Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 2, 2018, 677–691
DOI: 10.12775/TMNA.2018.034

LIPSCHITZ RETRACTIONS ONTO SPHERE VS SPHERICAL CUP IN A HILBERT SPACE

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Abstract. We prove that, in every infinite dimensional Hilbert space, there exists $t_0 > -1$ such that the smallest Lipscthiz constant of retractions from the unit ball onto its boundary is the same as the smallest Lipschitz constant of retractions from the unit ball onto its t-spherical cup for all $t \in [-1, t_0]$.

1. Introduction and preliminaries

For a given Banach space X, it is known that X has infinite dimension if and only if there exists a lipschitzian retraction from the closed unit ball B_X onto its boundary (the unit sphere) S_X (see [10] and [3]). Motivated by this fact, it is natural to ask for the value of the smallest Lipschitz constant of such a retraction, which is defined to be

$$k_0(X) := \inf\{k : \text{there exists a } k\text{-lipschitzian retraction from } B_X \text{ onto } S_X\}.$$

This question is generally regarded as the optimal retraction problem from the unit ball onto its sphere. Although the exact value of $k_0(X)$ has not been found even for one space X, some approximations are discovered. For example, see Table 1.

For the case of a Hilbert space H, the development of upper bounds of $k_0(H)$ in [8], [9], [4] and [2] is all based on the direct, yet very technical, constructions

2010 *Mathematics Subject Classification.* Primary: 47H09.
Key words and phrases. Lipschitz retraction; optimal retraction; spherical cup.

Infinite dimensional space X	Approximation	Reference(s)
Banach space	$k_0(X) \geq 3$	[8]
Hilbert space	$4.58 < k_0(X) < 28.99$	[5] and [2]
ℓ_1	$4 \leq k_0(X) \leq 8$	[8] and [1]
ℓ_∞	$3 \leq k_0(X) \leq 22.96$	[8] and [5]
$C[0,1], BC(\mathbb{R}), c$ and c_0	$3 \leq k_0(X) \leq 14.93$	[8] and [12]
$BC_0(\mathbb{R})$	$3 < k_0(X) \leq 6.83.$	[8] and [11]

Table 1. Some approximations of $k_0(X)$'s.

of retractions from B_H onto S_H. Until 2012, in [6], a new approach is introduced by considering a lipschitzian retraction from B_H onto its so-called spherical cup, which is a certain part of S_H, and it is shown that the smallest Lipschitz constant of such a retraction may help approximating $k_0(H)$. More precisely, let $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a (real) Hilbert space, $B = B_H$, $S = S_H$, e a unit vector in H and $E = (\mathrm{span}\{\mathrm{e}\})^\perp$. Then each element of H will be written in the form of $\alpha\mathrm{e} \oplus \beta x$, for some $\alpha, \beta \in \mathbb{R}$ and $x \in E \cap S$, with $\|\alpha\mathrm{e} \oplus \beta x\|^2 = \alpha^2 + \beta^2$. For each $t \in [-1, 1]$,

- the parallel hyperplane is $E_t := E + t\mathrm{e}$;
- the parallel ball section is $B_t := B \cap E_t$;
- the lense cut by E_t is $D_t := \{x \in B : \langle x, \mathrm{e}\rangle \geq t\}$;
- the spherical cup cut by E_t is $S_t := D_t \cap S$

(see Figure 1).

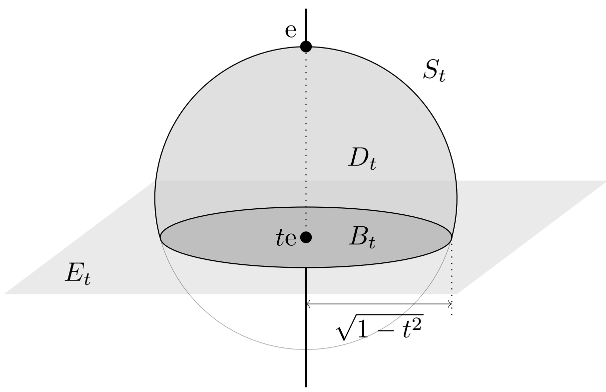


Figure 1. The sets E_t, B_t, D_t, and S_t.

Define $\kappa\colon [-1,1] \to \mathbb{R} \cup \{\infty\}$ by

$$\kappa(t) = \inf\{k : \text{there exists a } k\text{-lipschitzian retraction from } B \text{ onto } S_t\}.$$

Here are some properties of $\kappa(t)$ (see [6] and [7] for details):

- If $\dim H < \infty$,

$$\kappa(t) = \begin{cases} 0 & \text{if } t = 1; \\ \dfrac{\arccos t}{\sqrt{1-t^2}} & \text{if } -1 < t < 1; \\ \infty & \text{if } t = -1. \end{cases}$$

- If $\dim H = \infty$, then $\kappa(1) = 0$, $\kappa(-1) = k_0(H)$, $\kappa(t) \le (\arccos t)/\sqrt{1-t^2}$, and there exists $t_0 > -1$ such that

$$k_0(H) \le \kappa(t) \le \frac{3\sqrt{3}}{2}\, k_0(H) \quad \text{for all } -1 \le t \le t_0.$$

In particular, the last inequality sheds some light on a relationship between $\kappa(t)$ and $k_0(H)$, and suggests the quest for the sharper version. The aim of this work is to close the gap by showing that both $k_0(H)$ and $\kappa(t)$ indeed coincides for every $-1 \le t \le t_0$.

Fix $t \in (-1, 1)$. For each $x \in S \cap E$, let $B_{t,x} = B_t \cap \operatorname{span}\{\mathrm{e}, x\}$ and $S_{t,x} = S_t \cap \operatorname{span}\{\mathrm{e}, x\}$ (see Figure 2 (a)). That is,

- $B_{t,x}$ = the segment of length $2\sqrt{1-t^2}$ in 2-dimensional subspace $\operatorname{span}\{\mathrm{e}, x\}$;
- $S_{t,x}$ = the arc of length $2\arccos t$ from unit circle in 2-dimensional subspace $\operatorname{span}\{\mathrm{e}, x\}$.

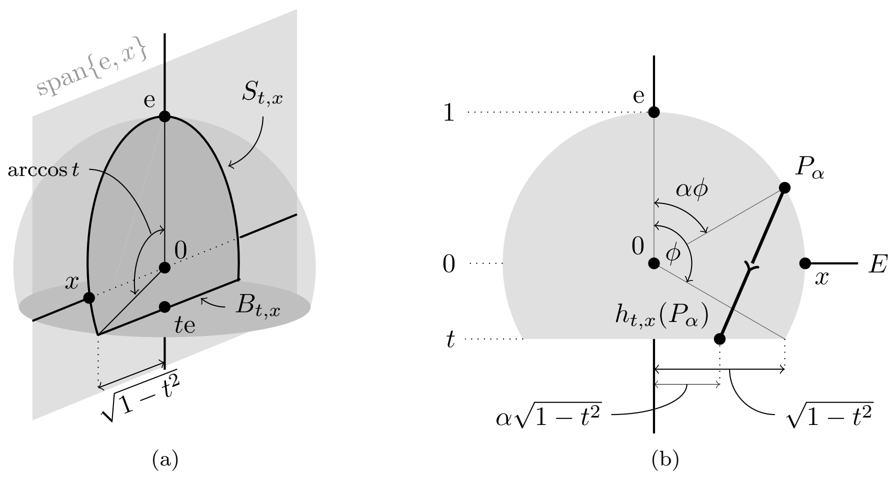


FIGURE 2. (a) The sets $B_{t,x}$ and $S_{t,x}$, and (b) the map $h_{t,x}$.

Obviously, there exists a natural nonexpansive homeomorphism $h_{t,x} \colon S_{t,x} \to B_{t,x}$ fixing $B_{t,x} \cap S_{t,x}$ by uniform scaling between arc and segment. To be precise, let $\phi = \arccos t$. Then, for each $P_\alpha = (\cos\alpha\phi)\,\mathrm{e} \oplus (\sin\alpha\phi)x \in S_{t,x}$ where $\alpha \in [-1,1]$,

$$h_{t,x}(P_\alpha) = h_{t,x}((\cos\alpha\phi)\,\mathrm{e} \oplus (\sin\alpha\phi)x) = t\,\mathrm{e} \oplus \Big(\alpha\sqrt{1-t^2}\Big)x \tag{1.1}$$

(see Figure 2 (b)). Then, by letting $h := \Big(\bigcup_{x\in S\cap E} h_{t,x}\colon S_t \to B_t\Big)$, h is also a nonexpansive homeomorphism and fixes $\bigcup_{x\in S\cap E} B_{t,x} \cap S_{t,x} = B_t \cap S_t$ (see [7] for details).

According to [7], there is a nonexpansive extension $\widetilde{h}\colon D_t \to B_t$ of h which is a retraction (see [7, cf. the map $(f|_{B_t})^{-1} \circ f$ with Properties 2.3 (iv) and (vi)]). With this construction in mind, we may assume in general throughout this work that there exists a nonexpansive retraction from any lense cut onto its parallel ball section whose restriction on its spherical cup is a homeomorphism.

2. Main result

By recalling from [6] that there exists a constant $t_0 > -1$ such that $\kappa(t) \ge k_0(H)$ for all $-1 \le t \le t_0$, the main result can be stated as follows.

Theorem 2.1. *In every infinite dimensional Hilbert space H, $\kappa(t) = k_0(H)$ for all $-1 \le t \le t_0$.*

According to the property of t_0 as above, the proof of this theorem can be accomplished by showing that $\kappa(t) \le k_0(H)$ for all $-1 \le t \le t_0$. Therefore it suffices to construct, for arbitrary $\varepsilon > 0$, a $(k_0(H)+\varepsilon)$-retraction $\varrho_{\varepsilon,t}\colon B \to S_t$, which can be done in the following way.

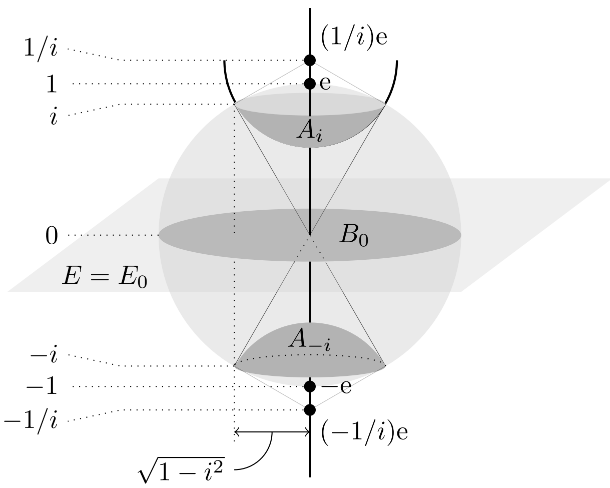


Figure 3. The set A_i.

For each $i \in [-1, 1]$, let

$$A_i = \begin{cases} S\left(\frac{1}{i}\mathrm{e}; \sqrt{\frac{1}{i^2} - 1}\right) \cap B & \text{if } i \neq 0; \\ B_0 & \text{if } i = 0, \end{cases}$$

and

$$C_i = \overline{\mathrm{co}}\, A_i = \overline{\{\alpha x + (1 - \alpha)y : 0 \leq \alpha \leq 1 \text{ and } x, y \in A_i\}},$$

where $B(p; r)$ and $S(p; r)$ denote the closed ball and the sphere centered at $p \in H$ with radius $r > 0$, respectively (see Figure 3).

Notice that, for each $i \neq 0$, since A_i and C_i are indeed homeomorphic to a spherical cup and a lense cut, respectively, of the ball $B(\mathrm{e}/i; \sqrt{1/i^2 - 1})$, there is a nonexpansive retraction $h_i \colon C_i \to B_i$ such that $h_{A_i} := (h_i|_{A_i} \colon A_i \to B_i)$ is a homeomorphism by the construction mentioned in the previous section. That is, by letting $\phi = \arccos \sqrt{1 - i^2}$ and applying (1.1) together with translation of center $(0 \to \mathrm{e}/i \oplus 0)$ and scaling of radius $(1 \to \sqrt{1/i^2 - 1})$, the formula of h_{A_i} is given as follows. For each

$$P_\alpha = \left(\frac{1}{i} \mp \sqrt{\frac{1}{i^2} - 1} \cdot \cos \alpha\phi\right) \mathrm{e} \oplus \left(\sqrt{\frac{1}{i^2} - 1} \cdot \sin \alpha\phi\right) x \in A_i$$

where $\alpha \in [0, 1]$ and $x \in S \cap E$ (here, '$-$' is applied if $i > 0$ while '$+$' is applied if $i < 0$ (see Figure 4 (a)),

$$\text{(2.1)} \quad \begin{aligned} h_{A_i}(P_\alpha) &= h_{A_i}\left(\left(\frac{1}{i} \mp \sqrt{\frac{1}{i^2} - 1} \cdot \cos \alpha\phi\right)\mathrm{e} \oplus \left(\sqrt{\frac{1}{i^2} - 1} \cdot \sin \alpha\phi\right)x\right) \\ &= i\,\mathrm{e} \oplus \left(\alpha\sqrt{1 - i^2}\right)x \end{aligned}$$

(see Figure 4 (b)). Additionally, let $h_0 = \mathrm{id}_{A_0} = \mathrm{id}_{B_0}$. Then $h_0^{-1} = h_0 \colon B_0 \to B_0$.

Recall the following propositions.

PROPOSITION 2.2. *Let A be a closed and convex subset of a Hilbert space H. Then the nearest point projection onto A, which is defined by*

$$\pi_A(x) = \{a \in A : \|x - a\| = d(x, A)\}$$

for all $x \in H$, is nonexpansive.

PROPOSITION 2.3. *Let $A_1, \ldots, A_n$ be subsets of a Hilbert space H such that $A = \bigcup_{i=1}^{n} A_i$ is convex, and $f \colon A \to H$. Denote by $[x, y]$ the segment joining x and y for every $x, y \in H$. Assume that $f|_{A_i}$ is k-lipschitzian for every $i = 1, \ldots, n$, and that for each $x, y \in A$, there are $a_0 = x, a_1, \ldots, a_{m-1}, a_m = y \in [x, y]$ such that for each $j = 1, \ldots, m$, $[a_{j-1}, a_j]$ is a subset of A_i for some $i = 1, \ldots, n$. Then f is also k-lipschitzian.*

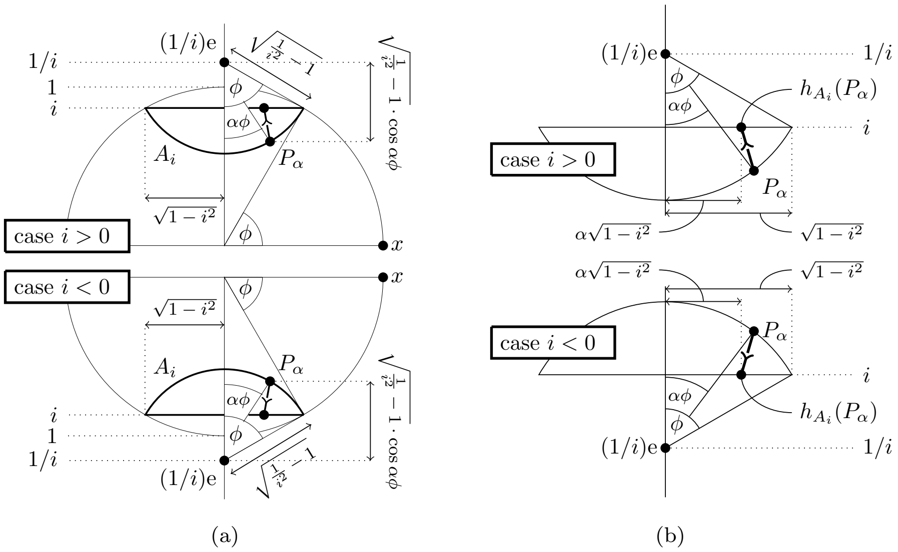


Figure 4. (a) The point P_α, and (b) the map h_{A_i}.

Now, observe that

$$\mathscr{P} := \bigg\{ \bigcup_{i>0} S\bigg(\frac{1}{i}\,\mathrm{e}; \sqrt{\frac{1}{i^2}-1}\bigg), E, \bigcup_{i<0} S\bigg(\frac{1}{i}\,\mathrm{e}; \sqrt{\frac{1}{i^2}-1}\bigg) \bigg\}$$

is a partition of H. For each $t \in (-1,0)$, we divide B into three parts as follows (see Figure 5):

(1) C_t;
(2) $L_t = \overline{B - D_t}$;
(3) $U_t = \overline{B - (C_t \cup L_t)}$.

Observe that U_t is the disjoint union $\coprod_{t\le i\le 1} A_i$. If $\alpha\mathrm{e} \oplus \beta x \in U_t$, then $t \le \alpha$ and $\alpha^2+\beta^2 \le 1$. This implies that $\alpha\mathrm{e} \oplus \beta x \in A_{2\alpha/(\alpha^2+\beta^2+1)}$ where $t \le 2\alpha/(\alpha^2+\beta^2+1) \le 1$ because

$$\begin{aligned}\bigg\| \bigg(\frac{1}{2\alpha/(\alpha^2+\beta^2+1)}\,\mathrm{e} \oplus 0\bigg) - (\alpha\mathrm{e} \oplus \beta x) \bigg\|^2 &= \bigg| \frac{\beta^2-\alpha^2+1}{2\alpha} \bigg|^2 + |\beta|^2 \\ &= \frac{\alpha^4+\beta^4+2\alpha^2\beta^2-2\alpha^2+2\beta^2+1}{4\alpha^2} \\ &= \frac{\alpha^4+\beta^4+2\alpha^2\beta^2+2\alpha^2+2\beta^2+1}{4\alpha^2} - 1 = \frac{1}{(2\alpha/(\alpha^2+\beta^2+1))^2} - 1\end{aligned}$$

and

$$t = \frac{(\alpha^2+\beta^2+1)t}{\alpha^2+\beta^2+1} \le \frac{2t}{\alpha^2+\beta^2+1} \le \frac{2\alpha}{\alpha^2+\beta^2+1} \le \frac{\alpha^2+1}{\alpha^2+\beta^2+1} \le 1.$$

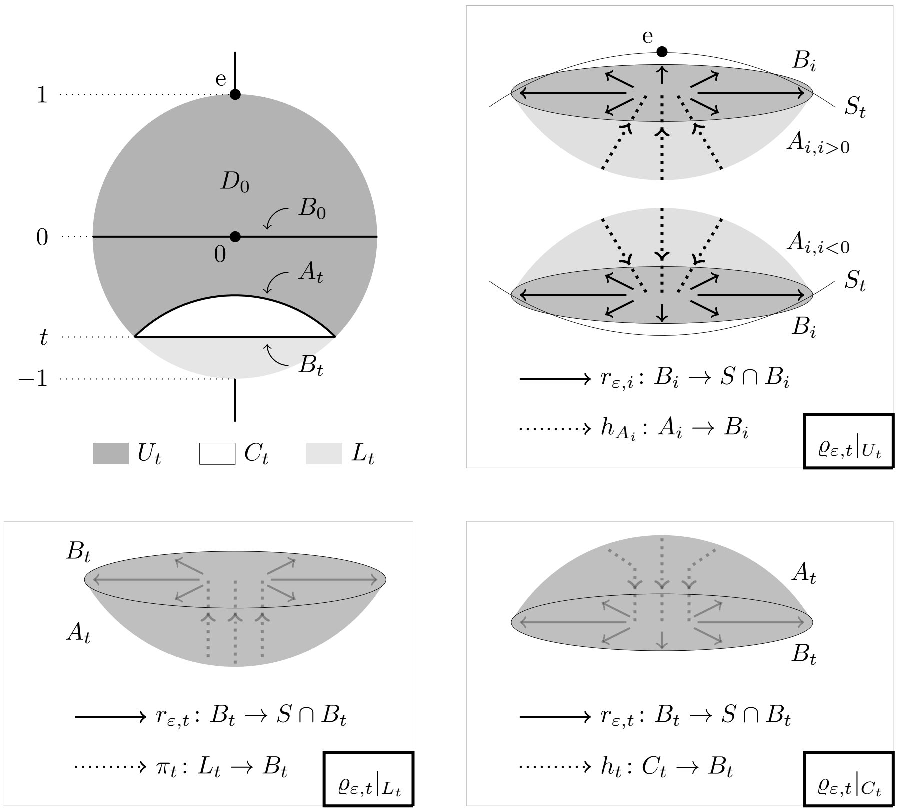


Figure 5. Subsets C_t, L_t and U_t of B, and the map $\varrho_{\varepsilon,t}$ in each case.

Moreover, $A_i \cap A_j = \emptyset$ for all $t \le i, j \le 1$ with $|i| < |j|$. This follows from two cases below:

(1) $ij \le 0$: since $\mathscr{P}$ forms a partition of H, this condition implies that A_i and A_j must lie on different parts of $\mathscr{P}$. Thus $A_i \cap A_j = \emptyset$;
(2) $ij > 0$: assume without loss of generality that $0 < i < j \le 1$. Then $\sqrt{1-j^2} \le 1$, which implies that

$$i - j + (j-i)\sqrt{1-j^2} = (j-i)\big(-1+\sqrt{1-j^2}\big) \le 0$$

(it is strict if $j < 1$). Thus, for $j < 1$,

$$\begin{aligned}\left\|\frac{1}{i}\mathrm{e} - \frac{1}{j}\mathrm{e}\right\| + \sqrt{\frac{1}{j^2}-1} &= \left|\frac{1}{i} - \frac{1}{j}\right| + \sqrt{\frac{1}{j^2}-1} \\ &= \sqrt{\left(\left|\frac{1}{i}-\frac{1}{j}\right| + \sqrt{\frac{1}{j^2}-1}\right)^2}\end{aligned}$$

$$= \sqrt{\frac{1}{i^2} + \frac{2}{j^2} - \frac{2}{ij} - 1 + \frac{2(j-i)}{ij}\sqrt{\frac{1}{j^2} - 1}}$$

$$= \sqrt{\frac{1}{i^2} - 1 + \frac{2}{ij^2}\big(i - j + (j-i)\sqrt{1-j^2}\big)} < \sqrt{\frac{1}{i^2} - 1},$$

while, for $j = 1$,

$$\left\|\frac{1}{i}\mathrm{e} - \frac{1}{j}\mathrm{e}\right\| + \sqrt{\frac{1}{j^2} - 1} = \frac{1}{i} - 1 < \sqrt{\frac{1}{i^2} - 1}.$$

Therefore

$$A_i \cap A_j \subseteq S\bigg(\frac{1}{i}\mathrm{e}; \sqrt{\frac{1}{i^2} - 1}\bigg) \cap S\bigg(\frac{1}{j}\mathrm{e}; \sqrt{\frac{1}{j^2} - 1}\bigg) = \emptyset.$$

Now, let $\varepsilon > 0$ and $k_\varepsilon = k_0(H) + \varepsilon$. By the definition of $k_0(H)$, there exists a k_ε-lipschitzian retraction $r_\varepsilon \colon B \to S$. For each $i \in [t,1]$, define a k_ε-lipschitzian retraction $r_{\varepsilon,i} \colon B_i \to S \cap B_i$ by

$$r_{\varepsilon,i}(i\mathrm{e} \oplus \beta x) = i\mathrm{e} \oplus \big(\sqrt{1-i^2}\big) r_\varepsilon\bigg(\frac{\beta}{\sqrt{1-i^2}}x\bigg), \quad i\mathrm{e} \oplus \beta x \in B_i. \tag{2.2}$$

Let $\pi_t \colon L_t \to B_t$ be the nearest point projection which is obviously nonexpansive, and define the map $\varrho_{\varepsilon,t} \colon B \to S_t$ by

$$\varrho_{\varepsilon,t}(P) = \begin{cases} r_{\varepsilon,t} \circ h_t(P) & \text{if } P \in C_t; \\ r_{\varepsilon,t} \circ \pi_t(P) & \text{if } P \in L_t; \\ r_{\varepsilon,i} \circ h_{A_i}(P) & \text{if } P \in A_i \subseteq U_t \text{ for some } i \in [t,1] \end{cases}$$

(see Figure 5 in frames). Clearly, $\varrho_{\varepsilon,t}$ is well-defined with $\varrho_{\varepsilon,t}|_{S_t} = \mathrm{id}_{S_t}$. Furthermore, the restrictions $\varrho_{\varepsilon,t}|_{C_t}$ and $\varrho_{\varepsilon,t}|_{L_t}$ are k_ε-lipschitzian. If $\varrho_{\varepsilon,t}|_{U_t}$ is k_ε-lipschitzian, then so is $\varrho_{\varepsilon,t}$ because $C_t \cup L_t \cup U_t = B$ is convex and satisfies the condition of Proposition 2.3.

According to the above argument, it remains to show that

Claim 2.4. *For each $t \in (-1,0)$ and $\varepsilon > 0$, $\varrho_{\varepsilon,t}|_{U_t}$ is k_ε-lipschitzian.*

3. Proof of Claim 2.4

In order to prove the Claim 2.4, the following lemma is required.

Lemma 3.1 (Ptolemy's Theorem, Isoceles Trapezoid Version). *Let $\square ABCD$ be an isosceles trapezoid with lengths l for both legs, a and b for other sides and d for the diagonal (see Figure 6). Then $d^2 = ab + l^2$.*

Proof. By letting $\phi = \angle DAB = \angle CBA$, the proof follows from the relation

$$d^2 = a^2 + l^2 - 2al\cos\phi = a(a - 2l\cos\phi) + l^2 = ab + l^2. \qquad \square$$

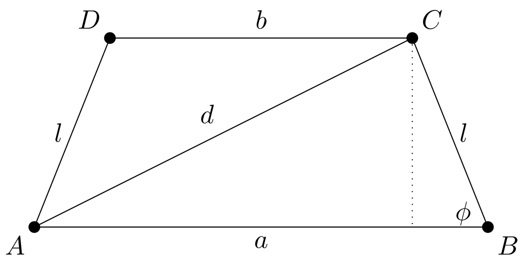
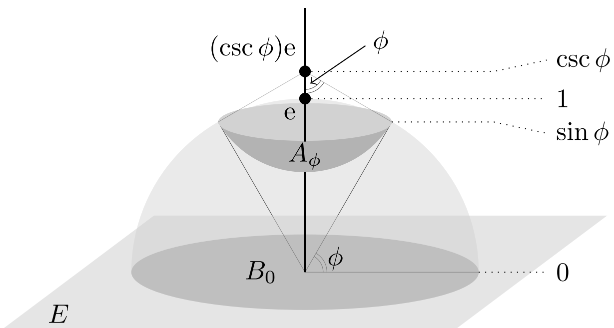
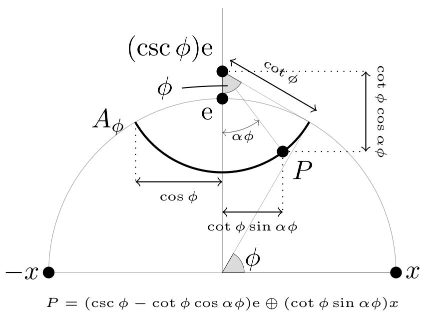


FIGURE 6. The Isosceles trapezoid in Lemma 3.1.



(a)



(b)

FIGURE 7. (a) The set A_ϕ, and (b) The representation of P on A_ϕ.

Let us begin the proof of claim with this observation: for each $i \in (0, 1)$, there is a unique $\phi \in (0, \pi/2)$ so that $\sin\phi = i$. Then, in this case, $\csc\phi = 1/i$, $\cot\phi = \sqrt{1/i^2 - 1}$ and A_i corresponds to

$$A_\phi = \{x \in B : \|x - (\csc\phi)\mathrm{e}\| = \cot\phi\}$$

(see Figure 7 (a)).

Similarly, B_i, h_i, $h_{A_i} = h_i|_{A_i}$ and $r_{\varepsilon,i}$ correspond to B_ϕ, h_ϕ, $h_{A_\phi} = h_\phi|_{A_\phi}$ and $r_{\varepsilon,\phi}$, respectively. Moreover, each element P of A_ϕ is of the form

$$P = (\csc\phi - \cot\phi\cos\alpha\phi)\,\mathrm{e} \oplus (\cot\phi\sin\alpha\phi)x =: P_{\alpha\phi,x},$$

for some $\alpha \in [0,1]$ and $x \in S \cap E$ (see Figure 7 (b)), and, according to (1.1) and (2.2), the explicit formulas of $h_{A_\phi}\colon A_\phi \to B_\phi$ and $r_{\varepsilon,\phi}\colon B_\phi \to S \cap B_\phi$, expressed in terms of this representation, are

$$h_{A_\phi}(P) = h_{A_\phi}((\csc\phi - \cot\phi\cos\alpha\phi)\,\mathrm{e} \oplus (\cot\phi\sin\alpha\phi)x) = (\sin\phi)\,\mathrm{e} \oplus (\alpha\cos\phi)x, \tag{3.1}$$

$$r_{\varepsilon,\phi}((\sin\phi)\,\mathrm{e} \oplus (\alpha\cos\phi)x) = (\sin\phi)\,\mathrm{e} \oplus (\cos\phi)r_\varepsilon(\alpha x) \tag{3.2}$$

where $\phi \in (0,\pi/2)$, $\alpha \in [0,1]$ and $x \in S$.

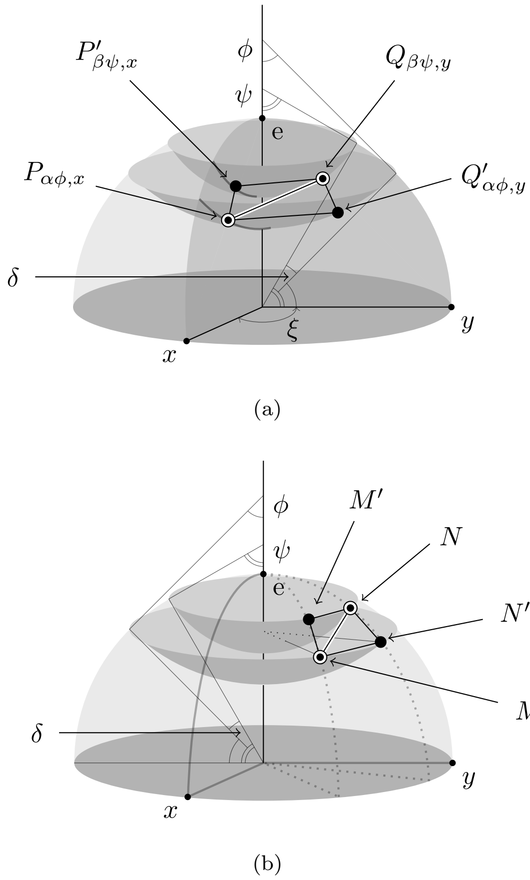


Figure 8. (a) Points P, Q, P' and Q', and (b) points M, N, M' and N'.

PROOF OF CLAIM 2.4. Let $t \in (-1, 0)$, $\varepsilon > 0$ and $U_t^+ = \bigcup_{0<i<1} A_i$. Firstly, we will show that $\varrho_{\varepsilon,t}|_{U_t^+}$ is k_ε-lipschitzian. Let

$$P = P_{\alpha\phi,x} = (\csc\phi - \cot\phi\cos\alpha\phi)\,\mathrm{e} \oplus (\cot\phi\sin\alpha\phi)x \in A_\phi,$$

$$Q = Q_{\beta\psi,y} = (\csc\psi - \cot\psi\cos\beta\psi)\,\mathrm{e} \oplus (\cot\psi\sin\beta\psi)y \in A_\psi,$$

for some $\phi, \psi \in (0, \pi/2)$ with $\phi \le \psi$, $\alpha, \beta \in [0, 1]$, and $x, y \in S \cap E$. In addition, let

$$P' = P'_{\beta\psi,x} = (\csc\psi - \cot\psi\cos\beta\psi)\,\mathrm{e} \oplus (\cot\psi\sin\beta\psi)x \in A_\psi,$$
$$Q' = Q'_{\alpha\phi,y} = (\csc\phi - \cot\phi\cos\alpha\phi)\,\mathrm{e} \oplus (\cot\phi\sin\alpha\phi)y \in A_\phi,$$

$\delta = \psi - \phi$ and $\xi = \angle(x, y) \in [0, \pi]$ (see Figure 8 (a)).

The case $\phi = \psi$ is trivial because $\varrho_{\varepsilon,t}|_{A_\phi \cup A_\psi} = \varrho_{\varepsilon,t}|_{A_\phi} = r_{\varepsilon,\phi} \circ h_{A_\phi}$, which is already k_ε-lipschitzian by nonexpansive-ness of h_{A_ϕ}. Assume $\phi < \psi$. Let us assume the following technical condition $(\star)$, which will be verified later.

$$(\star) \qquad (\alpha - \beta)^2 \cos\phi\cos\psi + \frac{1}{k_\varepsilon^2}(2 - 2\cos\delta) \le \|P - P'\|^2.$$

By straightforward calculation, $\square PP'QQ'$ is an isosceles trapezoid and it follows from Lemma 3.1 that

$$\begin{aligned}(3.3)\qquad \|P - Q\|^2 &= \|P - P'\|^2 + \|P - Q'\|\|Q - P'\| \\ &= \|P - P'\|^2 \\ &\quad + \sqrt{2\,(1 - \cos\xi)\,(\cot\phi\sin\alpha\phi)^2} \cdot \sqrt{2(1 - \cos\xi)(\cot\psi\sin\beta\psi)^2} \\ &= \|P - P'\|^2 + 2(1 - \cos\xi)\cot\phi\cot\psi\sin\alpha\phi\sin\beta\psi.\end{aligned}$$

Let

$$\begin{aligned}M = \varrho_{\varepsilon,t}(P) = r_{\varepsilon,\phi} \circ h_{A_\phi}(P_{\alpha\phi,x}) &\overset{(3.1)}{=} r_{\varepsilon,\phi}((\sin\phi)\,\mathrm{e} \oplus (\alpha\cos\phi)x) \\ &\overset{(3.2)}{=} (\sin\phi)\,\mathrm{e} \oplus (\cos\phi)r_\varepsilon(\alpha x)\end{aligned}$$

and

$$\begin{aligned}N = \varrho_{\varepsilon,t}(Q) = r_{\varepsilon,\psi} \circ h_{A_\psi}(Q_{\beta\psi,y}) &\overset{(3.1)}{=} r_{\varepsilon,\psi}((\sin\psi)\,\mathrm{e} \oplus (\beta\cos\psi)y) \\ &\overset{(3.2)}{=} (\sin\psi)\,\mathrm{e} \oplus (\cos\psi)r_\varepsilon(\beta y).\end{aligned}$$

Furthermore, let

$$M' = (\sin\psi)\,\mathrm{e} \oplus (\cos\psi)r_\varepsilon(\alpha x) \in S \cap A_\psi,$$
$$N' = (\sin\phi)\,\mathrm{e} \oplus (\cos\phi)r_\varepsilon(\beta y) \in S \cap A_\phi$$

(see Figure 8 (b)). Then $M, N' \in S \cap A_\phi$ and $N, M' \in S \cap A_\psi$. Again by straightforward calculation, $\square MM'NN'$ forms an isosceles trapezoid whose legs are MM' and $N'N$, where

- $\|M - N'\| = (\cos\phi)\|r_\varepsilon(\alpha x) - r_\varepsilon(\beta y)\| \le (\cos\phi)k_\varepsilon\|\alpha x - \beta y\|$;
- $\|N - M'\| = (\cos\psi)\|r_\varepsilon(\beta y) - r_\varepsilon(\alpha x)\| \le (\cos\psi)k_\varepsilon\|\alpha x - \beta y\|$;

- $\|M-M'\|^2 = \|N-N'\|^2 = (\cos\phi - \cos\psi)^2 + (\sin\phi - \sin\psi)^2 = 2 - 2\cos(\psi-\phi) = 2 - 2\cos\delta$.

Applying Lemma 3.1 to $\square MM'NN'$, we have

$$\begin{aligned}\|\varrho_{\varepsilon,t}(P) - \varrho_{\varepsilon,t}(Q)\|^2 &= \|M-N\|^2 \\ &= \|M-N'\|\|N-M'\| + \|M-M'\|^2 \\ &\leq k_\varepsilon^2 \cos\phi\cos\psi\|\alpha x - \beta y\|^2 + 2 - 2\cos\delta \\ &= k_\varepsilon^2\cos\phi\cos\psi(\alpha^2+\beta^2-2\alpha\beta\cos\xi) + 2 - 2\cos\delta \\ &= k_\varepsilon^2(\alpha-\beta)^2\cos\phi\cos\psi + 2k_\varepsilon^2(1-\cos\xi)\alpha\beta\cos\phi\cos\psi + 2 - 2\cos\delta.\end{aligned}$$

Combine this equation and $(\star)$ to obtain

$$\|\varrho_{\varepsilon,t}(P) - \varrho_{\varepsilon,t}(Q)\|^2 \leq k_\varepsilon^2\|P-P'\|^2 + 2k_\varepsilon^2\,(1-\cos\xi)\,\alpha\beta\cos\phi\cos\psi. \tag{3.4}$$

Since $\alpha,\beta\in[0,1]$, $0<\phi<\psi$ and the map $z\mapsto \sin z/z$ is decreasing on $(0,\pi/2)$, it follows that

$$\begin{aligned}\alpha\beta\cos\phi\cos\psi &= \frac{\alpha\phi}{\phi}\cdot\frac{\beta\psi}{\psi}\cdot\cos\phi\cos\psi \\ &\leq \frac{\sin\alpha\phi}{\sin\phi}\cdot\frac{\sin\beta\psi}{\sin\psi}\cdot\cos\phi\cos\psi = \cot\phi\cot\psi\sin\alpha\phi\sin\beta\psi.\end{aligned}$$

Therefore, according to equations (3.3) and (3.4), it can be concluded that

$$\begin{aligned}&\|\varrho_{\varepsilon,t}(P) - \varrho_{\varepsilon,t}(Q)\|^2 \\ &\quad\leq k_\varepsilon^2\|P-P'\| + 2k_\varepsilon^2(1-\cos\xi)\cot\phi\cot\psi\sin\alpha\phi\sin\beta\psi = k_\varepsilon^2\|P-Q\|^2.\end{aligned}$$

That is, $\rho_{\varepsilon,t}|_{U_t^+}$ is k_ε-lipschitzian and uniformly continuous. Hence, its extension on $\overline{U_t^+} = \bigcup_{0\leq i\leq 1} A_i$ must be k_ε-lipschitzian.

Now, let us observe that, for each $i\in[t,0]$, the map $\varrho_{\varepsilon,t}|_{A_i}$ is indeed the reflection through E of the map $\varrho_{\varepsilon,t}|_{A_{-i}}$ where $-i\in[0,|t|]\subseteq[0,1]$. Hence, according to the above conclusion, the map $\varrho_{\varepsilon,t}|_{\bigcup_{t\leq i\leq 0} A_i}$ must be k_ε-lipschitzian, also.

Finally, since $\varrho_{\varepsilon,t}|_{C_t\cup L_t}$ are already k_ε-lipschitzian and

$$\bigg(\bigcup_{0\leq i\leq 1} A_i\bigg)\cup\bigg(\bigcup_{t\leq i\leq 0} A_i\bigg)\cup C_t\cup L_t = B$$

is convex, it follows from Proposition 2.3 that $\varrho_{\varepsilon,t}|_B = \varrho_{\varepsilon,t}$ is k_ε-lipschitzian. Thus its restriction on U_t must be k_ε-lipschitzian as desired. □

Now, it remains to show $(\star)$, which states as follows.

$$(\alpha-\beta)^2\cos\phi\cos\psi + \frac{1}{k_\varepsilon^2}(2-2\cos\delta) \leq \|P-P'\|^2$$

where $P = P_{\alpha\phi,x}$ and $P' = P'_{\beta\psi,x}$ for some $\phi,\psi\in(0,\pi/2)$ with $\delta := \psi-\phi > 0$, $\alpha,\beta\in[0,1]$ and $x\in S\cap E$.

In order to prove ($\star$), let us recall that $\operatorname{span}\{\mathrm{e}, x\}$ is isometrically isomorphic to $\mathbb{R}^2$. Then, without loss of generality, let $\mathrm{e} = (0,1)$ and $x = (1,0)$. According to this representation,

$$P = P_{\alpha\phi} = (\cot\phi \sin\alpha\phi, \csc\phi - \cot\phi\cos\alpha\phi),$$
$$P' = P'_{\beta\psi} = (\cot\psi \sin\beta\psi, \csc\psi - \cot\psi\cos\beta\psi).$$

This form is more convenient for calculation.

PROOF OF ($\star$). Let $\tau = \beta - \alpha \in [-1,1]$. According to [8] and [2], $k_0 \in (4.5, 29)$. Define $\Delta_{\phi,\psi,\tau,\beta} \colon [4.5, 29] \to \mathbb{R}$ as follows.

$$\Delta_{\phi,\psi,\tau,\beta}(x) = \|P_{\alpha\phi} - P'_{\beta\psi}\|^2 - \frac{1}{x^2}(2 - 2\cos(\psi-\phi)) - \tau^2\cos\phi\cos\psi.$$

Since

$$\frac{d}{dx}\Delta_{\phi,\psi,\tau,\beta}(x) = \frac{8}{x^3}\sin^2\left(\frac{\delta}{2}\right) \geq 0 \quad \text{for all } x \in [4.5, 29],$$

$\Delta_{\phi,\psi,\tau,\beta}$ is non-decreasing. Thus

$$\min_{x\in[4.5,29]} \Delta_{\phi,\psi,\tau,\beta}(x) = \Delta_{\phi,\psi,\tau,\beta}(4.5).$$

It means that ($\star$) will follow if $\Delta_{\phi,\psi,\tau,\beta}(4.5) \geq 0$. Therefore, our aim is now proving the non-negativeness of $\Delta_{\phi,\psi,\tau,\beta}(4.5)$.

Let us restate $\Delta_{\phi,\psi,\tau,\beta}(4.5)$.

$$\begin{aligned}\Delta_{\phi,\psi,\tau,\beta}(4.5) &= \|P_{\alpha\phi} - P'_{\beta\psi}\|^2 - \frac{1}{4.5^2}(2 - 2\cos(\psi-\phi)) - \tau^2\cos\phi\cos\psi \\ &= (\csc\phi - \csc\psi)^2 + \cot^2\phi + \cot^2\psi \\ &\quad - 2\cot\phi\cot\psi\cos(\beta(\psi-\phi)+\tau\phi) \\ &\quad + 2(\csc\phi - \csc\psi)(\cot\psi\cos\beta\psi - \cot\phi\cos(\beta\phi - \tau\phi)) \\ &\quad - \frac{8}{81}(1-\cos(\psi-\phi)) - \tau^2\cos\phi\cos\psi\end{aligned}$$

where $\phi, \psi \in (0, \pi/2)$ with $\phi < \psi$, $\tau \in [-1,1]$ and $\beta \in [0,1]$.

For convenience, written by $\Delta_{\phi,\psi,\tau,\beta}$ the value $\Delta_{\phi,\psi,\tau,\beta}(4.5)$. Fix τ and β, and consider the following three cases:

(1) $0 < \phi < \psi \leq \pi/4$.
(2) $\pi/4 \leq \phi < \psi < \pi/2$.
(3) $0 < \phi < \pi/4 < \psi < \pi/2$.

Observe that if ($\star$) holds for the first two cases, then so does for the last case. This is because if $0 < \phi < \pi/4 < \psi < \pi/2$, there must be a point $P_0 \in A_{\pi/4} \cap [P, P']$, where $[P, P']$ is the segment joining P and P'; hence, ($\star$) follows from previous cases and Proposition 2.3. Furthermore, it is found by numerical optimization that

$$\min_{\substack{0<\phi<\psi\leq\pi/4\\ -1\leq\tau\leq1\\ 0\leq\beta\leq1}} \frac{\Delta_{\phi,\psi,\tau,\beta}}{\Delta_{\pi/2-\psi,\pi/2-\phi,\tau,\beta}} \approx 0.21016 > 0$$

and that

$$\max_{\substack{0<\phi<\psi\leq\pi/4\\ -1\leq\tau\leq1\\ 0\leq\beta\leq1}} \frac{\Delta_{\phi,\psi,\tau,\beta}}{\Delta_{\pi/2-\psi,\pi/2-\phi,\tau,\beta}} \approx 1.00000 < \infty.$$

Thus, by letting $\operatorname{sgn}\colon \mathbb{R} \to \{-1,0,1\}$ be the sign function, the above relations conclude

$$\operatorname{sgn}\Delta_{\phi,\psi,\tau,\beta} = \operatorname{sgn}\Delta_{\pi/2-\psi,\pi/2-\phi,\tau,\beta} \tag{3.5}$$

for all $\phi,\psi \in (0,\pi/4]$ with $\phi < \psi$, $\tau \in [-1,1]$ and $\beta \in [0,1]$.

Let $G_{\phi,\psi,\tau,\beta}$ be the polynomial

$$G_{\phi,\psi,\tau,\beta} = \frac{77}{81}(\phi^2+\psi^2) + \phi\psi\bigg(\frac{8}{81} - \tau^2 - 2\cos\bigg(\frac{\pi\tau}{2}\bigg)\bigg).$$

Again, by numerical optimization, it is found that

$$\min_{\substack{0<\phi<\psi\leq\pi/4\\ -1\leq\tau\leq1\\ 0\leq\beta\leq1}} \frac{\Delta_{\phi,\psi,\tau,\beta}}{G_{\phi,\psi,\tau,\beta}} \approx 0.12890 > 0.$$

This implies

$$\operatorname{sgn}\Delta_{\phi,\psi,\tau,\beta} = \operatorname{sgn}G_{\phi,\psi,\tau,\beta} \tag{3.6}$$

for all $\phi,\psi \in (0,\pi/4]$ with $\phi < \psi$, $\tau \in [-1,1]$ and $\beta \in [0,1]$. Therefore, by combining (3.5) and (3.6), it suffices to show $G_{\phi,\psi,\tau,\beta} \geq 0$.

Recall that the map

$$x \mapsto \begin{cases} 1 & \text{if } x = 0; \\ \dfrac{\sin x}{x} & \text{if } x \in (0,1] \end{cases}$$

is decreasing on $[0,1]$. Then

$$\frac{\sin(\pi\tau/2)}{\pi\tau/2} = \frac{\sin(\pi|\tau|/2)}{\pi|\tau|/2} \geq \frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi} > \frac{4}{\pi^2} \geq \frac{2|\tau|}{\pi(\pi|\tau|/2)} = \frac{2\tau}{\pi(\pi\tau/2)}$$

for all $\tau \in [-1,1]$. This implies that

$$\frac{\partial}{\partial\tau}G_{\phi,\psi,\tau,\beta} = \phi\psi\bigg(\pi\sin\bigg(\frac{\pi\tau}{2}\bigg) - 2\tau\bigg) \geq 0$$

for all $\tau \in [0,1]$, and that

$$\frac{\partial}{\partial\tau}G_{\phi,\psi,\tau,\beta} = \phi\psi\bigg(\pi\sin\bigg(\frac{\pi\tau}{2}\bigg) - 2\tau\bigg) \leq 0$$

for all $\tau \in [-1,0]$. With respect to τ, $G_{\phi,\psi,\tau,\beta}$ is increasing on $[0,1]$ but decreasing on $[-1,0]$. Thus $G_{\phi,\psi,\tau,\beta}$ attains its minimum at $\tau = 0$. That is,

$$G_{\phi,\psi,\tau,\beta} \geq G_{\phi,\psi,0,\beta} = \frac{77}{81}\left(\phi^2 + \psi^2 - 2\phi\psi\right) = \left(\frac{\sqrt{77}}{9}(\phi - \psi)\right)^2 \geq 0$$

for all $\phi, \psi \in (0, \pi/4]$ with $\phi < \psi$, $\tau \in [-1,1]$ and $\beta \in [0,1]$. □

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Manuscript received July 23, 2017

accepted June 5, 2018

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