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A CONTINUATION LEMMA AND THE EXISTENCE OF PERIODIC SOLUTIONS OF PERTURBED PLANAR HAMILTONIAN SYSTEMS WITH SUB-QUADRATIC POTENTIALS

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Abstract. In this paper, we study the existence of periodic solutions of perturbed planar Hamiltonian systems of the form

$$\begin{cases} x' = f(y) + p_1(t,x,y), \\ y' = -g(x) + p_2(t,x,y). \end{cases}$$

We prove a continuation lemma for a given planar system and further use it to prove that this system has at least one T-periodic solution provided that g has some sub-quadratic potentials.

1. Introduction

We are concerned with the existence of periodic solutions of a perturbed planar Hamiltonian system

$$\begin{cases} x' = f(y) + p_1(t,x,y), \\ y' = -g(x) + p_2(t,x,y), \end{cases} \tag{1.1}$$

where $f, g\colon \mathbb{R} \to \mathbb{R}$ are continuous, $p_i\colon \mathbb{R}^3 \to \mathbb{R}$ $(i = 1, 2)$ are continuous and T-periodic with respect to the first variable.

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In the case when $f(y) = y$, $p_1(t,x,y) \equiv 0$ and $p_2(t,x,y) = p(t)$, system (1.1) becomes

$$\begin{cases} x' = y, \\ y' = -g(x) + p(t), \end{cases}$$

which is equivalent to the second order differential equation

$$x'' + g(x) = p(t). \tag{1.2}$$

The periodic problems of equation (1.2) have been widely studied in literature. In [9], A.C. Lazer proved the existence of periodic solutions of equation (1.2) by assuming that $g(x)$ satisfies the sublinear condition

$$\lim_{|x|\to+\infty} \frac{g(x)}{x} = 0 \tag{1.3}$$

together with the sign condition

$$\operatorname{sgn}(x)(g(x) - \overline{p}) \geq 0, \quad |x| \geq d, \tag{1.4}$$

where

$$\overline{p} = \frac{1}{T}\int_0^T p(t)\,dt,$$

d is a positive constant. From then on, there have appeared many works which generalized condition (1.3) in different manners (see [3], [5], [12], [14], [15] and references therein). In particular, some one-sided growth conditions on $g(x)$ or its primitive $G(x)$, which is defined by

$$G(x) = \int_0^x g(s)\,ds,$$

were introduced to study the periodic solutions of equation (1.2). J. Mawhin and J.R. Ward proved in [12] the existence of periodic solutions of equation (1.2) under (1.4) and

$$\limsup_{x\to+\infty} \frac{g(x)}{x} < \left(\frac{\pi}{T}\right)^2. \tag{1.5}$$

L. Fernandes and F. Zanolin [5] generalized (1.5) to condition

$$\liminf_{x\to+\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2. \tag{1.6}$$

As to other nonresonance conditions on the potential $G(x)$ for periodic solutions of equation (1.2) one can check [13].

In the case when $f(y)$ is nonlinear and $p_1(t,x,y) \neq 0$, system (1.1) in general can not be transformed into (1.2). The periodic problem of planar system (1.1) is being studied with an increasing interests ([1], [2], [6]–[8], [16]). In [7], the existence of periodic solutions of planar Hamiltonian systems of the type

$x' = g_1(t, y)$, $y' = -g_2(t, x)$ was studied when $g_1, g_2 \colon \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are continuous, T-periodic with the first variable and satisfy the asymmetric nonlinear conditions,

$$\mu_\pm \le \liminf_{y\to\pm\infty} \frac{g_1(t,y)}{y} \le \limsup_{y\to\pm\infty} \frac{g_1(t,y)}{y} \le \mu^\pm,$$
$$\nu_\pm \le \liminf_{x\to\pm\infty} \frac{g_2(t,x)}{x} \le \limsup_{x\to\pm\infty} \frac{g_2(t,x)}{x} \le \nu^\pm,$$

uniformly for $t \in [0, T]$, where $\mu^\pm$, $\mu_\pm$, $\nu^\pm$ and $\nu_\pm$ are positive constants. It was proved in [7] that the given system has at least one T-periodic solution provided that the following condition holds,

$$[\tau_1, \tau_2] \cap \left\{ \frac{T}{n} : n \in \mathbf{N}_0 \right\} = \emptyset,$$

where $\mathbb{N}_0$ denotes the set of positive integers and

$$\tau_1 = \frac{\pi}{2}\left(\frac{1}{\sqrt{\mu^+}} + \frac{1}{\sqrt{\mu^-}}\right)\left(\frac{1}{\sqrt{\nu^+}} + \frac{1}{\sqrt{\nu^-}}\right),$$
$$\tau_2 = \frac{\pi}{2}\left(\frac{1}{\sqrt{\mu_+}} + \frac{1}{\sqrt{\mu_-}}\right)\left(\frac{1}{\sqrt{\nu_+}} + \frac{1}{\sqrt{\nu_-}}\right).$$

In the present paper, we shall study the existence of periodic solutions of system (1.1) when f satisfies the asymmetric nonlinear conditions and g has some sub-quadratic potential. Assume that f satisfies the conditions as follows,

$$(\mathrm{h}_1) \qquad a_+ \le \liminf_{y\to+\infty} \frac{f(y)}{y} \le \limsup_{y\to+\infty} \frac{f(y)}{y} \le a^+,$$

$$(\mathrm{h}_2) \qquad a_- \le \liminf_{y\to-\infty} \frac{f(y)}{y} \le \limsup_{y\to-\infty} \frac{f(y)}{y} \le a^-,$$

where $a^\pm$ and $a_\pm$ are positive constants. Moreover, assume that there exists a constant $M_0 > 0$ such that

$$(\mathrm{h}_3) \qquad |p_i(t,x,y)| \le M_0, \quad \text{for all } t, x, y \in \mathbb{R} \text{ and } i = 1, 2.$$

Meanwhile, there exists a positive constant d_1 such that

$$(\mathrm{g}_0) \qquad \operatorname{sgn}(x) g(x) > M_0, \quad |x| \ge d_1.$$

We prove the following results.

Theorem 1.1. *Assume that conditions* (h_i) $(i = 1, 2, 3)$ *and* (g_0) *hold. Then system* (1.1) *has at least one T-periodic solution provided that the inequality*

$$\liminf_{x\to+\infty} \frac{2G(x)}{x^2} < \left(\frac{c\pi}{T}\right)^2$$

holds, where c is a constant defined by

$$c = \frac{1}{2}\left(\frac{\sqrt{a_+}}{a^+} + \frac{\sqrt{a_-}}{a^-}\right).$$

Corollary 1.2. *Assume that conditions* (h_i) ($i = 1, 2, 3$) *and* (g_0) *hold. Then system* (1.1) *has at least one T-periodic solution provided that G satisfies*

$$\liminf_{x\to+\infty}\frac{2G(x)}{x^2}=0.$$

In particular, if g satisfies one-sided sublinear condition

$$\lim_{x\to+\infty}\frac{g(x)}{x}=0, \tag{1.7}$$

then system (1.1) *has at least one T-periodic solution.*

Remark 1.3. In [8], Landesman–Lazer conditions were introduced to ensure the existence of T-periodic solutions of planar systems

$$Ju'=F(t,u),\quad J=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

when, referring to equation (1.2), resonance occurs at the first eigenvalue ($\lambda_0 = 0$) of the T-periodic problem. When g satisfies (1.7), system (1.1) can be considered as a perturbation of the (possibly) resonant system $x' = f(y), y' = 0$. In case when f is piecewise linear, i.e. $f(y) = ay^+ - by^-$, where $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, a, b are two positive constants, we can write (1.1) in the form:

$$Ju'=\nabla H_0(u)+r(t,u),$$

with $u = (x, y)$ and

$$H_0(u)=\frac{1}{2}a{y^+}^2+\frac{1}{2}b{y^-}^2,\qquad r(t,u)=(g(x)-p_2(t,x,y),p_1(t,x,y)).$$

If (h_3), (g_0) hold and g satisfies double-sided sublinear condition

$$\lim_{|x|\to+\infty}\frac{g(x)}{x}=0,$$

then we can drive from [8, Corollary 3.1] that system (1.1) has at least one T-periodic solution. However, the latter conclusion of Corollary 1.2 can not be obtained from [8, Corollary 3.1] because g only satisfies one-sided sublinear condition (1.7).

Corollary 1.4. *Assume that* (h_3) *and* (g_0) *hold and there are constants $a > 0, b > 0$ such that*

$$\lim_{y\to+\infty}\frac{f(y)}{y}=a,\qquad \lim_{y\to-\infty}\frac{f(y)}{y}=b.$$

Then system (1.1) *has at least one T-periodic solution provided that G satisfies*

$$\liminf_{x\to+\infty}\frac{2G(x)}{x^2}<\left(\frac{l\pi}{T}\right)^2, \tag{1.8}$$

where $l = (1/\sqrt{a} + 1/\sqrt{b})/2$. *In particular, for any* $a > 0$, $b > 0$, *system*

$$\begin{cases} x' = ay^+ - by^- + p_1(t,x,y), \\ y' = -g(x) + p_2(t,x,y), \end{cases} \tag{1.9}$$

has at least one T*-periodic solution provided that* (1.8) *holds.*

REMARK 1.5. Corollary 1.4 can be regarded as a generalization of the result in [5]. In fact, if we take $a = b = 1$ and $p_1(t,x,y) \equiv 0$, $p_2(t,x,y) = p(t)$, then system (1.9) is equivalent to equation (1.2) and the condition (1.8) is exactly the condition (1.6).

2. A continuation lemma

It is well known that continuation results based on coincidence theory are widely used to study the existence of periodic solutions of differential equations. In this section, we shall prove a continuation lemma for system (1.1). At first, we imbed system (1.1) into a family of systems with one parameter $\lambda \in [0,1]$,

$$\begin{cases} x' = \lambda(f(y) + p_1(t,x,y)), \\ y' = \lambda(-g(x) + p_2(t,x,y)). \end{cases} \tag{2.1}$$

LEMMA 2.1. *Assume that conditions* (g_0), (h_3) *hold and there exists a positive constant* d_2 *such that*

$$\operatorname{sgn}(y) f(y) > M_0, \quad |y| \geq d_2, \tag{f_0}$$

where M_0 *is from condition* (h_3). *Suppose that there exists a constant* $\zeta > d_1$ *such that if* $(x_\lambda(t), y_\lambda(t))$ *is one* T*-periodic solution of* (2.1) *for some* $\lambda \in (0,1]$, *then*

$$\max\{x_\lambda(t) : t \in [0,T]\} \neq \zeta.$$

Then system (1.1) *has at least one* T*-periodic solution.*

We shall use a classical consequence of Mawhin's continuation theorem [11] to prove Lemma 2.1. For reader's convenience, we restate it here.

LEMMA 2.2. *Let* $\Psi = \Psi(t,z;\lambda)\colon [0,T] \times \mathbb{R}^m \times [0,1] \to \mathbb{R}^m$ *be a continuous function and let* $\Omega \subset \mathbb{R}^m$ *be a (non-empty) open bounded set (with boundary* $\partial\Omega$ *and closure* $\overline{\Omega}$*). Assume that the following conditions are satisfied*:

(a) *for any* T*-periodic solution* $z(t)$ *of* $z' = \lambda\Psi(t,z;\lambda)$ *with* $\lambda \in (0,1]$, *such that* $z(t) \in \overline{\Omega}$, *for all* $t \in [0,T]$, *it follows that* $z(t) \in \Omega$, *for all* $t \in [0,T]$;

(b) $\Psi_0(z) \neq 0$, *for each* $z \in \partial\Omega$ *and* $d_B(\Psi_0, \Omega, 0) \neq 0$, *where*

$$\Psi_0(z) = \frac{1}{T}\int_0^T \Psi(t,z;0), \quad \text{for } z \in \mathbb{R}^m.$$

Then the equation $z' = \Psi(t, z; 1)$ *has at least one* T*-periodic solution* $z(t) \in \overline{\Omega}$*, for all* $t \in [0, T]$.

Proof of Lemma 2.1. We first prove that there is a constant $M > 0$ such that, if $(x(t), y(t))$ is one T-periodic solution of (2.1) with $\lambda \in (0, 1]$ with $x(t) \le \zeta$, for $t \in [0, T]$, it follows that

$$|x(t)| + |y(t)| \le M, \quad t \in [0, T].$$

Let $(x(t), y(t))$ be any T-periodic solution of (2.1) satisfying $x(t) \le \zeta$, $t \in [0, T]$. Integrating both sides of $y' = \lambda(-g(x) + p_2(t, x(t), y(t)))$, we have

$$\int_0^T (g(x(t)) - p_2(t, x(t), y(t)))\, dt = 0. \tag{2.2}$$

Then we obtain

$$-\int_{I_1} g(x(t))\, dt = \int_{I_2} g(x(t))\, dt - \int_0^T p_2(t, x(t), y(t))\, dt,$$

where $I_1 = \{t \in [0, T] : x(t) \le -d_1\}$, $I_2 = \{t \in [0, T] : x(t) \ge -d_1\}$. Therefore, we have

$$\begin{aligned}\int_0^T |g(x(t))|\, dt &= -\int_{I_1} g(x(t))\, dt + \int_{I_2} |g(x(t))|\, dt \\ &\le 2\int_{I_2} |g(x(t))|\, dt + \int_0^T |p_2(t, x(t), y(t))|\, dt \le T(2M_1 + M_0),\end{aligned}$$

where $M_1 = \max\{|g(x)| : -d_1 \le x \le \zeta\}$. Consequently, we get

$$\begin{aligned}\int_0^T |y'(t)|\, dt &= \lambda \int_0^T |g(x(t)) - p_2(t, x(t), y(t))|\, dt \\ &\le \int_0^T |g(x(t))|\, dt + \int_0^T |p_2(t, x(t), y(t))|\, dt \le 2T(M_1 + M_0).\end{aligned} \tag{2.3}$$

Integrating both sides of $x'(t) = \lambda(f(y) + p_1(t, x(t), y(t)))$, we have

$$\int_0^T (f(y(t)) + p_1(t, x(t), y(t)))\, dt = 0.$$

According to the mean value theorem, we know that there is $t_* \in [0, T]$ such that

$$f(y(t_*)) + p_1(t_*, x(t_*), y(t_*)) = 0,$$

which implies $|f(y(t_*))| \le M_0$. From condition (f_0) we know $|y(t_*)| < d_2$. We infer from (2.3) that

$$|y(t)| \le |y(t_*)| + \int_0^T |y'(t)|\, dt < d_2 + 2T(M_0 + M_1) \triangleq c_1.$$

Set $M_2 = \max\{|f(y)| : -c_1 \le y \le c_1\}$. Then we have

$$\int_0^T |x'(t)|\,dt = \lambda \int_0^T |f(y(t)) + p_1(t, x(t), y(t))|\,dt \le T(M_0 + M_2).$$

From (2.2) we know that there is $t^* \in [0, T]$ such that

$$g(x(t^*)) - p_2(t^*, x(t^*), y(t^*)) = 0,$$

which implies that $|g(x(t^*))| \le M_0$. From (g_0) we have $|x(t^*)| < d_1$. Hence, we get

$$|x(t)| \le |x(t^*)| + \int_0^T |x'(t)|\,dt < d_1 + T(M_0 + M_2) \triangleq c_2.$$

Set $M = c_1 + c_2$. Let us define an open set $\Omega \subset \mathbb{R}^2$,

$$\Omega = \{(x, y) \in \mathbb{R}^2 : -M < x < \zeta,\ |y| < M\}.$$

If $(x(t), y(t))$ is any T-periodic solution of (2.1) satisfying $(x(t), y(t)) \in \overline{\Omega}$ for all $t \in [0, T]$, then $x(t) \le \zeta$, $t \in [0, T]$. From the conclusion above we know that $x(t) > -M$, $|y(t)| < M$ for $t \in [0, T]$. Since $\max\{x(t) : t \in [0, T]\} \ne \zeta$, we have $x(t) < \zeta$ for all $t \in [0, T]$. Therefore, $(x(t), y(t)) \in \Omega$ for all $t \in [0, T]$.

Let us define a function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$h(x, y) = (f(y) + \overline{p}_1(x, y), -g(x) + \overline{p}_2(x, y)),$$

where $\overline{p}_1, \overline{p}_2\colon \mathbb{R}^2 \to \mathbb{R}$ are defined by

$$\overline{p}_1(x, y) = \frac{1}{T}\int_0^T p_1(t, x, y)\,dt, \quad \overline{p}_2(x, y) = \frac{1}{T}\int_0^T p_2(t, x, y)\,dt.$$

Obviously, $|\overline{p}_1(x, y)| \le M_0$, $|\overline{p}_2(x, y)| \le M_0$, for all $(x, y) \in \mathbb{R}^2$. We claim that

$$d_B(h, \Omega, 0) = 1.$$

To prove this claim, we consider a map $H(x, y, \lambda)\colon \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$,

$$H(x, y, \lambda) = (f(y) + \lambda\overline{p}_1(x, y), -g(x) + \lambda\overline{p}_2(x, y)).$$

Since $M > \max\{d_1, d_2\}$ and $\zeta > d_1$, we know from (f_0) and (g_0) that, for $x \in [-M, \zeta]$ and $\lambda \in [0, 1]$,

$$f(M) + \lambda\overline{p}_1(x, M) \ne 0, \qquad f(-M) + \lambda\overline{p}_1(x, -M) \ne 0$$

and for $y \in [-M, M]$ and $\lambda \in [0, 1]$,

$$g(-M) - \lambda\overline{p}_2(-M, y) \ne 0, \qquad g(\zeta) - \lambda\overline{p}_2(\zeta, y) \ne 0.$$

Therefore, $H(x, y, \lambda) \ne 0$, for any $(x, y) \in \partial\Omega$, $\lambda \in [0, 1]$ and thus H is a homotopic mapping. From the homotopy invariance theorem of degree we know that

$$d_B(h, \Omega, 0) = d_B(H(\,\cdot\,, 1), \Omega, 0) = d_B(H(\,\cdot\,, 0), \Omega, 0).$$

Since $H(x,y,0)=(f(y),-g(x))$, we infer from (f_0) and (g_0) that

$$d_B(H(\,\cdot\,,0),\Omega,0)=1.$$

Thus, the claim is proved. According to Lemma 2.2, system (1.1) has at least one T-periodic solution $(x(t),y(t))\in\overline{\Omega}$ for all $t\in[0,T]$. □

REMARK 2.3. Assume that (g_0), (f_0) and (h_3) hold and there exists a constant $\zeta>d_2$ such that if $(x_\lambda(t),y_\lambda(t))$ is a T-periodic solution of (2.1) for some $\lambda\in(0,1]$, then $\max\{y_\lambda(t):t\in[0,T]\}\neq\zeta$. Then we can also prove by using the same method that system (1.1) has at least one T-periodic solution.

We now give an elementary application of Lemma 2.1.

EXAMPLE 2.4. Assume that conditions (g_0), (f_0) and (h_3) hold. Moreover, g, f also satisfy the following conditions

$$(\mathrm{g}_1)\qquad \limsup_{x\to+\infty}\frac{g(x)}{x}<+\infty,$$

$$(\mathrm{f}_1)\qquad \lim_{y\to+\infty}\frac{f(y)}{y}=0.$$

Then system (1.1) has at least one T-periodic solution.

PROOF. From (g_0) and (g_1) we know that there exist constants $a>0$, $b>0$ such that

$$0<g(x)\leq ax+b,\quad x\geq d_1.$$

According to (f_1), we have that, for any sufficiently small $\varepsilon>0$, there is $A_\varepsilon>0$ such that

$$|f(y)|\leq\varepsilon y+A_\varepsilon,\quad y\geq -d_2.$$

Let $(x(t),y(t))$ be any T-periodic solution of (2.1) with $x(t_*)=\max\{x(t):t\in[0,T]\}>d_1$. Then, there is an interval $[\alpha,\beta]$ containing t_*, with $0<\beta-\alpha<T$, such that

$$x(\alpha)=x(\beta)=d_1,\quad x(t)>d_1,\ t\in(\alpha,\beta).$$

Since $x'(t_*)=0$, we have that $f(y(t_*))+p_1(t_*,x(t_*),y(t_*))=0$. From condition (f_0) we know that $|y(t_*)|<d_2$. When $x(t)\geq d_1$, $t\in[\alpha,t_*]$, it follows from (g_0) that $y'(t)=\lambda(-g(x(t))+p_2(t,x(t),y(t)))\leq 0$ for $t\in[\alpha,t_*]$. Therefore, $y(t)\geq -d_2$, $t\in[\alpha,t_*]$. Integrating both sides of $x'(t)=\lambda(f(y(t))+p_1(t,x(t),y(t)))$ over the interval $[\alpha,t_*]$, we get

$$x(t_*)\leq A_1+\varepsilon\int_\alpha^{t_*}|y(t)|\,dt,\tag{2.4}$$

where $A_1 = d_1 + T(A_\varepsilon + M_0)$. On the other hand, we have that, for $t \in [\alpha, t_*]$,

$$(2.5)\qquad |y(t)| \le |y(t_*)| + \int_t^{t_*} |g(x(s))|\,ds + \int_t^{t_*} |p_2(s, x(s), y(s))|\,ds \le A_2 + a\int_t^{t_*} |x(s)|\,ds \le A_2 + aTx(t_*),$$

where $A_2 = d_2 + T(b + M_0)$. From (2.4) and (2.5) we get that

$$x(t_*) \le A_1 + \varepsilon T A_2 + \varepsilon a T^2 x(t_*).$$

Take a small $\varepsilon > 0$ such that $0 < \varepsilon a T^2 < 1$. Then we have

$$x(t_*) \le \frac{A_1 + \varepsilon T A_2}{1 - \varepsilon a T^2}.$$

Since $A_1 > d_1$ and $0 < \varepsilon a T^2 < 1$, we know that $(A_1 + \varepsilon T A_2)/(1 - \varepsilon a T^2) > d_1$. According to Lemma 2.1, system (1.1) has at least one T-periodic solution. $\square$

3. Periodic solutions of planar systems with sub-quadratic potentials

In this section, we shall use the continuation Lemma 2.1 to prove the existence of periodic solutions of (1.1) under sub-quadratic potentials.

PROOF OF THEOREM 1.1. From conditions (h_i) ($i = 1, 2$) we know that there exists $d_2 > 0$ such that (f_0) holds. Set

$$\varrho = \liminf_{x \to +\infty} \frac{2G(x)}{x^2} < \left(\frac{c\pi}{T}\right)^2.$$

Let us take a fixed constant $\eta > 0$ such that $\varrho_\eta = \varrho + \eta < (c\pi/T)^2$. From [4] we know that there is a sequence $\{x_n\}$ such that $\lim_{n \to +\infty} x_n = +\infty$ and

$$(3.1)\qquad 2(G(x_n) - G(x)) \le \varrho_\eta (x_n^2 - x^2), \quad x \in (0, x_n).$$

We claim that the condition of Lemma 2.1 holds with $\zeta = x_n$ for a sufficiently large n. Let $(x(t), y(t))$ be any T-periodic solution of system (2.1) and assume by contradicition that $\max_{t \in [0,T]} x(t) = x_n > d_1$ for a sufficiently large n.

Set $\max_{t \in [0,T]} x(t) = x(t_*)$, $t_* \in [0, T]$. Since

$$\int_0^T (g(x(t)) - p_2(t, x(t), y(t)))\,dt = 0,$$

there is $\widetilde{t} \in [0, T]$ such that $g(x(\widetilde{t})) = p_2(\widetilde{t}, x(\widetilde{t}), y(\widetilde{t}))$ and then $|g(x(\widetilde{t}))| \le M_0$. From condition ($\mathrm{g}_0$) we have that $|x(\widetilde{t})| < d_1$. Hence, there is an interval $[\alpha, \beta]$ containing t_*, with $0 < \beta - \alpha < T$, such that

$$x(\alpha) = x(\beta) = d_1, \quad x(t) > d_1, \quad t \in (\alpha, \beta),$$
$$x'(t_*) = 0, \quad y'(t) < 0, \quad t \in [\alpha, \beta].$$

From (2.1) we have

$$(3.2)\qquad (-g(x(t)) + p_2(t,x(t),y(t)))x'(t) = (f(y(t)) + p_1(t,x(t),y(t)))y'(t).$$

Integrating both sides of (3.2) over interval $[t,t_*]$ with $\alpha \le t \le t_*$, we get

$$(3.3)\qquad \begin{aligned} F(y(t)) - F(y(t_*)) &= G(x(t_*)) - G(x(t)) \\ &\quad + \int_t^{t_*} p_1(s,x(s),y(s))y'(s)\,ds - \int_t^{t_*} p_2(s,x(s),y(s))x'(s)\,ds. \end{aligned}$$

Since $y'(t) < 0$, $t \in [\alpha,t_*]$, we have that, for $t \in [\alpha,t_*]$,

$$(3.4)\qquad \begin{aligned} \bigg|\int_t^{t_*} p_1(s,x(s),y(s))y'(s)\,ds\bigg| &\le \int_t^{t_*} |p_1(s,x(s),y(s))||y'(s)|\,ds \\ &\le M_0 \int_t^{t_*} -y'(s)\,ds = M_0(y(t) - y(t_*)). \end{aligned}$$

Meanwhile,

$$\bigg|\int_t^{t_*} p_2(s,x(s),y(s))x'(s)\,ds\bigg| \le M_0 \int_t^{t_*} |x'(s)|\,ds.$$

Note that $x'(t_*) = 0$, we have $f(y(t_*)) + p_1(t_*,x(t_*),y(t_*)) = 0$. Hence, $|f(y(t_*))| \le M_0$. Furthermore, $|y(t_*)| < d_2$. Set $M = \sup\{|f(y)| + |p_1(t,x,y)| : |y| \le d_2,\ x \in \mathbb{R},\ t \in [0,T]\}$. If $t \in [\alpha,t_*]$ and $|y(s)| \le d_2$ for every $s \in [t,t_*]$, then

$$|f(y(s))| + |p_1(s,x(s),y(s))| \le M, \quad s \in [t,t_*].$$

In this case, we have

$$\int_t^{t_*} |x'(s)|\,ds \le \int_t^{t_*} (|f(y(s))| + |p_1(s,x(s),y(s))|)\,ds \le MT.$$

If $t \in [\alpha,t_*]$ and there is $\overline{t} \in [t,t_*]$ such that $y(s) \ge d_2$, $s \in [t,\overline{t}]$; $|y(s)| \le d_2$, $s \in [\overline{t},t_*]$, then we know from (f_0) that $x'(t) > 0$, $t \in [t,\overline{t}]$. In this case, we get that

$$\begin{aligned} \int_t^{t_*} |x'(s)|\,ds &= \int_t^{\overline{t}} |x'(s)|\,ds + \int_{\overline{t}}^{t_*} |x'(s)|\,ds \\ &\le x(\overline{t}) - x(t) + MT \le x(t_*) - x(t) + MT. \end{aligned}$$

It follows that, for $t \in [\alpha,t_*]$,

$$(3.5)\qquad \bigg|\int_t^{t_*} p_2(s,x(s),y(s))x'(s)ds\bigg| \le M_0 \int_t^{t_*} |x'(s)|ds \le M_0(x(t_*) - x(t)) + M_1,$$

where $M_1 = M_0MT$. From (3.3)–(3.5) we infer that, for $t \in [\alpha,t_*]$,

$$F(y(t)) - F(y(t_*)) \le G(x(t_*)) - G(x(t)) + M_0(y(t) - y(t_*)) + M_0(x(t_*) - x(t)) + M_1.$$

Since $|y(t_*)| < d_2$, there is a constant $D > 0$ such that $|F(y(t_*))| \le D$. Therefore, we get

$$(3.6)\qquad F(y(t)) - M_0 y(t) \le G(x(t_*)) - G(x(t)) + M_0(x(t_*) - x(t)) + M_2,$$

where $M_2 = M_1 + d_2 M_0 + D$. From (h_1) we know that, for any $0 < \varepsilon < a_+$, there is $M_\varepsilon > 0$ such that

$$(a_+ - \varepsilon)y - M_\varepsilon \le f(y) \le (a^+ + \varepsilon)y + M_\varepsilon, \quad y \ge -d_2. \tag{3.7}$$

Hence, there is $M'_\varepsilon > 0$ such that

$$F(y) \ge \frac{1}{2}(a_+ - \varepsilon)y^2 - M_\varepsilon y - M'_\varepsilon, \quad y \ge -d_2,$$

which, together with (3.6), implies that, for $t \in [\alpha, t_*]$,

$$\begin{aligned}&(a_+ - \varepsilon)y(t)^2 - 2(M_\varepsilon + M_0)y(t)\\ &\qquad\le 2(G(x(t_*)) - G(x(t))) + 2M_0(x(t_*) - x(t)) + 2(M_2 + M'_\varepsilon).\end{aligned}$$

It follows that

$$\left(y(t) - \frac{M_\varepsilon + M_0}{a_+ - \varepsilon}\right)^2 \le \frac{2}{a_+ - \varepsilon}(G(x(t_*)) - G(x(t)) + M_0(x(t_*) - x(t))) + c_1.$$

where

$$c_1 = \frac{2(M_2 + M'_\varepsilon)}{a_+ - \varepsilon} + \left(\frac{M_\varepsilon + M_0}{a_+ - \varepsilon}\right)^2.$$

Consequently, we obtain that, for $t \in [\alpha, t_*]$,

$$y(t) \le \sqrt{\frac{2}{a_+ - \varepsilon}(G(x(t_*)) - G(x(t)) + M_0(x(t_*) - x(t)))} + \frac{M_\varepsilon + M_0}{a_+ - \varepsilon} + \sqrt{c_1}.$$

Since

$$x'(t) = \lambda(f(y(t)) + p_1(t, x(t), y(t))),$$

we know from (3.7) that, for $t \in [\alpha, t_*]$,

$$x'(t) \le \frac{a^+ + \varepsilon}{\sqrt{a_+ - \varepsilon}}\sqrt{2(G(x(t_*)) - G(x(t)) + M_0(x(t_*) - x(t)))} + c_2, \tag{3.8}$$

where

$$c_2 = (a^+ + \varepsilon)\left(\frac{M_\varepsilon + M_0}{a_+ - \varepsilon} + \sqrt{c_1}\right) + M_\varepsilon + M_0.$$

As a result, we infer from (3.8) that

$$t_* - \alpha \ge \frac{\sqrt{a_+ - \varepsilon}}{a^+ + \varepsilon}\int_{d_1}^{x_*} \frac{dx}{\sqrt{2(G(x_*) - G(x)) + 2M_0(x_* - x)} + c_3},$$

where $x_* = x(t_*)$ and $c_3 = c_2\sqrt{a_+ - \varepsilon}/(a^+ + \varepsilon)$. From (3.1) we get

$$\begin{aligned}t_* - \alpha &\ge \frac{\sqrt{a_+ - \varepsilon}}{a^+ + \varepsilon}\int_{d_1}^{x_*} \frac{dx}{\sqrt{\varrho_\eta(x_*^2 - x^2) + 2M_0(x_* - x)} + c_3}\\ &= \frac{\sqrt{a_+ - \varepsilon}}{a^+ + \varepsilon}\int_0^1 \frac{dt}{\sqrt{\varrho_\eta(1 - t^2)}} + o(1) = \frac{\pi\sqrt{a_+ - \varepsilon}}{2(a^+ + \varepsilon)\sqrt{\varrho_\eta}} + o(1),\end{aligned}$$

where the notation $o(1)$ means an infinitesimal as $x_* \to +\infty$. Similarly, we can obtain

$$\beta - t_* \geq \frac{\pi\sqrt{a_- - \varepsilon}}{2(a^- + \varepsilon)\sqrt{\varrho_\eta}} + o(1).$$

Therefore, we have

$$\beta - \alpha \geq \frac{\pi}{2\sqrt{\varrho_\eta}}\bigg(\frac{\sqrt{a_+ - \varepsilon}}{a^+ + \varepsilon} + \frac{\sqrt{a_- - \varepsilon}}{a^- + \varepsilon}\bigg) + o(1).$$

Since

$$\rho_\eta < \left(\frac{c\pi}{T}\right)^2 \quad \text{and} \quad c = \frac{1}{2}\bigg(\frac{\sqrt{a_+}}{a^+} + \frac{\sqrt{a_-}}{a^-}\bigg),$$

we get

$$\lim_{\varepsilon \to 0^+} \frac{\pi}{2\sqrt{\varrho_\eta}}\bigg(\frac{\sqrt{a_+ - \varepsilon}}{a^+ + \varepsilon} + \frac{\sqrt{a_- - \varepsilon}}{a^- + \varepsilon}\bigg) = \frac{c\pi}{\sqrt{\varrho_\eta}} > T,$$

which implies that there is $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $\beta - \alpha > T$. This is a contradicition. Therefore, there is a sufficiently large $x_n > d_1$ such that, for any T-periodic solution $(x(t), y(t))$ of (2.1), we have $\max\{x(t) : t \in [0,T]\} \neq x_n$. According to Lemma 2.1, system (1.1) has at least one T-periodic solution. $\square$

Remark 3.1. In [10], the existence of periodic solutions of Rayleigh equations

$$x'' + f(t, x') + g(x) = p(t) \tag{3.9}$$

was studied, where $f\colon \mathbb{R} \times \mathbb{R}$ is continuous and T-periodic with the first variable, $g, p\colon \mathbb{R} \to \mathbb{R}$ are continuous and p is T-periodic. Obviously, equation (3.9) is equivalent to the planar system

$$\begin{cases} x' = y, \\ y' = -g(x) - f(t,y) + p(t), \end{cases} \tag{3.10}$$

which is of the form of system (1.1). Assume that the following conditions hold:

(c_1) There is a constant $d > 0$ such that

$$\operatorname{sgn}(x)g(x) > \sup\{|f(t,0)| + |p(t)| : t \in \mathbb{R}\}, \quad |x| \geq d;$$

(c_2) f satisfies sublinear condition:

$$\lim_{|y| \to +\infty} \frac{f(t,y)}{y} = 0 \quad \text{uniformly with } t \in [0,T];$$

(c_3) The primitive G of g satisfies

$$\liminf_{x \to +\infty} \frac{2G(x)}{x^2} < \left(\frac{\pi}{T}\right)^2.$$

It was proved in [10] that equation (3.9) has at least one T-periodic solution, which implies that system (3.10) has at least one T-periodic solution. We can see that there are similarities between the conditions of the result in [10] and the conditions of Theorem 1.1. However, it follows from (c_2) that $f(t,y)$ maybe unbounded and in this case, the conditions (h_3), (g_0) of Theorem 1.1 are not satisfied. Thus Theorem 1.1 does not include the result in [10].

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