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ON THE LYAPUNOV STABILITY THEORY FOR IMPULSIVE DYNAMICAL SYSTEMS

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Abstract. In this work, we establish necessary and sufficient conditions for the uniform and orbital stability of a special class of sets on impulsive dynamical systems. The results are achieved by means of Lyapunov functions.

1. Introduction

Impulsive systems describe phenomena where the continuous development of a process is interrupted by abrupt changes of state. It is already known that the study of impulsive dynamical systems is very challenging, especially when the impulses occur on the phase space and not in time. Many real world problems are described by impulsive systems. The reader may consult [1], [2], [8], [9], [11], [12], [14]–[16], [18] for more details about the theory of impulsive systems and some applications.

Qualitative properties of solutions as "asymptotic behavior" and "stability of sets" are very important in the study of trajectories on dynamical systems. Although there are many works of stability on impulsive dynamical systems, some questions concerning attraction and stability still lack answers. For instance, let $(X, \pi; M, I)$ be an impulsive system and $A \subset X$ be a nonempty set. There

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exist several examples where $A = \overline{A} \setminus M$ is stable "in some sense" but $\overline{A}$ is not stable, see Examples 3.4 and 3.5. The class of stable sets of the form $\overline{A} \setminus M$, $A \subset X$, is very important in impulsive systems since this class also includes global attractors in the sense of [2]. Thereby, we aim to show necessary and sufficient conditions to obtain results concerning the stability, orbital stability and the uniform stability for sets under impulsive systems.

In the next lines, we describe the organization of this paper. Section 2 deals with the basis of the theory of impulsive dynamical systems. In this section, we present the construction of an impulsive system, we discuss the continuity of a function which describes the times of meeting impulsive set and we exhibit the general hypotheses that will be considered in the main results.

Section 3 concerns the main results and is divided into two subsections. In Subsection 3.1, we establish results about the uniform stability, orbital stability and the attraction. Proposition 3.9 shows that a relatively compact set $A \subset X$ is uniformly $\widetilde{\pi}$-stable if and only if $\overline{A}$ is orbitally $\widetilde{\pi}$-stable. The characterization of the positive prolongation set of a relatively compact set is given in Proposition 3.11. In locally compact spaces, a relatively compact set $A \subset X$ is uniformly $\widetilde{\pi}$-stable if and only if the positive prolongation set of A coincides with its closure, see Theorem 3.13. In Theorem 3.15, we present sufficient conditions for a weakly $\widetilde{\pi}$-attractor set to be $\widetilde{\pi}$-attractor. The last result from this subsection, namely Proposition 3.16, exhibit conditions to show that the set $\overline{A} \setminus M$ is contained in the region of attraction of $A \subset X$.

In Subsection 3.2, we present conditions to obtain $\widetilde{\pi}$-stability and orbital $\widetilde{\pi}$-stability for sets of the form $\overline{A} \setminus M$, $A \subset X$. The results are achieved by means of Lyapunov functions. Theorem 3.19 concerns sufficient conditions for a set $\overline{A} \setminus M$ to be $\widetilde{\pi}$-stable. In the case of orbital stability, see Theorem 3.20, Corollaries 3.21 and 3.25. A result about instability is also presented in Theorem 3.22. The existence of Lyapunov functions under stability conditions is presented in Theorem 3.29.

2. Preliminaries

Let (X, d) be a metric space, $\mathbb{R}$ be the set of real numbers and $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all natural numbers. We denote by $\mathbb{R}_+$ the set of non-negative real numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given $A \subset X$ and $\varepsilon > 0$, let $B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ and $S(A, \varepsilon) = \{x \in X : d(x, A) = \varepsilon\}$. We use the notations ∂A and $\overline{A}$ to denote the boundary and the closure of A in X, respectively.

The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system* (or a *semiflow*) on X if the mapping $\pi\colon X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(x, t+s) = \pi(\pi(x, t), s)$ for all $x \in X$ and $t, s \in \mathbb{R}_+$. If $\mathbb{R}_+$ is replaced by $\mathbb{R}$ then the triple

$(X, \pi, \mathbb{R})$ will be called as a *dynamical system* (or a *flow*) on X. Along to this text, we shall denote the system $(X, \pi, \mathbb{R}_+)$ simply by (X, π).

The *positive orbit* of a point $x \in X$ is given by $\pi^+(x) = \{\pi(x,t) : t \in \mathbb{R}_+\}$. Let $A \subset X$ and $\Delta \subset \mathbb{R}_+$, we define

$$\pi(A, \Delta) = \{\pi(x,t) : x \in A \text{ and } t \in \Delta\} \quad \text{and} \quad \pi^+(A) = \{\pi^+(x) : x \in A\}.$$

For $x \in X$ and $t \in \mathbb{R}_+$, we write $F(x,t) = \{y \in X : \pi(y,t) = x\}$ and, for $D \subset X$ and $J \subset \mathbb{R}_+$, we define

$$F(D, J) = \{F(x,t) : x \in D \text{ and } t \in J\}.$$

A point $x \in X$ is called an *initial point* if $F(x,t) = \emptyset$ for all $t > 0$.

An *impulsive dynamical system* (IDS, for short) $(X, \pi; M, I)$ consists of a semidynamical system (X, π), a nonempty closed subset M of X such that for every $x \in M$ there exists $\varepsilon_x > 0$ satisfying

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \quad \text{and} \quad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset, \tag{2.1}$$

and a continuous function $I \colon M \to X$ whose action we explain below in the description of a positive impulsive trajectory. The set M is called the *impulsive set* and the function I is called the *impulse function*. We also define

$$M^+(x) = \left(\bigcup_{t>0} \pi(x,t) \right) \cap M \quad \text{for all } x \in X.$$

From the definition of $M^+(x)$, $x \in X$, and by condition (2.1), the mapping $\phi \colon X \to (0, +\infty]$ given by

$$\phi(x) = \begin{cases} s & \text{if } \pi(x,s) \in M \text{ and } \pi(x,t) \notin M \text{ for } 0 < t < s, \\ +\infty & \text{if } M^+(x) = \emptyset, \end{cases} \tag{2.2}$$

is well defined. Note that if $M^+(x) \neq \emptyset$, then $\phi(x)$ represents the least positive time for which the trajectory of $x \in X$ meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *positive impulsive trajectory* of x in $(X, \pi; M, I)$ is an X-valued function $\widetilde{\pi}_x$ defined in an interval $J_x \subset \mathbb{R}_+$, $0 \in J_x$, given inductively by the following rule: if $M^+(x) = \emptyset$, then $\widetilde{\pi}_x(t) = \pi(x,t)$ for all $t \in \mathbb{R}_+$ and $\phi(x) = +\infty$. However, if $M^+(x) \neq \emptyset$, then $\phi(x) < +\infty$, $\pi(x, \phi(x)) = x_1 \in M$ and $\pi(x,t) \notin M$ for $0 < t < \phi(x)$. Then we define $\widetilde{\pi}_x$ in $[0, \phi(x)]$ by

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x,t) & \text{if } 0 \leq t < \phi(x), \\ x_1^+ & \text{if } t = \phi(x), \end{cases}$$

where $x_1^+ = I(x_1)$. Since $\phi(x) < +\infty$, the process now continues from x_1^+ onwards and so on. Notice that $\widetilde{\pi}_x$ is defined on each interval $[t_n(x), t_{n+1}(x)]$, where

$$t_0(x) = 0 \quad \text{and} \quad t_{n+1}(x) = \sum_{i=0}^{n} \phi(x_i^+), \quad n = 0, 1, \ldots,$$

where $x_0^+ := x$. Hence, $\widetilde{\pi}_x$ is defined on $[0, t_{n+1}(x)]$.

The process above ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some $n \in \mathbb{N}_0$. However, it continues infinitely if $M^+(x_n^+) \neq \emptyset$ for all $n \in \mathbb{N}_0$, and in this case $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{+\infty} \phi(x_i^+)$.

The *impulsive positive orbit* of a point $x \in X$ in $(X, \pi; M, I)$ is defined by the set

$$\widetilde{\pi}^+(x) = \{\widetilde{\pi}(x, t) : t \in [0, T(x))\}.$$

Analogously to the non-impulsive case, an impulsive dynamical system satisfies the following standard properties: $\widetilde{\pi}(x, 0) = x$ and $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t + s)$ for all $x \in X$ and for all $t, s \in [0, T(x))$ such that $t + s \in [0, T(x))$.

In the next lines, we discuss the continuity of the function ϕ defined in (2.2). The reader may consult [8] for additional details.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x ($x \in X$) is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that:

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighbourhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*.

Now, let $(X, \pi; M, I)$ be an IDS. Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called TC-*tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write TC, if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have a *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write STC), if there exists a STC-tube $F(L, [0, 2\lambda])$ through x.

The following result concerns the continuity of ϕ which is accomplished outside of M.

Theorem 2.1 ([8, Theorem 3.8]). *Let $(X, \pi; M, I)$ be an* IDS. *Assume that no initial point in (X, π) belongs to the impulsive set M and that each element of M satisfies the condition* TC. *Then ϕ is continuous at x if and only if $x \notin M$.*

Let $(X, \pi; M, I)$ be an impulsive dynamical system. Throughout this work, we shall assume that the following conditions are satisfied:

(H1) No initial point in (X, π) belongs to the impulsive set M and each element of M satisfies STC, consequently ϕ is continuous on $X \setminus M$.
(H2) $M \cap I(M) = \emptyset$.

(H3) For each $x \in X$, the motion $\widetilde{\pi}(x,t)$ is defined for every $t \geq 0$, that is, $[0,+\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}(x,t)$.

Under conditions (H1)–(H3), the impulsive system $(X,\pi;M,I)$ behaves well and several important properties can be obtained, see [7]–[2].

Given $A \subset X$ and $\Delta \subset \mathbb{R}_+$, we denote

$$\widetilde{\pi}(A,\Delta) = \{\widetilde{\pi}(x,t) : x \in A,\ t \in \Delta\} \quad \text{and} \quad \widetilde{\pi}^+(A) = \bigcup_{x \in A} \widetilde{\pi}^+(x).$$

If $\widetilde{\pi}^+(A) \subset A$, we say that A is *positively $\widetilde{\pi}$-invariant.*

Next, we mention a result that will be useful later on.

Lemma 2.2 ([2, Corollary 3.9]). *Let $(X,\pi;M,I)$ be an IDS and $x \in X \setminus M$. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X which converges to x. Then, given $t \geq 0$, there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that*

$$\varepsilon_n \xrightarrow{n\to+\infty} 0 \quad \text{and} \quad \widetilde{\pi}(x_n, t+\varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(x,t).$$

In general, the closure of a positively $\widetilde{\pi}$-invariant set is not positively $\widetilde{\pi}$-invariant, see Examples 3.4 and 3.5 in the next section. However, we may obtain invariance of the closure of a positively $\widetilde{\pi}$-invariant set by excluding the points of M, see the next lemma.

Lemma 2.3 ([6, Lemma 3.37]). *Let $B \subset X$ be positively $\widetilde{\pi}$-invariant. Then $\overline{B} \setminus M$ is positively $\widetilde{\pi}$-invariant.*

The *positive limit set*, the *positive prolongation limit set* and the *positive prolongation set* of a subset $A \subset X$ are given respectively by

$$\widetilde{L}^+(A) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \widetilde{\pi}(A,\tau)}, \qquad \widetilde{J}^+(A) = \bigcap_{\varepsilon\geq 0} \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} \widetilde{\pi}(B(A,\delta),\tau)}$$

and

$$\widetilde{D}^+(A) = \bigcap_{\varepsilon > 0} \overline{\bigcup_{t\geq 0} \{\widetilde{\pi}(B(A,\varepsilon),t)}.$$

For each $x \in X$, we set $\widetilde{L}^+(x) = \widetilde{L}^+(\{x\})$, $\widetilde{J}^+(x) = \widetilde{J}^+(\{x\})$ and $\widetilde{D}^+(x) = \widetilde{D}^+(\{x\})$. It is clear that $\widetilde{L}^+(A)$, $\widetilde{J}^+(A)$ and $\widetilde{D}^+(A)$ are closed sets for all $A \subset X$. Moreover, by Lemma 2.3, we conclude that $\widetilde{L}^+(A) \setminus M$, $\widetilde{J}^+(A) \setminus M$ and $\widetilde{D}^+(A) \setminus M$ are positively $\widetilde{\pi}$-invariant sets. We have the following straightforward result about impulsive limit sets.

Lemma 2.4 ([4, Lemma 3.27]). *Let $A \subset X$. The following statements hold:*

(a) *$y \in \widetilde{L}^+(A)$ if and only if there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset A$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n\to+\infty} +\infty$ and $\widetilde{\pi}(x_n,t_n) \xrightarrow{n\to+\infty} y$;*

(b) *$y \in \widetilde{J}^+(A)$ if and only if there are sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$, $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $d(x_n,A) \xrightarrow{n\to+\infty} 0$, $t_n \xrightarrow{n\to+\infty} +\infty$ and $\widetilde{\pi}(x_n,t_n) \xrightarrow{n\to+\infty} y$;*

(c) $y \in \widetilde{D}^+(A)$ *if and only if there are sequences* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *such that* $d(x_n, A) \xrightarrow{n\to+\infty} 0$ *and* $\widetilde{\pi}(x_n, t_n) \xrightarrow{n\to+\infty} y$.

For more details about the structure of these types of impulsive systems, the reader may consult [2]–[5], [7]–[10], [13], [16], [17].

3. Main results

This section, which presents the main results, is divided into two subsections. The first subsection concerns results about the uniform and orbital stability and their relations. In the second subsection, we present results on the stability by the use of Lyapunov functions.

In order to present the main results, we start by defining the concepts of regions of attraction. Also, we recall the definitions of the stability, the uniform stability and the orbital stability for impulsive systems.

Definition 3.1. The *region of the weak attraction* of a set $A \subset X$ is defined by

$$\widetilde{P}_w(A) = \{x \in X : \widetilde{L}^+(x) \cap \overline{A} \neq \emptyset\}$$

and the *region of the attraction* of $A \subset X$ is defined by

$$\widetilde{P}(A) = \{x \in X : \widetilde{L}^+(x) \neq \emptyset \text{ and } \widetilde{L}^+(x) \subset \overline{A}\}.$$

We say that a set $A \subset X$ is a *weak $\widetilde{\pi}$-attractor* if $\widetilde{P}_w(A)$ is a neighbourhood of A and a *$\widetilde{\pi}$-attractor* if $\widetilde{P}(A)$ is a neighbourhood of A.

Note that $\widetilde{P}(A) \subset \widetilde{P}_w(A)$ for all $A \subset X$.

Lemma 3.2. *The sets* $\widetilde{P}(A)$, $\widetilde{P}_w(A)$, $\overline{\widetilde{P}(A)} \setminus M$ *and* $\overline{\widetilde{P}_w(A)} \setminus M$ *are positively $\widetilde{\pi}$-invariant for all* $A \subset X$.

Proof. Let $x \in X$ be such that $\widetilde{L}^+(x) \neq \emptyset$ and $t \geq 0$. Note that $\widetilde{L}^+(\widetilde{\pi}(x,t)) = \widetilde{L}^+(x)$. This shows that $\widetilde{P}(A)$ and $\widetilde{P}_w(A)$ are positively $\widetilde{\pi}$-invariant. Using 2.3, we conclude that the sets $\overline{\widetilde{P}(A)} \setminus M$ and $\overline{\widetilde{P}_w(A)} \setminus M$ are positively $\widetilde{\pi}$-invariant. □

Next, we present the concepts of stability of a set $A \subset X$. Although the definitions are stated in general situation and some results are proved in general form, the meaningful case of investigation appears for a set A with $A \setminus M$ closed in $X \setminus M$.

Definition 3.3. A subset $A \subset X$ is said to be:

(a) *$\widetilde{\pi}$-stable* if for every $\varepsilon > 0$ and $x \in A$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(A, \varepsilon)$;

(b) *uniformly $\widetilde{\pi}$-stable* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon)$;

(c) *orbitally $\widetilde{\pi}$-stable* if for every neighbourhood U of A there exists a positively $\widetilde{\pi}$-invariant neighbourhood V of A with $V \subset U$.

Let $A \subset X$ be a nonempty set and suppose that A is stable in some sense. Is the closure $\overline{A}$ stable too? In general, the answer is negative as the next examples show.

Example 3.4. Let $(\mathbb{R}, \pi; M, I)$ be an IDS, where the semidynamical system $(\mathbb{R}, \pi)$ is given by

$$\pi(x,t) = x + t, \quad x \in \mathbb{R} \quad \text{and} \quad t \in \mathbb{R}_+,$$

$M = \{1\}$ and $I\colon M \to \mathbb{R}$ is given by $I(1) = 0$. Let $A = [0,1)$. See Figure 1.

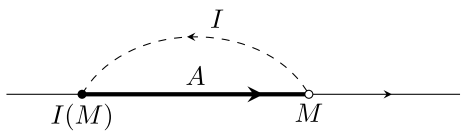

Figure 1. $\widetilde{\pi}^+(A) = A$ and $\widetilde{\pi}^+(\overline{A}) = [0, +\infty)$.

Note that A is a $\widetilde{\pi}$-attractor, orbitally $\widetilde{\pi}$-stable and $\widetilde{\pi}$-stable but it is not uniformly $\widetilde{\pi}$-stable. On the other hand, the set $\overline{A}$ is neither a $\widetilde{\pi}$-attractor, nor it is orbitally $\widetilde{\pi}$-stable and $\widetilde{\pi}$-stable.

Example 3.5. Let $(\mathbb{R}^2, \pi; M, I)$ be an IDS, where the semidynamical system $(\mathbb{R}^2, \pi)$ is given by

$$\pi((x,y),t) = (x+t, y), \quad (x,y) \in \mathbb{R}^2 \text{ and } t \geq 0,$$

$M = \{(x,y) \in \mathbb{R}^2 : x = 2\}$ and $I\colon M \to X$ is given by $I(x,y) = (0, y/2)$, $(x,y) \in M$. Let $A = [0,2) \times \{0\}$. See Figure 2.

The set A is $\widetilde{\pi}$-stable but it is neither orbitally $\widetilde{\pi}$-stable nor uniformly $\widetilde{\pi}$-stable. However, the set $\overline{A}$ is neither $\widetilde{\pi}$-stable, orbitally $\widetilde{\pi}$-stable nor uniformly $\widetilde{\pi}$-stable. Moreover, A is a $\widetilde{\pi}$-attractor with region of attraction $\widetilde{P}(A) = \{(x,y) \in \mathbb{R}^2 : x < 2\}$, while $\overline{A}$ is not a $\widetilde{\pi}$-attractor.

In many cases, the points of M are responsible for destroying the stability of $\overline{A}$ provided that A is stable in some sense. There are many unstable sets that become stable when we exclude their impulsive points, for instance, as presented in Examples 3.4 and 3.5, the set $\overline{A}$ is unstable while the set $\overline{A} \setminus M$ is stable. There is still no theory that characterizes the stability of sets of the form $\overline{A} \setminus M$. The class of the sets $\overline{A} \setminus M$, $A \subset X$, is very important in the theory of impulsive dynamical systems. For instance, a global attractor in the sense as presented in [2], belongs to this class. In this way, we shall present results concerning stability of sets of the form $\overline{A} \setminus M$, $A \subset X$. Moreover, we shall study results about uniform stability for relatively compact sets.

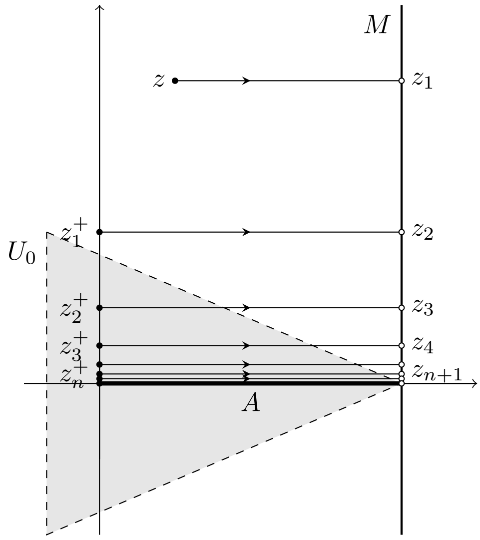

Figure 2. Impulsive trajectory of $z \in \mathbb{R}^2$. There is no positively $\widetilde{\pi}$-invariant neighbourhood V of A such that $V \subset U_0$, where U_0 is the hatched area.

3.1. Uniform and orbital stability. When X is locally compact and $A \subset X$ is compact, the concepts of the $\widetilde{\pi}$-stability, orbital $\widetilde{\pi}$-stability and the uniform $\widetilde{\pi}$-stability of A are equivalent. See the next result.

Theorem 3.6 ([7, Theorem 3.3], [9, Theorem 4.1]). *Let $(X, \pi; M, I)$ be an* IDS*, X be locally compact and A be a compact subset of X. Then the following conditions are equivalent*:

(a) *A is $\widetilde{\pi}$-stable*;
(b) *A is orbitally $\widetilde{\pi}$-stable*;
(c) *A is uniformly $\widetilde{\pi}$-stable*;
(d) $\widetilde{D}^+(A) = A$.

Remark 3.7. In Theorem 3.6, the implications (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) hold for any metric space X and for any nonempty set $A \subset X$.

In continuous dynamical systems theory, the positive invariance of the closure of a set is preserved provided this set is positively invariant. Nonetheless, this fact is not true in general for impulsive systems (see Example 3.4). Under stability condition, it is well-known that compact $\widetilde{\pi}$-stable sets are positively $\widetilde{\pi}$-invariant sets, see [9, Theorem 2.3]. But we may not assure that the closure of a non-compact positively $\widetilde{\pi}$-invariant set still positively $\widetilde{\pi}$-invariant, even if this set is $\widetilde{\pi}$-stable or orbitally $\widetilde{\pi}$-stable, see Example 3.4. Yet, in the case of uniform $\widetilde{\pi}$-stability, we have the following straightforward result.

Lemma 3.8. *Let* $(X, \pi; M, I)$ *be an* IDS *and* $A \subset X$*. If* $A \subset X$ *is uniformly* $\widetilde{\pi}$*-stable then* $\overline{A}$ *is positively* $\widetilde{\pi}$*-invariant.*

The concepts of the uniform $\widetilde{\pi}$-stability and the orbital $\widetilde{\pi}$-stability are not equivalent in general. However, we may relate the uniform $\widetilde{\pi}$-stability of a relatively compact set with the orbital $\widetilde{\pi}$-stability of its closure. See the next result.

Proposition 3.9. *Let* $(X, \pi; M, I)$ *be an* IDS *and* $A \subset X$ *be relatively compact. Then the set* A *is uniformly* $\widetilde{\pi}$*-stable if and only if* $\overline{A}$ *is orbitally* $\widetilde{\pi}$*-stable.*

Proof. First, let us assume that $\overline{A}$ is orbitally $\widetilde{\pi}$-stable. Given $\varepsilon > 0$ there exists a positively $\widetilde{\pi}$-invariant neighbourhood V of $\overline{A}$ such that $V \subset B(\overline{A}, \varepsilon)$. Since $\overline{A}$ is compact one can obtain $\delta = \delta(\varepsilon) > 0$ such that $B(\overline{A}, \delta) \subset V$. Consequently, we obtain

$$\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset \widetilde{\pi}(V, [0, +\infty)) \subset V \subset B(A, \varepsilon)$$

and we conclude that A is uniformly $\widetilde{\pi}$-stable.

Now, let us assume that A is uniformly $\widetilde{\pi}$-stable. Let U be a neighbourhood of $\overline{A}$. By compactness there is $\varepsilon > 0$ such that $B(\overline{A}, \varepsilon) \subset U$. Since A is uniformly $\widetilde{\pi}$-stable there exists $\delta = \delta(\varepsilon) > 0$ such that $\widetilde{\pi}(B(A, \delta), [0, +\infty)) \subset B(A, \varepsilon)$, that is,

$$\widetilde{\pi}(B(\overline{A}, \delta), [0, +\infty)) \subset B(\overline{A}, \varepsilon).$$

Taking $V = \widetilde{\pi}(B(\overline{A}, \delta), [0, +\infty))$, we conclude that $\overline{A}$ is orbitally $\widetilde{\pi}$-stable. □

Remark 3.10. According to the Proposition 3.9, we obtain the equivalence (b) ⇔ (c) from Theorem 3.6 for compact sets without assuming that X is locally compact.

Let $A \subset X$. In general, the positive prolongation set $\widetilde{D}^+(A)$ is not equal to the set $\bigcup \{\widetilde{D}^+(a) : a \in A\}$, see [4, Example 3.29]. But, when $A \subset X$ is compact we get the equality $\widetilde{D}^+(A) = \bigcup \{\widetilde{D}^+(a) : a \in A\}$, see [4, Proposition 3.30]. In the case of relatively compact sets, we have the following result.

Proposition 3.11. *If* $A \subset X$ *is relatively compact then*

$$\widetilde{D}^+(A) = \bigcup \{\widetilde{D}^+(a) : a \in \overline{A}\}.$$

Proof. It is enough to see that

$$\widetilde{D}^+(A) = \widetilde{D}^+(\overline{A}) \quad \text{and} \quad \widetilde{D}^+(\overline{A}) = \bigcup \{\widetilde{D}^+(a) : a \in \overline{A}\}$$

by [4, Proposition 3.30]. □

In Theorem 3.6, the equivalence (c) ⇔ (d) holds for compact sets in locally compact spaces. If we assume that $A \subset X$ is relatively compact and uniformly $\widetilde{\pi}$-stable, then we get the following result without assume that X is locally compact.

Theorem 3.12. *Let $A \subset X$ be relatively compact and uniformly $\widetilde{\pi}$-stable. Then $\widetilde{D}^+(A) = \overline{A}$.*

Proof. Since $\widetilde{D}^+(A)$ is closed we have $\overline{A} \subset \widetilde{D}^+(A)$. On the other hand, let $z \in \widetilde{D}^+(A)$. By Proposition 3.11 there is $a \in \overline{A}$ such that $z \in \widetilde{D}^+(a)$. Consequently, there are sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ and $\{a_n\}_{n\in\mathbb{N}} \subset X$ with $a_n \xrightarrow{n\to+\infty} a$ and

$$\widetilde{\pi}(a_n, t_n) \xrightarrow{n\to+\infty} z.$$

Let $\varepsilon > 0$ be arbitrary. By the uniform $\widetilde{\pi}$-stability of A, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\widetilde{\pi}(B(A,\delta), [0,+\infty)) \subset B(A,\varepsilon).$$

Then, for n sufficiently large, we have $\widetilde{\pi}(a_n, t_n) \in B(A,\varepsilon)$ which implies in

$$z \in \overline{B(A,\varepsilon)}.$$

Since $\varepsilon > 0$ is taken arbitrary, we have $z \in \bigcap_{\varepsilon>0} \overline{B(A,\varepsilon)} = \overline{A}$. Hence, $\widetilde{D}^+(A) = \overline{A}$. □

Assuming X is locally compact, we obtain the converse of Theorem 3.12.

Theorem 3.13. *Let $(X,\pi;M,I)$ be an* IDS*, X be locally compact and $A \subset X$ be relatively compact. Then the set A is uniformly $\widetilde{\pi}$-stable if and only if $\widetilde{D}^+(A) = \overline{A}$.*

Proof. Assume that $\widetilde{D}^+(A) = \overline{A}$. Since $\widetilde{D}^+(A) = \widetilde{D}^+(\overline{A})$ it follows by Theorem 3.6 that $\overline{A}$ is uniformly $\widetilde{\pi}$-stable. But it is equivalent to A be uniformly $\widetilde{\pi}$-stable. The necessary condition follows by Theorem 3.12. □

Theorem 3.15 establishes sufficient conditions for a relatively compact set to be $\widetilde{\pi}$-attractor. Before presenting Theorem 3.15 we exhibit an auxiliary result.

Lemma 3.14 ([4, Lemma 3.31]). *Let $x \notin M$ and $y \in \widetilde{L}^+(x)$, then $\widetilde{J}^+(x) \subset \widetilde{J}^+(y)$.*

Theorem 3.15. *Let $(X,\pi;M,I)$ be an* IDS *and $A \subset X$ be relatively compact. Assume that A is uniformly $\widetilde{\pi}$-stable and weakly $\widetilde{\pi}$-attractor. Then A is $\widetilde{\pi}$-attractor and $\overline{A} \subset \widetilde{P}(A)$.*

Proof. Since A is weakly $\widetilde{\pi}$-attractor then there is an open set $\mathcal{O}$ in X such that $A \subset \mathcal{O} \subset \widetilde{P}_w(A)$. We claim that $\mathcal{O} \subset \widetilde{P}(A)$. Indeed, let $x \in \mathcal{O}$ and take $v \in \widetilde{L}^+(x) \cap \overline{A}$.

First, let us assume that $x \notin M$. By Lemma 3.14 we have

$$\widetilde{L}^+(x) \subset \widetilde{J}^+(x) \subset \widetilde{J}^+(v) \subset \widetilde{D}^+(v) \subset \widetilde{D}^+(A),$$

where the last set inclusion follows by Proposition 3.11. By Theorem 3.12 we have $\widetilde{D}^+(A) = \overline{A}$. Therefore, $\widetilde{L}^+(x) \subset \overline{A}$ and $x \in \widetilde{P}(A)$, that is, $\mathcal{O} \subset \widetilde{P}(A)$.

Second, if $x \in M$, we may choose $\eta \in (0, \phi(x))$ such that $y = \widetilde{\pi}(x, \eta) = \pi(x, \eta) \in \mathcal{O} \setminus M$. Then $\widetilde{L}^+(y) \cap \overline{A} = \widetilde{L}^+(x) \cap \overline{A} \neq \emptyset$ and using the proof of the previous case we obtain $\widetilde{L}^+(x) = \widetilde{L}^+(y) \subset \overline{A}$. Thus $x \in \widetilde{P}(A)$, that is, $\mathcal{O} \subset \widetilde{P}(A)$. Therefore, A is $\widetilde{\pi}$-attractor.

At last, let us show that $\overline{A} \subset \widetilde{P}(A)$. Since A is uniformly $\widetilde{\pi}$-stable it follows that $\overline{A}$ is positively $\widetilde{\pi}$-invariant, see Lemma 3.8. Thus $\emptyset \neq \widetilde{L}^+(x) \subset \overline{A}$ for all $x \in \overline{A}$. □

As shown in Example 3.5 the set $\overline{A}$ is not contained in $\widetilde{P}(A)$ while $\overline{A} \setminus M \subset \widetilde{P}(A)$. In the next result, we present sufficient conditions for the set A to satisfy the property $\overline{A} \setminus M \subset \widetilde{P}(A)$.

Proposition 3.16. *Let $A \subset X$ be a relatively compact set such that $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable. Then $\overline{A} \setminus M \subset \widetilde{P}(A)$.*

Proof. Let $x \in \overline{A} \setminus M$ and $\varepsilon > 0$. Then there is $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x, \delta), [0, +\infty)) \subset B(\overline{A} \setminus M, \varepsilon)$. Thus,

$$\widetilde{\pi}^+(x) \subset \bigcap_{\varepsilon > 0} B\big(\overline{A} \setminus M, \varepsilon\big) \subset \overline{A}.$$

Since A is a relatively compact set we obtain $\emptyset \neq \widetilde{L}^+(x) \subset \overline{A}$ for all $x \in \overline{A} \setminus M$. Hence, $\overline{A} \setminus M \subset \widetilde{P}(A)$. □

3.2. Stability via Lyapunov functions. In this subsection, we present sufficient conditions for the $\widetilde{\pi}$-stability and the orbital $\widetilde{\pi}$-stability for sets of the form $\overline{A} \setminus M$, $A \subset X$, obtained by the use of Lyapunov functions (non-negative scalar functions of the state which decrease monotonically along trajectories). We also include a result about instability. First, we present an auxiliary result.

Lemma 3.17. *Let $(X, \pi; M, I)$ be an* IDS *and $\mathcal{O} \subset X$ be an open set such that $I(\overline{\mathcal{O}} \cap M) \subset \mathcal{O}$. Assume that there exist $x \in \mathcal{O}$ and $t_0 > 0$ such that $\widetilde{\pi}(x, t_0) \notin \mathcal{O}$. Then there exists $\tau \in (0, t_0]$ such that $\widetilde{\pi}(x, [0, \tau)) \subset \mathcal{O}$ and $\widetilde{\pi}(x, \tau) \in \partial\mathcal{O} \setminus M$.*

Proof. Let $\tau = \min\{t > 0 : \widetilde{\pi}(x, t) \notin \mathcal{O}\}$. By the openness of $\mathcal{O}$ there is $\varepsilon \in (0, \phi(x))$ such that $\widetilde{\pi}(x, [0, \varepsilon)) \subset \mathcal{O}$ and this fact shows that $\tau > 0$.

We claim that $\tau \neq \sum\limits_{i=0}^{n} \phi(x_i^+)$ for all $n \in \mathbb{N}_0$. In fact, if $\tau = \sum\limits_{i=0}^{n} \phi(x_i^+)$ for some $n \in \mathbb{N}_0$ then

$$\widetilde{\pi}(x, \tau) = x_{n+1}^+.$$

By the minimality of τ we have $\pi(x_n^+, (0, \phi(x_n^+))) \subset \mathcal{O}$ and, consequently, $x_{n+1} = \pi(x_n^+, \phi(x_n^+)) \in \overline{\mathcal{O}} \cap M$. Using the hypothesis $I(\overline{\mathcal{O}} \cap M) \subset \mathcal{O}$ we get $x_{n+1}^+ = I(x_{n+1}) \in \mathcal{O}$ which contradicts the definition of τ. Hence, $\tau \neq \sum\limits_{i=0}^{n} \phi(x_i^+)$ for all

$n \in \mathbb{N}_0$. Thus, there exists $k \in \mathbb{N}_0$ such that

$$\sum_{i=-1}^{k-1} \psi(x_i^+) < \tau < \sum_{i=-1}^{k} \psi(x_i^+),$$

where $\psi(x_{-1}^+) = 0$ and $\psi(x_i^+) = \phi(x_i^+)$ for $i = 0, \ldots, k$. Denote $\sum_{i=-1}^{k-1} \psi(x_i^+)$ by η_1 and $\sum_{i=-1}^{k} \psi(x_i^+)$ by η_2. Since

$$\widetilde{\pi}\big(x, (\eta_1, \tau)\big) = \pi\big(x_k^+, (0, \tau - \eta_1)\big) \subset \mathcal{O}, \qquad \widetilde{\pi}(x, \tau) = \pi\big(x_k^+, \tau - \eta_1\big) \notin \mathcal{O}$$

and π is continuous, it implies that $\widetilde{\pi}(x, \tau) = \pi(x_k^+, \tau - \eta_1) \in \partial\mathcal{O}$. Now, note that $\widetilde{\pi}(x, t) \notin M$ for all $t > 0$, as $I(M) \cap M = \emptyset$ by condition (H2). Hence, $\widetilde{\pi}(x, \tau) \in \partial\mathcal{O} \setminus M$. □

Remark 3.18. If $A \subset X$ is such that $d(I(x), A) < d(x, A)$ for all $x \in M$, then the set $\mathcal{O} = B(A, r)$ satisfies the condition $I(\overline{\mathcal{O}} \cap M) \subset \mathcal{O}$.

Note in the next result that we do not need any hypothesis on the boundedness of A.

Theorem 3.19. *Let $(X, \pi; M, I)$ be an* IDS*, $A \subset X$ and $r_0 > 0$ be a number such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Let $\gamma > 0$ and $\mathcal{U}$ be a neighbourhood of A with $B(A, \gamma) \subset \mathcal{U}$. Assume that there are $k > 0$ and a mapping $V: \mathcal{U} \to \mathbb{R}_+$ continuous on $\mathcal{U} \setminus M$ satisfying the following conditions:*

(a) *$V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n\to+\infty} 0$ implies $d(x_n, A) \xrightarrow{n\to+\infty} 0$;*

(b) *$V(\widetilde{\pi}(x, t)) \leq kV(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\widetilde{\pi}(x, [0, t]) \subset \mathcal{U}$.*

Then $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable.

Proof. Suppose to the contrary that there are $x \in \overline{A} \setminus M$, $0 < \varepsilon < \min\{\gamma, r_0\}$ and sequences $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $x_n \xrightarrow{n\to+\infty} x$ and

$$\widetilde{\pi}(x_n, t_n) \notin B(A, \varepsilon) \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

By the continuity of V in $\mathcal{U} \setminus M$ we conclude that $V(z) = 0$ for all $z \in \overline{A} \setminus M$. Since $x \in \overline{A} \setminus M$, we may assume that $x_n \notin M$ for all $n \in \mathbb{N}$ and, consequently, we have

$$V(x_n) \xrightarrow{n\to+\infty} V(x) = 0. \tag{3.2}$$

Let $n_0 \in \mathbb{N}$ be such that $x_n \in B(A, \varepsilon)$ for all $n \geq n_0$. By Lemma 3.17, there exists a sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$\tau_n \leq t_n, \ \widetilde{\pi}(x_n, [0, \tau_n)) \subset B(A, \varepsilon) \text{ and } \widetilde{\pi}(x_n, \tau_n) \in S(A, \varepsilon) \text{ for all } n \geq n_0. \tag{3.3}$$

Using condition (b) and (3.2) we obtain

$$V(\widetilde{\pi}(x_n, \tau_n)) \xrightarrow{n\to+\infty} 0.$$

Thus, by condition (a), we get $d(\widetilde{\pi}(x_n, \tau_n), A) \xrightarrow{n\to+\infty} 0$. But it contradicts the fact that $\widetilde{\pi}(x_n, \tau_n) \in S(A, \varepsilon)$ for all $n \geq n_0$. Therefore, $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable. □

Theorem 3.20. *Let $(X, \pi; M, I)$ be an* IDS*, $A \subset X$ and $r_0 > 0$ be a number such that $I(\overline{B(A,r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Let $\mathcal{U}$ be a neighbourhood of $\overline{A}$, $\mathcal{O}$ be an open set in X and $\alpha \in (0, r_0)$ be such that $\overline{A} \setminus M \subset \mathcal{O} \subset \mathcal{U}$ and $d(z, A) \geq \alpha$ for all $z \in X \setminus \overline{\mathcal{O}}$. Assume that there is a mapping $V : \mathcal{U} \to \mathbb{R}_+$ continuous on $\mathcal{U} \setminus M$ satisfying the following conditions*:

(a) *$V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n\to+\infty} 0$ implies $d(x_n, A) \xrightarrow{n\to+\infty} 0$;*

(b) *$V(\widetilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\widetilde{\pi}(x, [0, t]) \subset \mathcal{U}$.*

Then $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable.

Proof. Let $\mathcal{O}_1 = \{z \in \mathcal{O} : d(z, A) < \alpha\}$. Note that $\mathcal{O}_1$ is open and $\overline{A} \setminus M \subset \mathcal{O}_1$. Also, we have $I(\overline{\mathcal{O}}_1 \cap M) \subset \mathcal{O}_1$. In fact, if $x \in \overline{\mathcal{O}}_1 \cap M$ then $x \in \overline{\mathcal{O}} \cap M$ and $d(x, A) \leq \alpha$. By hypothesis we get $I(x) \in B(A, \alpha)$. Moreover, $I(x) \in \mathcal{O}$ since $d(z, A) \geq \alpha$ for all $z \in X \setminus \overline{\mathcal{O}}$.

Now, note that $d(z, A) = \alpha$ for all $z \in \partial\mathcal{O}_1 \setminus M$ and define

$$\mu = \inf\{V(z) : z \in \partial\mathcal{O}_1 \setminus M\}.$$

We assert that $\mu > 0$. Indeed, if there is a sequence $\{v_n\}_{n\in\mathbb{N}} \subset \partial\mathcal{O}_1 \setminus M$ such that $V(v_n) \xrightarrow{n\to+\infty} 0$ then $d(v_n, A) \xrightarrow{n\to+\infty} 0$ as we have condition (a). But it is a contradiction since $d(v_n, A) = \alpha$ for all $n \in \mathbb{N}$. Hence, $\mu > 0$.

Let $K = \{x \in \mathcal{O}_1 \setminus M : V(x) < \mu\}$. By the continuity of V in $\mathcal{U} \setminus M$ it is not difficult to see that $V(z) = 0$ for all $z \in A \cup (\partial A \setminus M)$. This shows that $\overline{A} \setminus M \subset K$.

Now, we claim that K is positively $\widetilde{\pi}$-invariant. In fact, let $x \in K$. First, let us show that $\widetilde{\pi}(x, t) \in \mathcal{O}_1$ for all $t \geq 0$. For that, suppose to the contrary that there is $t_* > 0$ such that $\widetilde{\pi}(x, t_*) \notin \mathcal{O}_1$. By Lemma 3.17, there is $\tau \in (0, t_*]$ such that $\widetilde{\pi}(x, [0, \tau)) \subset \mathcal{O}_1$ and $\widetilde{\pi}(x, \tau) \in \partial\mathcal{O}_1 \setminus M$. Then, using condition (b) and the definition of μ, we obtain

$$\mu \leq V(\widetilde{\pi}(x, \tau)) \leq V(x) < \mu$$

which is a contradiction. In conclusion, using again condition (b), we have $\widetilde{\pi}(x, t) \in K$ for all $x \in K$ and $t \geq 0$. Therefore, K is an open positively $\widetilde{\pi}$-invariant neighbourhood of $\overline{A} \setminus M$. □

For relatively compact sets, we have the following consequence of Theorem 3.20.

Corollary 3.21. *Let $(X,\pi;M,I)$ be an* IDS, *$A\subset X$ be compact and $r_0>0$ be such that $I(\overline{B(A,r)}\cap M)\subset B(A,r)$ for all $0<r<r_0$. Let $\mathcal{U}$ be a neighbourhood of A and assume that there is a mapping $V\colon\mathcal{U}\to\mathbb{R}_+$ continuous on $\mathcal{U}\setminus M$ satisfying the conditions*:

(a) *$V(x)=0$ for $x\in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{U}$ such that $V(x_n)\xrightarrow{n\to+\infty}0$ implies $d(x_n,A)\xrightarrow{n\to+\infty}0$*;

(b) *$V(\widetilde{\pi}(x,t))\le V(x)$ for all $x\in\mathcal{U}\setminus M$ and $t\ge 0$ such that $\widetilde{\pi}(x,[0,t])\subset\mathcal{U}$.*

Then $A\setminus M$ is orbitally $\widetilde{\pi}$-stable.

In the next result, we characterize the sets whose closure are unstable.

Theorem 3.22. *Let $A\subset X$ and $\mathcal{U}$ be a neighbourhood of A. Assume that there exists a mapping $V\colon\mathcal{U}\to\mathbb{R}_+$ continuous on $\mathcal{U}\setminus M$ satisfying the following conditions*:

(a) *$V(x)=0$ for $x\in A$*;

(b) *there exist $a\in\overline{A}$ and $s>0$ such that $V(\widetilde{\pi}(a,s))>0$.*

Then $\overline{A}$ is not $\widetilde{\pi}$-stable. In particular, $\overline{A}$ is neither orbitally $\widetilde{\pi}$-stable nor uniformly $\widetilde{\pi}$-stable.

Proof. Suppose to the contrary that $\overline{A}$ is $\widetilde{\pi}$-stable. Then given $\varepsilon>0$ there is $\delta>0$ such that $\widetilde{\pi}(B(a,\delta),[0,+\infty))\subset B(A,\varepsilon)$. Thus $\widetilde{\pi}^+(a)\subset\bigcap_{\varepsilon>0}B(A,\varepsilon)=\overline{A}$. The hypothesis (H2) implies $\widetilde{\pi}(a,s)\in\overline{A}\setminus M$. Since V is continuous on $\mathcal{U}\setminus M$ and we have condition (a), one can conclude that $V(\widetilde{\pi}(a,s))=0$. But this contradicts the condition (b). □

Example 3.23. Consider a linear differential equation $\dot{x}=Ax$ in the Hilbert space $H=L_2[0,1]$, with the continuous operator $A\colon L_2[0,1]\to L_2[0,1]$ defined by

$$(A\varphi)(\tau)=-\tau\varphi(\tau)\quad\text{for all } \tau\in[0,1] \text{ and } \varphi\in L_2[0,1].$$

Let $U(t)$ be given by

$$(U(t)\varphi)(\tau)=e^{-\tau t}\varphi(\tau)\quad\text{for all } t\in\mathbb{R} \text{ and } \varphi\in L_2[0,1].$$

Thus, the dynamical system generated by $\dot{x}=Ax$ is given by $(L_2[0,1],\pi,\mathbb{R})$, where

$$\pi(\varphi,t)=U(t)\varphi\quad\text{for all } \varphi\in L_2[0,1] \text{ and } t\in\mathbb{R}.$$

Consider the impulsive set

$$M=\left\{\psi\in L_2[0,1]:\int_0^1|\psi(s)|^2\,ds=1\right\}$$

and let $I\colon M\to L_2[0,1]$ be an impulse function such that

$$\|I(\psi)\|_2\le\alpha\|\psi\|_2\quad\text{for all } \psi\in M, \text{ where } 0<\alpha<1.$$

Thus, we have the associate impulsive system $(L_2[0,1], \pi; M, I)$. Note that $I(M) \cap M = \emptyset$ and each point of M satisfies STC.

Let $r > 2$ and $B_{L_2}(0,r)$ be the open ball in $L_2[0,1]$ with center 0 and radius r, where $\|\cdot\|_2$ is the usual norm in $L_2[0,1]$.

(1) The set $A_1 = \{\psi \in L_2[0,1] : 1 \leq \|\psi\|_2 \leq 2\}$ is not $\widetilde{\pi}$-stable.

In fact, define the mapping $V\colon B_{L_2}(0,r) \to \mathbb{R}_+$ by

$$V(x) = \begin{cases} \inf\limits_{a \in A_1} \|x - a\|_2 & \text{if } x \in B_{L_2}(0,r) \setminus A_1, \\ 0 & \text{if } x \in A_1, \end{cases}$$

which is continuous on $B_{L_2}(0,r) \setminus M$. Let $\varphi \in A_1 \cap M$. Since $\|\pi(\varphi,t)\|_2 < 1$ for all $t > 0$ we have $\phi(\varphi) = +\infty$ and

$$\|\widetilde{\pi}(\varphi,t)\|_2 = \|\pi(\varphi,t)\|_2 < 1, \quad \text{for all } t > 0.$$

Thus, for an arbitrary $s > 0$ we get $\widetilde{\pi}(\varphi,s) \notin A_1$ and

$$V(\widetilde{\pi}(\varphi,s)) = \inf_{a \in A_1} \|\widetilde{\pi}(\varphi,s) - a\|_2 \geq \inf_{a \in A_1} \|a\|_2 - \|\widetilde{\pi}(\varphi,s)\|_2 \geq 1 - \|\widetilde{\pi}(\varphi,s)\|_2 > 0.$$

By Theorem 3.22, we have A_1 is not $\widetilde{\pi}$-stable.

(2) Let $A_2 = \{\psi \in L_2[0,1] : \|\psi\|_2 \leq 1\}$. Then $A_2 \setminus M$ is orbitally $\widetilde{\pi}$-stable.

Indeed, consider the mapping $V\colon B_{L_2}(0,r) \to \mathbb{R}_+$ defined by

$$V(x) = \begin{cases} \|x\|_2 & \text{if } x \in B_{L_2}(0,r) \setminus A_2, \\ 0 & \text{if } x \in A_2. \end{cases}$$

Note that V is continuous on $B_{L_2}(0,r) \setminus M$ and $I(\overline{B_{L_2}(0,\mu)} \cap M) \subset B_{L_2}(0,\mu)$ for all $\mu > 0$. Let $\mathcal{U} = B_{L_2}(0,r)$ and $\mathcal{O} = B_{L_2}(0,(r+1)/2)$. As presented in [4, Example 3.53], we have $\|\widetilde{\pi}(\varphi,t)\|_2 \leq \|\varphi\|_2$ for all $\varphi \in L_2[0,1]$ and $t \geq 0$. Then $V(\widetilde{\pi}(\varphi,t)) \leq V(\varphi)$ for all $\varphi \in B_{L_2}(0,r)$ and $t \geq 0$. By Theorem 3.20, the set $A_2 \setminus M$ is orbitally $\widetilde{\pi}$-stable.

Proposition 3.24 presents sufficient conditions for a set $A \subset X$ to be a $\widetilde{\pi}$-attractor.

Proposition 3.24. *Let $(X, \pi; M, I)$ be an* IDS *and $A \subset X$ be a nonempty subset. Assume that there is a real valued function V defined on a neighbourhood $\mathcal{U}$ of $\overline{A}$ and continuous on $\mathcal{U} \setminus M$ satisfying:*

(a) *$V(\widetilde{\pi}(x,t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\widetilde{\pi}(x,[0,t]) \subset \mathcal{U}$.*

(b) *If $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.*

If there exist an open relatively compact positively $\widetilde{\pi}$-invariant set K and an open set $\mathcal{O} \subset X$ with $A \subset K \subset \overline{K} \subset \mathcal{O} \subset \mathcal{U}$, then A is $\widetilde{\pi}$-attractor.

Proof. Since K is a relatively compact positively $\widetilde{\pi}$-invariant set we have $\emptyset \neq \widetilde{L}^+(x) \subset \overline{K}$ for each $x \in K$. We claim that $K \subset \widetilde{P}(A)$. Indeed, let $x \in K$ and $y \in \widetilde{L}^+(x)$. Let $\{s_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ be a sequence such that $s_n \xrightarrow{n\to+\infty} +\infty$ and $\widetilde{\pi}(x, s_n) \xrightarrow{n\to+\infty} y$.

Case 1. $y \notin M$. Suppose to the contrary that $y \notin \overline{A}$. Let $t > 0$ be such that $\widetilde{\pi}(y, [0, t]) \cap \overline{A} = \emptyset$. Since $\widetilde{L}^+(x) \setminus M$ is positively $\widetilde{\pi}$-invariant we get $\widetilde{\pi}(y, [0, t]) \subset \widetilde{L}^+(x)$. Then there is a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $t_n \leq s_n$, for $n \in \mathbb{N}$, $t_n \xrightarrow{n\to+\infty} +\infty$ and

$$\widetilde{\pi}(x, t_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(y, t).$$

We may assume that $\widetilde{\pi}(x, s_n) \notin \overline{A}$ and $\widetilde{\pi}(x, t_n) \notin \overline{A}$ for all $n \in \mathbb{N}$. As $I(M)\cap M = \emptyset$ and $\overline{K} \subset \mathcal{O}$ we have $\widetilde{\pi}(x, s_n) \in \mathcal{U} \setminus (M \cup A)$ and $\widetilde{\pi}(x, t_n) \in \mathcal{U} \setminus (M \cup A)$ for all $n \in \mathbb{N}$. By the condition (b) and the continuity of V, we obtain

$$\begin{aligned} V(y) &= \lim_{n\to+\infty} V(\widetilde{\pi}(x, s_n)) = \lim_{n\to+\infty} V(\widetilde{\pi}(\widetilde{\pi}(x, t_n), s_n - t_n)) \\ &\leq \lim_{n\to+\infty} V(\widetilde{\pi}(x, t_n)) = V(\widetilde{\pi}(y, t)) \leq V(y), \end{aligned}$$

that is, $V(\widetilde{\pi}(y, t)) = V(y)$. But condition (b) implies that $V(\widetilde{\pi}(y, t)) \neq V(y)$, which is a contradiction. Hence, $y \in \overline{A}$ and $\widetilde{L}^+(x) \subset \overline{A}$.

Case 2. $y \in M$. Since M satisfies STC (see hypothesis (H1)) there exists a STC-tube $F(L, [0, 2\lambda])$ through y given by a section S. As the tube is a neighbourhood of y, there is $\eta > 0$ such that $B(y, \eta) \subset F(L, [0, 2\lambda])$.

Denote H_1 and H_2 by

$$H_1 = F(L, (\lambda, 2\lambda]) \cap B(y, \eta) \quad \text{and} \quad H_2 = F(L, [0, \lambda]) \cap B(y, \eta).$$

In the sequel, we consider only the cases:

- either $\{\widetilde{\pi}(x, s_n)\}_{n\in\mathbb{N}} \subset H_1$
- or $\{\widetilde{\pi}(x, s_n)\}_{n\in\mathbb{N}} \subset H_2$

(in the other cases we take subsequences).

If $\{\widetilde{\pi}(x, s_n)\}_{n\in\mathbb{N}} \subset H_2$ then $\widetilde{\pi}(x, s_n + \varepsilon) \xrightarrow{n\to+\infty} \widetilde{\pi}(y, \varepsilon) \in \widetilde{L}^+(x)$ for all $\varepsilon \in (0, \phi(y))$. Note that $\widetilde{\pi}(y, \varepsilon) \notin M$ for all $\varepsilon \in (0, \phi(y))$ since $I(M) \cap M = \emptyset$. As in the proof of Case 1, we may conclude that $\widetilde{\pi}(y, \varepsilon) \in \overline{A}$ for all $\varepsilon \in (0, \phi(y))$. Consequently, $y \in \overline{A}$ and $\widetilde{L}^+(x) \subset \overline{A}$.

If $\{\widetilde{\pi}(x, s_n)\}_{n\in\mathbb{N}} \subset H_1$, then $\phi(\widetilde{\pi}(x, s_n)) \xrightarrow{n\to+\infty} 0$. Assume that $s_n > \lambda > 0$ for all $n \in \mathbb{N}$ and consider the sequence $\{\widetilde{\pi}(x, s_n - \lambda/2)\}_{n\in\mathbb{N}} \subset K$. By the compactness of $\overline{K}$ we may assume that

$$\widetilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right) \xrightarrow{n\to+\infty} z.$$

Since

$$\widetilde{\pi}\left(\widetilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right), \frac{\lambda}{2}\right) = \widetilde{\pi}(x, s_n),$$

$$\phi(\widetilde{\pi}(x, s_n)) \xrightarrow{n\to+\infty} 0 \quad \text{and} \quad F(L, (\lambda, 2\lambda]) \cap M = \emptyset,$$

we have

$$\phi\left(\widetilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right)\right) = \frac{\lambda}{2} + \phi(\widetilde{\pi}(x, s_n)), \quad z \notin M$$

and

$$\widetilde{\pi}(x, s_n) = \pi\left(\widetilde{\pi}\left(x, s_n - \frac{\lambda}{2}\right), \frac{\lambda}{2}\right) \xrightarrow{n\to+\infty} \pi\left(z, \frac{\lambda}{2}\right) \in \widetilde{L}^+(x).$$

By the uniqueness, we get $y = \pi(z, \lambda/2)$. Now, note that $\widetilde{\pi}(z, [0, \lambda/2)) \subset \widetilde{L}^+(x) \setminus M$ as $z \in \widetilde{L}^+(x) \setminus M$ and $\widetilde{L}^+(x) \setminus M$ is positively $\widetilde{\pi}$-invariant. By the proof of Case 1, we have $\pi(z, [0, \lambda/2)) \subset \overline{A}$. Hence, $y \in \overline{A}$ and $\widetilde{L}^+(x) \subset \overline{A}$.

In conclusion, we have $\emptyset \neq \widetilde{L}^+(x) \subset \overline{A}$ for all $x \in K$, that is, $K \subset \widetilde{P}(A)$. Therefore, A is $\widetilde{\pi}$-attractor. □

As a consequence of Corollary 3.21 and Proposition 3.24, we can state the following result.

Corollary 3.25. *Let $(X, \pi; M, I)$ be an* IDS*, X be locally compact, $A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that there exists a non-negative real valued function V defined on a neighbourhood $\mathcal{U}$ of $\overline{A}$ and continuous on $\mathcal{U} \setminus M$ satisfying*:

- (a) *$V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n\to+\infty} 0$ implies $d(x_n, A) \xrightarrow{n\to+\infty} 0$;*
- (b) *$V(\widetilde{\pi}(x, t)) \le V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \ge 0$ such that $\widetilde{\pi}(x, [0, t]) \subset \mathcal{U}$.*
- (c) *If $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.*

Then $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable and A is $\widetilde{\pi}$-attractor.

Proof. By Corollary 3.21 the set $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable. On the other hand, since A is relatively compact and X is locally compact there exists $\alpha \in (0, r_0)$ such that $B(A, \alpha) \subset \mathcal{U}$ with $\overline{B(A, \alpha)}$ compact. Define $\mu = \inf\{V(z) : z \in S(A, \alpha/2)\}$. By condition(a) we have $\mu > 0$. Now, consider the set

$$K = \left\{x \in B\left(A, \frac{\alpha}{2}\right) : V(x) < \mu\right\}.$$

As in the proof of Theorem 3.20, we conclude that K is positively $\widetilde{\pi}$-invariant. Hence, K is an open relatively compact positively $\widetilde{\pi}$-invariant set with $A \subset K \subset \overline{K} \subset B(A, \alpha) \subset \mathcal{U}$. By Proposition 3.24, A is $\widetilde{\pi}$-attractor. □

Example 3.26. Consider an impulsive dynamical system

$$\begin{cases} \dot{x} = f(x), \\ x(0) = x_0, \\ I\colon M \to \mathbb{R}^n, \end{cases} \tag{3.4}$$

where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, $M \subseteq \mathbb{R}^n$ is the impulsive set and $I\colon M \to \mathbb{R}^n$ is the impulse function such that $\|I(x) - I(y)\| \le \eta\|x - y\|$ for all $x, y \in M$ with $0 < \eta < 1$. We assume that conditions (H1)–(H3) hold. Also, we assume that all the solutions of the non-impulsive system $\dot{x} = f(x)$, $x(0) = x_0$, are defined in the whole real line and give rise to a semigroup π on $\mathbb{R}^n$.

Now, let $V \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ be a function satisfying the following conditions:

(a) There exists a bounded subset $\mathcal{A} \subset \mathbb{R}^n$ such that $V(x) = 0$ if and only if $x \in \overline{\mathcal{A}}$;
(b) $\nabla V(x) \cdot f(x) \le -\alpha V(x)$ for all $x \in \mathbb{R}^n \setminus \mathcal{A}$, where $\alpha > 0$;
(c) $V(I(x)) < V(x)$ for all $x \in M \setminus \overline{\mathcal{A}}$ and $V(I(x)) = V(x)$ for all $x \in M \cap \overline{\mathcal{A}}$.

First, let us prove that $I(\overline{B(\mathcal{A}, r)} \cap M) \subset B(\overline{\mathcal{A}}, r)$ for all $r > 0$. It is enough to assume that $\overline{\mathcal{A}} \cap M \ne \emptyset$. According to the hypotheses (a) and (c), we have $I(\overline{\mathcal{A}} \cap M) \subset \overline{\mathcal{A}}$. Since I is a Lipschitz function, we have

$$I\bigg(B\bigg(\overline{\mathcal{A}} \cap M, \frac{r}{\eta}\bigg) \cap M\bigg) \subset B(I(\overline{\mathcal{A}} \cap M), r) \subset B(\overline{\mathcal{A}}, r) = B(\mathcal{A}, r) \quad \text{for all } r > 0.$$

Moreover, it is easy to see that

$$\overline{B(\mathcal{A}, r)} \cap M \subset \overline{B(\overline{\mathcal{A}} \cap M, r)} \cap M \subset B\bigg(\overline{\mathcal{A}} \cap M, \frac{r}{\eta}\bigg) \cap M.$$

Thus, the assertion is proved.

Let $\mu > 0$, $\gamma > 0$ and define $\mathcal{U} = \{x \in B(\mathcal{A}, \gamma) : V(x) < \mu\}$. Note that $\mathcal{U}$ is a neighbourhood of $\overline{\mathcal{A}}$. Let $\widetilde{\pi}(x_0, \cdot\,)$ be the impulsive solution of (3.4). It is not difficult to see that V satisfies the condition (a) from Theorem 3.25.

Now, for $x \in \mathcal{U} \setminus \mathcal{A}$ and $s > 0$ such that $\pi(x, [0, s]) \subset \mathcal{U} \setminus \mathcal{A}$, we have

$$\frac{d}{dt} V(\pi(x, t)) = \nabla V(\pi(x, t)) \cdot f(\pi(x, t)) \le -\alpha V(\pi(x, t)) \quad \text{for } t \in [0, s].$$

Therefore,

$$V(\pi(x, t)) \le e^{-\alpha t} V(x) < V(x) \quad \text{for all } t \in (0, s].$$

Since $V(I(x)) < V(x)$, $x \in M \setminus \overline{\mathcal{A}}$, we conclude that $V(\widetilde{\pi}(x, t)) < V(x)$ for all $x \in \mathcal{U} \setminus \mathcal{A}$ and $t > 0$ such that $\widetilde{\pi}(x, [0, t]) \subset \mathcal{U} \setminus \mathcal{A}$. This implies conditions (b) and (c) from Theorem 3.25. In conclusion, the set $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable and A is $\widetilde{\pi}$-attractor.

Our next aim is to show the existence of a Lyapunov function satisfying the conditions (a)–(c) of Corollary 3.25 provided that $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable and A is $\widetilde{\pi}$-attractor. For that, we present some auxiliary results.

LEMMA 3.27 ([3, Lemma 4.15]). *If* $\widetilde{L}^+(x) \neq \emptyset$ *for some* $x \in X$, *then* $\widetilde{L}^+(x) \setminus M \neq \emptyset$.

LEMMA 3.28. *Let* $(X, \pi; M, I)$ *be an* IDS *and* A *be a relatively compact subset of* X. *Let* A *be a* $\widetilde{\pi}$*-attractor and* $\overline{A} \setminus M$ *be a* $\widetilde{\pi}$*-stable set. Then the mapping* $W\colon \widetilde{P}(A) \to \mathbb{R}_+$ *given by*

$$W(x) = \begin{cases} \sup\limits_{t \in \mathbb{R}_+} d(\widetilde{\pi}(x,t), A) & \text{for } x \in \widetilde{P}(A) \setminus M, \\ d(x, A) & \text{for } x \in \widetilde{P}(A) \cap M, \end{cases}$$

is continuous on $\widetilde{P}(A) \setminus M$. *Moreover, if* $\overline{A}$ *is* $\widetilde{\pi}$*-stable then* W *is continuous on* $\widetilde{P}(A) \setminus (M \setminus \overline{A})$.

PROOF. Let $x \in \widetilde{P}(A) \setminus M$. Then $\emptyset \neq \widetilde{L}^+(x) \subset \overline{A}$. By Lemma 3.27 there exists $y \in \widetilde{L}^+(x) \setminus M \subset \overline{A} \setminus M$. Thus there is a sequence $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}_+$ such that $t_n \xrightarrow{n \to +\infty} +\infty$ and $\widetilde{\pi}(x, t_n) \xrightarrow{n \to +\infty} y$. Since $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable, for a given $\varepsilon_0 > 0$, there exists $\delta > 0$ such that

$$\widetilde{\pi}(B(y,\delta), [0, +\infty)) \subset B(A, \varepsilon_0).$$

Hence, there is $n_0 \in \mathbb{N}$ such that $\widetilde{\pi}(x, [t_{n_0}, +\infty)) \subset B(A, \varepsilon_0)$. Consequently, we have $W(x) \leq \sup\{d(\widetilde{\pi}(x, [0, t_{n_0}]), A), \varepsilon_0\} < +\infty$ and W is well defined.

Let us show the continuity of W in $\widetilde{P}(A) \setminus M$. Indeed, let $x \in \widetilde{P}(A) \setminus M$ and assume that $\phi(x_k^+) < \infty$ for all $k = 0, 1, \ldots$ Then we have

$$W(x) = \sup_{t \in \mathbb{R}_+} d(\widetilde{\pi}(x,t), A) = \sup_{k \in \mathbb{N}_0} \sup_{0 \leq t < \phi(x_k^+)} d(\pi(x_k^+, t), A). \tag{3.5}$$

Let $\{w_n\}_{n \in \mathbb{N}} \subset \widetilde{P}(A)$ be a sequence such that $w_n \xrightarrow{n \to +\infty} x$. Since M is closed, $I(M) \cap M = \emptyset$, I is continuous on M, ϕ is continuous on $X \setminus M$ and $x \notin M$, one can show that for each $k \in \mathbb{N}$ we have

$$(w_n)_k^+ = I(\pi((w_n)_{k-1}^+, \phi((w_n)_{k-1}^+))) \xrightarrow{n \to +\infty} I(\pi(x_{k-1}^+, \phi(x_{k-1}^+))) = x_k^+$$

and

$$\sup_{0 \leq t \leq \phi((w_n)_{k-1}^+)} d(\pi((w_n)_{k-1}^+, t), A) \xrightarrow{n \to +\infty} \sup_{0 \leq t \leq \phi(x_{k-1}^+)} d(\pi(x_{k-1}^+, t), A).$$

This shows that $W(w_n) \xrightarrow{n \to +\infty} W(x)$ and W is continuous on $\widetilde{P}(A) \setminus M$.

Now, suppose that $\overline{A}$ is $\widetilde{\pi}$-stable. It is enough to prove that W is continuous on $\overline{A} \cap M \cap \widetilde{P}(A)$ in order to conclude the continuity of W in $\widetilde{P}(A) \setminus (M \setminus \overline{A})$. For $x \in \overline{A} \cap M \cap \widetilde{P}(A)$ we have $W(x) = d(x, A) = 0$. Since $\overline{A}$ is $\widetilde{\pi}$-stable, given $\varepsilon > 0$ there is $\delta = \delta(x, \varepsilon) > 0$ such that $\widetilde{\pi}(B(x,\delta), [0, +\infty)) \subset B(A, \varepsilon)$. If $\{z_n\}_{n \in \mathbb{N}}$ is a sequence in $\widetilde{P}(A)$ such that $z_n \xrightarrow{n \to +\infty} x$, then there is a positive integer $n_0 > 0$ such that $z_n \in B(x, \delta)$ for all $n > n_0$. Consequently, $\widetilde{\pi}(z_n, [0, +\infty)) \subset$

$B(A,\varepsilon)$ for all $n > n_0$. Then $W(z_n) < \varepsilon$ for all $n > n_0$ and it implies that $W(z_n) \xrightarrow{n\to+\infty} 0 = W(x)$. Therefore, W is continuous in $\widetilde{P}(A) \setminus (M \setminus \overline{A})$. □

Theorem 3.29. *Let $(X,\pi;M,I)$ be an* IDS*, $A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A,r)} \cap M) \subset B(A,r)$ for all $0 < r < r_0$. Assume that $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable and A is $\widetilde{\pi}$-attractor. Then there exists a non-negative real valued function V defined on a neighbourhood $\mathcal{U}$ of A and continuous on $\mathcal{U} \setminus M$ satisfying*:

(a) *$V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n\to+\infty} 0$ implies $d(x_n, A) \xrightarrow{n\to+\infty} 0$*;
(b) *$V(\widetilde{\pi}(x,t)) < V(x)$ for all $x \in \mathcal{U} \setminus (M \cup \overline{A})$ and $t > 0$ such that $\widetilde{\pi}(x,[0,t]) \subset \mathcal{U} \setminus (M \cup \overline{A})$.*

Proof. Let $\mathcal{U} = \widetilde{P}(A)$ be a positively $\widetilde{\pi}$-invariant neighbourhood of A, see Lemma 3.2. By Proposition 3.16 we have $\overline{A} \setminus M \subset \widetilde{P}(A)$. Define a mapping $W \colon \widetilde{P}(A) \to \mathbb{R}_+$ by

$$W(x) = \begin{cases} \sup\limits_{t\in\mathbb{R}_+} d(\widetilde{\pi}(x,t), A) & \text{for } x \in \widetilde{P}(A) \setminus M, \\ d(x,A) & \text{for } x \in \widetilde{P}(A) \cap M. \end{cases}$$

By Lemma 3.28 the mapping W is well defined and continuous on $\widetilde{P}(A) \setminus M$. Note that $W(x) = 0$, for all $x \in A \setminus M$ since $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable, $W(x) = 0$ for $x \in A \cap M$ and $d(x,A) \le W(x)$ for all $x \in \widetilde{P}(A)$. Hence, W satisfies the condition (a).

Now, we assert that $W(\widetilde{\pi}(x,t)) \le W(x)$ for all $x \in \widetilde{P}(A) \setminus M$ and $t \ge 0$. Let us assume that $\phi(x_k^+) < \infty$ for all $k = 0,1,\ldots$ If $0 \le s < \phi(x)$ and $y = \widetilde{\pi}(x,s)$, then $x_k^+ = y_k^+$ for each $k = 1,2,\ldots$ Thus

$$W(\widetilde{\pi}(x,s)) = \sup_{t\in\mathbb{R}_+} d(\widetilde{\pi}(x,t+s), A) = \sup_{t\ge s} d(\widetilde{\pi}(x,t), A) \le W(x)$$

and

$$W(\widetilde{\pi}(x,\phi(x))) = W(x_1^+) = \sup_{t\in\mathbb{R}_+} d(\widetilde{\pi}(x_1^+,t), A) \le \sup_{t\in\mathbb{R}_+} d(\widetilde{\pi}(x,t), A) = W(x).$$

Now, for each $t \ge \phi(x)$, there is $n = n(t) \in \mathbb{N}$ such that $t = t_n(x) + r$ with $0 < r \le \phi(x_n^+)$. Thus

$$W(\widetilde{\pi}(x,t)) = W(\pi(x_n^+, r)) \le W(x_n^+). \tag{3.6}$$

In this way, we conclude that $W(\widetilde{\pi}(x,t)) \le W(x)$ for all $x \in \widetilde{P}(A) \setminus M$ and $t \ge 0$. However, the function W may not be strictly decreasing along to the trajectories in $\widetilde{P}(A) \setminus (\overline{A} \cup M)$. Thus, we define the mapping $V : \widetilde{P}(A) \to \mathbb{R}_+$ by

$$V(x) = \begin{cases} \displaystyle\int_0^{+\infty} W(\widetilde{\pi}(x,\tau))\exp(-\tau)d\tau & \text{for } x \in \widetilde{P}(A) \setminus M, \\ 0 & \text{for } x \in A \cap M, \\ 1 & \text{for } x \in (\widetilde{P}(A) \setminus A) \cap M. \end{cases}$$

It is not difficult to see that V satisfies the condition (a). It is also clear that $V(\widetilde{\pi}(x,t)) \leq V(x)$ for all $x \in \widetilde{P}(A) \setminus (\overline{A} \cup M)$ and $t \geq 0$. Suppose to the contrary that $V(\widetilde{\pi}(x_0,s)) = V(x_0)$ for some $x_0 \in \widetilde{P}(A) \setminus (\overline{A} \cup M)$ and $s > 0$ with $\widetilde{\pi}(x_0,[0,s]) \subset \widetilde{P}(A) \setminus (\overline{A} \cup M)$. Then

$$\int_0^{+\infty} [W(\widetilde{\pi}(x_0, s+\tau)) - W(\widetilde{\pi}(x_0,\tau))] \exp(-\tau)\, d\tau = 0,$$

that is, $W(\widetilde{\pi}(x_0,s+\tau)) = W(\widetilde{\pi}(x_0,\tau))$ for every $\tau \in [0,+\infty)$. In particular, $W(\widetilde{\pi}(x_0,ms)) = W(x_0)$ for all $m \in \mathbb{N}$.

We claim that $W(x_0) = 0$. In fact, given $\varepsilon > 0$ there is $t_0 > 0$ such that

$$\widetilde{\pi}(x_0,[t_0,+\infty)) \subset B(A,\varepsilon)$$

as A is $\widetilde{\pi}$-attractor, $\overline{A} \setminus M$ is $\widetilde{\pi}$-stable and we have Lemma 3.27. Thus the sequence $\{\widetilde{\pi}(x_0,ns)\}_{n\in\mathbb{N}}$ admits a convergent subsequence. We may assume that

$$\widetilde{\pi}(x_0,ns) \xrightarrow{n\to+\infty} a \in \widetilde{L}^+(x_0).$$

Case 1. $a \in \widetilde{L}^+(x_0) \setminus M$. In this case $a \in \overline{A} \setminus M$ as A is $\widetilde{\pi}$-atractor. Using the continuity of W in $\widetilde{P}(A) \setminus M$ we get $W(a) = 0$ and

$$W(x_0) = W(\widetilde{\pi}(x_0,ns)) \xrightarrow{n\to+\infty} W(a) = 0.$$

Hence, $W(x_0) = 0$.

Case 2. $a \in \widetilde{L}^+(x_0) \cap M$. By hypothesis (H1) the set M satisfies STC. Then there is a STC-tube $F(L,[0,2\lambda])$ through a given by a section S. Moreover, since the tube is a neighbourhood of a, there is $\eta > 0$ such that

$$B(a,\eta) \subset F(L,[0,2\lambda]).$$

Denote $H_1 = F(L,(\lambda,2\lambda]) \cap B(a,\eta)$ and $H_2 = F(L,[0,\lambda]) \cap B(a,\eta)$.

Next, we consider just two subcases since the others cases are analogous by taking subsequences.

Subcase 2.1. $\{\widetilde{\pi}(x_0,ns)\}_{n\in\mathbb{N}} \subset H_1$. Then $\phi(\widetilde{\pi}(x_0,ns)) \xrightarrow{n\to+\infty} 0$ and

$$\widetilde{\pi}(\widetilde{\pi}(x_0,ns),\phi(\widetilde{\pi}(x_0,ns))) \xrightarrow{n\to+\infty} I(a) \in \widetilde{L}^+(x_0) \setminus M \subset \overline{A} \setminus M.$$

Consequently,

$$W(\widetilde{\pi}(\widetilde{\pi}(x_0,ns),\phi(\widetilde{\pi}(x_0,ns)))) \xrightarrow{n\to+\infty} W(I(a)).$$

On the other hand, we have

$$W(\widetilde{\pi}(x_0, \phi(\widetilde{\pi}(x_0, ns)))) = W(\widetilde{\pi}(\widetilde{\pi}(x_0, ns), \phi(\widetilde{\pi}(x_0, ns))))$$

and

$$W(\widetilde{\pi}(x_0, \phi(\widetilde{\pi}(x_0, ns)))) \xrightarrow{n\to+\infty} W(x_0).$$

Hence, $W(x_0) = W(I(a)) = 0$ as $W(x) = 0$ for all $x \in \overline{A} \setminus M$.

Subcase 2.2. $\{\widetilde{\pi}(x_0, ns)\}_{n\in\mathbb{N}} \subset H_2$. By Lemma 2.2, there is a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ with $\varepsilon_n \xrightarrow{n\to+\infty} 0$ such that

$$\widetilde{\pi}(\widetilde{\pi}(x_0, ns), s+\varepsilon_n) \xrightarrow{n\to+\infty} \widetilde{\pi}(a, s).$$

Note that $\widetilde{\pi}(a, s) \in \widetilde{L}^+(x_0) \setminus M \subset \overline{A} \setminus M$. Since $W(\widetilde{\pi}(\widetilde{\pi}(x_0, ns), s+\varepsilon_n)) = W(\widetilde{\pi}(x_0, \varepsilon_n))$, $n \in \mathbb{N}$, we conclude that $W(x_0) = W(\widetilde{\pi}(a, s)) = 0$.

In conclusion, $W(x_0) = 0$ and it contradicts the fact that $x_0 \notin \overline{A}$. Therefore, V satisfies the condition (b). □

As a consequence of Theorem 3.29, we have a converse type of Corollary 3.25.

Corollary 3.30. *Let $(X, \pi; M, I)$ be an* IDS, *$A \subset X$ be a relatively compact set and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that $\overline{A} \setminus M$ is orbitally $\widetilde{\pi}$-stable and A is $\widetilde{\pi}$-attractor. Then there exists a non-negative real valued function V defined on a neighbourhood $\mathcal{U}$ of A and continuous on $\mathcal{U} \setminus M$ satisfying:*

(a) *$V(x) = 0$ for $x \in A$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $V(x_n) \xrightarrow{n\to+\infty} 0$ implies $d(x_n, A) \xrightarrow{n\to+\infty} 0$;*
(b) *$V(\widetilde{\pi}(x, t)) \leq V(x)$ for all $x \in \mathcal{U} \setminus M$ and $t \geq 0$ such that $\widetilde{\pi}(x, [0, t]) \subset \mathcal{U}$.*
(c) *If $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A) \neq \emptyset$ for some $x \in \mathcal{U}$, then V is not constant along trajectories in $\widetilde{L}^+(x) \cap (\mathcal{U} \setminus A)$.*

Finally we present a result that concerns the existence of a relatively compact positively $\widetilde{\pi}$-invariant set.

Proposition 3.31. *Let $(X, \pi; M, I)$ be an* IDS, *X be locally compact, $A \subset X$ be relatively compact and $r_0 > 0$ be such that $I(\overline{B(A, r)} \cap M) \subset B(A, r)$ for all $0 < r < r_0$. Assume that there is a function $V \colon \mathcal{U} \to \mathbb{R}_+$, where $\mathcal{U}$ is a neighbourhood of $\overline{A}$, continuous on $\mathcal{U} \setminus M$ satisfying the conditions* (a)–(c) *of Corollary* 3.30. *Then there is $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$ the set $G_\alpha = \{x \in \mathcal{U} \setminus M : V(x) < \alpha\}$ admits a relatively compact positively $\widetilde{\pi}$-invariant subset K_α such that K_α is a neighbourhood of $A \cup (\partial A \setminus M)$, $K_\alpha \subset \widetilde{P}(A)$ and*

$$K_\alpha \cap \big(\overline{G_\alpha \setminus K_\alpha}\big) = \emptyset.$$

Proof. Let $\varepsilon > 0$, $\varepsilon < r_0$, be such that $\overline{B(A, \varepsilon)} \subset \mathcal{U}$ is compact. Set

$$m(\varepsilon) = \inf\{V(x) : x \in S(A, \varepsilon)\}.$$

By condition (a), $m(\varepsilon) > 0$. Now let $0 < \alpha < m(\varepsilon)$ and define the set

$$K_\alpha = G_\alpha \cap B(A, \varepsilon).$$

By the continuity of V in $\mathcal{U} \setminus M$ we have $A \cup (\partial A \setminus M) \subset G_\alpha$. Then K_α is an open relatively compact neighbourhood of $A \cup (\partial A \setminus M)$. Using the last part of the proof of Theorem 3.20, we conclude that K_α is positively $\widetilde{\pi}$-invariant. Now, by the proof of Lemma 3.24, we get $K_\alpha \subset \widetilde{P}(A)$. Since $K_\alpha \cap \big(\overline{G_\alpha \setminus K_\alpha}\big) = \emptyset$, the proof is complete. □

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