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NONAUTONOMOUS CONLEY INDEX THEORY. THE HOMOLOGY INDEX AND ATTRACTOR-REPELLER DECOMPOSITIONS

Axel Jänig

Abstract. In a previous work, the author established a nonautonomous Conley index based on the interplay between a nonautonomous evolution operator and its skew-product formulation. This index is refined to obtain a Conley index for families of nonautonomous evolution operators. Different variants such as a categorial index, a homotopy index and a homology index are obtained. Furthermore, attractor-repeller decompositions and conecting homomorphisms are introduced for the nonautonomous setting.

In [4], the author defined a nonautonomous Conley index relying on the interplay between an evolution operator ([1]) and a skew-product formulation. Isolatted invariant sets obtained in the skew-product setting give rise to an index for a related nonautonomous evolution operator.

An important technical detail of defining the index is the class of index pairs under consideration. In [4], index pairs are always obtained in the skew-product formulation. In this paper, it will be proved that, roughly speaking, the same index can be defined using a broader class of index pairs based on the evolution operator instead of the skew-product formulation.

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([1]) or process

Firstly, we will formulate and prove an inclusion property for index pairs. A homotopy index, a categorial index and a homology Conley index will be introduced and, using the previously introduced inclusion properties, shown to be well-defined. Most of these concepts have evolved over decades and are only adapted (2) to the nonautonomous setting.

A powerful feature of Conley index theories is certainly its ability to reflect attractor-repeller decompositions obtained from the skew-product formulation. Passing to homology, an attractor-repeller decomposition gives rise to a long exact sequence [3], [2] and a so-called connecting homomorphism. These sequences contain information on the connections between attractor and repeller.

Usually this long exact sequence is obtained from so-called index triples. Using an appropriate adaption of index triples, these algebraic sequences and their connecting homomorphisms are shown to be available for the nonautonomous index, too.

Following a Preliminaries section, a notion of related index pairs is introduced in Section 2. Based on these results, a categorial index is defined in Section 3. Section 4 is devoted to attractor-repeller decompositions based on the notion of a homology Conley index defined there as well.

The reader who is interested in applications is referred to [4]. Continuation properties of Morse-decompositions and a uniformity property of the connecting homomorphism will be discussed in subsequent papers.

1. Preliminaries

For the convenience of the reader, we collect important definitions and terminology from other sources, mostly following the author's previous paper on the subject [4].

1.1. Quotient spaces.

Definition 1.1. Let X be a topological space, and $A, B \subset X$. Denote

$$A/B := A/R \cup \{A \cap B\},$$

where A/R is the set of equivalence classes with respect to the relation R on A which is defined by xRy if and only if $x = y$ or $x, y \in B$. We consider A/B as a topological space endowed with the quotient topology with respect to the canonical projection $q\colon A \to A/B$, that is, a set $U \subset A/B$ is open if and only if

$$q^{-1}(U) = \bigcup_{x \in U} x$$

is open in A.

(2) Each genuinely nonautonomous definition in this paper also applies to the autonomous setting. Therefore, a comparison is possible. Minor differences between our definition and other variants (such as [1], for instance) might occur.

Recall that the quotient topology is the final topology with respect to the projection q.

REMARK 1.2. The above definition is compatible with the definition used in [1] or [6]. The only difference occurs in the case $A \cap B = \emptyset$, where we add $\emptyset$, which is never an equivalence class, instead of an arbitrary point.

1.2. Evolution operators and semiflows. Let X be a metric space. Assuming that $\Diamond \notin X$, we introduce a symbol $\Diamond$, which means "undefined". The intention is to avoid the distinction if an evolution operator is defined for a given argument or not. Define $\overline{A} := A \dot{\cup} \{\Diamond\}$ whenever A is a set with $\Diamond \notin A$. Note that $\overline{A}$ is merely a set, the notation does not contain any implicit assumption on the topology.

DEFINITION 1.3. Let $\Delta := \{(t, t_0) \in \mathbb{R}^+ \times \mathbb{R}^+ : t \geq t_0\}$. A mapping $\Phi \colon \Delta \times \overline{X} \to \overline{X}$ is called an *evolution operator* if

(a) $\mathcal{D}(\Phi) := \{(t, t_0, x) \in \Delta \times X : \Phi(t, t_0, x) \neq \Diamond\}$ is open in $\mathbb{R}^+ \times \mathbb{R}^+ \times X$;
(b) Φ is continuous on $\mathcal{D}(\Phi)$;
(c) $\Phi(t_0, t_0, x) = x$ for all $(t_0, x) \in \mathbb{R}^+ \times X$;
(d) $\Phi(t_2, t_0, x) = \Phi(t_2, t_1, \Phi(t_1, t_0, x))$ for all $t_0 \leq t_1 \leq t_2$ in $\mathbb{R}^+$ and $x \in X$;
(e) $\Phi(t, t_0, \Diamond) = \Diamond$ for all $t \geq t_0$ in $\mathbb{R}^+$.

A mapping $\pi \colon \mathbb{R}^+ \times \overline{X} \to \overline{X}$ is called *semiflow* if $\widetilde{\Phi}(t+t_0, t_0, x) := \pi(t, x)$ defines an evolution operator. To every evolution operator Φ, there is an associated (skew-product) semiflow π on an extended phase space $\mathbb{R}^+ \times X$, defined by $(t_0, x)\pi t = (t_0 + t, \Phi(t + t_0, t_0, x))$. A function $u \colon I \to X$ defined on a subinterval I of $\mathbb{R}$ is called a *solution of* Φ if $u(t_1) = \Phi(t_1, t_0, u(t_0))$ for all $[t_0, t_1] \subset I$.

DEFINITION 1.4. Let X be a metric space, $N \subset X$ and π a semiflow on X.

(a) The set $\mathrm{Inv}_\pi^-(N) := \{x \in N$: there is a solution $u \colon \mathbb{R}^- \to N$ with $u(0) = x\}$ is called the *largest negatively invariant subset of* N.
(b) The set $\mathrm{Inv}_\pi^+(N) := \{x \in N : x\pi\mathbb{R}^+ \subset N\}$ is called the *largest positively invariant subset of* N.
(c) The set $\mathrm{Inv}_\pi(N) := \{x \in N$: there is a solution $u \colon \mathbb{R} \to N$ with $u(0) = x\}$ is called the *largest invariant subset of* N.

Let X and Y be metric spaces, and assume that $y \mapsto y^t$ is a global (3) semiflow on Y, to which we will refer as t-translation.

EXAMPLE 1.5. Let Z be a metric space, and let $Y := C(\mathbb{R}^+, Z)$ be a metric space such that a sequence of functions converges if and only if it converges uniformly on bounded sets. The translation can now be defined canonically by $y^t(s) := y(t + s)$ for $s, t \in \mathbb{R}^+$.

(3) defined for all $t \in \mathbb{R}^+$

A suitable abstraction of many non-autonomous problems is given by the concept of skew-product semiflows introduced below.

Definition 1.6. We say that $\pi = (\,\cdot^{\,t}, \Phi)$ is a skew-product semiflow on $Y \times X$ if $\Phi\colon \mathbb{R}^+ \times \overline{Y \times X} \to \overline{Y \times X}$ is a mapping such that

$$(t, y, x)\pi t := \begin{cases} (y^t, \Phi(t, y, x)) & \Phi(t, y, x) \neq \Diamond, \\ \Diamond & \text{otherwise}, \end{cases}$$

is a semiflow on $Y \times X$.

Definition 1.7. For $y \in Y$ let $\mathcal{H}^+(y) := \mathrm{cl}_Y\{y^t : t \in \mathbb{R}^+\}$ denote the positive hull of y. Let Y_c denote the set of all $y \in Y$ for which $\mathcal{H}^+(y)$ is compact.

Definition 1.8. Let $y_0 \in Y$ and $N \subset \mathcal{H}^+(y_0) \times X$ be a closed subset. N is called an *isolating neighbourhood* (for K in $\mathcal{H}^+(y_0) \times X$) if $\mathrm{Inv}N \subset \mathrm{int}_{\mathcal{H}^+(y_0) \times X}\, N$ (and $K = \mathrm{Inv}N$).

The following definition is a consequence of the slightly modified notion of a semiflow (Definition 1.3) but not a semantical change compared to [1], for instance.

Definition 1.9. We say that π explodes in $N \subset Y \times X$ if $x\pi[0, t[\, \subset N$ and $x\pi t = \Diamond$.

Following [5], we formulate the following asymptotic compactness condition.

Definition 1.10. A set $M \subset Y \times X$ is called *strongly admissible* provided the following holds: Whenever (y_n, x_n) is a sequence in M and $(t_n)_n$ is a sequence in $\mathbb{R}^+$ such that $(y_n, x_n)\pi\,[0, t_n] \subset M$, then the sequence $(y_n, x_n)\pi t_n$ has a convergent subsequence.

Definition 1.11. Let $\pi = (\,\cdot^{\,t}, \Phi)$ be a skew-product semiflow and $y \in Y$. Define

$$\Phi_y(t + t_0, t_0, x) := \Phi(t, y^{t_0}, x).$$

It is easily proved that Φ_y is an evolution operator in the sense of Definition 1.3.

2. Related index pairs

In this section we give a definition of a nonautonomous Conley index which is slightly different from the index defined in [4]. Essentially, the index is now purely based on nonautonomous index pairs which are subsets of $\mathbb{R}^+ \times X$, where X is an appropriate metric space. It is often more convenient to compute the index by using the modified definition of this section. The main results are Theorem 2.9 and its corollary.

We say that two index pairs for which the assumptions and thus also the conclusions of Theorem 2.9 hold are *related*. Roughly speaking, related index pairs define the same index ([4]).

Throughout this section, it is assumed that X and Y are metric spaces, and $\pi = \pi(\,\cdot\,^t, \Phi)$ is a skew-product semiflow on $Y \times X$. By $\chi := \chi_{y_0}$ we denote the canonical semiflow $(t,x)\chi_{y_0}s := (t+s, \Phi_{y_0}(s,0,x))$ on $\mathbb{R}^+ \times X$.

Definition 2.1. A pair (N_1, N_2) is called a *(basic) index pair* relative to a semiflow χ in $\mathbb{R} + \times X$ if

- (IP1) $N_2 \subset N_1 \subset \mathbb{R}^+ \times X$, N_1 and N_2 are closed in $\mathbb{R}^+ \times X$;
- (IP2) If $x \in N_1$ and $x\chi t \notin N_1$ for some $t \in \mathbb{R}^+$, then $x\chi s \in N_2$ for some $s \in [0,t]$;
- (IP3) If $x \in N_2$ and $x\chi t \notin N_2$ for some $t \in \mathbb{R}^+$, then $x\chi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [0,t]$.

The definition above establishes the core properties of an index pair and is taken from [4]. To obtain an index, we need to associate invariant sets with index pairs.

Definition 2.2. Let $y_0 \in Y$ and (N_1, N_2) be a basic index pair in $\mathbb{R}^+ \times X$ relative to χ_{y_0}. Define $r := r_{y_0} \colon \mathbb{R}^+ \times X \to \mathcal{H}^+(y_0) \times X$ by $r_{y_0}(t,x) := (y_0^t, x)$. et $K \subset \omega(y_0) \times X$ be an (isolated) invariant set. We say that (N_1, N_2) is a (strongly admissible) index pair ([5]) for (y_0, K) if:

- (IP4) there is a strongly admissible isolating neighbourhood N of K in $\mathcal{H}^+(y_0)$ $\times X$ such that $N_1 \setminus N_2 \subset r^{-1}(N)$;
- (IP5) there is a neighbuorhood W of K in $\mathcal{H}^+(y_0) \times X$ such that $r^{-1}(W) \subset N_1 \setminus N_2$.

Definition 2.3. We say that (y_0, K) is an *invariant pair* if $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$. An invariant pair (y_0, K) is called a *compact invariant pair* provided that K is compact.

Every FM-index pair relative to the skew-product semiflow induces an index pair. Therefore, the homotopy index defined here and the homotopy index from [4] agree ([6]).

Lemma 2.4. *Let $y_0 \in Y$ and let (N_1, N_2) be an* FM-*index pair for $K \subset \mathcal{H}^+(y_0) \times X$ such that N_1 is strongly admissible. Then*

$$(M_1, M_2) := (r_{y_0}^{-1}(N_1), r_{y_0}^{-1}(N_2))$$

is an index pair for (y_0, K).

([4]) This is not necessarily a homotopy index, so the vague language is intended.

([5]) Every index pair in the sense of Definition 2.2 is assumed to be strongly admissible.

([6]) A more detailed explanation can be found right after Theorem 2.9.

Proof. (M_1, M_2) is an index pair by Lemma 4.3 in [4]. We need to prove that the assumptions (IP4) and (IP5) of Definition 2.2 are satisfied.

(IP4) $N := \mathrm{cl}_{Y\times X}(N_1 \setminus N_2)$ is an isolating neighbourhood for K, and $M_1 \setminus M_2 = r^{-1}(N_1) \setminus r^{-1}(N_2) \subset r^{-1}(N)$.

(IP5) Let $W := \mathrm{int}_{\mathcal{H}^+(y_0)\times X}(N_1 \setminus N_2)$, which is a neighbourhood of K. We have $r^{-1}(W) \subset r^{-1}(N_1) \setminus r^{-1}(N_2)$. □

The following lemma is not much more than a restatement of Theorem 3.5 in [4].

Lemma 2.5. *Suppose that $(N_1, N_2) \subset (M_1, M_2)$ are index pairs for (y_0, K). The inclusion induced mapping $i\colon (N_1/N_2, N_2) \to (M_1/M_2, M_2)$ is a homotopy equivalence.*

Proof. By Definition 2.2, there is a neighbourhood W of K such that $r^{-1}(W) \subset (N_1 \setminus N_2) \cap (M_1 \setminus M_2)$. It follows from Definition 2.2 that the closure $\overline{W} := \mathrm{cl}_{Y\times X} W$ is strongly admissible, so by [4, Lemma 4.3], (N_1, N_2) and (M_1, M_2) are index pairs for $(\Phi_{y_0}, r_{y_0}^{-1}(W))$. The claim is now a direct consequence of Theorem 3.5 in [4]. □

Definition 2.6. Let (N_1, N_2) be an index pair in $\mathbb{R}^+ \times X$ (relative to the semiflow χ on $\mathbb{R}^+ \times X$). For $T \in \mathbb{R}^+$, we set

$$N_2^{-T} := N_2^{-T}(N_1) := \{(t, x) \in N_1 : \exists s \leq T \ (t, x)\chi s \in N_2\}.$$

Lemma 2.7. *Let (N_1, N_2) be an index pair for (y_0, K). Then so is (N_1, N_2^{-T}) for every $T \in \mathbb{R}^+$.*

Proof. We need to check the assumptions of Definitions 2.1 and 2.2.

(IP1) We need to show that N_2^{-T} is closed. Suppose that (s_n, x_n) is a sequence in N_2^{-T} with $(s_n, x_n) \to (s, x)$ in N_1. For every $n \in \mathbb{N}$, there is a $t_n \in [0, T]$ such that $(s_n, x_n)\chi t_n \in N_2$. We can assume without loss of generality that $t_n \to t \leq T$, so $(s, x)\chi t \in N_2$, which is closed. Thus it holds that $(s, x) \in N_2^{-T}$.

(IP2) Let $x \in N_2^{-T}$ but $x\chi t \notin N_1$ for some $t \in \mathbb{R}^+$. (N_1, N_2) is an index pair, so $x\chi s \in N_2 \subset N_2^{-T}$ for some $s \in [0, t]$.

(IP3) Suppose that $x \in N_2^{-T}$ and $x\chi t \notin N_2^{-T}$ for some $t \in \mathbb{R}^+$. Letting $t_0 := \sup\{s \in \mathbb{R}^+ : x\chi[0, s] \cap N_2 = \emptyset\}$, it follows that $t_0 \leq T$ and $x\chi t_0 \in N_2$. Furthermore, one has $x\chi[0, t_0] \subset N_2^{-T}$, so $t > t_0$. Since (N_1, N_2) is assumed to be an index pair, it follows that $x\chi s \in (\mathbb{R}^+ \times X) \setminus N_1$ for some $s \in [t_0, t]$.

(IP4) (N_1, N_2) is an index pair for (y_0, K), so there is an isolating neighbourhood N of K such that $N_1 \setminus N_2^{-T} \subset N_1 \setminus N_2 \subset r^{-1}(N)$.

(IP5) Let W be an open neighbourhood of K such that $r^{-1}(W) \subset N_1 \setminus N_2$. We consider the set $W^T := \{(y, x) \in W : (y, x)\pi[0, T] \subset W\}$.

If $(t,x) \in r^{-1}(W^T) \cap N_2^{-T}$, then $(t,x)\chi_{y_0}T \in r^{-1}(W) \cap N_2 = \emptyset$, so

$$r^{-1}(W^T) \subset N_1 \setminus N_2^{-T}.$$

We need to show that W^T is a neighbourhood of K. Suppose to the contrary that there is (7) a sequence $(y_n, x_n) \to (y_0', x_0) \in K$ in $N \setminus W^T$. For every $n \in \mathbb{N}$, there is a $t_n \in [0,T]$ with $(y_n,x_n)\pi t_n \in (\mathcal{H}^+(y_0) \times X) \setminus W$. We can assume, without loss of generality, that $t_n \to t_0$, so $(y_0', x_0)\pi t_0 \in (\mathcal{H}^+(y_0) \times X) \setminus W$, which is a closed set. However, $(y_0', x_0)\pi t_0 \in K \subset W$, a contradiction. □

One frequently needs to prove that a pair (N_1, N_2) is not only an index pair but also that it belongs to a certain pair (y_0, K). For this purpose and in conjunction with Lemma 2.7, the following — simple — "sandwich" lemma is useful.

Lemma 2.8. *Let $y_0 \in Y$, and let (N_1,N_2), (M_1,M_2) and (N_1',N_2') be index pairs with $N_1 \setminus N_2 \subset M_1 \setminus M_2 \subset N_1' \setminus N_2'$.*

If (N_1,N_2) and (N_1',N_2') are index pairs for (y_0,K), then so is (M_1,M_2).

Proof. One simply needs to check the assumptions of Definition 2.2.

(IP4) (N_1',N_2') is an index pair for (y_0,K), so there is a strongly admissible isolating neighbourhood N of K in $\mathcal{H}^+(y_0) \times X$ such that $M_1 \setminus M_2 \subset N_1' \setminus N_2' \subset r^{-1}(N)$.

(IP5) (N_1,N_2) is an index pair for (y_0,K), so there is a neighbourhood W of K in $\mathcal{H}^+(y_0) \times X$ such that $r^{-1}(W) \subset N_1 \setminus N_2 \subset M_1 \setminus M_2$. □

We are now in a position to formulate and prove the main result of this section.

Theorem 2.9. *Let there be given index pairs (N_1,N_2) and (M_1,M_2) for (y_0,K). Further, let $N \subset \mathcal{H}^+(y_0) \times X$ be a strongly admissible neighbourhood of K. Then there are a $t_0 \in \mathbb{R}^+$ and an index pair (L_1,L_2) such that*

$$(L_1,L_2) \subset (r^{-1}(N) \cap N_1 \cap M_1, N_2^{-t_0}(N_1) \cap M_2^{-t_0}(M_1)).$$

An important consequence of the theorem above is that the homotopy index of (y_0,K) can be defined as the pointed homotopy type of $(N_1/N_2, N_2)$, where (N_1,N_2) is an index pair for (y_0,K). It coincides (8) with Definition 4.1 in [4], so there is no need to redefine the homotopy index. We have merely extended the class of possible or good index pairs.

Corollary 2.10. *Under the assumptions of Theorem 2.9, the pointed homotopy types of $(N_1/N_2,N_2)$ and $(M_1/M_2,M_2)$ agree.*

(7) As a consequence of the admissibility assumption, K is compact.

(8) Under the assumptions of Theorem 2.9, it follows from [6] that there exists an isolating block for K in $\mathcal{H}^+(y_0) \times X$. This isolating block gives rise to an index pair for (y_0,K) as proved in Lemma 2.4.

Proof. By Theorem 2.9, there are an index pair and a constant $t_0 \in \mathbb{R}^+$ for which the following inclusions hold true.

$$(L_1, L_2) \subset (N_1, N_2^{-t_0}) \supset (N_1, N_2), \qquad (L_1, L_2) \subset (M_1, M_2^{-t_0}) \supset (M_1, M_2).$$

In view of Lemmas 2.5 and 2.7, this readily implies that $(N_1/N_2, N_2)$ and $(M_1/M_2, M_2)$ are isomorphic in the homotopy category of pointed spaces. □

The rest of this section is devoted to the proof of Theorem 2.9. The proof is similar to the proof of [1, Lemma 4.8], but instead of using isolating blocks, we will construct appropriate index pairs. In all subsequent lemmas, we will assume that the hypotheses of Theorem 2.9 hold.

Since N is a neighbourhood of K, there is an open (in $\mathcal{H}^+(y_0) \times X$) set U with $K \subset U \subset N$. Define $g^+, g^- : \mathcal{H}^+(y_0) \times X \to \mathbb{R}^+$ by

$$g^+(y,x) := \sup\{t \in \mathbb{R}^+ : (y,x)\pi[0,t] \subset U\},$$
$$g^-(y,x) := \sup\left\{d((y,x)\pi t, \operatorname{Inv}^-_\pi(N)) : t \int [0, g^+(y,x)]\right\}.$$

It is easy to see that both functions g^+ and g^- are continuous and monotone decreasing along solutions in U (resp. N), that is, if $u\colon [0,a] \to U$ (resp. $u\colon [0,a] \to N$) is a solution of π, then $t \mapsto g^+(u(t))$ (resp. $t \mapsto g^-(u(t))$) is continuous and monotone decreasing on $[0,a]$.

Lemma 2.11.

(a) *g^+ is lower-semicontinuous.*
(b) *g^- is lower-semicontinuous.*
(c) *$\{g^+ \le c\} := \{(y,x) \in N : g^+(y,x) \le c\}$ is closed.*
(d) *$\{g^- \le c\} := \{(y,x) \in N : g^-(y,x) \le c\}$ is closed.*
(e) *For all $c_1, c_2 > 0$, the set $(\{g^- \le c_1\} \cap \{g^+ > c_2\})$ is a neighbourhood of $K := \operatorname{Inv}(N)$.*

Proof. (a) Let $\varepsilon > 0$ and $(y,x) \in \mathcal{H}^+(y_0) \times X$. Suppose that $(y_n, x_n) \to (y,x)$ in $\mathcal{H}^+(y_0) \times X$ and $g^+(y_n,x_n) \le g^+(y,x) - \varepsilon$ for all $n \in \mathbb{N}$. We can assume, without loss of generality, that $g^+(y_n,x_n) \to t_0$.

First of all, as N is strongly admissible and $(y_n,x_n)\pi s \to (y,x)\pi s$, it follows that $(y,x)\pi s \in N$ for all $s \in [0,t_0]$. Secondly, one has $(y_n,x_n)\pi g^+(y_n,x_n) \in X \setminus U$, which is closed, so $(y,x)\pi t_0 \in X \setminus U$. However, $t_0 \le g^+(y,x) - \varepsilon$, which is a contradiction.

(b) Let $(y,x) \in \mathcal{H}^+(y_0) \times X$ and suppose that $(y_n,x_n) \to (y,x)$ but $g^-(y_n,x_n) \le g^-(y,x) - \varepsilon$ for some $\varepsilon > 0$.

Let $t \in [0, g^+(y,x)[$ be arbitrary. By the lower-semicontinuity of g^+, one has $g^+(y_n,x_n) \ge t$ provided that n is sufficiently large. Furthermore,

$$d((y,x)\pi t, \operatorname{Inv}^-(N)) \le d((y,x)\pi t, (y_n,x_n)\pi t) + d((y_n,x_n)\pi t, \operatorname{Inv}^-(N)),$$

so

$$d((y,x)\pi t, \mathrm{Inv}^-(N) \le g^-(y_n,x_n) \le g^-(y,x) - \varepsilon.$$

The last inequality holds for arbitrary $t \in [0, g^+(y,x)[$. We thus have $g^-(y,x) \le g^-(y,x) - \varepsilon$, which is a contradiction.

(c) and (d) follow immediately from the lower-semicontinuity of the respective function.

(e) Arguing by contradiction, we may assume that there are $(y_n,x_n) \to (y,x) \in K$ such that either $g^+(y_n,x_n) \le c_2$ or $g^-(y_n,x_n) > c_1$ for all $n \in \mathbb{N}$. In the first case, it follows that $g^+(y,x) \le c_2$ in contradiction to $(y,x) \in K$. In the second case, we can choose $t_n \in \mathbb{R}^+$ such that for all $n \in \mathbb{N}$, $t_n \le g^+(y_n,x_n)$ and

$$d((y_n,x_n)\pi t_n, \mathrm{Inv}^-(N)) \ge c_1 > 0. \tag{2.1}$$

Either $(t_n)_n$ has a convergent subsequence or $t_n \to \infty$. Suppose that $(t_{n(k)})_k$ is a subsequence with $t_{n(k)} \to t_0$ as $k \to \infty$. It follows that $d((y,x)\pi t_0, \mathrm{Inv}^-(N)) \ge c_1$, which is a contradiction to $(y,x) \in K$. Thus, one has $t_n \to \infty$, and using the admissibility of N, there is a subsequence $(y_{n(k)}, x_{n(k)})\pi t_{n(k)}$ which converges to a point $(y',x') \in \mathrm{Inv}^-(N)$, in contradiction to (2.1). □

Lemma 2.12. *For $c_1 > 0$ and $c_2 > 0$, set*

$$L_1^{c_1,c_2} := \{g^- \le c_1\} \cap \mathrm{cl}\{g^+ \ge c_2\}, \qquad L_2^{c_1,c_2} := L_1^{c_1,c_2} \cap \{g^+ \le c_2\}$$

and $\widehat{L}_i^{c_1,c_2} := r^{-1}(L_i^{c_1,c_2})$,for $i = 1,2$. Then, for c_1 small and c_2 large, one has

(a) *$L_1^{c_1,c_2} \subset U$, and*

(b) *$(L_1, L_2) := (\widehat{L}_1, \widehat{L}_2) := (\widehat{L}_1^{c_1,c_2}, \widehat{L}_2^{c_1,c_2})$ is an index pair for (y_0, K).*

Proof. (a) If $(y,x) \in \mathrm{cl}\{g^+ \ge c_2\}$, then $(y,x)\pi[0,c_2] \subset N$. Hence, if the claim does not hold, there is a point $(y',x') \in K \cap (N \setminus U) = \emptyset$.

(b) (IP1) It follows from Lemma 2.11 (c) and (d) that $L_1^{c_1,c_2}$ and $L_2^{c_1,c_2}$ are closed, so $\widehat{L}_1$ and $\widehat{L}_2$ are closed by the continuity of r.

(IP2) Let $x \in L_1^{c_1,c_2}$ and $x\pi t \notin L_1^{c_1,c_2}$ for some $t \ge 0$. The semiflow does not explode in N. Hence, there is a $t' \le t$ such that $x\pi t' \in (\mathcal{H}^+(y_0) \times X) \setminus L_1^{c_1,c_2}$. Choose a sequence $x_n \to x$ in $L_1^{c_1,c_2}$ with $g^+(x_n) \ge c_2$. We have $x_n\pi t \notin L_1^{c_1,c_2}$ for all n sufficiently large, so $x_n\pi s_n \in L_2^{c_1,c_2}$ for some $s_n \le t$ and all $n \in \mathbb{N}$. We can assume, without loss of generality, that $s_n \to s_0 \le t$, so $x\pi s_0 \in L_2^{c_1,c_2}$.

(IP3) Let $x \in L_2^{c_1,c_2}$ and $x\pi\,[0,t] \subset L_1^{c_1,c_2}$. We have $L_1^{c_1,c_2} \subset U$, so $g^+(x\pi s) \le g^+(x)$ for all $s \in [0,t]$. Hence, $x\pi[0,t] \subset L_2^{c_1,c_2}$.

Furthermore, one has $N \supset L_1^{c_1,c_2} \setminus L_2^{c_1,c_2} \supset W$, where $W := \{g^- \le c_1\} \cap \{g^+ > c_2\}$ is a neighbuorhood of K by Lemma 2.11 (e). Thus, $r^{-1}(N) \supset \widehat{L}_1^{c_1,c_2} \setminus \widehat{L}_2^{c_1,c_2} \supset r^{-1}(W)$, which shows that $(\widehat{L}_1, \widehat{L}_2)$ is an index pair for (y_0, K). □

Until now, our proof is based loosely on the respective proof in [6] concerning the existence of isolating blocks. However, our claim is significantly weaker, so the proof is, hopefully, easier to follow.

Since both (N_1, N_2) and (M_1, M_2) are index pairs for (y_0, K), we can assume without loss of generality that $r^{-1}(N) \subset N_1 \cap M_1$. Otherwise, one can simply replace N by a sufficiently small neighbourhood N', and thereby obtain a stronger result. In order to complete the proof of Theorem 2.9, we need

Lemma 2.13. *For every $d > 0$, one has $\widehat{L}_2^{c,d} \subset N_2^{-T}$ (resp. $\widehat{L}_2^{c,d} \subset M_2^{-T}$) provided that c is sufficiently small and T is sufficiently large.*

Proof. If the lemma is not true, then there are sequences $((t_n, x_n))_n$, $c_n \to 0$ and $T_n \to \infty$ such that $(t_n, x_n) \in \widehat{L}_2^{c_n,d}$ and $(t_n, x_n)\pi s \in N_1 \setminus N_2$ for all $s \le T_n$ and all $n \in \mathbb{N}$.

Taking subsequences and because $c_n \to 0$, we can assume without loss of generality that $(y_0^{t_n}, x_n) \to (y, x) \in \mathrm{Inv}^-(N)$, which is compact because N is strongly admissible. Since (N_1, N_2) is an index pair for K, there exists an isolating neighbourhood $\widetilde{N}$ for K with $N_1 \setminus N_2 \subset r_{y_0}^{-1}(\widetilde{N})$. The choice of the sequences implies that $(y, x) \in \mathrm{Inv}^+(\widetilde{N})$, so $(y, x) \in \mathrm{Inv}(\widetilde{N}) = K$. However, $\big(y_0^{t_n}, x_n\big)\pi g^+\big(y_0^{t_n}, x_n\big) \in N \setminus U$ for all $n \in \mathbb{N}$. Furthermore, $g^+\big(y_0^{t_n}, x_n\big) \le d$ by the choice of $\widehat{L}_2^{c,d}$. One may therefore assume without loss of generality that $g^+(y_0^{t_n}, x_n) \to t_0$. Consequently, one obtains $(y, x)\pi t_0 \in (N \setminus U) \cap K = \emptyset$, which is an obvious contradiction. □

By using Lemma 2.12, one can construct an index pair $(L_1, L_2) := \big(\widehat{L}_1^{c,d}, \widehat{L}_2^{c,d}\big)$ for (y_0, K) choosing c small and d large. In view of Lemma 2.13, one can find a possibly even smaller parameter $c > 0$ such that the conclusions of Theorem 2.9 hold for large t_0. The proof of Theorem 2.9 is complete. □

3. Categorial Conley index

A connected simple system is a small category with the following property: if A and B are objects, then there is exactly one morphism $A \to B$.

Understanding the Conley index as a connected simple system is perhaps the most elegant variant of the index. There is no loss of information, and other invariants such as a homotopy or (co)homology index can be derived by applying an appropriate functor. We will show in this section, that the nonautonomous extension of the Conley index defines a connected simple system as well.

Throughout this section, we will assume the hypotheses ([9]) at the beginning of the previous section.

Definition 3.1. Let $y_0 \in Y$, and let $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighbourhood. The categorial (nonautonomous) Conley index $\mathcal{C}(y_0, K)$ of (y_0, K) is the smallest subcategory of the homotopy category of pointed spaces with the following properties:

([9]) i.e. the spaces X, Y, the semiflows π, χ_{y_0} and the mapping $r := r_{y_0}$.

(a) Objects of $\mathcal{C}(y_0, K)$ are pairs $(N_1/N_2, N_2)$, where (N_1, N_2) is an index pair for (y_0, K).

(b) If (N_1, N_2) and (M_1, M_2) are index pairs for (y_0, K) with $(N_1, N_2) \subset (M_1, M_2)$, then the inclusion induced morphism

$$i\colon (N_1/N_2, N_2) \to (M_1/M_2, M_2)$$

in the homotopy category of pointed spaces is a morphism of $\mathcal{C}(y_0, K)$.

For brevity, we also write $[N_1, N_2] := (N_1/N_2, N_2)$.

THEOREM 3.2. *$\mathcal{C}(y_0, K)$ is (well-defined and) a connected simple system.*

The proof below can be sketched as follows: Given two arbitrary index pairs (N_1, N_2) and (M_1, M_2), one constructs a morphism $f\colon [N_1, N_2] \to [M_1, M_2]$ in $\mathcal{C}(y_0, K)$. This morphism f is a composition of inclusion induced morphisms or their inverse morphisms and therefore necessarily a morphism of $\mathcal{C}(y_0, K)$. These morphisms are then shown to be unique, that is, f depends only on (N_1, N_2) and (M_1, M_2), and invariant with respect to composition. In other words, the proof is nothing but an explicit construction.

PROOF. Let (N_1, N_2) and (M_1, M_2) be arbitrary index pairs for (y_0, K). By Theorem 2.9, there is an index pair (L_1, L_2) for (y_0, K) and a $T \in \mathbb{R}^+$ such that

$$(L_1, L_2) \subset (N_1 \cap M_1, N_2^{-T} \cap M_2^{-T}).$$

Each inclusion of index pairs gives rise to a morphism. We obtain the following diagram, the arrows of which denote isomorphisms (Lemma 2.5) (respectively the inverse morphim) of $\mathcal{C}(y_0, K)$.

$$[N_1, N_2] \longrightarrow [N_1, N_2^{-T}] \longleftarrow [L_1, L_2] \longrightarrow [M_1, M_2^{-T}] \longleftarrow [M_1, M_2] \tag{3.1}$$

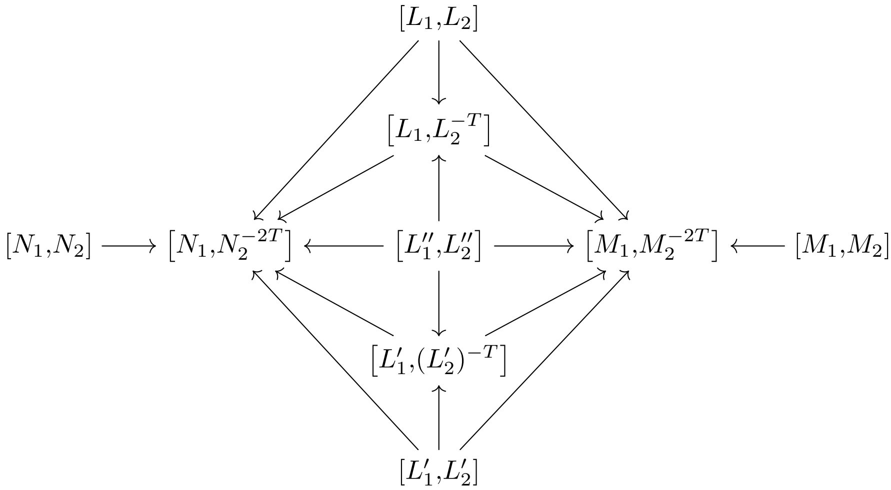
It follows that there is a morphism in $[N_1, N_2] \to [M_1, M_2]$ in $\mathcal{C}(y_0, K)$, namely the composition of the morphisms in the row above.

Next, we will show that the morphism obtained using this procedure is unique. Firstly, let $T_1 \geq T_2$ be positive real numbers. The following ladder with inclusion induced arrows is commutative.

$$\begin{array}{ccccccccc}
[N_1, N_2] & \longrightarrow & [N_1, N_2^{-T_1}] & \longleftarrow & [L_1, L_2] & \longrightarrow & [M_1, M_2^{-T_1}] & \longleftarrow & [M_1, M_2] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
[N_1, N_2] & \longrightarrow & [N_1, N_2^{-T_2}] & \longleftarrow & [L_1, L_2] & \longrightarrow & [M_1, M_2^{-T_2}] & \longleftarrow & [M_1, M_2]
\end{array}$$

Hence, the morphism $[N_1, N_2] \to [M_1, M_2]$ defined by (3.1) is independent of T. Secondly, one needs to consider the index pair (L_1, L_2). Suppose (L_1', L_2') is another index pair for (y_0, K) with $(L_1', L_2') \subset \big(N_1 \cap M_1, N_2^{-T} \cap M_2^{-T}\big)$. It follows again from Theorem 2.9 that there exist an index pair (L_1'', L_2'') for (y_0, K) and a constant $T > 0$ such that $(L_1'', L_2'') \subset \big(L_1 \cap L_1', L_2^{-T} \cap (L')_2^{-T}\big)$.

We obtain a commutative diagram below, where each arrow denotes an inclusion induced (iso)morphism.



The morphisms defined by (L_1, L_2) and (L_1', L_2') agree since each arrow in the above diagram denotes an isomorphism (Lemma 2.5).

Finally, we will show that the composition of two morphisms obtained from the above prodecure can be written as in (3.1). Suppose, we are given index pairs (N_1, N_2), (M_1, M_2) and (O_1, O_2) for (y_0, K). By Theorem 2.9, there are an index pair (L_1, L_2) for (y_0, K) and a $T \in \mathbb{R}^+$ such that

$$(L_1, L_2) \subset (N_1 \cap M_1 \cap O_1, N_2^{-T} \cap M_2^{-T} \cap O_2^{-T}).$$

For every two objects A, B in $\mathcal{C}(y_0, K)$, let $A \to B$ denote the unique morphism defined by (3.1). We also write $B \leftarrow A$ for the inverse (morphism) of $A \to B$. Given morphisms $A \to B$ and $B \to C$, we write $A \to B \to C$ to denote their composition. We need to prove that $A \to B \to C = A \to C$. One has

$$\begin{aligned}
&[N_1, N_2] \to [M_1, M_2] \to [O_1, O_2] \\
&\quad = [N_1, N_2] \to [N_1, N_2^{-T}] \leftarrow [L_1, L_2] \to [M_1, M_2^{-T}] \leftarrow [M_1, M_2] \\
&\qquad \to [M_1, M_2^{-T}] \leftarrow [L_1, L_2] \to [O_1, O_2^{-T}] \leftarrow [O_1, O_2] \\
&\quad = [N_1, N_2] \to [N_1, N_2^{-T}] \leftarrow [L_1, L_2] \to [M_1, M_2^{-T}] \leftarrow [L_1, L_2] \\
&\qquad \to [O_1, O_2^{-T}] \leftarrow [O_1, O_2] \\
&\quad = [N_1, N_2] \to [N_1, N_2^{-T}] \leftarrow [L_1, L_2] \to [O_1, O_2^{-T}] \leftarrow [O_1, O_2] \\
&\quad = [N_1, N_2] \to [O_1, O_2].
\end{aligned}$$

□

We will now introduce CSS($\mathcal{K}$), the category of connected simple systems in a given category $\mathcal{K}$. Objects of CSS($\mathcal{K}$) are subcategories of $\mathcal{K}$ which are connected simple systems. Let $\mathcal{A}$ and $\mathcal{B}$ be connected simple systems in $\mathcal{K}$.

A morphism $\mathcal{A} \to \mathcal{B}$ in CSS($\mathcal{K}$) is a family $(f_{A,B})_{(A,B)\in \mathrm{Obj}(\mathcal{A})\times \mathrm{Obj}(\mathcal{B})}$, where Obj(·) denotes the objects of a given category and each $f_{A,B}$ is a morphism $A \to B$ in $\mathcal{K}$ such that

$$\begin{array}{ccc} A & \xrightarrow{f_{A,B}} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow[f_{A',B'}]{} & B' \end{array}$$

is commutative. The vertical arrows denote the unique (inner) morphisms in $\mathcal{A}$, respectively $\mathcal{B}$.

If A is an object of $\mathcal{A}$, B is an object of $\mathcal{B}$, and $: A \to B$ is a morphism, then there is a unique morphism $F \in$ CSS($\mathcal{K}$) with $F = F(A, B) = f$. We say that $[f] := F$ is induced by f.

Now, set $\mathcal{K} = \mathcal{HT}$, the homotopy category of pointed spaces, and given an isolated invariant set $K \subset \mathcal{H}^+(y_0) \times X$ admitting a strongly admissible isolating neighbourhood, its index $\mathcal{C}(y_0, K)$ is an object of CSS($\mathcal{HT}$). The morphisms of $\mathcal{C}(y_0, K)$ are called *inner morphisms*.

4. Homology Conley index and attractor-repeller sequences

In this section, attractor-repeller decompositions of isolated invariant sets are studied. The main tool are long exact sequences in homology.

4.1. Attractor-repeller decompositions and index triples. Attractor-repeller decompositions with respect to semiflows are not exactly a new concept; in particular since they are applied to the skew-product formulation of the nonautonomous problem. The main goal of this section is to understand the implications of having an attractor-repeller decomposition in a space $\mathcal{H}^+(y_0) \times X$ on the index pairs respectively the index, living in the space $\mathbb{R}^+ \times X$.

First of all, α and ω-limes sets can be defined as usual.

$$\alpha(u) := \bigcap_{t\in\mathbb{R}^-} \mathrm{cl}_{\mathcal{H}^+(y_0)\times X}\, u(]-\infty, t]), \qquad \omega(u) := \bigcap_{t\in\mathbb{R}^+} \mathrm{cl}_{\mathcal{H}^+(y_0)\times X}\, u([t, \infty[).$$

Based on the above definitions, the notion of an attractor-repeller decomposition can be made precise.

Definition 4.1. Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set. (A, R) is an *attractor-repeller decomposition* of K if A, R are disjoint isolated invariant subsets of K and for every solution $u\colon \mathbb{R} \to K$ one of the following alternatives holds true.

(a) $u(\mathbb{R}) \subset A$,
(b) $u(\mathbb{R}) \subset R$,
(c) $\alpha(u) \subset R$ and $\omega(u) \subset A$.

We also say that (y_0, K, A, R) is an attractor-repeller decomposition.

Definition 4.2. Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighbourhood N. Suppose that (A, R) is an attractor-repeller decomposition of K. A triple (N_1, N_2, N_3) is called an *index triple* for (y_0, K, A, R) provided that:

(a) $N_3 \subset N_2 \subset N_1$,
(b) (N_1, N_3) is an index pair for (y_0, K),
(c) (N_2, N_3) is an index pair for (y_0, A).

Suppose we are given an isolated invariant set and an attractor-repeller decomposition thereof. Does there exist an index triple?

Lemma 4.3. *Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighbourhood N. Suppose that (A, R) is an attractor-repeller decomposition of K. Then there exists an index triple (N_1, N_2, N_3) for (y_0, K, A, R) such that $N_1 \subset r^{-1}(N)$.*

Proof. It is known that there exists an FM-index triple (N_1', N_2', N_3') (see [1]) with $N_1 \subset N$. By Lemma 2.4, $(r^-1(N_1'), r^{-1}(N_3'))$ is an index pair for (y_0, K) and $(r^{-1}(N_2'), r^{-1}(N_3'))$ is an index pair for (y_0, A). □

Lemma 4.4. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then, (N_1, N_2) is an index pair for (y_0, R).*

Proof. Firstly, we will show that (N_1, N_2) is a basic index pair, that is, we need to check Definition 2.1.

(IP2) Let $x \in N_1$ and $t \in \mathbb{R}^+$ such that $x\chi_{y_0}t \notin N_1$. It is known that (N_1, N_3) is an index pair, so $x\chi_{y_0}s \in N_3 \subset N_2$ for some $s \in [0, t]$.

(IP3) Let $x \in N_2$ and $t \in \mathbb{R}^+$ such that $x\chi_{y_0}t \notin N_2$. (N_2, N_3) is an index pair, so $x\chi_{y_0}s \in N_3$ for some $s \in [0, t]$. Since (N_1, N_3) is also an index pair, it follows that $x\chi_{y_0}s' \in X \setminus N_1$ for some $s' \in [s, t]$.

Recall the mapping $r := r_{y_0}$, which can be found in Definition 2.2. Since (N_1, N_3) (resp. (N_2, N_3)) is an index pair for (y_0, K) (resp. (y_0, A)), there is a strongly admissible isolating neighbourhood M_K (resp. M_A) such that $N_1 \setminus N_2 \subset r^{-1}(M_K)$ (resp. $N_2 \setminus N_3 \subset r^{-1}(M_A)$). There also exists an open neighbourhood W_K (resp. W_A) of K (resp. A) with $r^{-1}(W_K) \subset N_1 \setminus N_3$ (resp. $r^{-1}(W_A) \subset N_2 \setminus N_3$).

Recall that $A \cap R = \emptyset$ by the definition of an attractor-repeller decomposition, so there are disjoint open neighbourhoods U_A of A and U_R of R. We may assume without loss of generality that $W_A \subset U_A$. Setting $M_R := M_K \setminus W_A$, one has $\mathrm{Inv} M_R \subset R \subset U_R \subset M_R$, which means that M_R is an isolating neighbourhood for R. Moreover, one has

$$N_1 \setminus N_2 = (N_1 \setminus N_3) \setminus (N_2 \setminus N_3) \subset r^{-1}(M_K) \setminus r^{-1}(W_A) = r^{-1}(M_R).$$

Define $N'_A := \mathrm{cl}_{\mathcal{H}^+(y_0)\times X}\, r(N_2 \setminus N_3)$ and $W_R := W_K \setminus N'_A$. One has

$$N_1 \setminus N_2 \supset r^{-1}(W_K) \setminus (N_2 \setminus N_3) \supset r^{-1}(W_K) \setminus r^{-1}(N'_A) = r^{-1}(W_R).$$

The set $K \cap N'_A \subset M_A$ is positively invariant: Let $x \in K \cap N'_A$ and $x\pi s \in K \setminus N'_A$ for some $s \in \mathbb{R}^+$. There is a sequence (t_n, x_n) in $N_2 \setminus N_3 \subset \mathbb{R}^+ \times X$ such that $r(t_n, x_n) \to x$ as $n \to \infty$. We can assume that $r(t_n, x_n)\pi s \notin N'_A$ for all $n \in \mathbb{N}$, so, without loss of generality, there are reals $s_n \to s_0$ with $(t_n, x_n)\chi_{y_0} s_n \in N_3$ for all $n \in \mathbb{N}$. We have $r(t_n, x_n)\pi s_n \to x\pi s_0 \in K$, so $(t_n, x_n)\chi_{y_0} s_n \in r^{-1}(W_K)$ for all but finitely many n, which is a contradiction since $r^{-1}(W_K) \cap N_3 = \emptyset$. Hence, if $x \in K \cap N'_A$, then $\omega(x) \subset A$, implying that $R \cap N'_A = \emptyset$. Therefore W_R, which is obviously open, is a neighbourhood of R. □

Lemma 4.5. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then, for every $T \in \mathbb{R}^+$,*

$$(N_1, N_2^{-T}, N_3) := (N_1, N_2^{-T}(N_1), N_3)$$

and

$$(N_1, N_2^{-T}, N_3^{-T}) := (N_1, N_2^{-T}(N_1), N_3^{-T}(N_1))$$

are index triples for (y_0, K, A, R).

Proof. Lemma 2.7 implies that (N_1, N_2^{-T}) and (N_1, N_3^{-T}) are index pairs for (y_0, K) for every $T > 0$. Furthermore, assuming that (N_2^{-T}, N_3) is an index pair for (y_0, A), it follows from Lemma 2.7 ([10]) that (N_2^{-T}, N_3^{-T}) is an index pair for (y_0, A).

Hence, we only need to prove that (N_2^{-T}, N_3) is an index pair for (y_0, A).

(IP1) (N_1, N_2^{-T}) is an index pair, so N_2^{-T} is closed.

(IP2) Let $x \in N_2^{-T}$ and $x\chi_{y_0} t \notin N_2^{-T} \supset N_2$. We have $x\chi_{y_0} s' \in N_2$ for some $s' \le t$. Since (N_2, N_3) is an index pair, we must have $x\chi_{y_0} s \in N_3$ for some $s \in [s', t]$.

(IP3) Let $x \in N_3$ and $x\chi_{y_0} t \notin N_3$. (N_1, N_3) is an index pair, so $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus N_1 \subset (\mathbb{R}^+ \times X) \setminus N_2^{-T}$ for some $s \in [0, t]$.

(IP4) (N_1, N_2^{-T}) is an index pair for (y_0, R), so there is an open neighbourhood W_R of R such that $r^{-1}(W_R) \subset N_1 \setminus N_2^{-T}$. We may assume that $W_R \cap A = \emptyset$ because $A \cap R = \emptyset$. Let N_K be an isolating neighbourhood for K with $N_1 \setminus N_3 \subset r^{-1}(N_K)$. Then $N_A := N_K \setminus W_R$ is an isolating neighbourhood for A with

$$N_2^{-T} \setminus N_3 \subset (N_1 \setminus N_3) \setminus (N_1 \setminus N_2^{-T}) \subset r^{-1}(N_A).$$

(IP5) Since (N_2, N_3) is an index pair for (y_0, A), there is a neighbourhood W_A of A with $r^{-1}(W_A) \subset N_2 \setminus N_3$. One has

$$r^{-1}(W_A) \subset N_2 \setminus N_3 \subset N_2^{-T}(N_1) \setminus N_3. \qquad \square$$

([10]) $N_3^{-T}(N_1) = N_3^{-T}(N_2^{-T})$

4.2. Long exact sequences. The long exact sequence associated with an attractor-repeller sequence is usually defined using the concept of so-called weakly exact sequences (Definition 2.1 in [2]). Instead of weakly exact sequences, we use the long exact sequence of a triple as a starting point. The advantage is that our definition relies only on an axiomatic characterization of homology yet not necessarily on an underlying chain complex. It is therefore only assumed that $\mathrm{H}_* = (\mathrm{H}_q)_{q\in\mathbb{Z}}$ is a homology theory satisfying the Eilenberg–Steenrod axioms. Of course, H_* can also simply be read as the singular homology functor.

LEMMA 4.6. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Then, the projection $p\colon N_1/N_3 \to N_1/N_2$ induces an isomorphism*

$$\varrho\colon\ \mathrm{H}_*(N_1/N_3, N_2/N_3) \to \mathrm{H}_*(N_1/N_2, \{N_2\}).$$

The proof will be conducted in three steps, the first two being formulated as separate lemmas.

LEMMA 4.7. *Let (N_1, N_2) be an index pair for (y_0, K) and define $f\colon N_1 \to \mathbb{R}^+$ by $f(t, x) := \sup\{t_0 \in \mathbb{R}^+ : (t, x)\chi_{y_0} s \in \mathrm{cl}(N_1 \setminus N_2)$ for all $s \in [0, t_0]\}$. Then,*

(a) *f is upper semicontinuous, and*
(b) *bounded on N_2.*

PROOF. (a) Suppose that f is not upper semicontinuous. Then there is a sequence $(t_n, x_n) \to (t_0, x_0)$ in N_1 such that $f(t_n, x_n) > f(t_0, x_0) + \varepsilon$ for some $\varepsilon > 0$ and all $n \in \mathbb{N}$. By the definition of f, there is an $s \in [0, \varepsilon[$ with $(t_0, x_0)\chi_{y_0}(f(t_0, x_0) + s) \in (\mathbb{R}^+ \times X) \setminus (\mathrm{cl}(N_1 \setminus N_2))$. It follows that $(t_n, x_n)\chi_{y_0}(f(t_0, x_0) + s) \in (\mathbb{R}^+ \times X) \setminus (\mathrm{cl}(N_1 \setminus N_2))$ for all n sufficiently large. Hence, $f(t_n, x_n) < f(t_0, x_0) + \varepsilon$ for those n, which is a contradiction.

(b) (N_1, N_2) is an index pair for (y_0, K), so there is a strongly admissible isolating neighbourhood $N \subset \mathcal{H}^+(y_0) \times X$ for K such that $N_1 \setminus N_2 \subset r^{-1}(N)$. N is closed, so $\mathrm{cl}(N_1 \setminus N_2) \subset r^{-1}(N)$. Furthermore, there exists an open neighbourhood W of K with $r^{-1}(W) \subset N_1 \setminus N_2$. Now, suppose that f is unbounded on N_2. Then there is a sequence (t_n, x_n) in N_2 with $f(t_n, x_n) \to \infty$.

Because $f((t_n, x_n)\chi_{y_0} s) \neq 0$, we must have $(t_n, x_n)\chi_{y_0} s \in N_2 \cap (\mathrm{cl}_{\mathbb{R}^+\times X}(N_1 \setminus N_2))$ for all $s \in [0, f(t_n, x_n)[$ and all $n \in \mathbb{N}$, so $r(t_n, x_n)\pi s \in N \setminus W$ for all $s \in [0, f(t_n, x_n)]$.

Since N is strongly admissible, there is a solution $u\colon \mathbb{R} \to N \setminus W$ of π. However, $u(\mathbb{R}) \subset K$ because N is an isolating neighbourhood for K. This is a contradiction since $K \subset W$. □

LEMMA 4.8. *Let (N_1, N_2) be an index pair for (y_0, K). Then for all $T \in \mathbb{R}^+$ sufficiently large, $N_2^{-T} := N_2^{-T}(N_1)$ is a neighbourhood of N_2 in N_1.*

PROOF. By Lemma 4.7 (a), $W^T := f^{-1}([0, T[)$ is open for every $T \in \mathbb{R}^+$. If T is sufficiently large, then $W^T \supset N_2$ by Lemma 4.7 (b), so W^T is a neighbourhood of N_2 in N_1. We are going to show that $W^T \subset N_2^{-T}$, which implies that for large $T \in \mathbb{R}^+$, N_2^{-T} is a neighbourhood of N_2 as claimed.

In order to prove the inclusion $W^T \subset N_2^{-T}$, let $x \in W^T$ and $\varepsilon > 0$ be arbitrary. We have $x\chi t \notin \mathrm{cl}(N_1 \setminus N_2)$ for some $t \leq T + \varepsilon$ solely by the definition of f. Either $x\chi_{y_0} t \in N_1$ and thus $x\chi_{y_0} t \in N_2$ or $x\chi t' \in N_2$ for some $t' \leq t$ because (N_1, N_2) is an index pair. Since $\varepsilon > 0$ is arbitrary and N_2 closed, it follows that $x\chi t'' \in N_2$ for some $t'' \leq T$, so $x \in N_2^{-T}$. □

Proof of Lemma 4.6. Consider the following sequence of inclusion induced mappings.

$$\mathrm{H}_*(N_1/N_3, N_2/N_3) \xrightarrow{i} \mathrm{H}_*(N_1/N_3, N_2^{-T}/N_3) \xrightarrow{k} \mathrm{H}_*(N_1/N_2, N_2^{-T}/N_2) \xrightarrow{l} \mathrm{H}_*(N_1/N_2, N_2/N_2).$$

We will show that i, k, l are isomorphisms.

Firstly, we consider i. Define $\varphi_T\colon N_1/N_3 \to N_1/N_3$ by

$$\varphi_T([t,x]) := \begin{cases} [(t,x)\chi_{y_0} T] & (t,x)\chi_{y_0} s \in N_1 \setminus N_3 \text{ for all } s \in [0,T], \\ N_3 & \text{otherwise.} \end{cases}$$

It follows from Lemma 3.7 in [4] that φ_T and therefore its restriction to N_2^{-T}/N_3 are continuous. We conclude that $i^{-1} = \varphi_T$ up to homotopy, so i is indeed an isomorphism.

Secondly, choosing T sufficiently large, it follows from Lemma 4.8 that N_2^{-T} is a neighbourhood of $N_2 \supset N_3$. Hence, k is an isomorphism by the excision property of homology.

Thirdly, it follows as before that the one-point space N_2/N_2 is a deformation retract of N_2^{-T}/N_2. Hence, k must be an isomorphism as well, completing the proof. □

In view of Lemma 4.6, we can now make define long exact sequences associated with index triples. To keep the definition short, recall that the homology theory defines a boundary operator (connecting homomorphism) $\partial(X, A)$ for every topological pair (X, A). Let (X, A, B) be a triple of topological spaces, where $B \subset A \subset X$ are subspaces. There is a long exact sequence associated with (X, A, B) and its (natural) connecting homomorphism δ is given by $\delta := \mathrm{H}_*(k) \circ \partial(X, A)$, where $k\colon (A, \emptyset) \to (A, B)$ denotes the inclusion (see [7, Theorem 5 in Section 4.8]).

Definition 4.9. Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). Let $q\colon \mathrm{H}_*(N_1/N_3, N_2/N_3) \to \mathrm{H}_*(N_1/N_2, N_2/N_2)$ be inclusion induced and set $\partial = \delta \circ q^{-1}$, where δ is the connecting homomorphism associated with the triple

$(N_1/N_3, N_2/N_3, N_3/N_3)$. The long exact sequence associated with (N_1, N_2, N_3) is

$$\longrightarrow \mathrm{H}_*[N_2, N_3] \xrightarrow{i} \mathrm{H}_*[N_1, N_3] \xrightarrow{p} \mathrm{H}_*[N_1, N_2] \xrightarrow{\partial} \mathrm{H}_{*-1}[N_2, N_3] \longrightarrow \tag{4.1}$$

Here, we denote $\mathrm{H}_*[N_1, N_2] := \mathrm{H}_*(N_1/N_2, \{N_2\})$.

Lemma 4.10. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). The sequence* (4.1) *associated with (N_1, N_2, N_3) is exact.*

Proof. We rewrite (4.1) as follows.

$$\begin{array}{ccccccc} & & & \mathrm{H}_*[N_1, N_2] & & & \\ & & \nearrow^{p} & \uparrow{\scriptstyle q} & \searrow^{\partial} & & \\ \longrightarrow \mathrm{H}_*[N_2, N_3] & \xrightarrow{i} & \mathrm{H}_*[N_1, N_3] \longrightarrow & \mathrm{H}_*(N_1/N_3, N_2/N_3) & \xrightarrow{\delta} & \mathrm{H}_{*-1}[N_2, N_3] \longrightarrow & \end{array}$$

The lower row is the long exact sequence of the triple $(N_1/N_3, N_2/N_3, N_3/N_3)$. The result follows easily because q is an isomorphism by Lemma 4.6. □

Lemma 4.11. *Let (N_1, N_2, N_3) (resp. (N_1', N_2', N_3')) be an index triple for (y_0, K, A, R) (resp. (y_0', K', A', R')). The boundary operator ∂ is natural with respect to continuous mappings $f\colon (N_1, N_2, N_3) \to (N_1', N_2', N_3')$, that is, if ∂ and ∂' denote the respective boundary operators, then*

$$\begin{array}{ccc} \mathrm{H}_*[N_1, N_2] & \xrightarrow{\partial} & \mathrm{H}_*[N_2, N_3] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{H}_*[N_1', N_2'] & \xrightarrow[\partial']{} & \mathrm{H}_*[N_2', N_3'] \end{array}$$

is commutative.

Proof. This follows easily from Definition 4.9. The connecting homomorphisms of the long exact sequences associated with a triple are natural, and so are the projections q. □

4.3. The homology Conley index. The homotopy index of an invariant set is the homotopy type of an appropriate quotient space. A homology index could be defined similarly as equivalence class of graded modules or the like. However, to not loose the connecting homomorphisms introduced in the previous sections, requires a more sophisticated approach.

Definition 4.12. Let $y_0 \in Y$ and $K \subset \mathcal{H}^+(y_0) \times X$ be an isolated invariant set admitting a strongly admissible isolating neighbourhood, so its categorial Conley index $\mathcal{C}(y_0, K)$ is defined. The (categorial, nonautonomous) homology Conley index is obtained by applying the homology functor, that is:

(a) If A is an object of $\mathcal{C}(y_0, K)$, then $\mathrm{H}_*(A)$ is an object of $\mathrm{H}_*\,\mathcal{C}(y_0, K)$.

(b) If f is a morphism of $\mathcal{C}(y_0, K)$, then $\mathrm{H}_*(f)$ is a morphism of $\mathrm{H}_*\,\mathcal{C}(y_0, K)$.

As a consequence of the above definition, the homology Conley index is an object of CSS(gradMod), where gradMod denotes the category of graded modules. Suppose we are given a long sequence

$$\longrightarrow \mathcal{A}^n \xrightarrow{[f^n]} \mathcal{A}^{n+1} \longrightarrow$$

in CSS(gradMod). Choose objects A^n of $\mathcal{A}^n$ for every $n \in \mathbb{N}$. We say that the above sequence is exact if

$$\longrightarrow A^n \xrightarrow{f^n} A^{n+1} \longrightarrow$$

is exact. Note that this notion of exactness is independent of the particular choice of objects.

The above definition immediately leads to the following question: Does a connecting homomorphism ∂ which is defined by a particular index triple give rise to a unique morphism of the homology index? The answer is affirmative as we will see below.

Theorem 4.13. *Let (N_1, N_2, N_3) be an index triple for (y_0, K, A, R). The connecting homomorphism ∂ that is given by Definition* 4.9 *gives rise to a unique, i.e. independent of (N_1, N_2, N_3), morphism $[\partial]$ in* CSS(gradMod) *and*

$$(4.2)\quad \longrightarrow \mathrm{H}_*\,\mathcal{C}(y_0, A) \xrightarrow{[i]} \mathrm{H}_*\,\mathcal{C}(y_0, K) \xrightarrow{[p]} \mathrm{H}_*\,\mathcal{C}(y_0, R) \xrightarrow{[\partial]} \mathrm{H}_{*-1}\,\mathcal{C}(y_0, A) \longrightarrow$$

is a long exact sequence.

(4.2) is called the (*long exact*) *attractor-repeller sequence* of (y_0, K, A, R). We also say that $[\partial]$ is the connecting homomorphism of (y_0, K, A, R) respectively of the attractor-repeller sequence associated with (y_0, K, A, R).

It is an immediate consequence of Theorem 4.15 below that $[\partial]$ is well-defined. The proof that the morphisms $[i]$ and $[p]$ are well-defined is omitted.

Lemma 4.14. *Let (N_1, N_2, N_3) and (M_1, M_2, M_3) be index triples for (y_0, K, A, R). Then there is an index triple (L_1, L_2, L_3) such that, for some $T > 0$,*

$$(4.3)\quad (L_1, L_2, L_3) \subset (N_1 \cap M_1, N_2^{-T}(N_1) \cap M_2^{-T}(M_1), N_3^{-T}(M_1) \cap M_3^{-T}(M_1)).$$

Proof. By Theorem 2.9, there are index pairs $\big(\widetilde{L}_1, \widetilde{L}_3\big)$ for (y_0, K) and (L_2', L_3') for (y_0, A) which have the required inclusion properties, that is, for some $T' > 0$ it holds that

$$(\widetilde{L}_1, \widetilde{L}_3) \subset \big(N_1 \cap M_1, N_3^{-T'}(N_1) \cap M_3^{-T'}(M_1)\big),$$
$$(L_2', L_3') \subset \big(\widetilde{L}_1 \cap N_2 \cap M_2, N_3^{-T'}(N_2) \cap M_3^{-T'}(M_2)\big).$$

Assume for the moment that there is a constant $T'' > 0$ such that

(∗) $\big(L_2' \cup \widetilde{L}_3^{-T''}, \widetilde{L}_3^{-T''}\big)$ is an index pair for (y_0, A).

By Lemma 2.7, $\big(\widetilde{L}_1, \widetilde{L}_3^{-T''}\big)$ is an index pair for (y_0, K), so by $(*)$

$$(L_1, L_2, L_3) := \big(\widetilde{L}_1, L_2' \cup \widetilde{L}_3^{-T''}, \widetilde{L}_3^{-T''}\big)$$

is an index triple for (y_0, K, A, R). Furthermore, taking $T = T' + T''$, (4.3) is satisfied.

It is therefore sufficient to prove the assumption above. Firstly, we will show that $L_3' \subset \widetilde{L}_3^{-T''} := \widetilde{L}_3^{-T''}(\widetilde{L}_1)$ for T'' large enough. Suppose that $(t_n, x_n) \in L_3' \setminus \widetilde{L}_3^{-2n}(N_1)$ is a sequence. We have

$$(t_n, x_n)\chi_{y_0}[0, 2n] \subset N_3^{-T'}(N_2) \subset N_3^{-T'}(N_1) \tag{4.4}$$

for all $n \in \mathbb{N}$. $\big(N_1, N_3^{-T'}(N_1)\big)$ is an index pair for (y_0, K) by virtue of Lemma 2.7, so there is an admissible isolating neighbourhood N of K such that

$$r(t_n, x_n)\pi[0, 2n] \subset N \quad \text{for all } n \in \mathbb{N}.$$

We may assume without loss of generality that $r(t_n, x_n)\pi n \to (y, x) \in K$, $(t_n, x_n)\chi_{y_0} n \in N_1 \setminus N_3^{-T'}(N_1)$ provided that n is sufficiently large, in contradiction to (4.4).

To prove $(*)$, we need to check the assumptions of an index pair.

(IP1) It is clear that L_2 and L_3 are closed sets with $L_2 \subset L_3$.

(IP2) Let $x \in L_2 \setminus L_3$ and $x\chi_{y_0} t \notin L_2$ for some $t > 0$. It follows that $x\chi_{y_0} t \notin L_2'$, so $x\chi_{y_0} s \in L_3' \subset L_3$ for some $s \in [0, t]$.

(IP3) Suppose that $x \in L_3$, but $x\chi_{y_0} t \notin L_3$ for $t > 0$. (L_1, L_3) is an index pair by Lemma 2.7, so $x\chi_{y_0} s \in (\mathbb{R}^+ \times X) \setminus L_1 \subset (\mathbb{R}^+ \times X) \setminus L_2$ for some $s \in [0, t]$.

(IP4) (L_2', L_3') is an index pair for (y_0, A). Hence there is an admissible isolating neighbourhood $N \subset \mathcal{H}^+(y_0) \times X$ for A with $L_2 \setminus L_3 \subset L_2' \setminus L_3' \subset r^{-1}(N)$.

(IP5) There is a neighbourhood W of A in $\mathcal{H}^+(y_0) \times X$ such that $r^{-1}(W) \subset L_2' \setminus L_3'$. Since (L_1, L_3) is an index pair for (y_0, K), there is also a neighbourhood W_K of K with $r^{-1}(W_K) \subset L_1 \setminus L_3$. The intersection $W_0 := W \cap W_K$ is a neighbourhood of A, and $r^{-1}(W_0) \subset L_2 \setminus L_3$. □

Theorem 4.15. *Let (N_1, N_2, N_3) and (M_1, M_2, M_3) be index triples for (y_0, K, A, R). Then the following diagram is commutative.*

$$\begin{array}{ccccccc}
\longrightarrow & \mathrm{H}_*[N_1, N_2] & \xrightarrow{\partial} & \mathrm{H}_*[N_2, N_3] & \longrightarrow \\
& \downarrow & & \downarrow & \\
\longrightarrow & \mathrm{H}_*[M_1, M_2] & \xrightarrow[\partial]{} & \mathrm{H}_*[M_2, M_3] & \longrightarrow
\end{array}$$

Its rows represent the long exact sequences associated with the respective index triple, and the vertical arrows denote the respective inner morphism of the categorial Conley index.

Proof. Assuming that $(N_1, N_2, N_3) \subset (M_1, M_2, M_3)$, the inner morphisms are inclusion induced, so the theorem is merely a reformulation of Lemma 4.11. The general case follows from Lemma 4.14. Let the index triple (L_1, L_2, L_3) be given by that lemma. We have

$$(N_1, N_2, N_3) \subset \big(N_1, N_2^{-T}, N_3^{-T}\big) \supset (L_1, L_2, L_3),$$
$$(M_1, M_2, M_3) \subset \big(M_1, M_2^{-T}, M_3^{-T}\big) \supset (L_1, L_2, L_3),$$

for some $T > 0$.

By Lemma 4.5, the triples $\big(N_1, N_2^{-T}, N_3^{-T}\big)$ and $\big(M_1, M_2^{-T}, M_3^{-T}\big)$ in the middle are index triples. This reduces the general case to the special case covered by Lemma 4.11.

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Axel Jänig
Institut für Mathematik
Universität Rostock
18051 Rostock, GERMANY

E-mail address: axel.jaenig@uni-rostock.de