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# FINITE-TIME BLOW-UP IN A QUASILINEAR CHEMOTAXIS SYSTEM WITH AN EXTERNAL SIGNAL CONSUMPTION

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Abstract. This paper deals with a quasilinear chemotaxis system with an external signal consumption

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u) - \nabla \cdot (u\nabla v), & (x,t) \in \Omega \times (0,\infty), \\ 0 = \Delta v + u - g(x), & (x,t) \in \Omega \times (0,\infty), \end{cases}$$

under homogeneous Neumann boundary conditions in a ball $\Omega \subset \mathbb{R}^n$, where $\varphi(u)$ is a nonlinear diffusion function and $g(x)$ is an external signal consumption. Under suitable assumptions on the functions $\varphi$ and $g$, it is proved that there exists initial data such that the solution of the above system blows up in finite time.

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## 1. Introduction

In this paper, we consider the following quasilinear chemotaxis system with an external signal consumption under homogeneous Neumann boundary conditions

$$\begin{cases} u_t = \nabla \cdot (\varphi(u)\nabla u) - \nabla \cdot (u\nabla v), & (x,t) \in \Omega \times (0,\infty), \\ 0 = \Delta v + u - g(x), & (x,t) \in \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a ball, $\partial/\partial\nu$ denotes the differentiation with respect to the outward normal derivative on $\partial\Omega$, and $u = u(x,t)$ and $v = v(x,t)$ denotes the density of cells and the concentration of signal, respectively. Moreover, the initial data $u_0$ is a given nonnegative radially symmetric function, $\varphi(u)$ is a nonlinear diffusion function and $g(x)$ is an external signal consumption.

*Chemotaxis* is the directed movement of cells as a response to gradients of the concentration of the chemical signal substance in their environment. The biased movement is referred to as *chemoattraction* if the cells move toward the increasing signal concentration, while it is called *chemorepulsion* whenever the cells move away from the increasing signal concentration. The origin of chemotaxis model was introduced by Keller and Segel [20] and we refer the readers to the surveys [2], [12], [15], [16] where a comprehensive information of further examples illustrating the outstanding biological relevance of chemotaxis can be found. The general structure of chemotaxis models in [12] is as follows

$$\begin{cases} u_t = \nabla \cdot (k_1(u,v)\nabla u - k_2(u,v)u\nabla v) + k_3(u,v), & (x,t) \in \Omega \times (0,\infty), \\ v_t = \Delta v + k_4(u,v) - k_5(u,v)v, & (x,t) \in \Omega \times (0,\infty), \end{cases} \tag{1.2}$$

where $u = u(x,t)$ denotes the cell (or organism) density and $v = v(x,t)$ describes the concentration of chemical signals. The cell dynamics derive from population kinetics and movement, the latter comprising a diffusive flux modelling undirected (random) cell migration and an advective flux with velocity dependent on the gradient of the signal, modelling the contribution of chemotaxis. $k_1(u,v)$ describes the diffusivity of cells (sometimes also called motility), while $k_2(u,v)$ is the chemotactic sensitivity, both functions may depend on the levels of $u$ and $v$. $k_3(u,v)$ describes cell growth and death while the functions $k_4(u,v)$ and $k_5(u,v)$ are kinetic functions that describes production and degradation of the chemical signal, respectively.

In recent years, global existence and finite-time blow-up for special cases of the chemotaxis model (1.2) have been studied extensively by many authors (see [5], [7]–[9], [13]–[18], [21], [23]–[26], [36]–[40], [44], [46], [47] and the references therein). In particular, if chemicals diffuse much faster than cells move (see [19]), model (1.2) can be reduced to the following simplified parabolic–elliptic model

$$
\begin{cases}
u_t = \nabla \cdot (\varphi(u)\nabla u) - \nabla \cdot (\psi(u)\nabla v) + f(u), & (x,t) \in \Omega \times (0,\infty), \\
0 = \Delta v - v + u, & (x,t) \in \Omega \times (0,\infty), \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x), & x \in \Omega,
\end{cases} \tag{1.3}
$$

where $f(u) \le a - bu^k$ with $a \ge 0$, $b > 0$, $k > 1$ and $\Omega \subset \mathbb{R}^n$ $(n \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$. For the special case $\varphi(u) = 1$, $\psi(u) = \chi u$ and $k = 2$ in (1.3), Tello and Winkler [27] proved that the solutions to (1.3) are global and bounded provided that either $n \le 2$, or $n \ge 3$ and $b > (n-2)\chi/n$ with $\chi > 0$. Moreover, for any $n \ge 1$ and $b > 0$, the existence of global weak solution was shown under some additional conditions. Furthermore, if $k > 2 - 1/n$ in (1.3), some global very weak solutions of semilinear parabolic–elliptic model were constructed by Winkler [41]. When $\psi(u) = \chi u$, $\varphi(u) \ge c(u+1)^p$ with $p \in \mathbb{R}$, $k = 2$ and $b > (1 - 2/(n(1-p)_+))\chi$ with $\chi > 0$ in (1.3), Cao and Zheng [6] proved that system (1.3) has a unique global classical solution, which is uniformly bounded. Wang et al. [30] investigated the global boundedness and asymptotic behavior of solutions for (1.3) with the special case $\psi(u) = \chi u$ and $\varphi(u) \ge C_\varphi u^{m-1}$ $(m \ge 1)$ under other additional technical conditions, but didn't obtain the blow-up condition. When the second equation in (1.3) is replaced by $0 = \Delta v - m(t) + u$, where $m(t) = (1/|\Omega|)\int_\Omega u(x,t)\,dx$ denotes the time-dependent spatial mean of the cell density $u$.

In the absence of the logistic source (i.e. $f(u) = 0$) for (1.3), by the mass conservation, it is easy to see that $m(t) = (1/|\Omega|)\int_\Omega u_0(x)\,dx := M$. For the special case $\psi(u) = u$ in (1.3), Cieślak and Winkler [10] proved that if $\varphi(u) \ge c(1+u)^{-p}$ for all $u \ge 0$ holds with some $c > 0$ and $p < 2/n - 1$, all solutions are global and bounded; whereas if $\varphi(u) \le c(1+u)^{-p}$ for all $u \ge 0$ holds with some $c > 0$ and $p > 2/n - 1$ and $\Omega \subset \mathbb{R}^n$ is a ball, then the radially symmetric solution blows up in finite time.

Similarly, assume that $\varphi(u) \cong u^{-p}$ and $\psi(u) \cong u^q$ as $u \cong \infty$ with some $p \ge 0$ and $q \in \mathbb{R}$, Winkler and Djie [42] proved that all solutions of (1.3) are global and bounded if $p + q < 2/n$; while if $p + q > 2/n$ with $q > 0$ and $\Omega \subset \mathbb{R}^n$ is a ball, then the corresponding radial solution blows up in finite time. Moreover, the density of cells is assumed to be a priori bounded by the threshold which is considered by many authors [23], [31]–[34], [43]. For example, let $\psi(u) = uh(u)$, in [33], [34], Wang et al. proved that if $h(u)/\varphi(u) \ge c(1-u)^p$ with $p \in (0,1)$ and $c > 0$ is sufficiently large, the solutions attain the value $u = 1$ either in finite or infinite time, while if $\varphi(u) \ge c(1-u)^{-p}$ and $h(u) \le c(1-u)^q$ with $c > 0$ and

$p+q>1$, the corresponding solutions are bounded away from the threshold value $u=1$ uniformly for any $t>0$. T o the best of our knowledge, when $\varphi(u)\equiv 1$, $\psi(u)\equiv \chi u$ in (1.3), the only result obtained by Winkler in [35] for (1.3) with logistic source $f(u)$ is the finite-time blow-up in the higher-dimensional case under some additional conditions. Moreover, Zheng et al. [45] studied the global boundedness and blow-up of solutions for (1.3).

Recently, the chemotaxis models with an external production of chemoattractant are also considered as follows

$$\begin{cases} u_t=\Delta u-\nabla\cdot(u\nabla v), & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ 0=\Delta v+u+f(x), & (x,t)\in\mathbb{R}^n\times(0,\infty),\\ u(x,0)=u_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.4}$$

where $f(x)$ is an external signal production. External signal productions are motivated by the fact that in biological experiments artificial gradients of chemoattractants are introduced to observe the migration of cells (see [29]). Going further than just observing the response to external stimulants one could ask if and how a population of cells can be influenced in a desired way through these external signal, which could in particular be interesting for tumor treatment. Mathematically this would translate to an optimal control problem which, after a given time, nets a desired distribution of cells through adjustment of the external signals.

Since the occurrence of self-organizing patterns is closely linked to blow-up solutions, the first step before thinking of an appropriate optimal control formulation, is to verify for what class of external signal production functions blow-up may occur. When $n=2$ and $f(x)=f_0\delta(x)$ is a Dirac-distributed signal production, Tello and Winkler [28] proved that there exists a new critical mass phenomenon in system (1.4). It is shown that whenever $f_0>0$ and $u_0\not\equiv 0$, a measure-valued global weak solution can be constructed which blows up at $x=0$ immediately. Moreover, if the total mass of cells $M:=\int_{\mathbb{R}^2}u_0(x)\,dx$ is smaller than $8\pi-2f_0$, then the solution satisfies $u(x,t)\le C(\tau)|x|^{-f_0/(2\pi)}$ for $t>\tau>0$ and $|x|<1$ and hence doesn't blow up in $L^p_{\rm loc}(\mathbb{R}^2)$ for any $1\le p<4\pi/f_0$.

On the other hand, if $M>8\pi-2f_0$, then the mass will asymptotically completely concentrate at the origin, that is, $u(\,\cdot\,,t)$ converges to $M\delta$ as $t\to\infty$ in the sense of Radon measures. In higher dimensional radially symmetric setting, Black [4] proved that the generalized global measure-valued solutions of (1.4) blow up immediately for prototypical signal production functions $f$ satisfying

$$f(x):=\begin{cases} f_0|x|^{-\alpha} & \text{if } |x|\le R-\rho,\\ 0 & \text{if } |x|\ge R+\rho, \end{cases}$$

and smooth with some $1 > R > 0$, $n > \alpha > 2$, $\rho \in (0, R/2)$ and $f_0 > 2n(n-2) \cdot(n-\alpha)/\alpha$. Moreover, Black [3] studied the global boundedness of solutions for a fully parabolic chemotaxis system with external signal production. However, to the best of our knowledge, as for the external signal consumption, there are no any results about finite-time blow-up for chemotaxis system (1.1).

Motivated by the above works, the main purpose of the present paper is concerned with the interplay of the nonlinear diffusion $\varphi(u)$ and external signal consumption $g(x)$ for system (1.1). Throughout this paper, assume that the function $\varphi \in C^2([0,\infty))$ satisfies

$$0 < \varphi(s) \le c_1 s^{-p} \quad \text{with } c_1 > 0 \text{ and } p \in \mathbb{R} \text{ for all } s > 0 \tag{1.5}$$

and the external signal consumption function

$$g(x) := g(|x|) = g_0|x|^{\kappa} \quad \text{with constant } g_0 > 0 \text{ and } \kappa \in \mathbb{R}^+ \tag{1.6}$$

fulfills

$$\int_\Omega g(x)\,dx = \int_\Omega u_0(x)\,dx. \tag{1.7}$$

Our main result in this paper is stated as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$ be a ball centered at the origin. Assume that $\varphi \in C^2([0,\infty))$ satisfies (1.5) with $4/n - 1 < p < 1/2$ and $g$ satisfies (1.6) and (1.7) with $\kappa \ge -2/(2-p)$. Then, for all $m_0 > 0$ and each $T_0 > 0$, there exists radially symmetric positive nonincreasing initial data $u_0 \in C^\infty(\overline{\Omega})$ satisfying $(1/|\Omega|)\int_\Omega u_0(x)\,dx = m_0$ such that system (1.1) possesses a classical solution $(u,v)$ in $\Omega \times (0,T)$ for some $T \in (0,T_0)$, which satisfies*

$$\lim_{t\nearrow T} ||u(\,\cdot\,,t)||_{L^\infty(\Omega)} = \infty.$$

Remark 1.2. It follows from Theorem (1.1) that under the effect of nonlinear diffusion $\varphi$ and external signal consumption $g$, we can extend the blow-up result in [35] under the space dimension $n \ge 5$ into the physical domain $n \ge 3$ without logistic growth restriction.

Remark 1.3. When $g(x) \equiv M$ for some positive constant $M = (1/|\Omega|)\int_\Omega u_0\,dx$ in (1.1), Cieślak and Winkler [10] derived finite-time blow-up of solutions provided $p > 2/n - 1$. For the general external signal consumption $g(x)$, this paper asserts a finite-time blow-up in the interplay of the nonlinear diffusion $\varphi(u)$ and external signal consumption $g(x)$. However, for the cases $2/n - 1 < p \le 4/n - 1$ or $p \ge 1/2$, we have to leave an open problem here whether there exists a blow-up solution to (1.1).

This paper is organized as follows. In Section 2, we shall give some useful preliminaries and show the local-in-time existence of classical solution to system (1.1). In Section 3, we prove that the radially symmetric solutions blow up in finite time under suitable conditions. Let us emphasize that several calculations are similar to that done by Winkler in [35].

## 2. Preliminary lemmas

We first state a result concerning local-in-time existence of classical solutions to system (1.1).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Assume that the function $\varphi \in C^2([0,\infty))$ satisfies* (1.5) *and $g$ fulfills* (1.6) *and* (1.7). *Suppose that the nonnegative initial function $u_0 \in C^1(\overline{\Omega})$ satisfies*

$$m_0 = \frac{1}{|\Omega|}\int_\Omega u_0\,dx$$

*for positive constant $m_0$. Then there exist a maximal existence time $T_{\max} \in (0,\infty]$ and a pair function*

$$u \in C^0\big(\overline{\Omega}\times[0,T_{\max})\big) \cap C^{2,1}(\overline{\Omega}\times(0,T_{\max})) \quad \textit{and} \quad v \in C^0((0,T_{\max}); C^2(\overline{\Omega}))$$

*such that $(u,v)$ is a classical solution of system* (1.1) *in $\Omega\times(0,T_{\max})$. Moreover, we have the mass conservation that*

$$\int_\Omega u\,dx = \int_\Omega u_0\,dx \quad \textit{for all } t > 0.$$

*Finally, if $T_{\max} < +\infty$, then $\lim_{t\nearrow T_{\max}}\sup ||u(\,\cdot\,,t)||_{L^\infty(\Omega)} = \infty$.*

Proof. The local-in-time existence of classical solutions to system (1.1) is well-established by a fixed point theorem in the context of Keller-Segel-type chemotaxis systems. For the details, we refer the reader to [3], [4], [28]. □

Next, we give an elementary lemma of Gronwall's type, which will be used in the proof of finite-time blow-up for (1.1).

Lemma 2.2 (Lemma 2.4 in [35]). *Let $\varrho > 0$, $\delta > 0$ and $\gamma > 0$, and suppose that for some $T > 0$, $\xi \in C^0([0,T])$ is a nonnegative function and satisfies*

$$\xi(t) \geq \varrho + \delta\int_0^t \xi^{1+\gamma}(\tau)\,d\tau \quad \textit{for all } t \in (0,T).$$

*Then we have $T \leq 1/(\gamma\delta\varrho^\gamma)$.*

## 3. Finite-time blow-up

In this section, we show that the radially symmetric solutions blow up in finite time under suitable conditions. We set $\Omega = B_R(0)$ with some $R > 0$ and assume that the initial function $u_0 = u_0(r) \in C^1(\overline{\Omega})$ is a positive, nonincreasing and radially symmetric. In the radial framework, system (1.1) can be transformed into the following form

$$
\text{(3.1)}\qquad \begin{cases} u_t = r^{1-n}(r^{n-1}\varphi(u)u_r)_r - r^{1-n}(r^{n-1}uv_r)_r, & (r,t)\in(0,R)\times(0,\infty),\\ 0 = r^{1-n}(r^{n-1}v_r)_r + u - g(r), & (r,t)\in(0,R)\times(0,\infty),\\ u_r = v_r = 0, & r=R,\ t\in(0,\infty),\\ u(r,0)=u_0(r), & r\in(0,R). \end{cases}
$$

By Lemma 2.1, there exists a classical solution $(u,v)$ up to a maximal existence time $T_{\max}\in(0,\infty]$, where $(u,v)$ is also radially symmetric because of the uniqueness result. Without any danger of confusion, we shall write $u=u(r,t)$ and $v=v(r,t)$ with $r=|x|\in[0,R]$.

According to the ideas in [10], [19], [35], [42], we introduce

$$
\text{(3.2)}\qquad w(s,t) := \int_0^{s^{1/n}} \rho^{n-1}u(\rho,t)\,d\rho, \quad s=r^n\in[0,R^n],\ t\in[0,T_{\max}),
$$

then

$$
\text{(3.3)}\qquad w_s(s,t)=\frac{1}{n}u\big(s^{1/n},t\big) \quad\text{and}\quad w_{ss}=\frac{1}{n^2}s^{1/n-1}u_r\big(s^{1/n},t\big).
$$

Moreover, it follows from (1.6) and (3.1) that

$$
\begin{aligned}
\text{(3.4)}\qquad w_t(s,t) &= \int_0^{s^{1/n}} \rho^{n-1}u_t(\rho,t)\,d\rho\\
&= \int_0^{s^{1/n}} \big(\rho^{n-1}\varphi(u)u_\rho\big)_\rho\,d\rho - \int_0^{s^{1/n}} (\rho^{n-1}uv_\rho)_\rho\,d\rho\\
&= s^{1-1/n}\varphi\big(u\big(s^{1/n},t\big)\big)u_r\big(s^{1/n},t\big) - s^{1-1/n}u\big(s^{1/n},t\big)v_r\big(s^{1/n},t\big)\\
&= s^{1-1/n}\varphi\big(u\big(s^{1/n},t\big)\big)u_r\big(s^{1/n},t\big)\\
&\quad + u\big(s^{1/n},t\big)\int_0^{s^{1/n}} \rho^{n-1}(u(\rho,t)-g(\rho)\,)d\rho\\
&= s^{1-1/n}\varphi\big(u\big(s^{1/n},t\big)\big)u_r\big(s^{1/n},t\big) - \frac{g_0 s^{(n+\kappa)/n}}{n+\kappa}u\big(s^{1/n},t\big)\\
&\quad + u\big(s^{1/n},t\big)\int_0^{s^{1/n}} \rho^{n-1}u(\rho,t)\,d\rho.
\end{aligned}
$$

Collecting (3.2)–(3.4), we obtain that $w$ satisfies, for $s\in(0,R^n), t\in(0,T_{\max})$, the following scalar parabolic equation

$$
w_t = n^2 s^{2-2/n}\varphi(nw_s)w_{ss} + nww_s - \frac{ng_0}{n+\kappa}\,s^{(n+\kappa)/n}w_s.
$$

Hence, we deduce from (1.5) that the function $w(s,t)$ solves the following problem

$$
\begin{cases}
w_t \geq c_1 n^{2-p} s^{2-2/n} (w_s)^{-p} w_{ss} + n w w_s - \dfrac{n g_0 s^{(n+\kappa)/n}}{n+\kappa} w_s, \\
\qquad\qquad\qquad\qquad\qquad\qquad s \in (0,R^n),\ t \in (0,T_{\max}), \\
w(0,t) = 0, \\
w(R^n,t) = \displaystyle\int_0^R \rho^{n-1} u(\rho,t)\, d\rho \\
\qquad\qquad = \dfrac{1}{|\mathcal{S}_{n-1}|} \displaystyle\int_{B_R(0)} u(x,t)\, dx, \quad t \in (0,T_{\max}), \\
w(s,0) = w_0(s), \qquad\qquad\qquad s \in (0,R^n),
\end{cases}
\tag{3.5}
$$

with $|\mathcal{S}_{n-1}|$ representing the surface area of the unit sphere in $n$ dimensions, where

$$
w_0(s) = \int_0^{s^{1/n}} \rho^{n-1} u_0(\rho)\, d\rho \quad \text{for } s \in [0,R^n]. \tag{3.6}
$$

Due to the fact that the initial data $u_0$ is positive, it follows from the strong maximum principle that $u > 0$ in $\Omega \times (0,T_{\max})$. Thus, we infer from (3.3) that $w_s > 0$ in $\Omega \times (0,T_{\max})$. In particular, in view of the boundary condition at $s = R^n$, we have the following estimate

$$
w(s,t) \leq \frac{1}{|\mathcal{S}_{n-1}|} \int_{B_R(0)} u(x,t)\, dx \quad \text{for all } s \in [0,R^n],\ t \in [0,T_{\max}).
$$

According to the method used in [35], we shall focus on the time evolution of the functional $\int_0^{R^n} s^{-\alpha} w^\beta\, ds$ for suitable $\alpha > 1$ and $\beta < 1$, which will be determined later. To do this, we need the following lemmas, which still require much milder assumptions than needed for the proof of our main results.

Lemma 3.1. *Let $2/n - 1 < p < 1$ and suppose that $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$ and $n \geq 2$. Then for all $\alpha > 0$ and $\beta \in (0,1)$, the function $w$ defined by* (3.2) *satisfies the following estimate*

$$
\begin{aligned}
&\frac{1}{\beta} \int_0^{R^n} s^{-\alpha} w^\beta\, ds \\
&\quad + \frac{2c_1}{1-p} n^{1-p} (n-1) \int_0^t \int_0^{R^n} s^{1-2/n-\alpha} w^{\beta-1} (w_s)^{1-p}\, ds\, d\tau \\
&\quad + \frac{n g_0}{n+\kappa} \int_0^t \int_0^{R^n} s^{1+\kappa/n-\alpha} w^{\beta-1} w_s\, ds\, d\tau \\
&\geq \frac{1}{\beta} \int_0^{R^n} s^{-\alpha} w_0^\beta(s)\, ds \\
&\quad + \frac{c_1(1-\beta)}{1-p} n^{2-p} \int_0^t \int_0^{R^n} s^{2-2/n-\alpha} w^{\beta-2} (w_s)^{2-p}\, ds\, d\tau
\end{aligned}
\tag{3.7}
$$

$$+\frac{n}{2}\int_0^t\int_0^{R^n} s^{-\alpha}w^\beta w_s\,ds\,d\tau+\frac{n\alpha}{2(\beta+1)}\int_0^t\int_0^{R^n} s^{-\alpha-1}w^{\beta+1}\,ds\,d\tau$$

*for all $t\in(0,T_{\max})$, where $w_0(s)$ is defined by* (3.6).

Proof. Multiplying (3.5) by $(s+\varepsilon)^{-\alpha}w^{\beta-1}$ ($\varepsilon>0$ is an arbitrary constant) and integrating over $(0,R^n)$ with regard to $s$, we have

$$\begin{aligned}\frac{1}{\beta}\frac{d}{dt}&\int_0^{R^n}(s+\varepsilon)^{-\alpha}w^\beta\,ds\\ &\geq c_1n^{2-p}\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{-p}w_{ss}\,ds\\ &\quad+n\int_0^{R^n}(s+\varepsilon)^{-\alpha}w^\beta w_s\,ds\\ &\quad-\frac{ng_0}{n+\kappa}\int_0^{R^n} s^{1+\kappa/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}w_s\,ds=:\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\end{aligned}\tag{3.8}$$

for all $t\in(0,T_{\max})$. Integrating by parts, we obtain

$$\begin{aligned}\mathcal{I}_1&=c_1n^{2-p}\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{-p}w_{ss}\,ds\\ &=\frac{c_1}{1-p}n^{2-p}\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}((w_s)^{1-p})_s\,ds\\ &=\frac{c_1}{1-p}n^{2-p}s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{1-p}\Big|_0^{R^n}\\ &\quad-\frac{c_1}{1-p}n^{2-p}\int_0^{R^n}\frac{d}{ds}\big[s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}\big](w_s)^{1-p}\,ds.\end{aligned}\tag{3.9}$$

By the strong maximum principle, we have $u>0$ in $\overline{\Omega}\times(0,T_{\max})$, thus it follows that $w_s(s,t)=u(s^{1/n},t)/n$ is positive for $s\in[0,R^n]$ and $t\in(0,T_{\max})$. In particular, this implies that for all $t\in(0,T_{\max})$, we can find $c_1(t)>0$ such that $w(s,t)\geq c_1(t)s$ for all $s\in[0,R^n]$. Since $w_s(s,t)$ is bounded in $L^\infty((0,R^n))$ for any fixed $t\in(0,T_{\max})$, we derive

$$s^{2-2/n}w^{\beta-1}(s,t)(w_s)^{1-p}(s,t)\leq\frac{||w_s(\,\cdot\,,t)||^{1-p}_{L^\infty((0,R^n))}}{c_1^{1-\beta}(t)}\,s^{\beta+1-2/n}\to 0\tag{3.10}$$

as $s\to 0$, due to the fact that $\beta+1-2/n>1-2/n\geq 0$ and $n\geq 2$. By $2/n-1<p<1$ and the positivity of $w_s$, we deduce from (3.10) that

$$\frac{c_1}{1-p}n^{2-p}s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{1-p}\Big|_0^{R^n}\geq 0.\tag{3.11}$$

Since

$$\begin{aligned}\tag{3.12} \int_0^{R^n} &\frac{d}{ds}\big[s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}\big](w_s)^{1-p}\,ds \\ &= (\beta-1)\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-2}(w_s)^{2-p}\,ds \\ &\quad + \int_0^{R^n}\bigg[\bigg(2-\frac{2}{n}\bigg)s^{1-2/n}(s+\varepsilon)^{-\alpha} \\ &\qquad - \alpha s^{2-2/n}(s+\varepsilon)^{-\alpha-1}\bigg]w^{\beta-1}(w_s)^{1-p}\,ds \\ &\le (\beta-1)\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-2}(w_s)^{2-p}\,ds \\ &\quad + \bigg(2-\frac{2}{n}\bigg)\int_0^{R^n} s^{1-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{1-p}\,ds,\end{aligned}$$

it follows from (3.9) and (3.11) that, for all $t \in (0, T_{\max})$,

$$\begin{aligned}\tag{3.13} \mathcal{I}_1 \ge{}& \frac{c_1(1-\beta)}{1-p}n^{2-p}\int_0^{R^n} s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-2}(w_s)^{2-p}\,ds \\ &- \frac{2c_1}{1-p}n^{1-p}(n-1)\int_0^{R^n} s^{1-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{1-p}\,ds.\end{aligned}$$

As for $\mathcal{I}_2$, we split $\mathcal{I}_2 = \mathcal{I}_2/2 + \mathcal{I}_2/2$ and integrate by parts to obtain

$$\begin{aligned}\tag{3.14} \frac{\mathcal{I}_2}{2} &= \frac{n}{2(\beta+1)}\int_0^{R^n}(s+\varepsilon)^{-\alpha}\big(w^{\beta+1}\big)_s\,ds \\ &= \frac{n}{2(\beta+1)}(s+\varepsilon)^{-\alpha}w^{\beta+1}\Big|_0^{R^n} + \frac{n\alpha}{2(\beta+1)}\int_0^{R^n}(s+\varepsilon)^{-\alpha-1}w^{\beta+1}\,ds.\end{aligned}$$

According to the fact that $w(0,t) = 0$ for all $t \in (0, T_{\max})$, it is easy to see that for all $t \in (0, T_{\max})$ we have

$$\tag{3.15} \frac{n}{2(\beta+1)}(s+\varepsilon)^{-\alpha}w^{\beta+1}\Big|_0^{R^n} \ge 0.$$

Inserting (3.15) into (3.14), we derive

$$\tag{3.16} \frac{\mathcal{I}_2}{2} \ge \frac{n\alpha}{2(\beta+1)}\int_0^{R^n}(s+\varepsilon)^{-\alpha-1}w^{\beta+1}\,ds.$$

Therefore, integrating over $(0,t)$, it follows from (3.8), (3.13) and (3.16) that

$$\begin{aligned}\tag{3.17} \frac{1}{\beta}\int_0^{R^n}&(s+\varepsilon)^{-\alpha}w^{\beta}\,ds + \frac{ng_0}{n+\kappa}\int_0^t\int_0^{R^n} s^{1+\kappa/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}w_s\,ds\,d\tau \\ &+ \frac{2c_1}{1-p}\,n^{1-p}(n-1)\int_0^t\int_0^{R^n} s^{1-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-1}(w_s)^{1-p}\,ds\,d\tau\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\beta}\int_0^{R^n}(s+\varepsilon)^{-\alpha}w_0^\beta(s)\,ds\\
&\quad+\frac{c_1(1-\beta)}{1-p}\,n^{2-p}\int_0^t\int_0^{R^n}s^{2-2/n}(s+\varepsilon)^{-\alpha}w^{\beta-2}(w_s)^{2-p}\,ds\,d\tau\\
&\quad+\frac{n}{2}\int_0^t\int_0^{R^n}(s+\varepsilon)^{-\alpha}w^\beta w_s\,ds\,d\tau\\
&\quad+\frac{n\alpha}{2(\beta+1)}\int_0^t\int_0^{R^n}(s+\varepsilon)^{-\alpha-1}w^{\beta+1}\,ds\,d\tau.
\end{aligned}
$$

Taking $\varepsilon\searrow 0$ in (3.17) and applying the monotone convergence theorem, we can obtain the desired estimate (3.7). The proof of Lemma 3.1 is complete. □

With the above statement, it follows from Lemma 3.1 that we can proceed to derive a favorable integral inequality for $\int_0^{R^n}s^{-\alpha}w^\beta(s,t)\,ds$ for suitable $\alpha>1$ and $\beta<1$.

LEMMA 3.2. *Let $n\geq 3$, $\kappa\geq -2/(2-p)$ and $4/n-1<p<1/2$. Suppose that $u_0=u_0(r)$ is positive in $\Omega=B_R(0)\subset\mathbb{R}^n$ such that $(1/|\Omega|)\int_\Omega u_0\,dx=m_0$ for some positive constant $m_0$. Then there exist $\alpha>1$, $\beta\in(0,1)$, $\delta>0$ and $C>0$ such that $w$ defined by (3.2) satisfying the following estimate*

$$
\begin{aligned}
\int_0^{R^n}s^{-\alpha}w^\beta(s,t)\,ds \geq{}& \int_0^{R^n}s^{-\alpha}w_0^\beta(s)\,ds\\
&+\delta\int_0^t\left(\int_0^{R^n}s^{-\alpha}w^\beta(s,\tau)\,ds\right)^{(\beta+1)/\beta}d\tau-Ct,
\end{aligned}
\tag{3.18}
$$

*for all $t\in(0,T_{\max})$, where $w_0$ is given by (3.6).*

PROOF. According to the conditions that $n\geq 3$ and $p\in(4/n-1,1/2)$, we can fix $\alpha\in(1,(2+2p-4/n)/(1+p))$, which ensures that $2/n+(1+p)(\alpha-1)<1+p-2/n$. Hence, we can finally choose $\beta\in(0,1)$ satisfying

$$
\beta>\max\left\{\frac{2/n+(1+p)(\alpha-1)}{1+p-2/n},p\right\}. \tag{3.19}
$$

We now suppose that $u_0=u_0(r)$ is positive with $(1/|\Omega|)\int_\Omega u_0\,dx=m_0$, and let $w$ and $w_0$ be defined by (3.2) and (3.6), respectively. Then it follows from Lemma 3.1 that

$$
\begin{aligned}
&\frac{1}{\beta}\int_0^{R^n}s^{-\alpha}w^\beta\,ds\\
&\geq\frac{1}{\beta}\int_0^{R^n}s^{-\alpha}w_0^\beta(s)\,ds+C_1\int_0^t\int_0^{R^n}s^{2-2/n-\alpha}w^{\beta-2}(w_s)^{2-p}\,ds\,d\tau\\
&\quad+C_1\int_0^t\int_0^{R^n}s^{-\alpha}w^\beta w_s\,ds\,d\tau+C_1\int_0^t\int_0^{R^n}s^{-\alpha-1}w^{\beta+1}\,ds\,d\tau
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&- \frac{2c_1}{1-p} n^{1-p}(n-1)\int_0^t\int_0^{R^n} s^{1-2/n-\alpha} w^{\beta-1}(w_s)^{1-p}\,ds\,d\tau\\
&- \frac{ng_0}{n+\kappa}\int_0^t\int_0^{R^n} s^{1+\kappa/n-\alpha} w^{\beta-1} w_s\,ds\,d\tau\\
&=: J_1+J_2+J_3+J_4-J_5-J_6
\end{aligned}
$$

for all $t\in(0,T_{\max})$, where

$$
C_1=\min\bigg\{\frac{c_1(1-\beta)}{1-p}\,n^{2-p},\frac{n}{2},\frac{n\alpha}{2(\beta+1)}\bigg\}.
$$

By Young's inequality, there exists $C_2>0$ such that

$$
\begin{aligned}
J_5 \le \frac{C_1}{2}\int_0^t\int_0^{R^n} s^{2-2/n-\alpha} w^{\beta-2}(w_s)^{2-p}\,ds\,d\tau \\
+ C_2\int_0^t\int_0^{R^n} s^{-2/n-\alpha+p} w^{\beta-p}\,ds\,d\tau.
\end{aligned}
\tag{3.21}
$$

We deduce from Young's inequality again that

$$
\begin{aligned}
C_2\int_0^t\int_0^{R^n} s^{-2/n-\alpha+p} w^{\beta-p}\,ds\,d\tau \le \frac{C_1}{3}\int_0^t\int_0^{R^n} s^{-\alpha-1} w^{\beta+1}\,ds\,d\tau\\
+ C_3\int_0^t\int_0^{R^n} s^{(-2/n-(1+p)\alpha+(1+p-2/n)\beta)/(1+p)}\,ds\,d\tau,
\end{aligned}
\tag{3.22}
$$

where $C_3>0$. According to the first restriction contained in (3.19), it is easy to see that $(-2/n-(1+p)\alpha+(1+p-2/n)\beta)/(1+p)>-1$. Thus, we have

$$
\begin{aligned}
C_3\int_0^t\int_0^{R^n} s^{(-2/n-(1+p)\alpha+(1+p-2/n)\beta)/(1+p)}\,ds\,d\tau\\
=\frac{C_3(1+p)R^{n\cdot(-2/n-(1+p)(\alpha-1)+(1+p-2/n)\beta)/(1+p)}}{-2/n-(1+p)(\alpha-1)+(1+p-2/n)\beta}t:=C_4 t.
\end{aligned}
\tag{3.23}
$$

Combining (3.21)–(3.23), we obtain

$$
J_5\le\frac{1}{2}J_2+\frac{1}{3}J_4+C_4t\quad\text{for all } t\in(0,T_{\max}).
\tag{3.24}
$$

Next, we estimate $J_6$. By applying Young's inequality twice as before, we can find $C_5>0$ and $C_6>0$ satisfying

$$
\begin{aligned}
J_6&=\frac{ng_0}{n+\kappa}\int_0^t\int_0^{R^n} s^{1+\kappa/n-\alpha} w^{\beta-1} w_s\,ds\,d\tau\\
&\le\frac{C_1}{2}\int_0^t\int_0^{R^n} s^{2-2/n-\alpha} w^{\beta-2}(w_s)^{2-p}\,ds\,d\tau\\
&\quad+C_5\int_0^t\int_0^{R^n} s^{(((2-p)\kappa+2)/n-\alpha+p(\alpha-1))/(1-p)} w^{(\beta(1-p)+p)/(1-p)}\,ds\,d\tau\\
&\le\frac{C_1}{2}\int_0^t\int_0^{R^n} s^{2-2/n-\alpha} w^{\beta-2}(w_s)^{2-p}\,ds\,d\tau
\end{aligned}
\tag{3.25}
$$

$$+\frac{C_1}{3}\int_0^t\int_0^{R^n} s^{-\alpha-1}w^{\beta+1}\,ds\,d\tau$$
$$+C_6\int_0^t\int_0^{R^n} s^{(((2-p)\kappa+2)(\beta+1))/(n(1-2p))+\beta-\alpha}\,ds\,d\tau.$$

According to (3.19), $p\in(4/n-1,1/2)$ and $\kappa\geq -2/(2-p)$, we have

$$\frac{[(2-p)\kappa+2](\beta+1)}{n(1-2p)}+\beta-\alpha>-1,$$

so that we can find $C_7>0$ satisfying

$$\begin{aligned}C_6\int_0^t\int_0^{R^n} s^{((2-p)\kappa+2)(\beta+1)/((n(1-2p))+\beta-\alpha}\,ds\,d\tau \\ =\frac{C_6R^{((2-p)\kappa+2)(\beta+1)/(1-2p)+n(\beta-\alpha+1)}}{((2-p)\kappa+2)(\beta+1)/(n(1-2p))+\beta-\alpha+1}\,t:=C_7t.\end{aligned}\tag{3.26}$$

Therefore, it follows from (3.25) and (3.26) that

$$J_6\leq\frac{1}{2}J_2+\frac{1}{3}J_4+C_7t\quad\text{for all } t\in(0,T_{\max}).\tag{3.27}$$

Combining (3.20), (3.24) and (3.27), it follows from $J_3>0$ that

$$\begin{aligned}\int_0^{R^n} s^{-\alpha}w^\beta(s,t)\,ds\geq\int_0^{R^n} s^{-\alpha}w_0^\beta(s)\,ds \\ +\frac{\beta C_1}{3}\int_0^t\int_0^{R^n} s^{-\alpha-1}w^{\beta+1}\,ds\,d\tau-C_8t\end{aligned}\tag{3.28}$$

for all $t\in(0,T_{\max})$, where $C_8=C_4+C_7>0$. By using Hölder's inequality, we have

$$\begin{aligned}\int_0^{R^n} s^{-\alpha}w^\beta(s,t)\,ds&=\int_0^{R^n} s^{-\alpha+\beta(\alpha+1)/(\beta+1)}\left(s^{-\alpha-1}w^{\beta+1}\right)^{\beta/(\beta+1)}ds \\ &\leq\left(\int_0^{R^n} s^{-\alpha-1}w^{\beta+1}\,ds\right)^{\beta/(\beta+1)}\left(\int_0^{R^n} s^{-\alpha+\beta}\,ds\right)^{1/(\beta+1)}.\end{aligned}\tag{3.29}$$

According to (3.19), we see that $-\alpha+\beta>-1$, thus it follows that

$$\int_0^{R^n} s^{-\alpha+\beta}\,ds=\frac{R^{n(1+\beta-\alpha)}}{1+\beta-\alpha}.\tag{3.30}$$

Collecting (3.29) and (3.30), we have

$$\int_0^{R^n} s^{-\alpha-1}w^{\beta+1}\,ds\geq\left(\frac{1+\beta-\alpha}{R^{n(1+\beta-\alpha)}}\right)^{1/\beta}\left(\int_0^{R^n} s^{-\alpha}w^\beta(s,t)\,ds\right)^{(\beta+1)/\beta}.\tag{3.31}$$

Therefore, it follows from (3.28) and (3.31) that

$$\int_0^{R^n} s^{-\alpha} w^\beta(s,t)\,ds \geq \int_0^{R^n} s^{-\alpha} w_0^\beta(s)\,ds - C_8 t + \frac{\beta C_1}{4}\int_0^t \left(\frac{1+\beta-\alpha}{R^{n(1+\beta-\alpha)}}\right)^{1/\beta}\left(\int_0^{R^n} s^{-\alpha} w^\beta(s,t)\,ds\right)^{(\beta+1)/\beta} d\tau$$

for all $t \in (0, T_{\max})$. The proof of Lemma 3.2 is complete. □

Proof of Theorem 1.1. We fix $n \geq 3$ and may assume that $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$. Then for given $\kappa \geq -2/(2-p)$, $p \in (4/n - 1, 1/2)$ and $m_0 > 0$, we let $\alpha > 1$, $\beta \in (0,1)$, $\delta > 0$ and $C > 0$ be as provided by Lemma 3.2. Now, for fixed $T > 0$, we pick $\varrho > 0$ large such that

$$\varrho > \left(\frac{\beta}{\delta T}\right)^\beta. \tag{3.32}$$

Next, according to the methods used by Winkler in [35], we let

$$\phi_\varepsilon(s) := \frac{m_0}{n}\cdot\frac{R^n+\varepsilon}{s+\varepsilon}\cdot s, \quad s \in [0, R^n],\ \varepsilon > 0,$$

it is obvious to see that $\phi_\varepsilon(s)$ is nonnegative and satisfies

$$\phi_\varepsilon(s) \nearrow \frac{m_0 R^n}{n} \quad \text{for all } s \in [0, R^n] \text{ as } \varepsilon \searrow 0.$$

By the monotone convergence theorem, we have

$$\int_0^{R^n} s^{-\alpha}\phi_\varepsilon^\beta(s)\,ds \to +\infty \quad \text{as } \varepsilon \searrow 0.$$

Hence, we can find some sufficiently small $\varepsilon > 0$ such that

$$\int_0^{R^n} s^{-\alpha}\phi_\varepsilon^\beta(s)\,ds \geq \varrho + CT. \tag{3.33}$$

With the value of fixed $\varepsilon$, we let

$$w_0(s) := \phi_\varepsilon(s), \quad s \in [0, R^n], \tag{3.34}$$

then it follows that $w_0$ belongs to $C^\infty([0,R^n])$ and satisfies $w_0(0) = 0$, $w_0(R^n) = m_0 R^n/n$ and $w_{0s}(s) > 0$ for all $s \in [0, R^n]$. Accordingly, the function $u_0$ defined by $u_0(x) := n w_{0s}(|x|^n)$ for $x \in \overline{\Omega}$ is radially symmetric, smooth and positive in $\overline{\Omega}$ with $(1/|\Omega|)\int_\Omega u_0(x)\,dx = m_0$. Next, we claim that the maximal existence time $T_{\max}$ of the corresponding solution $(u,v)$ satisfies $T_{\max} < T$. In fact, it follows from (3.33), (3.34) and Lemma 3.2 that

$$\begin{aligned}(3.35)\quad \int_0^{R^n} s^{-\alpha} w^\beta(s,t)\,ds \\ \geq \int_0^{R^n} s^{-\alpha} w_0^\beta(s)\,ds + \delta \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^\beta(s,\tau)\,ds \right)^{(\beta+1)/\beta} d\tau - Ct \\ \geq \varrho + \delta \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^\beta(s,\tau)\,ds \right)^{(\beta+1)/\beta} d\tau\end{aligned}$$

for all $t \in (0, T_{\max})$. Let

$$\xi(t) := \int_0^{R^n} s^{-\alpha} w^\beta(s,t)\,ds, \quad t \in (0, T_{\max}),$$

then (3.35) can be transformed into the following form

$$\xi(t) \geq \varrho + \delta \int_0^t \xi^{1+1/\beta}(\tau)\,d\tau \quad \text{for all } t \in (0, T_{\max}).$$

By Lemma 2.2, it is easy to see that $T_{\max} \leq 1/\big(\delta \varrho^{1/\beta}/\beta\big)$. Therefore, it follows from (3.32) that $T_{\max} < T$. □

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## References


[1] N.D. Alikakos, *$L^p$ bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations **4** (1979), 827–868.

[2] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, *Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues*, Math. Models Methods Appl. Sci. **25** (2015), 1663–1763.

[3] T. Black, *Boundedness in a Keller–Segel system with external signal production*, J. Math. Anal. Appl. **446** (2017), 436–455.

[4] T. Black, *Blow-up of weak solutions to a chemotaxis system under influence of an external chemoattractant*, Nonlinearity **29** (2016), 1865–1886.

[5] J. Burczak, T. Cieślak and C. Morales-Rodrigo, *Global existence vs. blow-up in a fully parabolic quasilinear* 1D *Keller–Segel system*, Nonlinear Anal. **75** (2012), 5215–5228.

[6] X. Cao and S. Zheng, *Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source*, Math. Meth. Appl. Sci. **37** (2014), 2326–2330.

[7] T. Cieślak, *Quasilinear nonuniformly parabolic system modelling chemotaxis*, J. Math. Anal. Appl. **326** (2007), 1410–1426.

[8] T. Cieślak and C. Stinner, *Finite-time blowup and global-in-time unbounded solutions to a parabolic–parabolic quasilinear Keller–Segel system in higher dimensions*, J. Differential Equations**252** (2012), 5832–5851.

[9] T. Cieślak and C. Stinner, *Finite-time blowup in a supercritical quasilinear parabolic–parabolic Keller–Segel system in dimension* 2, Acta Appl. Math. **129** (2014), 135–146.

[10] T. Cieślak and M. Winkler, *Finite-time blow-up in a quasilinear system of chemotaxis*, Nonlinearity **21** (2008), 1057–1076.

[11] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[12] T. Hillen and K.J. Painter, *A user's guide to PDE models for chemotaxis*, J. Math. Biol. **58** (2009), 183–217.

[13] D. Horstmann, *On the existence of radially symmetric blow-up solutions for the Keller–Segel model*, J. Math. Biol. **44** (2002), 463–478.

[14] D. Horstmann, *Generalizing the Keller–Segel model: Lyapunov functionals, steady state analysis, and blow-up results for multi-species chemotaxis models in the presence of attraction and repulsion between competitive interacting species*, J. Nonlinear Sci. **21** (2011), 231–270.

[15] D. Horstmann, *From* 1970 *until present: The Keller–Segel model in chemotaxis and its consequences* I, Jahresber. Dtsch. Math. -Ver. **105** (2003), 103–165.

[16] D. Horstmann, *From* 1970 *until present: the Keller–Segel model in chemotaxis and its consequences* II, Jahresber. Dtsch. Math.-Ver. **106** (2004), 51–69.

[17] D. Horstmann and G. Wang, *Blow-up in a chemotaxis model without symmetry assumptions*, European J. Appl. Math. **12** (2001), 159–177.

[18] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Differential Equations **215** (2005), 52–107.

[19] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819–824.

[20] E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theoret. Biol. **26** (1970), 399–415.

[21] C. Mu, L. Wang, P. Zheng and Q. Zhang, *Global existence and boundedness of classical solutions to a parabolic–parabolic chemotaxis system*, Nonlinear Anal. Real World Appl. **14** (2013), 1634–1642.

[22] L. Nirenberg, *An extended interpolation inequality*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) **20** (1966), 733–737.

[23] K.J. Painter and T. Hillen, *Volume-filling and quorum-sensing in models for chemosensitive movement*, Can. Appl. Math. Q. **10** (2002), 501–543.

[24] Y. Tao and Z.A. Wang, *Competing effects of attraction vs. repulsion in chemotaxis*, Math. Models Methods Appl. Sci. **1** (2013), 1–36.

[25] Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity*, J. Differential Equations **252** (2012), 692–715.

[26] Y. Tao and M. Winkler, *Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant*, J. Differential Equations **252** (2012), 2520–2543.

[27] J.I. Tello and M. Winkler, *A chemotaxis system with logistic source*, Comm. Partial Differential Equations **32** (2007), no. 6, 849–877.

[28] J.I. Tello and M. Winkler, *Reduction of critical mass in a chemotaxis system by external application of chemoattractant*, Ann. Sci. Norm. Super. Pisa Cl. Sci. **12** (2013), 833–862.

[29] W. Wagner, et al., *Hematopoietic progenitor cells and cellular microenvironment: behavioral and molecular changes upon interaction*, Stem Cells **23** (2015), 1180–1191.

[30] L. Wang, C. Mu and P. Zheng, *On a quasilinear parabolic-elliptic chemotaxis system with logistic source*, J. Differential Equations **256** (2014), 1847–1872.

[31] Z.A. Wang, *On chemotaxis models with cell population interactions*, Math. Model. Nat. Phenom. **5** (2010), 173–190.

[32] Z.A. Wang and T. Hillen, *Classical solutions and pattern formation for a volume filling chemotaxis model*, Chaos **17** (2007), 037108.

[33] Z.A. Wang, M. Winkler and D. Wrzosek, *Singularity formation in chemotaxis systems with volume-filling effect*, Nonlinearity **24** (2011), 3279–3297.

[34] Z.A. Wang, M. Winkler and D. Wrzosek, *Global regularity vs. infinite-time singularity formation in a chemotaxis model with volume-filling effect and degenerate diffusion*, SIAM J. Math. Anal. **44** (2012), 3502–3525.

[35] M. Winkler, *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl. **384** (2011), 261–272.

[36] M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system*, J. Math. Pures Appl. **100** (2013), 748–767.

[37] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model*, J. Differential Equations **248** (2010), 2889–2905.

[38] M. Winkler, *Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source*, Comm. Partial Differential Equations **35** (2010), 1516–1537.

[39] M. Winkler, *Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity*, Math. Nachr. **283** (2010), 1664–1673.

[40] M. Winkler, *Does a 'volume-filling effect' always prevent chemotactic collapse*?, Math. Methods Appl. Sci. **33** (2010), 12–24.

[41] M. Winkler, *Chemotaxis with logistic source: very weak global solutions and their boundedness properties*, J. Math. Anal. Appl. **348** (2008), 708–729.

[42] M. Winkler and K.C. Djie, *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*, Nonlinear Anal. **72** (2010), 1044–1064.

[43] D. Wrzosek, *Model of chemotaxis with threshold density and singular diffusion*, Nonlinear Anal. **73** (2010), 338–349.

[44] P. Zheng and C. Mu, *Global existence of solutions for a fully parabolic chemotaxis system with consumption of chemoattractant and logistic source*, Math. Nachr. **288** (2015), 710–720.

[45] P. Zheng, C. Mu and X. Hu, *Boundedness and blow-up for a chemotaxis system with generalized volume-filling effect and logistic source*, Discrete Contin. Dyn. Syst. **35** (2015), 2299–2323.

[46] P. Zheng, C. Mu, X. Hu and Y. Tian, *Boundedness of solutions in a chemotaxis system with nonlinear sensitivity and logistic source*, J. Math. Anal. Appl. **424** (2015), 509–522.

[47] P. Zheng, C. Mu, L. Wang and L. Li, *Boundedness and asymptotic behavior in a fully parabolic chemotaxis-growth system with signal-dependent sensitivity*, J. Evol. Equ. **17** (2017), 909–929.

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