed nonlinear Schrödinger equation in $\mathbb{R}^N$. Employing the non-degeneracy result of [14], they deduced the existence of various types of spike solution such that each of local maxima concentrates on a critical points of $V$.

In [19], [20], the Brézis–Nirenberg problem is also considered when the fractional Laplace operator is given by (1.4). In particular, Felmer, Quaas and Tan in [13] show that: for every $U \in H^1_{0,L}(\mathbb{R}^{N+1}_+)$, it holds that

$$S(s,N)\bigg(\int_{\mathbb{R}^N} |U(x,0)|^{2N/(N-2s)}\bigg)^{(N-2s)/N} \leq \int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla U|^2\,dx\,dy. \tag{1.8}$$

The best constant takes the exact value

$$S(s,N) = \frac{2\,\pi^s\Gamma(1-s)\Gamma\Big(\dfrac{N+2s}{2}\Big)\Big(\Gamma\Big(\dfrac{N}{2}\Big)\Big)^{2s/N}}{\Gamma(s)\Gamma\Big(\dfrac{N-2s}{2}\Big)(\Gamma(N))^s}$$

and can be achieved when $U_\varepsilon = E_s(u_\varepsilon)$ takes the form

$$u_\varepsilon(x) = \frac{\varepsilon^{(N-2s)/2}}{(\varepsilon^2 + |x|^2)^{(N-2s)/2}}, \quad \varepsilon > 0. \tag{1.9}$$

The paper is organized as follows. In Section 2, we introduce the variational setting of the problem and present some preliminary results. In Section 3, some properties of the fractional operator are discussed, and we given the proof of Theorem 1.1. Finally, the proof of Theorem 1.2 is given in Section 4.

In the end of this section, we fix some notations that will be used in the sequel.

**Notations.**

- $L^p(\Omega)$, $1 < p \leq \infty$, denote Lebesgue spaces with norm $|\cdot|_p$.
- The dual space of a Banach space $E$ will be denoted by $E^{-1}$.
- $|\Omega|$ is the Lebesgue measure of $\Omega$. $B_r(x)$ is the ball at $x$ with radius $r$.
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- $C$, $C_i$, $c_i$ $(i = 1, 2, \ldots)$ will denote various positive constants which may vary from line to line.

## 2. Preliminaries

In this section, we collect some preliminary facts in order to establish the functional setting. We refer the reader to [1], [8], [6], [12], [17] and to the reference therein.

For $s > 0$, $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : |\xi|^s \widehat{u}(\xi) \in L^2\big(\mathbb{R}^N\big)\big\},$$

where $\widehat{u}$ denots the Fourier transform of $u$, with norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \big\|\big(1 + |\xi|^{2s}\big)\widehat{u}(\xi)\big\|_{L^2(\mathbb{R}^N)}.$$

This norm is equivalent to

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \bigg(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy\bigg)^{1/2}.$$

Given a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and $0 < s < 1$, the space $H^s(\Omega)$ is defined as the set of functions $u \in L^2(\Omega)$ for which the following norm is finite

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + \bigg(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy\bigg)^{1/2}.$$

An equivalent construction consists of restrictions of functions in $H^s(\mathbb{R}^N)$. We define $H_0^s(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. It is well-known from that for $0 < s \leq 1/2$, $H_0^s(\Omega) = H^s(\Omega)$, which for $1/2 < s < 1$ the inclusion $H_0^s(\Omega) \subseteq H^s(\Omega)$ is strict.

The space $\mathbb{H}^s(\Omega)$ define (1.2) is the interpolation space $(H_0^2(\Omega), L^2(\Omega))_{s,2}$, see [1], [16]. It was shown in [16] that $(H_0^2(\Omega), L^2(\Omega))_{s,2} = H_0^s(\Omega)$ for $0 < s < 1$, $s \neq 1/2$, while

$$\big(H_0^2(\Omega), L^2(\Omega)\big)_{1/2,2} = H_{0,0}^{1/2}(\Omega)$$

where

$$H_{0,0}^{1/2}(\Omega) = \bigg\{u \in H^{1/2}(\Omega) : \int_\Omega \frac{u(x)^2}{d(x)}\,dx < +\infty\bigg\},$$

and $d(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

An important feature of the operator $(-\Delta)^s$ is its nonlocal character, which is best seen by realizing the fractional Laplacian as the boundary operator of suitable existence in the half-cylinder $\Omega \times (0, +\infty)$. Such an interpretation was demonstrated in [8] for the fractional Laplacian in $\mathbb{R}^N$. Their construction can easily be extended to the case of bounded domains as described below.

Let us define

$$\mathcal{C}_\Omega = \Omega \times (0, +\infty), \qquad \partial_L \mathcal{C}_\Omega = \partial\Omega \times [0, +\infty).$$

We know from [4], see also [23], that for any $u \in H_0^s(\Omega)$, letting $U \in H_{0,L}^1(\mathcal{C}_\Omega)$ be the extension of $u$ defined in (1.5), then the mapping $u \mapsto U$ is an isometry between $H_0^s(\Omega)$ and $H_{0,L}^1(\mathcal{C}_\Omega)$, that is,

$$\|u\|_{H_0^s(\Omega)} = \|U\|_{H_{0,L}^1(\mathcal{C}_\Omega)}, \quad \text{for all } u \in H_0^s(\Omega).$$

Now we are looking for the solutions of problem (1.6). First we consider the Nehari minimization problem, i.e. for $\lambda > 0$,

$$m_I = \inf\{I(U) : U \in \mathcal{N}\}, \quad \text{where } \mathcal{N} = \big\{U \in H_{0,L}^1(\mathcal{C}_\Omega)\backslash\{0\} : \langle I'(U), U\rangle = 0\big\}.$$

Define

$$\Psi(U) = \langle I'(U), U\rangle = k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ - \lambda \int_\Omega f(x)|U(x,0)|^q\,dx - \int_\Omega |U(x,0)|^{2_s^*}\,dx.$$

Then, for any $U \in \mathcal{N}$,

$$\langle \Psi'(U), U\rangle = 2k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ - \lambda\, q \int_\Omega f(x)|U(x,0)|^q\,dx - 2_s^* \int_\Omega |U(x,0)|^{2_s^*}\,dx.$$

Similarly to the method used in [26] and [27], we split $\mathcal{N}$ into three parts:

$$\mathcal{N}^+ = \{U \in \mathcal{N} : \langle \Psi'(U), U\rangle > 0\};$$
$$\mathcal{N}^0 = \{U \in \mathcal{N} : \langle \Psi'(U), U\rangle = 0\};$$
$$\mathcal{N}^- = \{U \in \mathcal{N} : \langle \Psi'(U), U\rangle < 0\}.$$

Then we have the following results.

Lemma 2.1. *Let $\theta := 2_s^*/(2_s^* - q)$ and*

$$\lambda_1 = \left(\frac{2_s^* - 2}{2_s^* - q}\right)\left(\frac{2-q}{2_s^* - q}\right)^{(2-q)/(2_s^*-q)} (k_s S(s,N))^{(2_s^*-q)/(2_s^*-2)} |f|_\theta^{-1}.$$

*Then, for every $U \in H_{0,L}^1(\mathcal{C}_\Omega)$, $U \neq 0$ and $\lambda \in (0, \lambda_1)$, there exist unique $t^+(U)$ and $t^-(U)$ such that:*

(a) $0 \le t^+(U) < t_{\max} = \left(\dfrac{(2-q)k_s \displaystyle\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy}{(2_s^*-q)\displaystyle\int_\Omega |U(x,0)|^{2_s^*}\,dx}\right)^{1/(2_s^*-2)} < t^-(U);$

(b) $t^-(U)U \in \mathcal{N}^-$ *and* $t^+(U)U \in \mathcal{N}^+$;

(c) $\mathcal{N}^- = \left\{U \in H_{0,L}^1(\mathcal{C}_\Omega) \setminus \{0\} : t^-\left(\dfrac{U}{\|U\|_{H_{0,L}^1(\mathcal{C}_\Omega)}}\right) = \|U\|_{H_{0,L}^1(\mathcal{C}_\Omega)}\right\}$;

(d) $I(t^-U) = \max\limits_{t \ge t_{\max}} I(tU)$ *and* $I(t^+U) = \min\limits_{t \in [0,t^-]} I(tU)$.

*Moreover, $t^+(U) > 0$ if and only if*

$$\int_\Omega f(x)|U(x,0)|^q\,dx > 0.$$

Proof. The proof is almost the same as that in [26]. We need only to define

$$g(t) = t^{2-q}k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - t^{2_s^*-q}\int_\Omega |U(x,0)|^{2_s^*}\,dx.$$

Thus, we omit the details here. □

Lemma 2.2. *There exists $\lambda_2 > 0$ such that $\mathcal{N}^0 = \{0\}$ for each $\lambda \in (0, \lambda_2)$.*

Proof. Suppose the contrary, there exists a $U \in \mathcal{N}^0 \setminus \{0\}$, such that

$$\langle \Psi'(U), U\rangle = 0. \tag{2.1}$$

Then, we consider the following two cases.

*Case 1.* $\int_\Omega f(x)|U(x,0)|^q\,dx = 0$. Thus

$$\begin{aligned}\langle \Psi'(U), U\rangle &= 2k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ &\quad - \lambda q \int_\Omega f(x)|U(x,0)|^q\,dx - \frac{2N}{N-2s}\int_\Omega |U(x,0)|^{2_s^*}\,dx \\ &= 2k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{2N}{N-2s}k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ &= -\frac{4s}{N-2s}\|U\|^2_{H_{0,L}^1(\mathcal{C}_\Omega)} < 0.\end{aligned}$$

So, in this case $U \in \mathcal{N}^-$.

*Case 2.* $\int_\Omega f(x)|U(x,0)|^q dx \neq 0$. From (2.1), we get

$$\begin{aligned}0 &= 2k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \lambda\, q \int_\Omega f(x)|U(x,0)|^q\,dx - 2_s^* \int_\Omega |U(x,0)|^{2_s^*}\,dx \\ &= (2-q)k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - (2_s^*-q)\int_\Omega |U(x,0)|^{2_s^*}\,dx,\end{aligned}$$

which implies that

$$\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = \frac{2_s^*-q}{2-q}\int_\Omega |U(x,0)|^{2_s^*}\,dx. \tag{2.2}$$

Thus

$$\begin{aligned}\lambda\int_\Omega f(x)|U(x,0)|^q\,dx &= k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \int_\Omega |U(x,0)|^{2_s^*}\,dx \\ &= k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{2-q}{2_s^*-q}k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ &= \frac{2_s^*-2}{2_s^*-q}\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}.\end{aligned} \tag{2.3}$$

Therefore, by (2.3) and the Hölder inequality, we obtain

$$\|U\|^{2-q}_{H^1_{0,L}(\mathcal{C}_\Omega)} \le \lambda\bigg(\frac{2_s^*-q}{2_s^*-2}\bigg)(k_s S(s,N))^{-q/2}|f|_\theta. \tag{2.4}$$

Let $K\colon \mathcal{N}\to\mathbb{R}$ be given by

$$K(U) = C\|U\|^{(2(2_s^*-1))/(2_s^*-2)}_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigg(\int_\Omega |U(x,0)|^{2_s^*}\,dx\bigg)^{1/(2-2_s^*)} - \lambda\int_\Omega f(x)|U(x,0)|^q\,dx,$$

where

$$C = \bigg(\frac{2_s^*-2}{2-q}\bigg)\bigg(\frac{2-q}{2_s^*-q}\bigg)^{(2_s^*-1)/(2_s^*-2)}.$$

Then $K(U)=0$ for all $U\in\mathcal{N}^0$. Indeed, by (2.2) and (2.3),

$$\begin{aligned}K(U) &= C\|U\|^{2(2_s^*-1)/(2_s^*-2)}_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigg(\int_\Omega |U(x,0)|^{2_s^*}\,dx\bigg)^{1/(2-2_s^*)} - \lambda\int_\Omega f(x)|U(x,0)|^q\,dx \\ &= \bigg(\frac{2_s^*-2}{2-q}\bigg)\bigg(\frac{2-q}{2_s^*-q}\bigg)^{(2_s^*-1)/(2_s^*-2)}\bigg(\frac{2_s^*-q}{2-q}\bigg)^{1/(2_s^*-2)}\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} \\ &\quad - \lambda\int_\Omega f(x)|U(x,0)|^q\,dx \\ &= \frac{2_s^*-2}{2_s^*-q}\,\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_\Omega f(x)|U(x,0)|^q\,dx = 0.\end{aligned}$$

On the other hand, from (2.3) and (2.4),

$$
\begin{aligned}
(2.5)\qquad K(U) &\geq C\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}^{2(2^*_s-1)/(2^*_s-2)}\bigg(\int_\Omega |U(x,0)|^{2^*_s}\,dx\bigg)^{1/(2-2^*_s)} \\
&\quad - \lambda|f|_\theta|U(x,0)|^q_{L^{2^*_s}} \\
&\geq C(k_s\,S(s,N))^{(N+2s)/(4s)}\bigg(\int_\Omega |U(x,0)|^{2^*_s}\,dx\bigg)^{1/(2^*_s)} \\
&\quad - \lambda|f|_\theta|U(x,0)|^q_{L^{2^*_s}} \\
&\geq \lambda|f|_\theta|U(x,0)|^q_{2^*_s}\bigg[\frac{1}{(\lambda|f|_\theta)^{1/(2-q)}}\bigg(\frac{2^*_s-2}{2^*_s-q}\bigg)^{1/(2-q)} \\
&\quad \cdot (2-q)^{(n-2s)/(4s)}(k_s\,S(s,N))^{(q-1)/(2-q)+(N+2s)/(4s)} - 1\bigg].
\end{aligned}
$$

This implies that there exists

$$
\lambda_2 := |f|_\theta^{-1}\bigg(\frac{2^*_s-2}{2^*_s-q}\bigg)(2-q)^{(N-2s)(2-q)/(4s)}(k_s\,S(s,N))^{(4s(q-1)+(N+2s)(2-q))/(4s)}
$$

such that, for each $\lambda \in (0, \lambda_2)$, we have $K(U) > 0$ for all $U \in \mathcal{N}^0 \setminus \{0\}$, which yields a contradiction. Thus, we can conclude that $\mathcal{N}^0 = \{0\}$ for all $\lambda \in (0, \lambda_2)$. □

LEMMA 2.3. *If $U \in \mathcal{N}^+$, then*

$$
\int_\Omega f(x)|U(x,0)|^q\,dx > 0.
$$

PROOF. From $U \in \mathcal{N}^+$, we have

$$
\begin{aligned}
2k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy &> \lambda q\int_\Omega f(x)|U(x,0)|^q\,dx + 2^*_s\int_\Omega |U(x,0)|^{2^*_s}\,dx \\
&= qk_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy + (2^*_s-q)\int_\Omega |U(x,0)|^{2^*_s}\,dx,
\end{aligned}
$$

that is

$$
k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy > \frac{2^*_s-q}{2-q}\int_\Omega |U(x,0)|^{2^*_s}\,dx.
$$

Then, we have

$$
\begin{aligned}
\lambda\int_\Omega f(x)|U(x,0)|^{q+1}\,dx &= k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \int_\Omega |U(x,0)|^{2^*_s}\,dx \\
&> \frac{2^*_s-2}{2-q}\int_\Omega |U(x,0)|^{2^*_s}\,dx > 0.
\end{aligned}
$$

This completes the proof. □

The following lemma shows that the minimizers on $\mathcal{N}$ are actually the critical points of functional $I$.

LEMMA 2.4. *For $\lambda \in (0,\lambda_2)$. If $U \in H^1_{0,L}(\mathcal{C}_\Omega)$ is a local minimizer for $I$ on $\mathcal{N}$, then $I'(U) = 0$ in $H^{-1}(\mathcal{C}_\Omega)$, where $H^{-1}(\mathcal{C}_\Omega)$ denotes the dual space of $H^1_{0,L}(\mathcal{C}_\Omega)$.*

PROOF. If $U_0$ is a local minimizer of $I$ on $\mathcal{N}$, then $U_0$ is a nontrivial solution of the optimization problem:

$$\text{minimize } I(U) \text{ subject to } \langle I'(U), U\rangle = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $I'(U_0) = \theta\Psi'(U_0)$ in $H^{-1}$, which implies that

$$\langle I'(U_0), U_0\rangle = \theta\langle \Psi'(U_0), U_0\rangle. \tag{2.6}$$

Then, by Lemma 2.2, for every $U_0 \neq 0$, we have $\langle \Psi'(U_0), U_0\rangle \neq 0$ and so, by (2.6), $\theta = 0$. □

LEMMA 2.5. *The functional $I$ is coercive and bounded from below on $\mathcal{N}$.*

PROOF. For $U \in \mathcal{N}$, we have

$$\begin{aligned} I(U) &= \bigg(\frac{1}{2} - \frac{1}{2^*_s}\bigg) k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \lambda\bigg(\frac{1}{q} - \frac{1}{2^*_s}\bigg)\int_\Omega f(x)|U(x,0)|^q\,dx \\ &\geq \frac{s}{N}\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\bigg(\frac{2^*_s - q}{q2^*_s}\bigg)|f|_\theta (k_s\,S(s,N))^{-q/2}\|U\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)} \\ &\geq \frac{q-2}{2}\bigg(\frac{N}{2s}\bigg)^{q/(2-q)}(\lambda C)^{2/(2-q)}, \end{aligned}$$

where

$$C = \bigg(\frac{2^*_s - q}{q2^*_s}\bigg)|f|_\theta k_s^{-q/2}\,S(s,N)^{-q/2}.$$

This tell us that $I$ is coercive and bounded from below on $\mathcal{N}$. □

In the end of this section, we will use the idea of [25] to get the property of $\mathcal{N}$.

LEMMA 2.6. *Let $\lambda \in (0,\lambda_2)$. For each $U \in \mathcal{N}\setminus\{0\}$, there exists $r > 0$ and a differentiable function $t = t(V)$ such that $t(V) > 0$ for all $V \in \big\{U \in H^1_{0,L}(\mathcal{C}_\Omega) : \|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} < \varepsilon\big\}$ satisfying*

$$t(0) = 1, \qquad t(V)(U - V) \in \mathcal{N} \quad \text{and} \quad \langle t'(0), V\rangle = \frac{A(U,V)}{B(U,U)}$$

*for all $V \in H^1_{0,L}(\mathcal{C}_\Omega)$, where*

$$\begin{aligned} A(U,V) = 2k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\nabla U\nabla V\,dx\,dy - q\lambda\int_\Omega f(x)|U(x,0)|^{q-2}U(x,0)V(x,0)\,dx \\ - 2^*_s\int_\Omega |U(x,0)|^{2^*_s-2}U(x,0)V(x,0)\,dx \end{aligned}$$

*and*

$$B(U,U) = (2-q)k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - (2_s^*-q)\int_\Omega |U(x,0)|^{2_s^*}\,dx.$$

Proof. Define $\mathscr{F}\colon \mathbb{R}\times H^1_{0,L}(\mathcal{C}_\Omega)\to\mathbb{R}$ as follows:

$$\begin{aligned}\mathscr{F}(t,V) &= \langle I'(t(U-V)), t(U-V)\rangle \\ &= t^2 k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla(U-V)|^2\,dx\,dy \\ &\quad - \lambda\, t^q \int_\Omega f(x)|U(x,0)-V(x,0)|^q\,dx - t^{2_s^*}\int_\Omega |U(x,0)-V(x,0)|^{2_s^*}\,dx,\end{aligned}$$

for all $V\in H^1_{0,L}(\mathcal{C}_\Omega)$.

Since $\mathscr{F}(1,0)=\langle I'(U),U\rangle = 0$ and by Lemma 2.2, we obtain

$$\begin{aligned}\mathscr{F}_t'(1,0) &= 2k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ &\quad - \lambda q\int_\Omega f(x)|U(x,0)|^q\,dx - 2_s^*\int_\Omega |U(x,0)|^{2_s^*}\,dx \\ &= (2-q)k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - (2_s^*-q)\int_\Omega |U(x,0)|^{2_s^*}\,dx \neq 0.\end{aligned}$$

Applying the implicit function theorem at the point $(1,0)$, we get that there exist $\varepsilon>0$ small and a function $t=t(V)$ satisfying $t(0)=1$ and

$$\langle t'(0),V\rangle = \frac{A(U,V)}{B(U,U)}.$$

Moreover, there is a $t(V)$ such that $\mathscr{F}(t(V),V)=0$ for all $V\in\{U\in H^1_{0,L}(\mathcal{C}_\Omega): \|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}<\varepsilon\}$, which is equivalent to $\langle I'(t(V)(U-V)), t(V)(U-V)\rangle = 0$, that is, $t(V)(U-V)\in\mathcal{N}$. □

## 3. Proof of Theorem 1.1

Since the energy functional $I$ associated with the problem (1.6) is not bounded on $H^1_{0,L}(\mathcal{C}_\Omega)$, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N} = \{U\in H^1_{0,L}(\mathcal{C}_\Omega)\setminus\{0\} : \langle I'(U),U\rangle = 0\}.$$

It is clear that all critical points of $I$ must lie on $\mathcal{N}$ and, as the results in Section 2, local minimizers on $\mathcal{N}$ are actually critical points of $I$.

**3.1. The minimizer solution on $\mathcal{N}^+$.** Let

$$\lambda_* = \min\{\lambda_1,\lambda_2\}. \tag{3.1}$$

By Lemmas 2.2 and 2.5, for $\lambda\in(0,\lambda_*)$, we know that $\mathcal{N}=\mathcal{N}^+\cup\mathcal{N}^-$ and $I$ is bounded from below on $\mathcal{N}$ and so on $\mathcal{N}^+$, $\mathcal{N}^-$. Therefore, we may define

$$m_I = \inf\{I(U): U\in\mathcal{N}\},$$

$$m^+ = \inf\{I(U) : U \in \mathcal{N}^+\}, \qquad m^- = \inf\{I(U) : U \in \mathcal{N}^-\}.$$

In this subsection, we will show that problem (1.6) has a position solution if $\lambda < \lambda_*$, which is the minimizer of $I$ on $\mathcal{N}^+$.

Now we consider the following auxiliary equation:

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla U) = 0 & \text{in } \mathcal{C}_\Omega, \\ U = 0 & \text{on } \partial_L \mathcal{C}_\Omega, \\ k_s y^{1-2s} \dfrac{\partial U}{\partial \nu} = \lambda f(x)|U(x,0)|^{q-2}U(x,0) & \text{on } \Omega. \end{cases} \tag{3.2}$$

In this case, we use the notation $F$ and $\mathcal{M}$ respectively, for the energy functional and the natural constrain, namely,

$$F(U) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{\lambda}{q}\int_\Omega f(x)|U(x,0)|^q\,dx,$$
$$\mathcal{M} = \big\{U \in H^1_{0,L}(\mathcal{C}_\Omega)\setminus\{0\} : \langle F'(U), U\rangle = 0\big\}.$$

Setting $m_\lambda = \inf\{F(U) : U \in \mathcal{M}\}$ we have the following result.

Theorem 3.1. *For each $\lambda > 0$, problem* (3.2) *has a positive solution $U_0$ such that $F(U_0) = m_\lambda < 0$.*

Proof. We start by showing that $F$ is coercive, bounded from below on $\mathcal{M}$ and $m_\lambda < 0$. Indeed, for any $U \in \mathcal{M}$, we have

$$\begin{aligned} k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy &= \lambda\int_\Omega f(x)|U(x,0)|^q\,dx \\ &\le \lambda|f|_\theta\,(k_sS(s,N))^{-q/2}\|U\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)}. \end{aligned} \tag{3.3}$$

This implies

$$F(U) \ge \frac{1}{2}\,\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{1}{q}\,\lambda\,|f|_\theta\big(k_sS(s,N)\big)^{-q/2}\|U\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)},$$

and therefore, we easily derive the coerciveness for $1 < q < 2$. Moreover, (3.3) implies

$$\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \le \big(\lambda|f|_\theta(k_sS(s,N))^{-q/2}\big)^{1/(2-q)}. \tag{3.4}$$

Hence, for all $U \in \mathcal{M}$,we have

$$F(U) = \left(\frac{1}{2} - \frac{1}{q}\right)\|U\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} \ge -\frac{2-q}{2q}\big(\lambda|f|_\theta\,(k_sS(s,N))^{-q/2}\big)^{2/(2-q)}.$$

So $F$ is bounded from below on $\mathcal{M}$ and $m_\lambda < 0$.

Let $\{U_n\} \subset H^1_{0,L}(\mathcal{C}_\Omega)$ be a minimizing sequence of $F$ on $\mathcal{M}$. Then, by (3.4) and the compact imbedding theorem, there exists a subsequence of $\{U_n\}$, still denoted by $\{U_n\}$, and $U_0$ such that

$$\begin{aligned} U_n &\rightharpoonup U_0 && \text{weakly in } H^1_{0,L}(\mathcal{C}_\Omega); \\ U_n(\,\cdot\,,0) &\to U_0(\,\cdot\,,0) && \text{strongly in } L^p(\Omega) \text{ for } 1 < p < 2^*_s; \\ U_n(\,\cdot\,,0) &\to U_0(\,\cdot\,,0) && \text{a.e. in } \Omega. \end{aligned} \tag{3.5}$$

Now, we claim that

$$\int_\Omega f(x)|U_0(x,0)|^q\,dx > 0.$$

If not, by (3.5) we obtain

$$\int_\Omega f(x)|U_0(x,0)|^q\,dx = 0 \quad \text{and} \quad \int_\Omega f(x)|U_n(x,0)|^q\,dx \to 0 \quad \text{as } n\to\infty.$$

Hence

$$\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_n|^2\,dx\,dy \to 0 \quad \text{and} \quad F(U_n)\to 0 \quad \text{as } n\to\infty$$

which contradicts $F(U_n)\to m_\lambda < 0$ as $n\to\infty$. Therefore, we have

$$\int_\Omega f(x)|U_0(x,0)|^q\,dx > 0.$$

In particular $U_0 \not\equiv 0$ in $\Omega$.

Next, we prove $U_n \to U_0$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$. Let us suppose on the contrary that

$$\|U_0\|_{H^1_{0,L}(\mathcal{C}_\Omega)} < \liminf_{n\to\infty}\|U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \quad \text{as } n\to\infty$$

and

$$\int_\Omega f(x)|U_n(x,0)|^q\,dx \to \int_\Omega f(x)|U_0(x,0)|^q\,dx \quad \text{as } n\to\infty.$$

So

$$\begin{aligned} &\|U_0\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_\Omega f(x)|U_0(x,0)|^q\,dx \\ &\qquad < \liminf_{n\to\infty}\bigg(\|U_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\int_\Omega f(x)|U_n(x,0)|^q\,dx\bigg) = 0. \end{aligned} \tag{3.6}$$

On the other hand, from $\int_\Omega f(x)|U_0(x,0)|^q\,dx > 0$ and (3.6), we known that the function

$$F(t\,U_0) = \frac{t^2}{2}\,k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_0|^2\,dx\,dy - \frac{\lambda\,t^q}{q}\int_\Omega f(x)|U_0(x,0)|^q\,dx$$

is initially decreasing and eventually increasing on $t$ with a single turning point $t_0 \neq 1$ such that $t_0U_0 \in \mathcal{M}$. Thus, from $t_0U_n \rightharpoonup t_0U_0$ and (3.6) we get that

$$F(t_0U_0) < F(U_0) < \liminf_{n\to\infty} F(U_n) = m_\lambda$$

which is a contradiction. Hence $U_n \to U_0$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$. This implies $U_0 \in \mathcal{M}$ and $F(U_0) = m_\lambda$. Moreover, it follows from $F(U_0) = F(|U_0|)$ and $|U_0| \in \mathcal{M}$ that $U_0$ is a nonnegative weak solution to (3.2). Then, by the strong maximum principle [23], we have $U_0 > 0$ in $\mathcal{C}_\Omega$, that is, $U_0$ is a positive solution of problem (3.2). □

Now, we establish the existence of a minimum for $I$ on $\mathcal{N}^+$.

PROPOSITION 3.2. *For each $\lambda \in (0, \lambda_*)$, the functional $I$ has a minimizer $U_1$ in $\mathcal{N}$.*

PROOF. From Lemma 2.5, it is easily derived the coerciveness and the lower boundedness of $I$ on $\mathcal{N}$. Clearly, by the Ekeland's variational principle applying for the minimization problem $\inf_{\mathcal{N}} I(U)$, there exists a minimizing sequence $\{U_n\} \subset \mathcal{N}$ such that

$$I(U_n) < m_I + \frac{1}{n}, \tag{3.7}$$

$$I(W) \geq I(U_n) - \frac{1}{n} \|U_n - W\|_{H^1_{0,L}(\mathcal{C}_\Omega)}, \quad \text{for all } W \in \mathcal{N}. \tag{3.8}$$

Let $U_0$ be a positive solution of (3.2) satisfying $F(U_0) = m_\lambda < 0$. Then

$$\begin{aligned} m_\lambda = F(U_0) = \frac{k_s}{2} \int_{\mathcal{C}_\Omega} y^{1-2s} |U_0|^2 \, dx \, dy \\ - \frac{\lambda}{q} \int_\Omega f(x) |U_0(x,0)|^q \, dx = \left( \frac{1}{2} - \frac{1}{q} \right) \|U_0\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}, \end{aligned}$$

that is,

$$\|U_0\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = \frac{2q}{q-2} m_\lambda > 0. \tag{3.9}$$

By Lemma 2.5 in [26], for $U_0$, there exists a positive constant $t_1$ such that $t_1 U_0 \in \mathcal{N}^+$, i.e.

$$\int_\Omega |t_1 U_0(x,0)|^{2^*_s} \, dx < \frac{2-q}{2^*_s - q} k_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla (t_1 U_0)|^2 \, dx \, dy. \tag{3.10}$$

Then, from (3.9) and (3.10),

$$\begin{aligned} I(t_1 U_0) &= \frac{q-2}{2q} k_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla (t_1 U_0)|^2 \, dx \, dy + \frac{2^*_s - q}{q \, 2^*_s} \int_\Omega |t_1 U_0(x,0)|^{2^*_s} \, dx \\ &< \left( \frac{1}{2} - \frac{1}{2^*_s} \right) \frac{q-2}{q} \|t_1 U_0\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = \frac{s}{N} \frac{q-2}{q} t_1^2 \frac{2q}{q-2} m_\lambda = \frac{2s}{N} t_1^2 m_\lambda < 0. \end{aligned}$$

This yields

$$m_I \leq m^+ < 0. \tag{3.11}$$

So (3.7), (3.11) and the coerciveness of $I$ imply that the minimizer sequence $\{U_n\}$ is bounded, and so there exists a subsequence of $\{U_n\}$, still denoted by $\{U_n\}$, and $U_1$ such that

$$\begin{aligned} U_n &\rightharpoonup U_1 && \text{weakly in } H^1_{0,L}(\mathcal{C}_\Omega);\\ U_n(\,\cdot\,,0) &\to U_1(\,\cdot\,,0) && \text{strongly in } L^p(\Omega) \text{ for } 1 \le p < 2^*_s;\\ U_n(\,\cdot\,,0) &\to U_1(\,\cdot\,,0) && \text{a.e. in } \Omega. \end{aligned}$$

Now, we claim that $U_1 \not\equiv 0$. In fact, suppose on the contrary that $U_1 \equiv 0$. Since $U_n \in \mathcal{N}$, we deduce

$$\begin{aligned} I(U_n) &= \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_n|^2\,dx\,dy\\ &\quad - \frac{\lambda}{q}\int_\Omega f(x)|U_n(x,0)|^q\,dx - \frac{1}{2^*_s}\int_\Omega |U_n(x,0)|^{2^*_s}\,dx\\ &= \frac{2s}{N}\|U_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\,\frac{2^*_s-q}{2^*_s q}\int_\Omega f(x)|U_n(x,0)|^q\,dx\\ &> -\lambda\,\frac{2^*_s-q}{q2^*_s}\int_\Omega f(x)|U_n(x,0)|^q\,dx, \end{aligned}$$

which and (3.7) implies that

$$\int_\Omega f(x)|U_n(x,0)|^q\,dx > -\frac{q2^*_s}{\lambda(2^*_s-q)}\,I(U_n) \ge -\frac{q2^*_s}{\lambda(2^*_s-q)}\left(m_I+\frac{1}{n}\right) > 0$$

as $n\to\infty$, which clearly shows that $U_1 \not\equiv 0$.

Next, we will show that $\|I'(U_n)\| \to 0$ as $n\to\infty$. Exactly the same as in Lemma 2.6 we may apply suitable function $t_n\colon B_\varepsilon(0)\to\mathbb{R}^+$ for some $\varepsilon>0$ small such that

$$t_n(V)(U_n - V) \in \mathcal{N}, \quad \text{for all } V \in H^1_{0,L}(\mathcal{C}_\Omega) \text{ with } \|V\|_{H^1_{0,L}(\mathcal{C}_\Omega)} < \varepsilon.$$

Set $\eta_n = t_n(V)(U_n - V)$. Since $\eta_n \in \mathcal{N}$, we deduce from (3.8) that

$$I(\eta_n) - I(U_n) \ge -\frac{1}{n}\,\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}.$$

By the mean value theorem, we have

$$\langle I'(U_n), \eta_n - U_n\rangle \ge -\frac{1}{n}\,\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} + o\big(\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\big). \tag{3.12}$$

Thus, from $\eta_n - U_n = (t_n(V)-1)(U_n - V) - V$ and (3.12), we get

$$\begin{aligned} \langle I'(U_n), -V\rangle &+ (t_n(V)-1)\langle I'(U_n), U_n - V\rangle\\ &\ge -\frac{1}{n}\,\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} + o\big(\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\big). \end{aligned} \tag{3.13}$$

Let $V = r\,U_1/\|U_1\|_{H^1_{0,L}(\mathcal{C}_\Omega)}$, $0<r<\varepsilon$. Substituting into (3.13), we have

$$\begin{aligned} \left\langle I'(U_n), \frac{U_1}{\|U_1\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\right\rangle &\le \frac{1}{n\,r}\,\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\\ &\quad + \frac{1}{r}\,o\big(\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\big) + \frac{(t_n(V)-1)}{r}\,\langle I'(U_n), U_n - V\rangle. \end{aligned} \tag{3.14}$$

Since

$$\|\eta_n - U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} = \|(t_n(V)-1)\,U_n - t_n(V)\,V\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \leq \varepsilon|t_n(V)| + |t_n(V)-1|\|U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \tag{3.15}$$

and

$$\lim_{r\to 0}\frac{|t_n(V)-1|}{r} = \lim_{r\to 0}\frac{|\langle t_n'(0), V\rangle|}{r} \leq \|t_n'(0)\|_{H^1_{0,L}(\mathcal{C}_\Omega)}. \tag{3.16}$$

If we let $r \to 0$ in the right hand of (3.14) for a fixed $n$, then by (3.15), (3.16) and the boundedness of $U_n$, we can find a constant $C > 0$ such that

$$\left\langle I'(U_n), \frac{U_1}{\|U_1\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\right\rangle \leq \frac{C}{n}\big(1 + \|t_n'(0)\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\big). \tag{3.17}$$

We are done once we show that $\|t_n'(0)\|_{H^1_{0,L}(\mathcal{C}_\Omega)}$ is uniformly bounded in $n$. Since

$$\langle t_n'(0), V\rangle = \frac{A(U_n,V)}{B(U_n,U_n)},$$

we have by the boundness of $U_n$,

$$\|t_n'(0)\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \leq \frac{C_1}{\left|(2-q)\|U_n\|^2 - (2^*_s - q)\displaystyle\int_\Omega |U_n(x,0)|^{2^*_s}\,dx\right|}, \tag{3.18}$$

for some suitable positive constant $C_1$. We next only need to show that

$$\left|(2-q)\|U_n\|^2_{H^s_{0,L}(\mathcal{C}_\Omega)} - (2^*_s - q)\int_\Omega |U_n(x,0)|^{2^*_s}\,dx\right| \geq c > 0 \tag{3.19}$$

for some $c > 0$ and $n$ large enough. Arguing by contradiction, assume that there exists a subsequence $\{U_n\}$ such that

$$(2-q)\|U_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - (2^*_s - q)\int_\Omega |U_n(x,0)|^{2^*_s}\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

Then

$$\lim_{n\to\infty}\int_\Omega |U_n(x,0)|^{2^*_s}\,dx = \lim_{n\to\infty}\frac{2-q}{2^*_s - q}\,\|U_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} \geq \frac{2-q}{2^*_s - q}\,\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} > 0. \tag{3.21}$$

Therefore, we can find a constant $C_2 > 0$ such that

$$\int_\Omega |U_n(x,0)|^{2^*_s}\,dx > C_2 \tag{3.22}$$

for $n$ large enough. In addition, (3.20) and the fact that $U_n \in \mathcal{N}$ also give as

$$\lambda\int_\Omega f(x)|U_n(x,0)|^q\,dx = \|U_n\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \int_\Omega |U_n(x,0)|^{2^*_s}\,dx = \frac{4s}{(N-2s)(2-q)}\int_\Omega |U_n(x,0)|^{2^*_s}\,dx + o(1)$$

and

$$\|U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \le \left[\lambda\,\frac{(2^*_s - q)(N-2s)}{4s}\,|f|_\theta\, S^{-N/(N-2s)}\right]^{1/(2-q)} + o(1). \tag{3.23}$$

This implies $K(U_n) = o(1)$, where $K$ is given in Section 2. However, by (3.22), (3.23), similar to the calculation of (2.5), for each $\lambda \in (0, \lambda_*)$, there is a $C_3 > 0$ such that $K(U_n) > C_3$, which is impossible. Hence, from (3.17)–(3.19),

$$\left\langle I'(U_n), \frac{U_1}{\|U_1\|_{H^1_{0,L}(\mathcal{C}_\Omega)}} \right\rangle \le \frac{C}{n}$$

for some $C > 0$. Taking $n \to \infty$, we get $\|I'(U_n)\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \to 0$. This shows that $\{U_n\}$ is a (PS) sequence of functional $I$.

Finally, we prove that $U_n \to U_1$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$. Since $U_n \rightharpoonup U_1$ weakly in $H^1_{0,L}(\mathcal{C}_\Omega)$, it follows that

$$\begin{aligned}
m_I \le I(U_1) &= \frac{1}{2}\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{\lambda}{q}\int_\Omega f(x)|U_1(X,0)|^q\,dx - \frac{1}{2^*_s}\int_\Omega |U_1(x,0)|^{2^*_s}\,dx \\
&= \frac{1}{2}\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{\lambda}{q}\int_\Omega f(x)|U_1(X,0)|^q\,dx \\
&\quad - \frac{1}{2^*_s}\left(\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \int_\Omega f(x)|U_1(X,0)|^q\,dx\right) \\
&= \frac{s}{N}\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\,\frac{2^*_s - q}{q2^*_s}\int_\Omega f(x)|U_1(x,0)|^q\,dx \le \lim_{n\to\infty} I(U_n) = m_I.
\end{aligned}$$

Consequently, $U_n \to U_1$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$ and $I(U_1) = m_I$. □

Theorem 3.3. *For each $\lambda \in (0, \lambda_*)$, the problem* (1.6) *admits a positive solution in $\mathcal{N}^+$.*

Proof. From Proposition 3.2, we have that $U_1$ is a nontrivial solution of problem (1.6). Moreover, we have $U_1 \in \mathcal{N}^+$. In fact, if $U_1 \in \mathcal{N}^-$, by Lemma 2.1, there exists a unique $t^-(U_1) > 0$, $t^+(U_1) > 0$ such that $t^-(U_1)\,U_1 \in \mathcal{N}^-$, then we have $t^-(U_1) = 1$ and $t^+(U_1) < 1$. Since $I(t^+(U_1)\,U_1) = \min_{t\in[0,t^-(U_1)]} I(t\,U_1)$, we can find a $t_0 \in (t^+(U_1), t^-(U_1))$ such that

$$I(t^+(U_1)U_1) < I(t_0U_1) \le I(t^-(U_1)\,U_1) = I(1\cdot U_1) = m_I,$$

which implies that $U_1 \in \mathcal{N}^+$. Since $I(U_1) = I(|U_1|)$ and $|U_1| \in \mathcal{N}^+$, we can take $U_1 \ge 0$. By the strong maximum principle [23], we get $U_1 > 0$ in $H^1_{0,L}(\mathcal{C}_\Omega)$. Hence, $U_1$ is a positive solution of problem (1.6) and $I(U_1) = m^+$. □

REMARK 3.4. For $U_1 \in \mathcal{N}^+$, by the Hölder inequality and the Young inequality, we have

$$\begin{aligned}0 > I(U_1) &= \frac{s}{N}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_1|^2\,dx\,dy - \lambda\,\frac{2_s^*-q}{q\,2_s^*}\int_\Omega f(x)|U_1(x,0)|^q\,dx\\ &\geq \frac{s}{N}\,\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \lambda\,\frac{2_s^*-q}{q\,2_s^*}\,|f|_\theta\,\big(k_sS(s,N)\big)^{-q/2}\|U_1\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)}\\ &\geq -\lambda\,\frac{2-q}{q\,2_s^*}\big(|f|_\theta\,k_sS(s,N)^{-q/2}\big)^{2/(2-q)}.\end{aligned}$$

So, we deduce that $I(U_1)\to 0$ as $\lambda\to 0$.

**3.2. The minimizer solution on $\mathcal{N}^-$.** In the following, we prove that problem (1.6) has a solution in $\mathcal{N}^-$. Since $I$ is coercive and bounded from below on $\mathcal{N}^-$, there exists a minimizing sequence $\{U_n\}\subset\mathcal{N}^-$ such that

$$I(U_n)\to m^- \quad \text{as } n\to\infty. \tag{3.24}$$

First, we establish the following result.

LEMMA 3.5. *The set $\mathcal{N}^-$ is closed.*

PROOF. Suppose that there are some $U_n\in\mathcal{N}^-$ and $U_n\to U_0\notin\mathcal{N}^-$, then $U_0\in\mathcal{N}^0=\{0\}$. For $U_n\in\mathcal{N}^-$, we have

$$0\leq(2-q)k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_n|^2\,dx\,dy < (2_s^*-q)\int_\Omega |U_n(x,0)|^{2_s^*}\,dx\to 0.$$

This implies that

$$\lim_{n\to\infty} k_s\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_n|^2\,dx\,dy=0.$$

Note that if $U_n\in\mathcal{N}^-$, then $\|U_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\geq\gamma$ for a suitable $\gamma>0$. This is a contradiction. Hence we have $U_0\in\mathcal{N}^-$, and so $\mathcal{N}^-$ is closed. $\square$

Next, we will use the trace inequality (1.8) to the family of minimizers $U_\varepsilon = E_s(u_\varepsilon)$, where $u_\varepsilon$ is given in (1.9). Note that $f$ is a indefinite continuous function on $\Omega$ and $f^+\not\equiv 0$, where $f^+=\max\{f(x),0\}$, then the set $\Sigma:=\{x\in\Omega : f(x) > 0\}\subset\Omega$ is an open set with positive measure. Without loss of generality, we may assume that $\Sigma$ is a domain.

Let $\eta\in C_0^\infty(\mathcal{C}_\Sigma)$, $0\leq\eta\leq 1$ (for all $(x,y)\in\Sigma\times(0,\infty)$), be a positive function satisfying

$$\big(\mathrm{supp} f^+\times\{y>0\}\big)\cap\{(x,y)\in\mathcal{C}_\Sigma : \eta=1\}\neq\emptyset.$$

Moreover, for small fixed $\rho>0$,

$$\eta(x,y)=\begin{cases}1 & \text{if } (x,y)\in\mathcal{B}_\rho,\\ 0 & \text{if } (x,y)\notin\overline{\mathcal{B}}_{2\rho},\end{cases}$$

where $\mathcal{B}_\rho=\{(x,y) : |(x,y)|<\rho,\ y>0,\ x\in\Sigma\}$. We take $\rho$ small enough such that $\overline{\mathcal{B}_{2\rho}}\subset\overline{\mathcal{C}_\Sigma}$. Note that $\eta\,U_\varepsilon\in H^1_{0,L}(\mathcal{C}_\Omega)$.

Let $\lambda_* > 0$ be as in (3.1). Then for $\lambda \in (0, \lambda_*)$ we have the following result.

LEMMA 3.6. *Let $U_1$ be the local minimum in Proposition 3.2. Then, for $\varepsilon > 0$ small enough,*

$$\sup_{t\geq 0} I(U_1 + t\eta U_\varepsilon) < m_I + \frac{s}{N}\big(k_s S(s,N)\big)^{N/(2s)}.$$

PROOF. First, we have

$$\begin{aligned}(3.25)\qquad I(U_1 + t\eta U_\varepsilon) &= \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla(U_1 + t\eta U_\varepsilon)|^2\,dx\,dy \\ &\quad - \frac{\lambda}{q}\int_\Omega f(x)|(U_1 + t\eta U_\varepsilon)(x,0)|^q dx \\ &\quad - \frac{1}{2_s^*}\int_\Omega |(U_1 + t\eta U_\varepsilon)(x,0)|^{2_s^*}\,dx \\ &= \frac{1}{2}\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + \frac{t^2}{2}\|\eta U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + t\langle U_1, \eta U_\varepsilon\rangle \\ &\quad - \frac{\lambda}{q}\int_\Omega f(x)|(U_1 + t\eta U_\varepsilon)(x,0)|^q\,dx \\ &\quad - \frac{1}{2_s^*}\int_\Omega |(U_1 + t\eta U_\varepsilon)(x,0)|^{2_s^*}\,dx.\end{aligned}$$

It follows from $U_1$ is a solution of problem (1.6) that

$$(3.26)\quad \frac{1}{2}\|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} = I(U_1) + \frac{\lambda}{q}\int_\Omega f(x)|U_1(x,0)|^q\,dx + \frac{1}{2_s^*}\int_\Omega |U_1(x,0)|^{2_s^*}\,dx,$$

and

$$\begin{aligned}(3.27)\quad t\langle U_1, \eta U_\varepsilon\rangle = t\lambda\int_\Omega f(x)|U_1(x,0)|^{q-1}\eta\, U_\varepsilon(x,0)\,dx \\ + t\int_\Omega |U_1(x,0)|^{2_s^*-1}\,\eta\, U_\varepsilon(x,0)\,dx.\end{aligned}$$

Moreover, by direct computation, we get that

$$\begin{aligned}(3.28)\qquad \int_\Omega |(U_1 + t\eta U_\varepsilon)(x,0)|^{2_s^*}\,dx &= \int_\Omega |U_1(x,0)|^{2_s^*}\,dx \\ &+ 2_s^* t\int_\Omega |U_1(x,0)|^{2_s^*-2}U_1(x,0)\eta U_\varepsilon(x,0)\,dx \\ &+ t^{2_s^*}\int_\Omega |\eta U_\varepsilon(x,0)|^{2_s^*}\,dx \\ &+ 2_s^* t^{2_s^*-1}\int_\Omega |\eta U_\varepsilon(x,0)|^{2_s^*-2}\eta U_\varepsilon(x,0)U_1(x,0)\,dx + o(\varepsilon^{(N-2s)/2}),\end{aligned}$$

and

$$\begin{aligned}(3.29)\quad &\int_\Sigma f(x)\big(|(U_1+t\eta U_\varepsilon)(x,0)|^q-|U_1(x,0)|^q+qt|U_1(x,0)|^{q-1}\eta U_\varepsilon(x,0)\big)\,dx\\ &=q\int_\Sigma f^+(x)\bigg(\int_0^{t\eta U_\varepsilon(x,0)}(|U_1(x,0)+\tau|^{q-1}+|U_1(x,0)|^{q-1}\tau)\,d\tau\bigg)\,dx\\ &\ge q\int_\Sigma f^+(x)\bigg(\int_0^{t\eta U_\varepsilon(x,0)}(|U_1(x,0)+\tau|^{q-1}+|U_1(x,0)|^{q-1}\tau)\,d\tau\bigg)\,dx\ge 0.\end{aligned}$$

Substituting (3.26)–(3.29) in (3.25) and using the fact that $\eta\in C_0^\infty(\mathcal{C}_\Sigma)$, we obtain

$$\begin{aligned}&I(U_1+t\eta U_\varepsilon)\\ &=I(U_1)-\frac{\lambda}{q}\int_\Omega f(x)(|U_1(x,0)+t\eta\,U_\varepsilon(x,0)|^q-|U_1(x,0)|^q)\,dx\\ &\quad+t\langle U_1,\eta\,U_\varepsilon\rangle-t\int_\Omega|U_1(x,0)|^{2_s^*-1}\eta U_\varepsilon(x,0)\,dx\\ &\quad+\frac{t^2}{2}\,\|\eta U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2_s^*}}{2_s^*}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*}\,dx\\ &\quad-t^{2_s^*-1}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*-1}w_1\,dx+o\big(\epsilon^{(N-2s)/2}\big)\\ &=I(U_1)-\frac{\lambda}{q}\int_\Sigma f(x)(|U_1(x,0)+t\eta\,U_\varepsilon(x,0)|^q-|U_1(x,0)|^q\\ &\quad+qt|U_1(x,0)|^{q-1}\eta\,U_\varepsilon(x,0))\,dx+\frac{t^2}{2}\,\|\eta\,U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2_s^*}}{2_s^*}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*}\,dx\\ &\quad-t^{2_s^*-1}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*-1}U_1(x,0)\,dx+o\big(\epsilon^{(N-2s)/2}\big)\\ &\le I(U_1)+\frac{t^2}{2}\,\|\eta\,U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}-\frac{t^{2_s^*}}{2_s^*}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*}\,dx\\ &\quad-t^{2_s^*-1}\int_\Omega|\eta\,U_\varepsilon(x,0)|^{2_s^*-1}U_1(x,0)\,dx+o\big(\epsilon^{(N-2s)/2}\big).\end{aligned}$$

Since

$$\begin{aligned}\int_\Omega|\eta U_\varepsilon(x,0)|^{2_s^*-1}\,dx&=\int_\Omega\bigg[\frac{\eta\varepsilon^{(N-2s)/2}}{(\varepsilon^2+|x|^2)^{(N-2s)/2}}\bigg]^{(N+2s)/(N-2s)}dx\\ &=\int_{\mathbb{R}^N}\frac{\varepsilon^{(N+2s)/2}}{\varepsilon^{N+2s}(1+|z|^2)^{(N+2s)/2}}\;\varepsilon^N\,dz\\ &=C\,\varepsilon^{(N-2s)/2}\int_0^{+\infty}\frac{1}{(1+r^2)^{(N+2s)/2}}\le C\,\varepsilon^{(N-2s)/2},\end{aligned}$$

and from [3] and [22], we have

$$\|\eta U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}=\|U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}+O(\varepsilon^{N-2s}),$$

$$\int_\Omega |\eta U_\varepsilon(x,0)|^{2_s^*}\,dx = \int_{\mathbb{R}^N} \bigg(\frac{\varepsilon}{\varepsilon^2+|x|^2}\bigg)^N dx + O(\varepsilon^N).$$

Thus

$$\begin{aligned}(3.30)\quad I(U_1+t\eta U_\varepsilon) \le I(U_1) + \frac{t^2}{2}\,\|U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |U_\varepsilon(x,0)|^{2_s^*}\,dx \\ + O(\varepsilon^N) - C\,\varepsilon^{(N-2s)/2} + o\big(\epsilon^{(N-2s)/2}\big).\end{aligned}$$

Let

$$h(t) = \frac{t^2}{2}\,\|U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |U_\varepsilon(x,0)|^{2_s^*}\,dx, \quad \text{for all } t \ge 0.$$

Since $h(t)\to-\infty$ as $t\to+\infty$, $\sup_{t\ge0} h(t)$ is achieved at some $t_\varepsilon>0$ with $h'(t_\varepsilon)=0$. That is

$$0 = \|U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - t_\varepsilon^{2_s^*-2}\int_\Omega |U_\varepsilon(x,0)|^{2_s^*}\,dx.$$

Therefore,

$$(3.31)\quad h(t) \le h(t_\varepsilon) = \bigg(\frac{1}{2}-\frac{1}{2_s^*}\bigg)\|U_\varepsilon\|^{(2_s^*)/(2_s^*-2)}_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigg(\int_\Omega |U_\varepsilon(x,0)|^{2_s^*}\,dx\bigg)^{-2/(2_s^*-2)}.$$

On the other hand, since $U_\varepsilon$ are minimizers of the trace inequality of (1.8), we have that

$$(3.32)\qquad \|U_\varepsilon\|^2_{H^1_{0,L}(\mathbb{R}^{N+1})} = k_s\,S(s,N)\bigg(\int_{\mathbb{R}^N}|U_\varepsilon(x,0)|^{2_s^*}\,dx\bigg)^{2/(2_s^*)}.$$

Hence, as from [3] and (3.30)–(3.32), we obtain

$$\begin{aligned}I(U_1+t\eta U_\varepsilon) &\le I(U_1) + \frac{s}{N}\big(k_s\,S(s,N)\big)^{N/(2s)} + O\big(\varepsilon^N\big) \\ &\quad - C\varepsilon^{(N-2s)/2} + o\big(\varepsilon^{(N-4)/2}\big) < m_I + \frac{s}{N}\big(k_s\,S(s,N)\big)^{N/(2s)},\end{aligned}$$

for $\varepsilon>0$ sufficiently small. □

The following proposition provides a precise description of the (PS)-sequence of $I$.

Proposition 3.7. *If every minimizing sequence $\{U_n\}$ of $I$ on $\mathcal{N}^-$ satisfies*

$$m_I \le I(U_n) < m_I + \frac{s}{N}(k_sS)^{N/(2s)},$$

*then $\{U_n\}$ satisfies the* (PS)*-condition on $\mathcal{N}^-$.*

Proof. By (3.24) and $\{U_n\}\subset\mathcal{N}^-$, it is easy to prove that the sequence $\{U_n\}$ is bounded in $H^1_{0,L}(\mathcal{C}_\Omega)$. Them we can extract a subsequence, still denoted by $\{U_n\}$, and $U_2\in\mathcal{N}^-$ such that, as $n\to\infty$,

$$(3.33)\qquad \begin{aligned} U_n &\rightharpoonup U_2 && \text{weakly in } H^1_{0,L}(\mathcal{C}_\Omega);\\ U_n(\,\cdot\,,0) &\to U_2(\,\cdot\,,0) && \text{strongly in } L^p(\Omega), \text{ for all } 1\le p<2_s^*;\\ U_n(\,\cdot\,,0) &\to U_2(\,\cdot\,,0) && \text{a.e. in } \Omega.\end{aligned}$$

Since $\{U_n\} \subset \mathcal{N}^-$ is a minimizing sequence, by the Lagrange multiplier method, we get that $I'(U_n) \to 0$ as $n \to \infty$. Consequently, by (3.33) we have

$$\langle I'(U_2), \Phi \rangle = 0, \quad \text{for all } \Phi \in H^1_{0,L}(\mathcal{C}_\Omega).$$

Then $U_2$ is a solution in $H^1_{0,L}(\mathcal{C}_\Omega)$ for problem (1.6), and $I(U_2) \geq m_I$.

First, we claim that $U_2 \not\equiv 0$. If not, by (3.33) we have

$$\int_\Omega f(x)|U_2(x,0)|^q \, dx \to 0 \quad \text{as } n \to \infty.$$

Thus, from $I'(U_n) \to 0$, we obtain that

$$k_s \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla U_n|^2 \, dx \, dy = \int_\Omega |U_n(x,0)|^{2^*_s} \, dx + o_n(1). \tag{3.34}$$

and

$$\begin{aligned} I(U_n) &= \frac{k_s}{2} \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla U_n|^2 \, dx \, dy \\ &\quad - \frac{\lambda}{q} \int_\Omega f(x)|U_n(x,0)|^q \, dx - \frac{1}{2^*_s} \int_\Omega |U_n(x,0)|^{2^*_s} \, dx \\ &= \frac{s}{N} \int_\Omega |U_n(x,0)|^{2^*_s} \, dx < m_I + \frac{s}{N} (k_s \, S(s,N))^{N/(2s)} \\ &< \frac{s}{N} (k_s \, S(s,N))^{N/(2s)} \quad (\text{since } m_I < 0). \end{aligned}$$

So, we get

$$\int_\Omega |U_n(x,0)|^{2^*_s} \, dx < (k_s S(s,N))^{N/(2s)}. \tag{3.35}$$

On the other hand, from (3.34) and (1.8), we have that

$$\int_\Omega |U_n(x,0)|^{2^*_s} \, dx \geq (k_s S(s,N))^{N/(2s)}.$$

This contradicts (3.35). Then $U_2 \not\equiv 0$ and $I(U_2) \geq m_I$.

We write $\widehat{U}_n = U_n - U_2$ with $\widehat{U}_n \rightharpoonup 0$ weakly in $H^1_{0,L}(\mathcal{C}_\Omega)$. By the Brezis–Lieb Lemma, we have

$$\begin{aligned} \int_\Omega \left|\widehat{U}_n(x,0)\right|^{2^*_s} dx &= \int_\Omega |U_n(x,0) - U_2(x,0)|^{2^*_s} \, dx \\ &= \int_\Omega |U_n(x,0)|^{2^*_s} \, dx - \int_\Omega |U_2(x,0)|^{2^*_s} \, dx + o_n(1). \end{aligned}$$

Hence, for $n$ large enough, we can conclude that

$$\begin{aligned} m_I + \frac{s}{N}(k_s S(s,N))^{N/(2s)} &> I\big(U_2 + \widehat{U}_n\big) \\ &= I(U_2) + \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \frac{1}{2_s^*}\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx + o_n(1) \\ &\geq m_I + \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \frac{1}{2_s^*}\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx + o_n(1), \end{aligned}$$

this is,

$$\begin{aligned} \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \frac{1}{2_s^*}\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx \\ < \frac{s}{N}(k_s S(s,N))^{N/(2s)} + o_n(1). \end{aligned} \tag{3.36}$$

Since $I'(U_n) \to 0$ as $n \to \infty$, $\{U_n\}$ is uniformly bounded and $U_2$ is a solution of (1.6), it follows

$$\begin{aligned} o_n(1) &= \langle I'(U_n), U_n\rangle \\ &= I'(U_2) + k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx + o_n(1) \\ &= k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}dx + o_n(1), \end{aligned}$$

we obtain

$$k_s\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy = \int_\Omega |\widehat{U}_n(x,0)|^{2_S^*}\,dx + o_n(1) \quad \text{as } n \to \infty. \tag{3.37}$$

We claim that (3.36) and (3.37) can hold simultaneously only if $\{\widehat{U}_n\}$ admits a subsequence which converges strongly to zero. If not, then $\|\widehat{U}_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)}$ is bounded away from zero, that is, $\|\widehat{U}_n\|_{H^1_{0,L}(\mathcal{C}_\Omega)} > c > 0$. From (3.37) and (1.8) then it follows

$$\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_S^*}\,dx \geq (k_s S(s,N))^{N/(2s)} + o_n(1). \tag{3.38}$$

On the other hand, by (3.36)–(3.38), for $n$ large enough, we have

$$\begin{aligned} \frac{s}{N}(k_s S(s,N))^{N/2s} &\leq \frac{s}{N}\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx + o_n(1) \\ &= \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}\big|\nabla \widehat{U}_n\big|^2\,dx\,dy - \frac{1}{2_s^*}\int_\Omega \big|\widehat{U}_n(x,0)\big|^{2_s^*}\,dx + o_n(1) \\ &< \frac{s}{N}(k_s S(s,N))^{N/(2s)}, \end{aligned}$$

which is a contradiction. Consequently, $U_n \to U_2$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$ and $U_2 \in \mathcal{N}^-$. □

Next, we establish the existence of a local minimum for $I$ on $\mathcal{N}^-$.

PROPOSITION 3.8. *For any* $\lambda \in (0,\lambda_*)$, *the functional* $I$ *has a minimizer* $U_2 \in \mathcal{N}^-$ *such that*

$$I(U_2) = m^- < m_I + \frac{s}{N}(k_s S(s,N))^{N/(2s)}.$$

PROOF. For every $U \in H^1_{0,L}(\mathcal{C}_\Omega)$, by Lemma 2.1, we can find a unique $t^-(U) > 0$ such that $t^-(U)U \in \mathcal{N}^-$. Define

$$\mathbb{W}_1 = \bigg\{U : U = 0 \text{ or } t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) > \|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigg\},$$
$$\mathbb{W}_2 = \bigg\{U : t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) < \|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigg\}.$$

Then $\mathcal{N}^-$ disconnects $H^1_{0,L}(\mathcal{C}_\Omega)$ in two connected components $\mathbb{W}_1$ and $\mathbb{W}_2$, and $H^1_{0,L}(\mathcal{C}_\Omega)\setminus\mathcal{N}^- = \mathbb{W}_1 \cup \mathbb{W}_2$.

For each $U \in \mathcal{N}^+$, there exist unique

$$t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) > 0 \quad\text{and}\quad t^+\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) > 0$$

such that

$$t^+\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) < t_{\max} < t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg);$$
$$t^+\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg)\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}} \in \mathcal{N}^+;$$

and

$$t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg)\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}} \in \mathcal{N}^-.$$

Since $U \in \mathcal{N}^+$, we have

$$t^+\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg)\frac{1}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}} = 1.$$

By the fact that

$$t^+\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) < t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg),$$

we get

$$t^-\bigg(\frac{U}{\|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) > \|U\|_{H^1_{0,L}(\mathcal{C}_\Omega)},$$

and then $\mathcal{N}^+ \subset \mathbb{W}_1$. In particular, $U_1 \in \mathbb{W}_1$ is the minimizer of $I$ in $\mathcal{N}^+$.

Now, we claim that there exists $l_0 > 0$ such that $U_1 + l_0\eta U_\varepsilon \in \mathbb{W}_2$. First, we find a constant $c > 0$ such that

$$0 < t^-\bigg(\frac{U_1 + l\eta\, U_\varepsilon}{\|U_1 + l\eta\, U_\varepsilon\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\bigg) < c \quad \text{for each } l > 0. \tag{3.39}$$

Otherwise, there exists a sequence $\{l_n\}$ such that $l_n \to \infty$ and

$$t^-\biggl(\frac{U_1 + l_n\eta\, U_\varepsilon}{\|U_1 + l_n\eta\, U_\varepsilon\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\biggr) \to \infty \quad \text{as } n \to \infty.$$

Let $\widetilde{U}_n = (U_1 + l_n\eta U_\varepsilon)/(\|U_1 + l_n\eta U_\varepsilon\|_{H^1_{0,L}(\mathcal{C}_\Omega)})$. By Lemma 2.1, we obtain $t^-(\widetilde{U}_n)\widetilde{U}_n \in \mathcal{N}^-$, and

$$\begin{aligned}\int_\Omega \bigl|\widetilde{U}_n(x,0)\bigr|^{2^*_s}\,dx &= \frac{1}{\|U_1 + l_n\eta\, U_\varepsilon\|^{2^*_s}_{H^1_{0,L}(\mathcal{C}_\Omega)}} \int_\Omega \bigl|(U_1 + l_n\eta\, U_\varepsilon)(x,0)\bigr|^{2^*_s}\,dx \\ &= \frac{1}{\|U_1/l_n + \eta\, U_\varepsilon\|^{2^*_s}_{H^1_{0,L}(\mathcal{C}_\Omega)}} \int_\Omega \biggl|\biggl(\frac{U_1}{l_n} + \eta\, U_\varepsilon\biggr)(x,0)\biggr|^{2^*_s}\,dx \\ &\to \frac{1}{\|\eta\, U_\varepsilon\|^{2^*_s}_{H^1_{0,L}(\mathcal{C}_\Omega)}} \int_\Omega |\eta\, U_\varepsilon(x,0)|^{2^*_s}\,dx > 0\end{aligned}$$

as $n \to \infty$. Thus

$$\begin{aligned}I\bigl(t^-(\widetilde{U}_n)\widetilde{U}_n\bigr) = \frac{1}{2}\bigl[t^-(\widetilde{U}_n)\bigr]^2 - \frac{\lambda}{q}\bigl[t^-(\widetilde{U}_n)\bigr]^q \int_\Omega f(x)\bigl|\widetilde{U}(x,0)\bigr|^q_n\,dx \\ - \frac{\bigl[t^-(\widetilde{U}_n)\bigr]^{2^*_s}}{2^*_s} \int_\Omega \bigl|\widetilde{U}_n(x,0)\bigr|^{2^*_s}\,dx \to -\infty\end{aligned}$$

as $n \to \infty$. This contradicts that $I$ is bounded below on $\mathcal{N}$. Let

$$l_0 = \frac{\sqrt{\bigl|c^2 - \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}\bigr|}}{\|\eta\, U_\varepsilon\|_{H^1_{0,L}(\mathcal{C}_\Omega)}} + 1.$$

It follows that $U_1$ is a nontrivial solution of (1.6) and from the definition of $\eta$, we have

$$\begin{aligned}\langle U_1, \eta U_\varepsilon\rangle &= \lambda \int_\Omega f(x)|U_1(x,0)|^{q-1}\eta(x,0)U_\varepsilon(x,0)\,dx \\ &\quad + \int_\Omega |U_1(x,0)|^{2^*_s-1}\eta(x,0)U_\varepsilon(x,0)\,dx \\ &= \lambda \int_{\mathcal{B}_{2\rho}\cap\{y=0\}} f(x)|U_1(x,0)|^{q-1}\eta(x,0)U_\varepsilon(x,0)\,dx \\ &\quad + \int_{\mathcal{B}_{2\rho}\cap\{y=0\}} |U_1(x,0)|^{2^*_s-1}\eta(x,0)U_\varepsilon(x,0)\,dx \\ &\geq \int_{\mathcal{B}_{\rho}\cap\{y=0\}} |U_1(x,0)|^{2^*_s-1}U_\varepsilon(x,0)\,dx > 0.\end{aligned} \tag{3.40}$$

Then, from (3.39) and (3.40), we obtain

$$\begin{aligned}\|U_1 + l_0\eta U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} &= \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + l_0^2\|\eta U_\varepsilon\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + 2l_0\langle U_1, \eta U_\varepsilon\rangle \\ &\geq \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + \left|c^2 - \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}\right| + 2l_0\langle U_1, \eta U_\varepsilon\rangle \\ &\geq \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} + \left|c^2 - \|U_1\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}\right| \\ &\geq c^2 > \left[t^-\left(\frac{U_1 + l_0\eta U_\varepsilon}{\|U_1 + l_0\eta U_\varepsilon\|_{H^1_{0,L}(\mathcal{C}_\Omega)}}\right)\right]^2,\end{aligned}$$

that is, $U_1 + l_0\eta U_\varepsilon \in \mathbb{W}_2$. Now, we define

$$\beta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1_{0,L}(\mathcal{C}_\Omega)) : \gamma(0) = U_1 \text{ and } \gamma(1) = U_1 + l_0\eta U_\varepsilon\}$.

Define a path $\gamma(t) = U_1 + tl_0\eta U_\varepsilon$ for $t \in [0,1]$, and we have $\gamma(0) \in \mathbb{W}_1$, $\gamma(1) \in \mathbb{W}_2$. Then there exists $t_0 \in (0,1)$ such that $\gamma(t_0) \in \mathcal{N}^-$, and we have $\beta > m^-$. Therefore, by Lemma 3.6, we get

$$m^- \leq \beta < m_I + \frac{s}{N}\big(k_s S(s,N)\big)^{N/(2s)}.$$

Analogously to the proof of Proposition 3.2, one can show that Ekeland's variational principle gives a sequence $\{U_n\} \in \mathcal{N}^-$ which satisfies

$$I(U_n) \to m^- \quad \text{and} \quad I'(U_n) \to 0 \quad \text{as } n \to \infty.$$

Since $m^- < m_I + s(k_s S(s,N))^{N/(2s)}/N$, by Proposition 3.7 and Lemma 3.5, there exist a subsequence $\{U_n\}$ and $U_2$ such that $U_n \to U_2$ strongly in $H^1_{0,L}(\mathcal{C}_\Omega)$, $U_2 \in \mathcal{N}^-$ and $I(U_2) = m^-$.

Moreover, since $I(U_2) = I(|U_2|)$ and $|U_2| \in \mathcal{N}^-$, we can always take $U_2 \geq 0$. By the maximum principle [23], we get $U_2 > 0$ in $H^1_{0,L}(\mathcal{C}_\Omega)$. Hence, $U_2$ is a positive solution of problem (1.6). □

Proof of Theorem 1.1. By Theorem 3.3 and Proposition 3.8, the equation (1.6) has two positive solutions $U_1$ and $U_2$ such that $U_1 \in \mathcal{N}^+$ and $U_2 \in \mathcal{N}^-$. Since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$. This implies that problem (1.6) has at least two positive solutions. □

## 4. Concentration behavior

In this section, we give the proof of Theorem 1.2. For every $\mu > 0$, we define

$$J_\mu(U) = \frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{\mu}{2^*_s}\int_\Omega |U(x,0)|^{2^*_s}\,dx;$$

$$\mathcal{O}_\mu = \{U \in H^1_{0,L}(\mathcal{C}_\Omega) : U \not\equiv 0 \text{ and } \langle J'_\mu(U), U\rangle = 0\}.$$

We have the following lemmas.

Lemma 4.1. *For every* $U \in \mathcal{N}^-$, *there is a unique* $t(U) > 0$ *such that* $t(U)U \in \mathcal{O}_1$ *and*

$$1 - \lambda|f|_\theta \bigg(\frac{2_s^* - q}{S_0(2-q)}\bigg)^{(2_s^*-q)/(2_s^*-2)} \leq t^{2_s^*-2}(U) \leq 1 + \lambda|f|_\theta \bigg(\frac{2_s^* - q}{S_0(2-q)}\bigg)^{(2_s^*-q)/(2_s^*-2)}, \tag{4.1}$$

*where* $S_0 = k_s S(s,N)$.

Proof. For each $U \in \mathcal{N}^-$, we have

$$k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \lambda \int_\Omega f(x)|U(x,0)|^q\,dx - \int_\Omega |U(x,0)|^{2_s^*}\,dx = 0 \tag{4.2}$$

and

$$0 < (2-q)k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy < (2_s^* - q)\int_\Omega |U(x,0)|^{2_s^*}\,dx. \tag{4.3}$$

Thus, from (4.3), the functional

$$J_1(tU) = t^2\,\frac{k_s}{2}\int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy - \frac{t^{2_s^*}}{2_s^*}\int_\Omega |U(x,0)|^{2_s^*}\,dx$$

with respect to $t$ is initially increasing and eventually decreasing and with a single turning point $t(U)$ such that $t(U)U \in \mathcal{O}_1$. So

$$t^2(U)k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy = t^{2_s^*}(U)\int_\Omega |U(x,0)|^{2_s^*}\,dx. \tag{4.4}$$

Then, from (4.2), (4.4) and the Hölder inequality

$$\begin{aligned} 1 - \lambda|f|_\theta\,|U(x,0)|_{2_s^*}^{-(2_s^*-q)} \leq t^{2_s^*-2}(U) &= \frac{k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy}{\int_\Omega |U(x,0)^{2_s^*}\,dx} \\ &= 1 + \frac{\lambda \int_\Omega f(x)|U(x,0)|^q\,dx}{\int_\Omega |U(x,0)|^{2_s^*}\,dx} \leq 1 + \lambda|f|_\theta\,|U(x,0)|_{2_s^*}^{-(2_s^*-q)}. \end{aligned} \tag{4.5}$$

On the other hand, by (1.8) and (4.3), we get

$$\begin{aligned} \int_\Omega |U(x,0)|^{2_s^*}\,dx &> \frac{2-q}{2_s^*-q}\,k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U|^2\,dx\,dy \\ &\geq \frac{2-q}{2_s^*-q}\,k_s S(s,N)\bigg(\int_\Omega |U(x,0)|^{2_s^*}\,dx\bigg)^{2/2_s^*}, \end{aligned}$$

that is

$$|U(x,0)|_{2_s^*} > \left( \frac{(2-q)k_s S(s,N)}{2_s^* - q} \right)^{1/(2_s^*-2)}. \tag{4.6}$$

Hence, from (4.6) and (4.5), we obtain (4.1). □

Remark 4.2. From (4.1), it is easy to see that $t(U) \to 1$ as $\lambda \to 0$.

Proof the Theorem 1.2. Suppose that $\{\lambda_n\}$ is a sequence of positive number such that $\lambda_n \to 0$ as $n \to +\infty$. Let $U_{1,n} \in \mathcal{N}^+$ and $U_{2,n} \in \mathcal{N}^-$ are position solutions of equation (1.6) corresponding to $\lambda = \lambda_n$. We have two following results:

(a) By Remark 3.4, for every $U_{1,n} \in \mathcal{N}^+$, we can conclude that

$$\|U_{1,n}\|_{H^1_{0,L}(\mathcal{C}_\Omega)} \to 0 \quad \text{as } n \to \infty.$$

(b) By Lemma 4.1 and Remark 4.2, for every $U_{2,n} \in \mathcal{N}^-$, there is a unique $t(U_{2,n}) > 0$ such that

$$t(U_{2,n})\, U_{2,n} \in \mathcal{O}_1 \quad \text{and} \quad t(U_{2,n}) \to 1 \quad \text{as } n \to \infty.$$

For case (b). For each $U_{2,n} \in \mathcal{N}^-$, let

$$g(t) = J_\mu(tU_{2,n}) = t^2 \frac{k_s}{2} \int_{\mathcal{C}_\Omega} y^{1-2s} |\nabla U_{2,n}|^2 \, dx\, dy - t^{2_s^*} \frac{\mu}{2_s^*} \int_\Omega |U_{2,n}(x,0)|^{2_s^*} \, dx,$$

for all $t \geq 0$. Since $g(t) \to -\infty$ as $t \to +\infty$, $\sup\limits_{t\geq 0} g(t)$ is achieved at some $\widetilde{t} > 0$ with $h'(\widetilde{t}) = 0$, which is

$$h'(\widetilde{t}) = \widetilde{t} \left( \|U_{2,n}\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \widetilde{t}^{\,2_s^*-2} \mu \int_\Omega |U_{2,n}(x,0)|^{2_s^*} \, dx \right) = 0.$$

Let

$$\widetilde{t} = \left( \frac{\|U_{2,n}\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}}{\mu \int_\Omega |U_{2,n}(x,0)|^{2_s^*} \, dx} \right)^{1/(2_s^*-2)}.$$

Then $\widetilde{t}\, U_{2,n} \in \mathcal{O}_\mu$ and

$$\sup_{t\geq 0} J_\mu(t\, U_{2,n}) = J_\mu(\widetilde{t}\, U_{2,n}) = \frac{s}{N} \left( \frac{\|U_{2,n}\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)}}{\mu \int_\Omega |U_{2,n}(x,0)|^{2_s^*} \, dx} \right)^{(N-2s)/2}. \tag{4.7}$$

On the other hand, by Hölder inequality and Young inequality, for $\mu \in (0,1)$, we have

$$\begin{aligned} \int_\Omega f(x) \big|\widetilde{t}\, U_{2,n}(x,0)\big|^q \, dx &\leq |f|_\theta \left( \int_\Omega \big|\widetilde{t}\, U_{2,n}(x,0)\big|^{2_s^*} \, dx \right)^{q/2_s^*} \\ &\leq |f|_\theta \big(k_s\, S(s,N)\big)^{-q/2}\, \widetilde{t}^{\,q}\, \|U_{2,n}\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)} \end{aligned}$$

$$\leq \frac{2-q}{2}\big(|f|_\theta(k_s\,S(s,N)\,\mu)^{-q/2}\big)^{2/(2-q)} + \mu\,\frac{q}{2}\big(\widetilde{t}^q\,\|U_{2,n}\|^q_{H^1_{0,L}(\mathcal{C}_\Omega)}\big)^{2/q}$$
$$= \frac{2-q}{2}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)} + \frac{\mu q}{2}\big\|\widetilde{t}\,U_{2,n}\big\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)},$$

where $\theta = 2^*_s/(2^*_s - q)$. Then we get

$$\begin{aligned}
(4.8)\qquad I(\widetilde{t}\,U_{2,n}) &= \frac{1}{2}\,\big\|\widetilde{t}\,,U_{2,n}\big\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} \\
&\quad - \frac{\lambda}{q}\int_\Omega f(x)|\widetilde{t}\,U_{2,n}(x,0)|^q\,dx - \frac{1}{2^*_s}\int_\Omega |\widetilde{t}\,U_{2,n}(x,0)|^{2^*_s}\,dx \\
&\geq \frac{1-\lambda\mu}{2}\big\|\widetilde{t}\,U_{2,n}\big\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} \\
&\quad - \frac{\lambda(2-q)}{2q}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N)\big)^{-q/2})^{2/(2-q)} \\
&\quad - \frac{1}{2^*_s}\int_\Omega |\widetilde{t}\,U_{2,n}(x,0)|^{2^*_s}\,dx \\
&= (1-\lambda\mu)\bigg(\frac{1}{2}\|\widetilde{t}\,U_{2,n}\|^2_{H^1_{0,L}(\mathcal{C}_\Omega)} - \frac{1/(1-\lambda\mu)}{2^*_s}\int_\Omega |\widetilde{t}\,U_{2,n}(x,0)|^{2^*_s}\,dx\bigg) \\
&\quad - \frac{\lambda(2-q)}{2q}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)} \\
&= (1-\lambda\mu)\,J_{1/(1-\lambda\mu)}(\widetilde{t}\,U_{2,n}) \\
&\quad - \frac{\lambda(2-q)}{2q}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)} \\
&= (1-\lambda\mu)^{(N-2s+2)/2}\,J_1\big(\widetilde{t}\,U_{2,n}\big) \\
&\quad - \frac{\lambda(2-q)}{2q}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)}.
\end{aligned}$$

Therefore, corresponding to $\lambda = \lambda_n$, from (4.8), Remark 4.2 and the fact

$$I(U_{2,n}) < m_I + \frac{s}{N}(k_sS(s,N))^{N/(2s)},$$

we obtain

$$\begin{aligned}
J_1(\widetilde{t}\,U_{2,n}) &\leq \bigg(\frac{1}{1-\lambda_n\mu}\bigg)^{(N-2s+2)/2} \\
&\quad\cdot\bigg[I(\widetilde{t}\,U_{2,n}) + \frac{\lambda_n(2-q)}{2q}\,\mu^{c-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)}\bigg] \\
&< \bigg(\frac{1}{1-\lambda_n\mu}\bigg)^{(N-2s+2)/2}\bigg[m_I + \frac{s}{N}(k_sS(s,N))^{N/(2s)} \\
&\quad + \frac{\lambda_n(2-q)}{2q}\,\mu^{-q/(2-q)}\big(|f|_\theta(k_s\,S(s,N))^{-q/2}\big)^{2/(2-q)}\bigg].
\end{aligned}$$

Since $m_I \to 0$, $\widetilde{t} \to 1$ as $n \to \infty$, it is easy to see that

$$\limsup_{n\to\infty} J_1(U_{2,n}) \leq \frac{s}{N}(k_sS(s,N))^{N/(2s)}.$$

This and (4.7) tell us

$$\lim_{n\to\infty} J_1(U_{2,n}) = \frac{s}{N}(k_s S(s,N))^{N/(2s)}.$$

We can conclude that $\{U_{2,n}\}$ is a minimizing sequence for $J_1$ in $\mathcal{O}_1$. Then

$$k_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla U_{2,n}|^2\,dx\,dy - \int_\Omega |U_{2,n}(x,0)|^{2_s^*}\,dx \to 0$$

and

$$J_1(U_{2,n}) \to \frac{s}{N}(k_s S(s,N))^{N/(2s)}$$

as $n\to\infty$. This implies that $\{U_{2,n}\}$ is a $(\mathrm{PS})_c$-sequence for $J_1$ at level $c = s(k_s S(s,N))^{N/(2s)}/N$. Clearly, $\{U_{2,n}\}$ is bounded, and then there exists a subsequence $\{U_{2,n}\}$ and $U_0 \in H^1_{0,L}(\mathcal{C}_\Omega)$ such that $U_{2,n} \rightharpoonup U_0$ weakly in $H^1_{0,L}(\mathcal{C}_\Omega)$. Since $\Omega$ is bounded, we have $U_0 = 0$. Moreover, by the concentration-compactness principle (see [17] or [18]), there exist two sequence $\{x_n\}\subset\Omega$, $\{\sigma_n\}\subset(0,\infty)$ such that $\sigma_n\to\infty$ and $\|U_{2,n}-\rho_{x_n,\sigma_n}(W)\|_{H^1_{0,L}(\mathbb{R}^{N+1}_+)} \to 0$ as $n\to\infty$, where $W$ is a positive solution of (1.7). $\square$

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