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NONAUTONOMOUS CONLEY INDEX THEORY. CONTINUATION OF MORSE-DECOMPOSITIONS

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Abstract. In previous works the author established a nonautonomous Conley index based on the interplay between a nonautonomous evolution operator and its skew-product formulation. In this paper, the treatment of attractor–repeller decomposition is refined. The more general concept of partially ordered Morse-decompositions is used. It is shown that, in the nonautonomous setting, these Morse-decompositions persist under small perturbations. Furthermore, a continuation property for these Morse decompositions is established. Roughly speaking, the index of every Morse-set and every connecting homomorphism continue as the nonautonomous problem, depending continuously on a parameter, changes.

In previous works [5], [6] the author developed a nonautonomous Conley index theory. The index relies on the interplay between a skew-product semiflow and a nonautonomous evolution operator. It can be applied to various nonautonomous problems, including ordinary differential equations and semilinear parabolic equations (see [5]).

There are multiple variants such as a homotopy index, a homology Conley index or a categorial index. In [6], also attractor–repeller decompositions of isolated invariant sets are introduced. In particular, every attractor–repeller decomposition of an isolated invariant set gives rise to a long exact sequence

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involving the homology Conley index. The connecting homomorphism of this sequence contains information on the connections between repeller and attractor. In particular, the connecting homomorphism vanishes if a connecting orbit does not exist.

Partially ordered Morse-decomposition have been developed as a generalization of attractor–repeller decompositions (see [4], [3] or [1]). They are required to define concepts such as homology index braids or the connection matrix. In this paper, the theory of partially ordered Morse-decomposition, their persistence under perturbations and a perturbation property are proved for the nonautonomous Conley index.

The paper is structured as follows. We begin with a Preliminaries section reviewing the required material and introducing weak index filtrations, which slightly differ from the concept of index filtrations used by other authors. In Section 2, a notion of convergence for Morse-decompositions is defined. Additionally, it is proved that Morse-decompositions persists under small perturbations of the nonautonomous problem (Theorem 2.2). In Section 3, the continuation of the index, the convergence of connecting homomorphisms and even the convergence of homology index braids are subsumed under a new concept, called *continuation class*. The main result of this section is Theorem 3.4 stating an abstract continuation principle. A continuation principle for attractor–repeller decompositions is recovered as a special case in Corollary 3.5.

1. Preliminaries

The section starts with a collection of useful definitions and terminology, mainly from [5] and [6]. Thereafter, we review the concept of partially ordered Morse-decompositions as used in [3], [4] or [1] and adapt it to the current nonautonomous setting. Finally, the notion of a *weak index filtration* is introduced.

1.1. Quotient spaces.

Definition 1.1. Let X be a topological space, and $A, B \subset X$. Denote

$$A/B := A/R \cup \{A \cap B\},$$

where A/R is the set of equivalence classes with respect to the relation R on A which is defined by xRy if and only if $x = y$ or $x, y \in B$.

We consider A/B as a topological space endowed with the quotient topology with respect to the canonical projection $q\colon A \to A/B$, that is, a set $U \subset A/B$ is open if and only if

$$q^{-1}(U) = \bigcup_{x \in U} x$$

is open in A.

Recall that the quotient topology is the final topology with respect to the projection q.

Remark 1.2. The above definition is compatible with the definition used in [1] or [7]. The only difference occurs in the case $A \cap B = \emptyset$, where we add $\emptyset$, which is never an equivalence class, instead of an arbitrary point.

1.2. Evolution operators and semiflows. Let X be a metric space. Assuming that $\Diamond \notin X$, we introduce a symbol $\Diamond$, which means "undefined". The intention is to avoid the distinction if an evolution operator is defined for a given argument or not. Define $\overline{A} := A \dot{\cup} \{\Diamond\}$ whenever A is a set with $\Diamond \notin A$. Note that $\overline{A}$ is merely a set, the notation does not contain any implicit assumption on the topology.

Definition 1.3. Let $\Delta := \{(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+ : t \geq t_0\}$. A mapping $\Phi\colon \Delta \times \overline{X} \to \overline{X}$ is called an *evolution operator* if

(a) $\mathcal{D}(\Phi) := \{(t, t_0, x) \in \Delta \times X : \Phi(t, t_0, x) \neq \Diamond\}$ is open in $\mathbb{R}^+ \times \mathbb{R}^+ \times X$;
(b) Φ is continuous on $\mathcal{D}(\Phi)$;
(c) $\Phi(t_0, t_0, x) = x$ for all $(t_0, x) \in \mathbb{R}^+ \times X$;
d) $\Phi(t_2, t_0, x) = \Phi(t_2, t_1, \Phi(t_1, t_0, x))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}^+$ and $x \in X$;
(e) $\Phi(t, t_0, \Diamond) = \Diamond$ for all $t \geq t_0$ in $\mathbb{R}^+$.

A mapping $\pi\colon \mathbb{R}^+ \times \overline{X} \to \overline{X}$ is called *semiflow* if $\widetilde{\Phi}(t + t_0, t_0, x) := \pi(t, x)$ defines an evolution operator. To every evolution operator Φ, there is an associated (skew-product) semiflow π on an extended phase space $\mathbb{R}^+ \times X$, defined by $(t_0, x)\pi t = (t_0 + t, \Phi(t + t_0, t_0, x))$.

A function $u\colon I \to X$ defined on a subinterval I of $\mathbb{R}$ is called a *solution of* (*with respect to*) Φ if $u(t_1) = \Phi(t_1, t_0, u(t_0))$ for all $[t_0, t_1] \subset I$.

Definition 1.4. Let X be a metric space, $N \subset X$ and π a semiflow on X. The set $\mathrm{Inv}_\pi^-(N) := \{x \in N : \text{ there is a solution } u\colon \mathbb{R}^- \to N \text{ with } u(0) = x\}$ is called the *largest negatively invariant subset of* N.

The set $\mathrm{Inv}_\pi^+(N) := \{x \in N : x\pi\mathbb{R}^+ \subset N\}$ is called the *largest positively invariant subset of* N.

The set $\mathrm{Inv}_\pi(N) := \{x \in N : \text{ there is a solution } u\colon \mathbb{R} \to N \text{ with } u(0) = x\}$ is called the *largest invariant subset of* N.

Let X and Y be metric spaces, and assume that $y \mapsto y^t$ is a global (1) semiflow on Y, to which we will refer as t-translation.

Example 1.5. Let Z be a metric space, and let $Y := C(\mathbb{R}^+, Z)$ be a metric space such that a sequence of functions converges if and only if it converges uniformly on bounded sets. The translation can now be defined canonically by $y^t(s) := y(t + s)$ for $s, t \in \mathbb{R}^+$.

(1) defined for all $t \in \mathbb{R}^+$

A suitable abstraction of many non-autonomous problems is given by the concept of skew-product semiflows introduced below.

Definition 1.6. We say that $\pi = (\,\cdot\,^t, \Phi)$ is a skew-product semiflow on $Y \times X$ if $\Phi\colon \mathbb{R}^+ \times \overline{Y \times X} \to \overline{Y \times X}$ is a mapping such that

$$(t,y,x)\pi t := \begin{cases} (y^t, \Phi(t,y,x)) & \text{for } \Phi(t,y,x) \neq \Diamond, \\ \Diamond & \text{otherwise}, \end{cases}$$

is a semiflow on $Y \times X$.

A skew-product semiflow gives rise to evolution operators.

Definition 1.7. Let $\pi = (\,\cdot\,^t, \Phi)$ be a skew-product semiflow and $y \in Y$. Define

$$\Phi_y(t + t_0, t_0, x) := \Phi(t, y^{t_0}, x).$$

It is easily proved that Φ_y is an evolution operator in the sense of Definition 1.3.

Definition 1.8. For $y \in Y$ let $\mathcal{H}^+(y) := \operatorname{cl}_Y \{y^t : t \in \mathbb{R}^+\}$ denote the positive hull of y. Let Y_c denote the set of all $y \in Y$ for which $\mathcal{H}^+(y)$ is compact.

Definition 1.9. Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an invariant set. A closed set $N \subset Y \times X$ (resp. $N \subset \mathcal{H}^+(y_0) \times X$) is called an *isolating neighbourhood* for (y_0, K) (in $Y \times X$) (resp. in $\mathcal{H}^+(y_0) \times X$) provided that:

(a) $K \subset \mathcal{H}^+(y_0) \times X$
(b) $K \subset \operatorname{int}_{Y \times X} N$ (resp. $K \subset \operatorname{int}_{\mathcal{H}^+(y_0) \times X} N$)
(c) K is the largest invariant subset of $N \cap (\mathcal{H}^+(y_0) \times X)$

The following definition is a consequence of the slightly modified notion of a semiflow (Definition 1.3) but not a semantical change compared to [1], for instance.

Definition 1.10. We say that π explodes in $N \subset Y \times X$ if $x\pi[0,t[\,\subset N$ and $x\pi t = \Diamond$.

Following [7], we formulate the following asymptotic compactness condition.

Definition 1.11. A set $M \subset Y \times X$ is called *strongly admissible* provided the following holds: whenever (y_n, x_n) is a sequence in M and $(t_n)_n$ is a sequence in $\mathbb{R}^+$ such that $(y_n, x_n)\pi[0, t_n] \subset M$, then the sequence $(y_n, x_n)\pi t_n$ has a convergent subsequence.

A very similar concept is called *skew-admissibility* and formulated as Definition 5.1 in [5].

Definition 1.12. A subset $M \subset Y \times X$ is called *skew-admissible* provided that the following holds: whenever $(y_n, x_n)_n$ in N and $(t_n)_n$ in $\mathbb{R}^+$ are sequences such that $t_n \to \infty$, $y_n^{t_n} \to y_0$ in Y and $(y_n, x_n)\pi[0, t_n] \subset N$, the sequence $\Phi(t_n, y_n, x_n)$ has a convergent subsequence. M is called *strongly skew-admissible* if it is skew-admissible and π does not explode in M.

1.3. Index pairs and index triples. The notion of (basic) index pairs relies on [5] and was introduced in [6].

Definition 1.13. A pair (N_1, N_2) is called a (*basic*) *index pair* relative to a semiflow χ in $\mathbb{R} + \times X$ if

- (IP1) $N_2 \subset N_1 \subset \mathbb{R}^+ \times X$, N_1 and N_2 are closed in $\mathbb{R}^+ \times X$.
- (IP2) If $x \in N_1$ and $x\chi t \notin N_1$ for some $t \in \mathbb{R}^+$, then $x\chi s \in N_2$ for some $s \in [0, t]$.
- (IP3) If $x \in N_2$ and $x\chi t \notin N_2$ for some $t \in \mathbb{R}^+$, then $x\chi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [0, t]$.

Definition 1.14. Let $y_0 \in Y$ and (N_1, N_2) be a basic index pair in $\mathbb{R}^+ \times X$ relative to χ_{y_0}. Define $r := r_{y_0} : \mathbb{R}^+ \times X \to \mathcal{H}^+(y_0) \times X$ by $r_{y_0}(t, x) := (y_0^t, x)$. Let $K \subset \omega(y_0) \times X$ be an (isolated) invariant set. We say that (N_1, N_2) is a (strongly admissible) index pair ([2]) for (y_0, K) if:

- (IP4) there is a strongly admissible isolating neighbourhood N of K in $\mathcal{H}^+(y_0) \times X$ such that $N_1 \setminus N_2 \subset r^{-1}(N)$.
- (IP5) There is a neighbourhood W of K in $\mathcal{H}^+(y_0) \times X$ such that $r^{-1}(W) \subset N_1 \setminus N_2$.

Concepts such as attractor–repeller decompositions or connecting homomorphism rely on index triples as defined below.

Definition 1.15. Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighbourhood N. Suppose that (A, R) is an attractor–repeller decomposition of K. A triple (N_1, N_2, N_3) is called an *index triple* for (y_0, K, A, R) provided that:

- (a) $N_3 \subset N_2 \subset N_1$,
- (b) (N_1, N_3) is an index pair for (y_0, K),
- (c) (N_2, N_3) is an index pair for (y_0, A).

1.4. Partially ordered Morse-decompositions. Let $(P, \prec)$ be a strictly partially ordered set, that is, $\prec$ is a relation on P which is irreflexive and transitive. Using the partial order $\prec$, one defines intervals and attracting intervals. A subset $I \subset P$ is an interval, $I \in \mathcal{I}(P, \prec)$, if $i, j, k \in P$, $i, k \in I$ and $i \prec j \prec k$ implies $j \in I$. An interval $I \in \mathcal{I}(P, \prec)$ is called attracting, $I \in \mathcal{A}(P, \prec)$, if $i, j \in P$, $j \in I$ and $i \prec j$ implies $i \in I$.

([2]) Every index pair in the sense of Definition 1.14 is assumed to be strongly admissible.

Definition 1.16. Let (y_0, K) be a compact invariant pair. A family $(M_p)_{p\in P}$ is called a *$\prec$-ordered Morse-decomposition* (for (y_0, K)) provided that the following holds:

(a) The sets M_p, $p \in P$ are closed, invariant and pairwise disjoint.
(b) For every solution $u\colon \mathbb{R} \to K$, either $u(\mathbb{R}) \subset M_p$ for some $p \in P$, or there are $p, q \in P$ such that $p \prec q$, $\omega(u) \subset M_p$ and $\alpha(u) \subset M_q$.

Given an interval $I \in \mathcal{I}(P, \prec)$, let $M(I)$ denote the maximal invariant subset of K such that $(M_p)_{p\in I}$ is a $\prec$-ordered Morse-decomposition. In other words, $M(I)$ contains every Morse-set M_p with $p \in I$ and every connecting orbit (3) between Morse-sets M_p and M_q with $p, q \in I$. The sets $M(I)$ are closed (Corollary 1.19) and hence isolated compact invariant sets.

An element p in $(P, \prec)$ is maximal (resp. minimal) if $p \prec q$ (resp. $q \prec p$) does hold fory any $q \in P$.

Lemma 1.17. *Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a $(P, \prec)$-ordered Morse-decomposition. Let $I \subset P$ be an interval and $p \in P \setminus I$ be a maximal or minimal element with respect to $(P, \prec)$. Then*

$$(\mathrm{cl}_{Y\times X} M(I)) \cap M_p = \emptyset.$$

Proof. Suppose that the intersection is not empty. We will prove that p can neither be minimal nor maximal.

If $(\mathrm{cl}_{Y\times X} M(I)) \cap M_p \neq \emptyset$, there is a sequence $u_n\colon \mathbb{R} \to M(I)$ of solutions converging pointwise to a solution $u_0\colon \mathbb{R} \to K$ with $u_0(0) \in M_p$. Let

$$s_n^- := \inf\{s \leq 0 : u_n([s, 0] \subset N_p\},$$
$$s_n^+ := \sup\{s \geq 0 : u_n([0, s] \subset N_p\},$$

where N_p is an isolating neighbourhood for M_p in $\mathcal{H}^+(y_0) \times X$.

It is easy to see that $u_n(s_n^-) \in \partial N_p$ for all $n \in \mathbb{N}$. Taking subsequences, we can assume without loss of generality that either $s_n^- \to s_0$ or $s_n^- \to -\infty$. Setting $v_n(t) := u_n(t + s_n)$ and using the compactness of K, we can assume that there is a solution $v\colon \mathbb{R} \to K$ and $v_n(t) \to v(t)$ pointwise for all $t \in \mathbb{R}$. It follows that $\omega(v) \subset M_p$ as well as $v(0) \in \partial N_p$. Hence, v is a solution connecting a Morse set M_q to M_p. Since $(M_p)_{p\in P}$ is a $(P, \prec)$-ordered Morse-decomposition, one has $p \prec q$, which means that p is not maximal.

Analogously, using s_n^+, one obtains that p is not minimal. □

Lemma 1.18. *Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a $(P, \prec)$-ordered Morse-decomposition. Let $p \in P$ be a maximal or minimal element with respect to $\prec$. Then $M(P \setminus \{p\})$ is closed (compact).*

(3) If $u\colon \mathbb{R} \to K$ is a solution with $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$ for some $p, q \in P$, then $u(\mathbb{R})$ is a connecting orbit between M_p and M_q.

Proof. For brevity, we consider only the case that p is maximal. One can argue analogously if p is minimal.

Let $u_n\colon \mathbb{R} \to M(P \setminus \{p\})$ be a sequence of solutions converging pointwise to a solution $u_0\colon \mathbb{R} \to K$ with $u_0(0) \notin M(P \setminus \{p\})$. It follows that $\alpha(u_0) \subset M_p \cap \mathrm{cl}_{Y\times X} M(P \setminus \{p\})$, so $M_p \cap \mathrm{cl}_{Y\times X} M(P \setminus \{p\}) \neq \emptyset$, in contradiction to Lemma 1.17. □

Corollary 1.19. *Let $N \subset Y \times X$ be an isolating neighbourhood for a compact invariant pair (y_0, K), and let $(M_p)_{p\in P}$ be a $(P, \prec)$-ordered Morse-decomposition. For every $I \in \mathcal{I}(P, \prec)$, the set $M(I)$ is closed* (*compact*).

Proof. The proof is conducted by induction on the number of elements of $P \setminus I$. If $P \neq I$, there is a maximal or a minimal element p in $P \setminus I$. It follows from Lemma 1.18 that $M(P')$ is compact where we set $P' := P \setminus \{p\}$. Moreover, restricting $\prec$ to P' yields a $(P', \prec)$-ordered Morse-decomposition of $M(P')$. By induction, it follows that $M(I)$ is closed. □

Corollary 1.20. *Let $N \subset Y \times X$ be an isolating neighbourhood for a compact invariant pair (y_0, K), and let $(M_p)_{p\in P}$ be a $(P, \prec)$-ordered Morse-decomposition. For every $I \in \mathcal{I}(P, \prec)$, there is an isolating neighbourhood $N(I) \subset Y \times X$ for $(y_0, M(I))$ such that $M_p \cap N(I) = \emptyset$ for all $p \in P \setminus I$.*

Proof. Since $M(I)$ is compact, there exists a closed neighbourhood $N(I) \subset N$ of $M(I)$ which is disjoint from M_p for every $p \in P \setminus I$. Let $u\colon \mathbb{R} \to N(I)$ be a solution. It follows that $u(\mathbb{R}) \subset K$. Because $(M_p)_{p\in P}$ is a Morse-decomposition of K, we must have $u(\mathbb{R}) \subset M(I)$. □

We are now in a position to introduce the notion of *weak* ([4]) *index filtrations* (for the nonautonomous index).

Definition 1.21. Let (y_0, K) be a compact invariant pair, and let $(M_p)_{p\in P}$ be a Morse-decomposition for (y_0, K). A *weak index filtration* for $(y_0, K, (M_p)_{p\in P})$ is a family $(N(A))_{A\in\mathcal{A}(P,\prec)}$ of closed subsets of $\mathbb{R}^+ \times X$ such that:

(a) For all $A \in \mathcal{A}(P, \prec)$, $(N(A), N(\emptyset))$ is an index pair for $(y_0, M(A))$.
(b) $A, B \in \mathcal{A}(P, \prec)$ and $A \subset B$ implies $N(A) \subset N(B)$.

For some $n \in \mathbb{N}$, let $I_1, \ldots, I_n$ be intervals. We say that the tuple $(I_1, \ldots, I_n)$ is *increasingly ordered* if the order imposed by the indices is compatible with $\prec$, that is, there do not exist $0 \le l < k \le n$ and $(p, q) \in I_l \times I_k$ such that $q \prec p$. If it holds additionally that $I_1, \ldots, I_n$ are pairwise disjoint and $I_1 \ldots I_n := I_1 \cup \ldots \cup I_n$ is an interval, we write $(I_1, \ldots, I_n) \in \mathcal{I}_n(P, \prec)$.

([4]) Compare this to the definition of an index filtration given in [1].

Lemmas 1.22 and 1.23 together imply that a weak index filtration gives rise to index triples index triples (5) for every attractor–repeller decomposition $(y_0, M(IJ), M(I), M(J))$ with $(I,J) \in \mathcal{I}_2(P,\prec)$.

Lemma 1.22. *Let* $(N(A))_{A\in\mathcal{A}(P,\prec)}$ *be a weak index filtration,* $(I,J,K) \in \mathcal{I}_3(P,\prec)$ *and* $IJK \in \mathcal{A}(P,\prec)$. *Then,* $(M(J), M(K))$ *is an attractor–repeller decomposition of* $M(JK)$*, and* $(N(IJK), N(IJ), N(I))$ *is an index triple for* $(y_0, M(JK), M(J), M(K))$.

Proof. Suppose that $u\colon \mathbb{R} \to M(JK)$ is a solution. Either $u(\mathbb{R}) \subset M_p$ for some $p \in P$, or there are $p \prec q$ such that $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$. Suppose that neither $u(\mathbb{R}) \subset M(J)$ nor $u(\mathbb{R}) \subset M(K)$ hold. (J,K) is increasingly ordered, so there are $q \in K$ and $p \in J$ such that $\alpha(u) \subset M_q$ and $\omega(u) \subset M_p$. The sets $M(J)$ and $M(K)$ are disjoint by definition and closed (hence compact) by Corollary 1.19. We have proved that $(M(J), M(K))$ is an attractor–repeller decomposition of $M(JK)$.

It is easy to see that IJ is an attracting interval. Hence, $(N(IJK), N(IJ), N(\emptyset))$ is an index triple simply because $(N(A))_{A\in\mathcal{A}(P,\prec)}$ is a weak index filtration. It follows from Lemma 4.4 in [6] that $(N(IJK), N(IJ))$ is an index pair for $(y_0, M(K))$. Analogously, one obtains that $(N(IJ), N(I))$ is an index pair for $(y_0, M(J))$, whence it follows immediately (Definition 1.15) that $(N(IJK), N(IJ), N(I))$ is an index triple for $(y_0, M(JK), M(J), (K))$. □

Lemma 1.23. *Let* $J \in \mathcal{I}(P,\prec)$*,* $A := \{p \in P\colon p \preceq q$ *for some* $q \in J\}$ *and* $I := A \setminus J$*, where* $p \preceq q$ *if* $p \prec q$ *or* $p = q$. *Then,* $I \in \mathcal{A}(P,\prec)$*,* (I,J) *is increasingly ordered and* $IJ \in \mathcal{A}(P,\prec)$.

Proof. Let $p,q,r \in P$. If $q \in A$ and $r \prec q$, then $q \preceq q'$ for some $q' \in J$, so $r \prec q'$ and thus $r \in A$, showing that A is an attracting interval.

Suppose that (I,J) is not increasingly ordered, that is, there are $q \in I$ and $p \in J$ such that $p \prec q$. We have $q \prec q'$ for some $q' \in J$, so $q \in J$ since J is an interval. As $q \in I \cap J = \emptyset$ cannot hold, (I,J) must be increasingly ordered.

If I is not an attracting interval, there are $q \in I$ and $r \in P$ such that $r \prec q$ but $r \notin I$. The interval A, however, is attracting, so $r \in J$. Since (I,J) is increasingly ordered, such r and q cannot exist, showing that I is an attracting interval. □

Fix a weak index filtration $(N(A))_{A\in\mathcal{A}(P,\prec)}$. Suppose (I,J) is an increasingly ordered tuple of intervals and IJ is an attracting interval. According to Lemma 1.22, $(N(IJ), N(J))$ is an index pair for $(y_0, M(I))$. Now let (A_0, I, J, K) be increasingly ordered intervals such that A_0IJK is an attracting interval (6), let

(5) not necessarily unique

(6) A_0 always exists in view of Lemma 1.23.

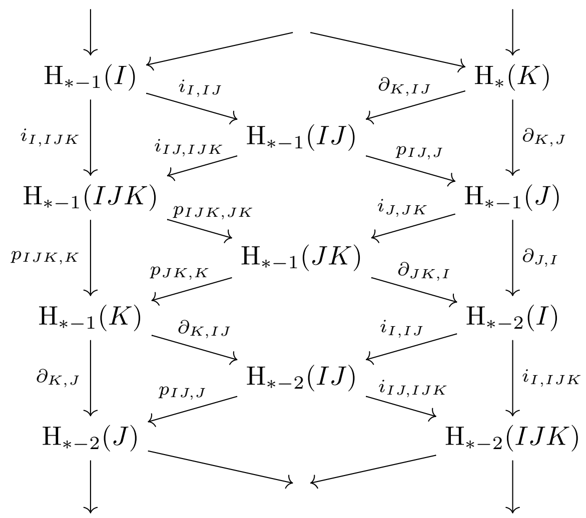
$(A_0, I, J, K) =: (I_1, \ldots, I_4)$ and set $\mathrm{H}_*(I_k \ldots I_l) := \mathrm{H}_*[N(I_1 \ldots I_l), N(I_1 \ldots I_{k-1})]$ for brevity. One obtains a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{H}_*(I) & \xrightarrow{i_{I,IJ}} & \mathrm{H}_*(IJ) & \xrightarrow{p_{IJ,J}} & \mathrm{H}_*(J) & \xrightarrow{\partial_{J,I}} & \mathrm{H}_{*-1}(I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_*(I) & \xrightarrow{i_{I,IJK}} & \mathrm{H}_*(IJK) & \xrightarrow{p_{IJK,JK}} & \mathrm{H}_*(JK) & \xrightarrow{\partial_{JK,I}} & \mathrm{H}_{*-1}(I) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_*(IJ) & \xrightarrow[i_{IJ,IJK}]{} & \mathrm{H}_*(IJK) & \xrightarrow[p_{IJK,K}]{} & \mathrm{H}_*(K) & \xrightarrow[\partial_{K,IJ}]{} & \mathrm{H}_{*-1}(IJ) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{H}_*(J) & \xrightarrow[i_{J,JK}]{} & \mathrm{H}_*(JK) & \xrightarrow[p_{JK,K}]{} & \mathrm{H}_*(K) & \xrightarrow[\partial_{K,J}]{} & \mathrm{H}_{*-1}(J)
\end{array} \tag{1.1}$$

Except for the connecting homorphism, every morphism is inclusion induced and every row is a part of the long exact sequence associated with the respective index triple (Definition 4.13 in [6]).

An aesthetically more pleasing variant of the above diagram is the so-called *homology index braid*, which is depicted in (1.2). Using (1.1), one easily proves its commutativity.

(1.2)



2. Persistence of attractor–repeller-decompositions

As in the previous section, let P be a finite set equipped with a strict partial order $\prec$.

We will now formulate the main result of this section: the convergence (or persistence) of Morse-decompositions.

Definition 2.1. For all $n \in \mathbb{N}$, let $y_n \in Y_c$ (resp. $y_0 \in Y_c$), $K_n \subset \mathcal{H}^+(y_n) \times X$ (resp. $K \subset \mathcal{H}^+(y_0) \times X$) be an isolated invariant set, and for each $n \in \mathbb{N}$ let $(M_{n,p})_p$ (resp. $(M_p)_p$) be a $(P, \prec)$-ordered Morse-decomposition of K_n (resp. K).

We say that the Morse-decompositions converge, i.e. $(y_n, K_n, (M_{n,p})_p) \to (y_0, K, (M_p)_p)$, provided that the following holds: There is an isolating neighbourhood N (resp. N_p) of (y_0, K) (resp. (y_0, M_p)) in $Y \times X$, and there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, it holds that N (resp. N_p) is an isolating neighbourhood for (y_n, K_n) (resp. $(y_n, M_{n,p})$).

The following theorem concerning the convergence of Morse-decompositions is the main result of this section. Roughly speaking, we claim that a convergence of y_n in the sense of (2.1) below implies the convergence of a certain Morse decomposition ([7]).

Theorem 2.2. *Suppose that $(y_n)_n$ is a sequence in Y, $y_0 \in Y_c$ and* ([8])

$$d\big(y_n^t, \mathcal{H}^+(y_0)\big) \to 0 \quad \text{as } t, n \to \infty. \tag{2.1}$$

Let N (resp. N_p, $p \in P$) be a strongly skew-admissible isolating neighbourhood for (y_0, K) (resp. (y_0, M_p), $p \in P$), and let $(M_p)_p$ be a $(P, \prec)$-ordered Morse-decomposition of K. Then there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$:

(a) *There is an invariant subset $K_n \subset \mathcal{H}^+(y_n) \times X$ (resp. $M_{n,p} \subset \mathcal{H}^+(y_n) \times X$) such that N (resp. N_p) is an isolating neighbourhood for (y_n, K_n) (resp. $(y_n, M_{n,p})$).*

(b) *$(M_{n,p})_p$ is a $(P, \prec)$-ordered Morse-decomposition of K_n.*

The proof uses two auxiliary lemmas stated and proved below.

Lemma 2.3. *Assume the hypotheses of Theorem 2.2, let $A \in \mathcal{A}(P, \prec)$, and let N_A be a strongly skew-admissible isolating neighbourhood for $(y_0, M(A))$. Then, for all $n \in \mathbb{N}$ sufficiently large,*

$$N_A^T := \{(y, x) \in N_A : (y, x)\pi[0, T] \subset N\} \tag{2.2}$$

is an isolating neighbourhood for (y_n, K_n') and $(y_0, M(A))$, where $K_n' := (\mathrm{Inv} N_A^T) \cap (\mathcal{H}^+(y_n) \times X)$. Moreover, $N_A^T \cap K_n$ is positively invariant provided that T and n are sufficiently large.

Proof. First of all, it is easy to see that N_A^T is closed for every $T > 0$. Fix some $T > 0$, and assume that N_A^T is not an (isolating) neighbourhood for

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([7]) Replacing isolated invariant sets by their respective isolating neighbourhoods, one could define Morse-decompositions of isolating neigbourhoods. Theorem 2.2 could now be rephrased in the following way: The Morse-decomposition with respect to y_0 is a Morse-decomposition with respect to y_n for all $n \in \mathbb{N}$ sufficiently large.

([8]) By an abuse of notation, we write $d(y, \mathcal{H}^+(y_0)) := \inf_{\widetilde{y} \in \mathcal{H}^+(y_0)} d(y, \widetilde{y})$.

$(y_0, M(A))$. There must be a sequence $(y'_n, x_n) \in N_A \setminus N_A^T$ with

$$d((y'_n, x_n), M(A)) \to 0 \quad \text{as } n \to \infty.$$

Consequently, one has $(y'_n, x_n)\pi s_n \in \partial N_A$ for some $s_n \leq T$. Taking subsequences, we may assume without loss of generality that $(y'_n, x_n) \to (y, x) \in M(A)$ and $s_n \to s_0$. We thus have $(y, x)\pi s_0 \in M(A) \cap \partial N_A$, but N_A is an isolating neighbourhood for $M(A)$, so (y, x) cannot exist. We have proved that N_A^T is an isolating neighbourhood for $(y_0, M(A))$. Subsequently, Theorem 5.6 in [5] implies that N_A^T is an isolating neighbourhood for (y_n, K'_n) provided that n is sufficiently large. This proves our first claim.

We still need to prove the positive invariance property for T and n large. Suppose to the contrary that we are given sequences $T_n \to \infty$ and $(y'_n, x_n) \in K_n \cap N_A^{T_n}$ such that $s_n := \sup\{s \in \mathbb{R}^+ : (y'_n, x_n)\pi[0, s] \subset N_A^{T_n}\} < \infty$ for all $n \in \mathbb{N}$. We must have $(y'_n, x_n)\pi(s_n + T_n) \in \partial N_A$ by the choice of s_n.

In view of Lemma 5.7 in [5], there is a solution $u\colon \mathbb{R}^- \to K \cap N_A$ with $(y'_n, x_n)\pi(s_n + T_n) \to u(0)$, so $u(0) \in \partial N_A$. Hence, u can be extended to a full solution $u'\colon \mathbb{R} \to K$ with $\alpha(u) \subset M(A)$. Since A is an attracting interval, we conclude that $u'(\mathbb{R}) \subset M(A)$, which is a contradiction since N_A is an isolating neighbourhood for $(y_0, M(A))$. □

Lemma 2.4. *Assume the hypotheses of Theorem 2.2, let $A \in \mathcal{A}(P, \prec)$, and let N_A be an isolating neighbourhood for $(y_0, M(A))$. Define N_A^T by (2.2). Then, for all $n \in \mathbb{N}$ and $T \in \mathbb{R}^+$ sufficiently large as well as for every solution $u\colon \mathbb{R} \to (\mathcal{H}^+(y_n) \times X) \cap N$, it holds that either* (9) $\alpha(u) \cap \operatorname{int}_{Y \times X} N_A^T = \emptyset$ *or* $u(\mathbb{R}) \subset N_A^T$.

Proof. Choose n and T large enough that the conclusions of Lemma 2.3 hold. In particular, $K_n \cap N_A^T$ is positively invariant. If $\alpha(u) \cap \operatorname{int}_{Y\times X} N_A^T \neq \emptyset$, then there is a sequence $t_n \to -\infty$ such that $u(t_n) \in N_A^T \cap K_n$. It follows from the positive invariance of $N_A^T \cap K_n$ that $u(t_n + s) \in N_A^T \cap K_n$ for all $s \in \mathbb{R}^+$, and thus $u(t) \in N_A^T \cap K_n$ for all $t \in \mathbb{R}$ because $t_n \to -\infty$. □

Proof of Theorem 2.2. (a) This is barely more than a restatement of Theorem 5.6 in [5].

(b) In view of (a), one can assume without loss of generality that for all $p \in P$, the set $N_p \subset Y \times X$ is an isolating neighbourhood for $(y_n, M_{n,p})$ for all $n \in \mathbb{N}$.

We are going to prove that for $n \in \mathbb{N}$ large, $(M_{n,p})_{p\in P}$ is a $(P, \prec)$-ordered Morse-decomposition of K_n by induction on the cardinality of P. Let $p_0 \in P$

(9) It is easy to see that the two following conditions are mutually exclusive for large n. Namely, $u(\mathbb{R}) \subset N_A^T$ implies that $\alpha(u) \subset N_A^T$. If the latter set is an isolating neighbourhood, one immediately obtains that $\alpha(u) \subset \operatorname{int}_{Y\times X} N_A^T$.

be a maximal element, so $A := P \setminus \{p_0\}$ is an attracting interval. By Corollary 1.20 and Lemma 5.5 in [5], there exists a strongly skew-admissible isolating neigbhorhood $N_A \subset N$ for $(y_0, M(A))$ such that $N_A \cap M_{p_0} = \emptyset$.

Thus, we can assume by induction that, for all $n \geq n_0(A)$, $(M_{n,p})_{p\in A}$ is a Morse-decomposition of $K'_n := (\mathrm{Inv} N_A) \cap (\mathcal{H}^+(y_n) \times X)$. Replacing N_A by N_A^T and choosing T large, we can additionally assume that $N_A \cap K_n$ is positively invariant, and N_A satisfies the conclusions of Lemma 2.4 for all $n \geq n_0 := n_0(P) \geq n_0(A)$. Set $N'_{p_0} := N \setminus \mathrm{int}_{Y\times X} N_A$ and $M_{n,p_0} := (\mathrm{Inv} N_{p_0}) \cap (\mathcal{H}^+(y_n) \times X)$. It is easy to see that N'_{p_0} is another isolating neighbourhood for (y_0, M_{p_0}). Hence, there exists an $n_0 = n_0(p_0, A) \geq n_0(A)$ such that for all $n \geq n_0$, N'_{p_0} is an isolating neighbourhood for M_{n,p_0}.

Let $n \geq n_0(p_0, A)$, and let $u\colon \mathbb{R} \to N \cap (\mathcal{H}^+(y_n) \times X)$ be a solution. Either $u(\mathbb{R}) \subset N_A^T$, in which case the induction argument applies, or $\alpha(u) \cap \mathrm{int}_{Y\times X} N_A^T = \emptyset$.

In the second case, one has $\alpha(u) \subset N'_{p_0}$, so $\alpha(u) \subset M_{n,p_0}$ by the choice of n_0. Either $u(\mathbb{R}) \subset N'_{p_0}$, implying that $u(\mathbb{R}) \subset M_{n,p_0}$, or $\omega(u) \subset N_A$ since $N_A \cap K_n$ is positively invariant for all $n \geq n_0$. Hence, for $n \geq n_0(p_0, A)$, $(M_{n,p})_p$ is a $(P, \prec)$-ordered Morse-decomposition of K_n. □

3. Continuation

Let P be a finite set and $\prec$ a strict partial order on P. Consider an isolated invariant set and a $(P, \prec)$-ordered Morse-decomposition of this invariant set. A continuous change of a dynamical system which preserves the invariant set and its Morse-decomposition preserves the categorial Conley index — and thus also the homotopy index and every other index which can be obtained from it. It also preserves the homology index braid and, in particular, its homomorphisms.

We will make the following standing assumptions on Y:

(L1) Y is a linear metric space (over the reals), and the metric d on Y is invariant, that is, $d(y_1, y_2) = d(y_1 - y2, 0)$ for all $y_1, y_2 \in Y$.

(L2) The translation $y \mapsto y^t$ is linear, that is, $(y_1 + y_2)^t = y_1^t + y_2^t$ for all $y_1, y_2 \in Y$, $t \in \mathbb{R}^+$ and $\lambda y^t = (\lambda y)^t$ for all $\lambda \in \mathbb{R}$, $y \in Y$, $t \in \mathbb{R}^+$.

We will now, mutatis mutandis, proceed as in the proof of the continuation property given in [5]. The first step is to find an appropriate replacement for the homotopy index, which is called *continuation class*. Note the mixture between homotopy and homology in the definition below.

Definition 3.1. Let Ω be a set. Consider the set $\mathcal{P}$ of all tuples (Y, X, π, y_0, K, M), where $X, Y \subset \Omega$ are metric spaces, π is a skew-product semiflow on $Y \times X$, $y_0 \in Y_c$, $N \subset Y \times X$ is a skew-admissible isolating neighbourhood for (y_0, K) and $(M_p)_{p\in P}$ a $(P, \prec)$-ordered Morse-decomposition. Define an equivalence relation on $\mathcal{P}$ as follows: (Y, X, π, y_0, K, M) and $(Y', X', \pi', y_0', K', M')$ are related if there exists a family $(\theta_I)_{I\in\mathcal{I}(P,\prec)}$ such that:

(a) $\theta(I)\colon \mathcal{C}(y_0, M(I)) \to \mathcal{C}(y_0', M'(I))$ is an isomorphism.

(b) For every $(I, J) \in \mathcal{I}_2(P, \prec)$, the following ladder, the rows of which are attractor–repeller sequences, is commutative.

$$\begin{array}{ccccccccc} \to \mathrm{H}_* \mathcal{C}(y_0, M(I)) & \to & \mathrm{H}_* \mathcal{C}(y_0, M(IJ)) & \to & \mathrm{H}_* \mathcal{C}(y_0, M(J)) & \to & \mathrm{H}_{*-1} \mathcal{C}(y_0, M(I)) \to \\ \downarrow \mathrm{H}_*\,\theta(I) & & \downarrow \mathrm{H}_*\,\theta(IJ) & & \downarrow \mathrm{H}_*\,\theta(J) & & \downarrow \mathrm{H}_{*-1}\,\theta(I) \\ \to \mathrm{H}_* \mathcal{C}(y_0', M'(I)) & \to & \mathrm{H}_* \mathcal{C}(y_0', M'(IJ)) & \to & \mathrm{H}_* \mathcal{C}(y_0', M'(J)) & \to & \mathrm{H}_{*-1} \mathcal{C}(y_0', M'(I)) \to \end{array} \tag{3.1}$$

The *continuation class*

$$\mathrm{ContCl}(y_0, K, M) := \mathrm{ContCl}(\pi, y_0, K, M) := \mathrm{ContCl}(Y, X, \pi, y_0, K, M)$$

is the equivalence class of (Y, X, π, y_0, K, M) under the above relation. The notation of Y, X and π is omitted whenever possible.

Usually the skew-product semiflow π remains unchanged. As before, there is a technically motivated exception, stated below.

Lemma 3.2. *Let $Y' := \big(\{1/n : n \in \mathbb{N}\} \cup \{0\}\big) \times Y$ and π' a skew-product semiflow on $Y' \times X$ such that $((\lambda, y), x)\pi' t := ((\lambda, y^t), \Phi(t, (\lambda, y), x))$. Define semiflows π_n on $Y \times X$ by $(y, x)\pi_n t := (y, \Phi(t, (\lambda, y), x))$ for $\lambda = 1/n$ if $n \in \mathbb{N}$ and $\lambda = 0$ if $n = 0$. Suppose that $y_0 \in Y_c$ and*

$$((1/n, y_0), K_n, (M_{n,p})_{p\in P}) \to ((0, y_0), K_0, (M_p)_{p\in P}).$$

Then, there is an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, one has

$$\mathrm{ContCl}(\pi_n, y_0, K_n, (M_{n,p})_{p\in P}) = \mathrm{ContCl}(\pi_0, y_0, K_0, (M_p)_{p\in P}).$$

Proof. This lemma serves primarily as an interface to the results of [2]. By Theorem 3.4 in [2], there is an $n_0 \in \mathbb{N}$ such that every $n \geq n_0$, there are index filtrations $\big(\widehat{N}_n(I)\big)_{I\in\mathcal{I}(P,\prec)}$ and $\big(\widehat{N}_n'(I)\big)_{I\in\mathcal{I}(P,\prec)}$ for $(\pi_n, K_n, (M_{n,p})_{p\in P})$ (resp. $(\pi_0, K_0, (M_p)_{p\in P})$) possessing the required nesting property, that is,

$$\widehat{N}_n(I) \subset \widehat{N}_0(I) \subset \widehat{N}_n'(I) \subset \widehat{N}_0'(I), \quad A \in \mathcal{A}(P, \prec). \tag{3.2}$$

By using weak index filtrations, we may limit our attention to attracting intervals.

For every $A \in \mathcal{A}(P, \prec)$, define

$$N(A) := \{(t, x) : (y_0^t, x) \in \widehat{N}(A)\}, \qquad N'(A) := \{(t, x) : (y_0^t, x) \in \widehat{N}'(A)\}$$

For all $A \in \mathcal{A}(P, \prec)$, it holds that $\big(\widehat{N}_n(A), \widehat{N}_n(\emptyset)\big)$ $\big($resp. $\big(\widehat{N}_n'(A), \widehat{N}_n'(\emptyset)\big)\big)$ is an FM-index pair for $M_n(A)$, so by Lemma 2.4 in [6] $(N_n(A), N_n(\emptyset))$ and $(N_n'(A), N_n'(\emptyset))$ are index pairs for $(y_0, M_n(A))$ – with respect to π_n.

Let $(I,J)\in\mathcal{I}_2(P,\prec)$ and $IJ, I\in\mathcal{A}(P,\prec)$. By Lemma 1.22, $(N_n(IJ),N_n(I))$ and $(N'_n(IJ),N'_n(I))$ (resp. $(N(IJ),N(I))$ and $(N'(IJ),N'(I))$) are index pairs for $(y_0,M_n(I)):=(y_0,M(I))$ with respect to the skew-product semiflow π_n (resp. $(y_0,M(I))$ with respect to the skew-product semiflow π_0). From (3.2), we obtain the following inclusion induced morphisms:

$$[N_n(IJ),N_n(I)]\xrightarrow{i}[N(IJ),N(I)]\xrightarrow{j}[N'_n(IJ),N'_n(I)]\xrightarrow{k}[N'(IJ),N'(I)].$$

Lemma 2.5 in [6] implies that each of the morphisms $j\circ i$, $k\circ j$ is a homotopy equivalence. Hence, i, j, k are homotopy equivalences, and

$$\theta(J):=[j]\colon\ \mathcal{C}(\pi_0,y_0,M_n(J))\to\mathcal{C}(\pi_n,y_0,M(J))$$

is an isomorphism in the homotopy category of pointed spaces.

In view of Lemma 1.23, we can always find an interval I such that the above construction of $\theta(J)$ is possible. The next step is to prove that $\theta(J)$ is well-defined, that is, independent of I. Suppose that $(I',J)\in\mathcal{I}_2(P,\prec)$ and $I'J, I'\in\mathcal{A}(P,\prec)$. Then $I_0:=I'\cap I$ is again an attracting interval and so is $I_0J=I'J\cap IJ$. Thus it is s sufficient to prove that the morphisms $\theta(J)$ defined by I_0 and I agree. This follows easily from the commutativity of the diagram below because the vertical (inclusion-induced) morphisms are inner morphisms of the categorial Conley index,

$$\begin{array}{ccc}
[N(I_0J),N(I_0)] & \xrightarrow{j_0} & [N'_n(I_0J),N'_n(I_0)] \\
\downarrow & & \downarrow \\
[N(IJ),N(I)] & \xrightarrow[j]{} & [N'_n(IJ),N'_n(I)]
\end{array}$$

so $[j]=[j_0]$.

Finally, let $(I,J,K)\in\mathcal{I}_3(P,\prec)$ and consider (3.3), where every morphism is inclusion induced except for the connecting homomorphism of the respective attractor–repeller sequence.

$$\begin{array}{ccccccc}
\mathrm{H}_*[N(IJ),N(I)] & \longrightarrow & \mathrm{H}_*[N(IJK),N(I)] & \longrightarrow & \mathrm{H}_*[N(IJK,N(IJ)] & \xrightarrow{\partial_n} & \mathrm{H}_{*-1}[N(IJ),N(I)] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_*[N'_n(IJ),N'_n(I)] & \to & \mathrm{H}_*[N'_n(IJK),N'_n(I)] & \to & \mathrm{H}_*[N'_n(IJK,N'_n(IJ)] & \xrightarrow{\partial} & \mathrm{H}_{*-1}[N'_n(IJ),N'_n(I)]
\end{array} \tag{3.3}$$

It is clear that inclusion induced morphisms commute. From Lemma 4.11 in [6], one obtains that the square with the connecting homomorphisms is commutative as well. The commutativity of (3.1) follows. □

For the rest of this section, we will make the following assumptions. Let P be a finite set and $\prec$ a strict partial order on P, Γ a metric space and $f=(y(\gamma),K(\gamma),(M_p(\gamma))_{p\in P})_{\gamma\in\Gamma}$ a family such that, for all $\gamma\in\Gamma$,

(1) $y(\gamma) \in Y_c$
(2) there is a strongly-skew-admissible isolating neighbourhood for $(y(\gamma), K(\gamma))$,
(3) $(M_p(\gamma))_{p\in P}$ is a $(P,\prec)$-ordered Morse-decomposition of $K(\gamma)$

Definition 3.3. We say that f is continuous at $\gamma_0 \in \Gamma$ if whenever $\gamma_n \to \gamma_0$ in Γ:

(C1) $(y(\gamma_n), K(\gamma_n), (M_p(\gamma_n))_{p\in P}) \to (y(\gamma_0), K(\gamma_0), (M_p(\gamma_0))_{p\in P})$.
(C2) $d(y^t_{\gamma_n}, y^t_{\gamma_0}) \to 0$ as $n, t \to \infty$.

f is continuous if it is continuous in every point $\gamma_0 \in \Gamma$.

The rest of this section is devoted to the proof of the following continuation theorem.

Theorem 3.4. *Suppose that* Γ *is connected and*

$$f = (y(\gamma), K(\gamma), (M_p(\gamma))_{p\in P})_{\gamma\in\Gamma}$$

is continuous. Then $\mathrm{ContCl} \circ f$ *is constant.*

The reason behind the definition of a continuation class and the significance of Theorem 3.4 can be seen in the following corollary since its proof is almost trivial. Roughly speaking, if the continuation class agrees, then all connecting homomorphisms agree up to conjugacy.

Corollary 3.5. *Suppose that* Γ *is connected and*

$$f = (y(\gamma), K(\gamma), (M_p(\gamma))_{p\in P})_{\gamma\in\Gamma}$$

is continuous. Let $I \in \mathcal{I}(P,\prec)$ *and let* $M(\gamma, I)$ *denote the set* $M(I)$ *relative to* $(y(\gamma), K(\gamma)$ *and the Morse-decomposition* $(M_p)_{p\in P}$*. Let* $\partial(y(\gamma_i), J, I)$ *denote the connecting homomorphism of the attractor repeller sequence of* $(y(\gamma_i), K(\gamma_i), M(\gamma_i, I), M(\gamma_i, J))$ *as defined in* [6]*, following Theorem* 4.13*. Then, for all* $\gamma_1, \gamma_2 \in \Gamma$ *and all* $(I, J) \in \mathcal{I}_2(P,\prec)$*, there is a commutative diagram*:

$$\begin{array}{ccc}
\mathrm{H}_* \,\mathcal{C}(y(\gamma_1), M(\gamma_1, J)) & \xrightarrow{\partial_*(\gamma_1,J,I)} & \mathrm{H}_{*-1}\,\mathcal{C}(y(\gamma_1), M(\gamma_1, I)) \\
\Big\downarrow {\scriptstyle \mathrm{H}_*\,\theta(J)} & & \Big\downarrow {\scriptstyle \mathrm{H}_{*-1}\,\theta(I)} \\
\mathrm{H}_* \,\mathcal{C}(y(\gamma_2), M(\gamma_2, J)) & \xrightarrow[\partial_*(\gamma_2,J,I)]{} & \mathrm{H}_{*-1}\,\mathcal{C}(y(\gamma_2), M(\gamma_2, I))
\end{array}$$

Furthermore, $\mathrm{H}_*(\theta(I))$ *and* $\mathrm{H}_*(\theta(J))$ *are isomorphisms.*

Proof. This follows from Theorem 3.4 and (3.1). □

At first glance, the following lemma appears technical. However, it perfectly outlines the strategy of the following proof of Theorem 3.4 since it reduces the problem to the local constantness of the continuation class. Moreover, the definition of $y(\gamma, \lambda)$, which is possible due to assumptions (L1) and (L2), allows for an application of Lemma 3.2.

LEMMA 3.6. *Let $\gamma_0 \in \Gamma$, and let N (resp. N_p) be a strongly skew-admisible isolating neighbourhood for $(y(\gamma_0), K(\gamma_0))$ (resp. $(y(\gamma_0), (M_p(\gamma_0))_{p\in P})$). Then there is a neighbourhood U of γ_0 in Γ such that for all $\gamma \in U$ and all $\lambda \in [0, 1]$:*

(a) *There is a set $K_{\gamma,\lambda}$ (resp. $M_{\gamma,\lambda,p}$, $p \in P$) such that N (resp. N_p, $p \in P$) is an isolating neighbourhood for*

$$(\underbrace{\lambda y(\gamma_0) + (1 - \lambda) y(\gamma)}_{=:y(\gamma,\lambda)}, K_{\gamma,\lambda})$$

(resp. $(y(\gamma, \lambda), M_{\gamma,\lambda,p})$, $p \in P$).

(b) *$(M_{\gamma,\lambda,p})_{p\in P}$ is a $(P, \prec)$-ordered Morse-decomposition of $K_{\gamma,\lambda}$*

(c) *$f(\gamma) = (y(\gamma, 0), K_{\gamma,0}, (M_{\gamma,0,p})_{p\in P})$.*

We need an auxiliary lemma to prove Lemma 3.6.

LEMMA 3.7. *Let N and N' be strongly admissible isolating neighbourhoods for (y_0, K), and let*

$$d\big(y_n^t, y_0^t\big) \to 0 \quad \text{as } t, n \to \infty.$$

Then $K_n := \operatorname{Inv}(N) \cap (\mathcal{H}^+(y_n) \times X) = \operatorname{Inv}(N') \cap (\mathcal{H}^+(y_n) \times X) =: K'_n$ for all but finitely many $n \in \mathbb{N}$.

PROOF. Arguing by contradiction, we can assume without loss of generality that there exists a sequence $x_n \in K_n \setminus K'_n$. By using Theorem 5.6 in [5], one obtains that N' is an isolating neighbourhood for (y_n, K'_n) for all but finitely many $n \in \mathbb{N}$. Hence, the solution through x_n leaves N' at least once. Therefore, one can choose a sequence $(x'_n)_{n\geq n_0}$ with $x'_n \in K_n \setminus N'$ for all $n \geq n_0$. By Lemma 5.7 in [5], there is a convergent subsequence $x''_n \to x_0 \in \operatorname{Inv}(N)$.

Because N is an isolating neighbourhood for K, one has $x_0 \in K$. On the other hand, $x''_n \in (Y \times X) \setminus N'$ for all $n \in \mathbb{N}$ implies that $x_0 \in (Y \times X) \setminus \operatorname{int} N'$. However, $K \subset \operatorname{int} N'$, which is a contradiction. □

PROOF OF LEMMA 3.6. (a), (b) Otherwise, there are sequences $\gamma_n \to \gamma_0$ and $\lambda_n \in [0, 1]$ such that (a) or (b) are not satisfied. However,

$$d(y(\gamma_n, \lambda_n)^t, y(\gamma_0)^t) \to 0 \quad \text{as } t, n \to \infty,$$

so in particular $d(y(\gamma_n, \lambda_n)^t, \mathcal{H}^+(y(\gamma_0))) \to 0$ as $t, n \to \infty$. This is a contradiction to Theorem 2.2.

(c) First of all, it is clear that $y(\gamma, 0) = y(\gamma)$. By (C1), there is an isolating neighbourhood N' (resp. N'_p, $p \in P$) for $(y(\gamma), K(\gamma))$ (resp. $(y(\gamma), M_p(\gamma))$, $p \in P$). It follows from Lemma 3.7 that $K_{\gamma,0} = K(\gamma)$ (resp. $M_{\gamma,0,p} = M_p(\gamma)$, $p \in P$) for all γ in a sufficiently small neighbourhood of γ_0. □

Lemma 3.8. *Let $\gamma_0 \in \Gamma$, and let N (resp. N_p, $p \in P$) be a strongly skew-admisible isolating neighbourhood for $(y(\gamma_0), K(\gamma_0))$ (resp. $(y(\gamma_0), M_p)$). Let $U \subset \Gamma$ as well as $K_{\gamma,\lambda}$ and $M_{\gamma,\lambda,p}$ be given by Lemma 3.6. Let $\gamma \in U$, $y_0 := y(\gamma_0)$, $h := y(\gamma) - y(\gamma_0)$, and consider the spaces $Y' := \mathcal{H}^+(y_0) \times \mathcal{H}^+(h)$ and $Y'' := [0,1] \times Y'$. On $Y' \times X$, define a family $(\pi_\lambda)_{\lambda\in[0,1]}$ of semiflows by*

$$(y, h, x)\pi_\lambda t := (y^t, h^t, \Phi(t, y + \lambda h, x)).$$

On $Y'' \times X$, one defines $(\lambda, y, h, x)\pi' t := (\lambda, (y, h, x)\pi_\lambda t)$. Define $j\colon \mathcal{H}^+(y_0) \times X \to Y \times X$ by $j(y_0, h, x) := (y_0 + h, x)$, and fix an arbitrary $\gamma \in U$. Then:

(a) *The sets $(M'_{\lambda,p})_p := (j^{-1}(M_{\gamma,\lambda,p}))_p$ form a Morse-decomposition of $K'_\lambda := j^{-1}(K_{\gamma,\lambda})$ relative to π_λ. Furthermore,*

$$((\lambda_n, y_0, h), K'_{\lambda_n}, (M'_{\lambda_n,p})_p) \to ((\lambda, y_0, h), K'_\lambda, (M'_{\lambda,p})_p)$$

whenever $\lambda_n \to \lambda$ in $[0,1]$.

(b) $\mathrm{ContCl}(\pi_\lambda, (y_0, h), K'_\lambda, (M'_{\lambda,p})_p)$ *is independent of $\lambda \in [0,1]$.*

(c) $\mathrm{ContCl}(\pi, f(\gamma)) = \mathrm{ContCl}(\pi_1, (y_0, h), K'_1, (M'_{1,p})_p) = \mathrm{ContCl}(\pi_0, (y_0, h), K'_0, (M'_{0,p})_p) = \mathrm{ContCl}(\pi, f(\gamma_0))$.

Proof. By Lemma 3.6 respectively the choice of U, $N \subset Y \times X$ (resp. $N_p \subset Y \times X$) is an isolating neighbourhood for $(y(\gamma, \lambda), K_{\gamma,\lambda})$ (resp. $(y(\gamma, \lambda), M_{\gamma,\lambda,p})$) for all $\lambda \in [0,1]$ and all $\gamma \in U$.

It is easy to see ([10]) that for all $\lambda \in [0,1]$, the closed set $N' := j^{-1}(N)$ (resp. $N'_p := j^{-1}(N_p)$) is an isolating neighbourhood for $(\lambda, y_0, h, K'_\lambda)$ (resp. $(\lambda, y_0, h, M'_{\lambda,p})$).

(a) Let $u\colon \mathbb{R} \to K'_\lambda$ be a solution of π_λ. It follows that $j \circ u$ is a solution of π, so either $j \circ u \subset M_{\gamma,\lambda,p}$ for some $p \in P$ or there are $p \prec q$ in P such that $\alpha(j \circ u) \subset M_{\gamma,\lambda,q}$ and $\omega(j \circ u) \subset M_{\gamma,\lambda,p}$. In the first case, we immediately conclude that $u \subset M'_{\lambda,p}$. In the second case, one has $u(t) \in N'_{\lambda,q}$ for all t sufficiently small and $u(t) \in N'_{\lambda,p}$ for all t sufficiently large. Since N'_q and N'_p are isolating neighbourhoods for $(\lambda, y_0, h, M'_{\lambda,q})$ and $(\lambda, y_0, h, M'_{\lambda,p})$, it follows that $\alpha(u) \subset M'_{\lambda,q}$ and $\omega(u) \subset M'_{\lambda,p}$. Since N' and N'_p are isolating neighbourhoods regardless of $\lambda \in [0,1]$, one trivially has $((\lambda_n, y_0, h), K'_{\lambda_n}, (M'_{\lambda_n,p})_p) \to ((\lambda, y_0, h), K'_\lambda, (M'_{\lambda,p})_p)$ whenever $\lambda_n \to \lambda$ as claimed.

(b), (c) Recall the notation of Lemma 3.6. For every interval $I \subset P$, there is an isolated invariant subset $M_{y(\gamma,\lambda)}(I)$. If (N_1, N_2) is an index pair for $(y(\gamma, \lambda), M_{y(\gamma,\lambda)}(I))$ relative to π, then (N_1, N_2) is also an index pair for $((y_0, h), M'_\lambda(I))$ relative to π_λ, where $M'_\lambda(I)$ is defined with respect to the Morse-decomposition $(M'_{\lambda,p})_{p\in P}$. Consequently,

$$\mathrm{ContCl}(\pi, y(\gamma, \lambda), K_{\gamma,\lambda}, (M_{\gamma,\lambda,p})_p) = \mathrm{ContCl}(\pi_\lambda, (y_0, h), K'_\lambda, (M'_{\lambda,p})_p).$$

([10]) Firstly, the set $j^{-1}(K_\lambda)$ is invariant because $\mathcal{H}^+(y_0, h)$ is compact. Secondly, $\mathrm{Inv}N' \cap (\mathcal{H}^+(y_0, h) \times X) \subset j^{-1}(K_\lambda)$. One can argue analogously for $M_{\lambda,p}$, $p \in P$.

Suppose that $\chi(\lambda) := \mathrm{ContCl}(\pi_\lambda, (y_0, h), K_\lambda, (M_{\lambda,p})_p)$ is not constant for $\lambda \in [0,1]$. There must exist a sequence $\lambda_n \to \lambda_0$ such that $\chi(\lambda_n) \neq \chi(\lambda_0)$, in contradiction to Lemma 3.2. □

Proof of Theorem 3.4. Firstly, we will prove that $\mathrm{ContCl} \circ f$ is locally constant. Let $\gamma_0 \in \Gamma$, and let the isolating neighbourhoods N and N_p, $p \in P$ be determined by (C1). It follows from Lemma 3.8 above, that $\mathrm{ContCl} \circ f$ is constant in a neighbourhood U of γ_0.

We have shown that $\mathrm{ContCl} \circ f$ is locally constant. Moreover, Γ is connected, so the proof is complete. □

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