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GLOBAL EXISTENCE OF A DIFFUSION LIMIT WITH DAMPING FOR THE COMPRESSIBLE RADIATIVE EULER SYSTEM COUPLED TO AN ELECTROMAGNETIC FIELD

Xavier Blanc — Bernard Ducomet — Šárka Nečasová

Dedicated to the memory of Professor Marek Burnat

Abstract. We study the Cauchy problem for a system of equations corresponding to a singular limit of radiative hydrodynamics, namely the 3D radiative compressible Euler system coupled to an electromagnetic field through the MHD approximation. Assuming the presence of damping together with suitable smallness hypotheses for the data, we prove that this problem admits a unique global smooth solution.

1. Introduction

In [4], following the study of Buet and Després [5] we considered a singular limit for a compressible inviscid radiative flow. The motion of the fluid was governed by the Euler system with damping for the evolution of the density $\varrho = \varrho(t,x)$, the velocity field $\overrightarrow{u} = \overrightarrow{u}(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \mathbb{R}^3$. A damping term was added to the momentum equation. We proved that the

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associated Cauchy problem admits a unique global smooth solution, provided that the data are small enough.

In the present work we study the coupling of the previous model with an electromagnetic field through the so-called magnetohydrodynamic approximation (MHD) [3].

Let us recall briefly that Maxwell's electromagnetic theory relies on the Ampère-Maxwell equation and Faraday's law. The first one reads

$$\partial_t \overrightarrow{D} + \overrightarrow{J} = \mathrm{curl}_x \overrightarrow{H} \tag{1.1}$$

where $\overrightarrow{D} = \epsilon \overrightarrow{E}$ is the electric induction and $\overrightarrow{H}$ is the magnetic field. The second one reads

$$\partial_t \overrightarrow{B} + \mathrm{curl}_x \overrightarrow{E} = 0, \tag{1.2}$$

where $\overrightarrow{B} = \mu \overrightarrow{H}$ is the magnetic induction. Here, the constant $\mu > 0$ stands for the permeability of free space.

The two last laws are Coulomb's law

$$\mathrm{div}_x \overrightarrow{D} = q, \tag{1.3}$$

where q is the electric charge density, and Gauss's law

$$\mathrm{div}_x \overrightarrow{B} = 0. \tag{1.4}$$

We assume that the electric current density $\overrightarrow{J}$ is related to the electric field $\overrightarrow{E}$ and the macroscopic fluid velocity $\overrightarrow{u}$ via Ohm's law

$$\overrightarrow{J} = \sigma(\overrightarrow{E} + \overrightarrow{u} \times \overrightarrow{B}), \tag{1.5}$$

where σ is the electrical conductivity of the fluid.

The magnetic force acting on the fluid (Lorentz's force) $\overrightarrow{f}_m$ and the magnetic energy supply E_m are given by

$$\overrightarrow{f}_m = \overrightarrow{J} \times \overrightarrow{B}, \qquad E_m := \overrightarrow{J} \cdot \overrightarrow{E}. \tag{1.6}$$

The MHD approximation consists in neglecting the displacement current $\partial_t \overrightarrow{D}$ (for the electric induction given by $\overrightarrow{D} = \epsilon \overrightarrow{E}$) in Ampère–Maxwell equation (1.1) and supposing that the charge q is negligible, so we obtain

$$\mu \overrightarrow{J} = \mathrm{curl}_x \overrightarrow{B}, \quad \mu > 0, \tag{1.7}$$

where, as mentioned above, the constant μ is the permeability of free space.

Moreover, using (1.5), equation (1.2) can be written [6] in the form

$$\partial_t \overrightarrow{B} + \mathrm{curl}_x(\overrightarrow{B} \times \overrightarrow{u}) + \mathrm{curl}_x(\lambda\, \mathrm{curl}_x \overrightarrow{B}) = 0, \tag{1.8}$$

where $\lambda = (\mu\sigma)^{-1}$ is the magnetic diffusivity of the fluid.

Finally, from Faraday's law we get

$$\partial_t \left(\frac{1}{2\mu} \, |\overrightarrow{B}|^2 \right) + \overrightarrow{J} \cdot \overrightarrow{E} = \mathrm{div}_x \left(\frac{1}{\mu} \, \overrightarrow{B} \times \overrightarrow{E} \right). \tag{1.9}$$

Concerning radiation, we consider the non equilibrium diffusion regime where radiation appears through an extra equation of parabolic type for the radiative temperature which is *a priori* different from the fluid temperature.

More specifically the system of equations to be studied for the five unknowns $(\varrho, \overrightarrow{u}, \vartheta, E_r, \overrightarrow{B})$ reads

$$\partial_t \varrho + \mathrm{div}_x(\varrho \overrightarrow{u}) = 0, \tag{1.10}$$

$$\partial_t(\varrho \overrightarrow{u}) + \mathrm{div}_x(\varrho \overrightarrow{u} \otimes \overrightarrow{u}) + \nabla_x(p + p_r) + \frac{1}{\mu}\overrightarrow{B} \times \mathrm{curl}_x \overrightarrow{B} + \nu \overrightarrow{u} = 0, \tag{1.11}$$

$$\begin{aligned} \partial_t(\varrho E) + \mathrm{div}_x((\varrho E + p)\overrightarrow{u}) + \overrightarrow{u} \cdot \nabla_x p_r + \frac{1}{\mu}\big(\overrightarrow{u} \times \overrightarrow{B}\big) \cdot \mathrm{curl}_x \overrightarrow{B} \\ = \mathrm{div}_x(\kappa \nabla_x \vartheta) - \sigma_a(a\vartheta^4 - E_r) + \frac{\lambda}{\mu}\left|\mathrm{curl}_x \overrightarrow{B}\right|^2, \end{aligned} \tag{1.12}$$

$$\partial_t E_r + \mathrm{div}_x(E_r \overrightarrow{u}) + p_r \mathrm{div}_x \overrightarrow{u} = \mathrm{div}_x\left(\frac{1}{3\sigma_s}\ \nabla_x E_r\right) - \sigma_a(E_r - a\vartheta^4), \tag{1.13}$$

$$\partial_t \overrightarrow{B} + \mathrm{curl}_x\big(\overrightarrow{B} \times \overrightarrow{u}\big) + \mathrm{curl}_x\big(\lambda\, \mathrm{curl}_x\, \overrightarrow{B}\big) = 0, \tag{1.14}$$

where $\overrightarrow{B}$ is a divergence-free vector field, $E = |\overrightarrow{u}|^2/2 + e(\varrho, \vartheta)$, E_r is the radiative energy related to the radiation temperature T_r by $E_r = aT_r^4$ and p_r is the radiative pressure given by $p_r = aT_r^4/3 = E_r/3$, with $a > 0$. We have also supposed for simplicity that μ, σ_a, σ_s, σ and a are positive constants. This implies in particular that

$$\mathrm{curl}_x\left(\frac{1}{\sigma}\ \mathrm{curl}_x\left(\frac{1}{\mu}\ \overrightarrow{B}\right)\right) = -\frac{1}{\sigma\mu}\,\Delta \overrightarrow{B}.$$

Extending the analysis of [4] and using stability arguments introduced by Beauchard and Zuazua in [1], our goal is to prove the global existence of solutions for the system (1.10)–(1.14) when data are sufficiently close to an equilibrium state.

The plan of the paper is as follows: in Section 2 we state our main result (Theorem 2.1) then, in Section 3, we study the MHD model and prove Theorem 2.1.

2. Main result

Hypotheses imposed on the constitutive relations are motivated by the general existence theory for the Euler–Fourier system developed in [20], [21]. Hypotheses on the transport coefficients are physically relevant for the radiative part [17], [19]. We impose that the pressure $p(\varrho, \vartheta) > 0$, the internal energy $e(\varrho, \vartheta) > 0$ and the specific entropy $s(\varrho, \vartheta)$ are smooth functions of their arguments. Moreover, we impose the following monotony assumptions:

$$\frac{\partial p}{\partial \varrho}(\varrho, \vartheta) > 0, \qquad \frac{\partial e}{\partial \vartheta}\,(\varrho, \vartheta) > 0, \tag{2.1}$$

for all $\vartheta > 0$ and all $\varrho > 0$.

In our simplified setting, the transport coefficients κ, σ_a, σ_s and the Planck's coefficient a are supposed to be fixed positive numbers. Finally the damping term with coefficient $\nu > 0$ of the Darcy type can be interpreted here as a diffusion of a light gas into a heavy one.

We are going to prove that, under the above structural assumptions on the equation of state, system (1.10)–(1.14) has a global smooth solution close to any equilibrium state.

THEOREM 2.1. *Let $\big(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\overrightarrow{B}}\big)$ be a constant state with $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$ and $\overline{E_r} = a\overline{\vartheta}^4 > 0$. Consider $d > 7/2$. There exists $\varepsilon > 0$ such that, for any initial state $\big(\varrho_0, \overrightarrow{u}_0, \vartheta_0, E_r^0, \overrightarrow{B}_0\big)$ satisfying*

$$\big\|\big(\varrho_0, \overrightarrow{u}_0, \vartheta_0, E_r^0, \overrightarrow{B}_0\big) - \big(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E_r}, \overline{\overrightarrow{B}}\big)\big\|_{H^d(\mathbb{R}^3)} \le \varepsilon, \tag{2.2}$$

there exists a unique global solution $\big(\varrho, \overrightarrow{u}, \vartheta, E_r, \overrightarrow{B}\big)$ to (1.10)–(1.14), such that

$$\big(\varrho - \overline{\varrho}, \overrightarrow{u}, \vartheta - \overline{\vartheta}, E_r - \overline{E_r}, \overrightarrow{B} - \overline{\overrightarrow{B}}\big) \in C\big([0, +\infty); H^d(\mathbb{R}^3)\big) \cap C^1\big([0, +\infty); H^{d-1}(\mathbb{R}^3)\big).$$

In addition, this solution satisfies the following energy inequality:

$$\begin{aligned} &\big\|(\varrho(t) - \overline{\varrho}, \overrightarrow{u}(t), \vartheta(t) - \overline{\vartheta}, E_r(t) - \overline{E_r}, \overrightarrow{B}(t) - \overline{\overrightarrow{B}})\big\|_{H^d(\mathbb{R}^3)} \\ &\quad + \int_0^t \big\|\nabla_x\big(\varrho, \overrightarrow{u}, \vartheta, E_r, \overrightarrow{B}\big)(s)\big\|^2_{H^{d-1}(\mathbb{R}^3)}\,ds \\ &\quad + \int_0^t \big(\big\|\nabla_x \vartheta(s)\big\|^2_{H^d(\mathbb{R}^3)} + \|\nabla_x E_r(s)t\|^2_{H^d(\mathbb{R}^3)} + \big\|\nabla_x \overrightarrow{B}(s)\big\|^2_{H^d(\mathbb{R}^3)}\big)\,ds \\ &\le C\big\|\big(\varrho_0 - \overline{\varrho}, \overrightarrow{u}_0, \vartheta_0 - \overline{\vartheta}, E_r^0 - \overline{E_r}, \overrightarrow{B}_0 - \overline{\overrightarrow{B}}\big)\big\|^2_{H^d(\mathbb{R}^3)}, \end{aligned} \tag{2.3}$$

for some constant $C > 0$ which does not depend on t.

3. The Euler-MHD system

3.1. The linearized Euler-MHD system. Multiplying (1.11) by $\overrightarrow{u}$ and using (1.10) we get

$$\partial_t\bigg(\frac{1}{2}\,\varrho|\overrightarrow{u}|^2\bigg) + \operatorname{div}_x\bigg(\frac{1}{2}\,\varrho|\overrightarrow{u}|^2\,\overrightarrow{u}\bigg) + \nabla_x(p + p_r)\cdot\overrightarrow{u} + \nu|\overrightarrow{u}|^2 = \overrightarrow{f}_m \cdot \overrightarrow{u}.$$

Subtracting this relation to (1.12), using the definition $C_v = \partial_\vartheta e$ and the thermodynamical identity $\partial_\varrho e = (p - \vartheta\partial_\vartheta p)/\varrho^2$ (Maxwell's relation), equation (1.12) can be replaced by the equation for temperature

$$\begin{aligned} &\varrho C_v\big(\partial_t\vartheta + \overrightarrow{u}\cdot\nabla_x\vartheta\big) + \vartheta p_\vartheta \operatorname{div}_x\overrightarrow{u} - \nu\overrightarrow{u}^2 \\ &\qquad = \operatorname{div}_x(\kappa\nabla_x\vartheta) - \sigma_a(a\vartheta^4 - E_r) + E_m - \overrightarrow{f}_m\cdot\overrightarrow{u}. \end{aligned} \tag{3.1}$$

Linearizing the system (1.10), (1.11), (3.1), (1.13), (1.14) around the constant state $\big(\overline{\varrho}, 0, \overline{\vartheta}, \overline{E}_r, \overrightarrow{\overline{B}}\big)$ with the compatibility condition $\overline{E}_r = a\overline{\vartheta}^4$ and setting

$$\varrho = r + \overline{\varrho}, \qquad \vartheta = T + \overline{\vartheta}, \qquad E_r = e_r + \overline{E}_r \quad \text{and} \quad \overrightarrow{B} = \overrightarrow{b} + \overrightarrow{\overline{B}}$$

we get

$$\partial_t r + \overline{\varrho}\, \mathrm{div}_x \overrightarrow{u} = 0, \tag{3.2}$$

$$\partial_t \overrightarrow{u} + \frac{\overline{p}_\varrho}{\overline{\varrho}} \nabla_x r + \frac{\overline{p}_\vartheta}{\overline{\varrho}} \nabla_x T + \frac{1}{3\overline{\varrho}} \nabla_x e_r + \frac{1}{\mu} \overrightarrow{\overline{B}} \times \big(\mathrm{curl}_x \overrightarrow{b}\big) + \nu \overrightarrow{u} = 0, \tag{3.3}$$

$$\partial_t T + \frac{\overline{\vartheta}\,\overline{p}_\vartheta}{\overline{\varrho}\,\overline{C}_v} \mathrm{div}_x \overrightarrow{u} = \mathrm{div}_x \left(\frac{\kappa}{\overline{\varrho}\,\overline{C}_v} \nabla_x T \right) - \frac{\sigma_a}{\overline{\varrho}\,\overline{C}_v} \big(4a\overline{\vartheta}^3 T - e_r\big), \tag{3.4}$$

$$\partial_t e_r + \frac{4}{3} \overline{E}_r \,\mathrm{div}_x \overrightarrow{u} = \mathrm{div}_x \left(\frac{1}{3\sigma_s} \nabla_x e_r \right) - \sigma_a \big(e_r - 4a\overline{\vartheta}^3 T\big), \tag{3.5}$$

$$\partial_t \overrightarrow{b} + \overrightarrow{\overline{B}}\, \mathrm{div}_x \overrightarrow{u} - \big(\overrightarrow{\overline{B}} \cdot \nabla_x\big) \overrightarrow{u} = \lambda\, \Delta \overrightarrow{b}. \tag{3.6}$$

Using the vector notation

$$U := \begin{pmatrix} r \\ u_1 \\ u_2 \\ u_3 \\ T \\ e_r \\ b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

the linearized system (3.2)–(3.6) maybe rewritten as

$$\partial_t U + \sum_{j=1}^{3} \mathcal{A}_j \partial_j U = \mathcal{D} \Delta U - \mathcal{B} U, \tag{3.7}$$

with

$$\mathcal{A}_1 = \begin{pmatrix} 0 & \overline{\varrho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha' & 0 & 0 & 0 & \beta' & \beta'' & 0 & \overline{B}_2/\mu & \overline{B}_3/\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_1/\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_1/\mu \\ 0 & \gamma' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_2 & -\overline{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_3 & 0 & -\overline{B}_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_2 = \begin{pmatrix} 0 & 0 & \overline{\varrho} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_2/\mu & 0 & 0 \\ \alpha' & 0 & 0 & 0 & \beta' & \beta'' & \overline{B}_1/\mu & 0 & \overline{B}_3/\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_2/\mu \\ 0 & 0 & \gamma' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma'' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\overline{B}_2 & \overline{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{B}_3 & -\overline{B}_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 & \overline{\varrho} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_3/\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_3/\mu & 0 \\ \alpha' & 0 & 0 & 0 & \beta' & \beta'' & \overline{B}_1/\mu & \overline{B}_2/\mu & 0 \\ 0 & 0 & 0 & \gamma' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma'' & 0 & 0 & 0 & 0 & 0 \\ 0 & -\overline{B}_3 & 0 & \overline{B}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\overline{B}_3 & \overline{B}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\mathcal{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \delta'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$\mathcal{B} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & -\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\pi & \sigma_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\alpha' = \frac{\overline{p}_\varrho}{\overline{\varrho}}, \qquad \beta' = \frac{\overline{p}_\vartheta}{\overline{\varrho}}, \qquad \beta'' = \frac{1}{3\overline{\varrho}}, \qquad \gamma' = \frac{\overline{p}_\vartheta}{\overline{C}_v}, \qquad \delta' = \frac{\kappa}{\overline{\varrho}\,\overline{C}_v},$$

$$\gamma'' = \frac{4}{3}\,\overline{E}_r, \qquad \delta'' = \frac{1}{3\sigma_s}, \qquad \zeta = \frac{4a\sigma_a\overline{\vartheta}^3}{\overline{\varrho}\overline{C}_v}, \qquad \eta = \frac{\sigma_a}{\overline{\varrho}\,\overline{C}_v}, \qquad \pi = 4a\sigma_a\overline{\vartheta}^3.$$

In order to apply the Kreiss theory we have to put the system (3.7) in a symmetric form [2]. For that purpose it is sufficient to consider a diagonal symmetrizer

$$\widetilde{\mathcal{A}}_0 = \begin{pmatrix} \mu\alpha'/\overline{\varrho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu\beta'/\gamma' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu\beta''/\gamma'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

Multiplying the first equation (3.7) by $\widetilde{\mathcal{A}}_0$ on the left, we get

$$\widetilde{\mathcal{A}}_0\,\partial_t U + \sum_{j=1}^{3} \widetilde{\mathcal{A}}_j\,\partial_j U = \widetilde{\mathcal{D}}\Delta U - \widetilde{\mathcal{B}}U, \tag{3.9}$$

where the matrices $\widetilde{\mathcal{A}}_j = \widetilde{\mathcal{A}}_0\mathcal{A}_j$ are symmetric, for all $j = 1, 2, 3$. More specifically,

$$\widetilde{\mathcal{A}}_1 = \begin{pmatrix} 0 & \mu\alpha' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu\alpha' & 0 & 0 & 0 & \mu\beta' & \mu\beta'' & 0 & \overline{B}_2 & \overline{B}_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_1 \\ 0 & \mu\beta' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu\beta'' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_2 & -\overline{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_3 & 0 & -\overline{B}_1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
\widetilde{\mathcal{A}}_2 = \begin{pmatrix}
0 & 0 & \mu\alpha' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_2 & 0 & 0 \\
\mu\alpha' & 0 & 0 & 0 & \mu\beta' & \mu\beta'' & \overline{B}_1 & 0 & \overline{B}_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_2 \\
0 & 0 & \mu\beta' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu\beta'' & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\overline{B}_2 & \overline{B}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \overline{B}_3 & -\overline{B}_2 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\widetilde{\mathcal{A}}_3 = \begin{pmatrix}
0 & 0 & 0 & \mu\alpha' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{B}_3 & 0 \\
\mu\alpha' & 0 & 0 & 0 & \mu\beta' & \mu\beta'' & \overline{B}_1 & \overline{B}_2 & 0 \\
0 & 0 & 0 & \mu\beta' & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu\beta'' & 0 & 0 & 0 & 0 & 0 \\
0 & -\overline{B}_3 & 0 & \overline{B}_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\overline{B}_3 & \overline{B}_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

The hyperbolic part of system (3.9) is now symmetric while its symmetric dissipative part is given by

$$
\widetilde{\mathcal{D}} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu\beta'\delta'/\gamma' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu\beta''\delta''/\gamma'' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}, \tag{3.10}
$$

$$\widetilde{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu\nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu\nu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu\beta'\zeta/\gamma' & -\mu\beta'\eta/\gamma' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mu\beta''\pi/\gamma'' & \mu\sigma_a\beta''/\gamma'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.11}$$

A simple computation proves that $\widetilde{\mathcal{B}}$ is a semi-definite positive matrix, that is,

$${}^tX\widetilde{\mathcal{B}}X \geq 0, \quad \text{for any vector } X \in \mathbb{R}^9.$$

Applying the Fourier transform in x to (3.9) we get

$$\widetilde{\mathcal{A}}_0\partial_t\widehat{U} + i\sum_{j=1}^{3}\xi_j\widetilde{\mathcal{A}}_j\widehat{U} = -|\xi|^2\widetilde{\mathcal{D}}\widehat{U} - \widetilde{\mathcal{B}}\widehat{U}, \tag{3.12}$$

or

$$\widetilde{\mathcal{A}}_0\partial_t\widehat{U} = E(\xi)\widehat{U}, \tag{3.13}$$

with $E(\xi) = -\mathcal{B}(\xi) - i\mathcal{A}(\xi)$, where

$$\mathcal{A}(\xi) = \sum_{j=1}^{3}\xi_j\widetilde{\mathcal{A}}_j = \tag{3.14}$$

$$\begin{pmatrix} 0 & \mu\alpha'\xi_1 & \mu\alpha'\xi_2 & \mu\alpha'\xi_3 & 0 & 0 & 0 & 0 & 0 \\ \mu\alpha'\xi_1 & 0 & 0 & 0 & \mu\beta'\xi_1 & \mu\beta''\xi_1 & -\overline{B}_2\xi_2-\overline{B}_3\xi_3 & \overline{B}_2\xi_1 & \overline{B}_3\xi_1 \\ \mu\alpha'\xi_2 & 0 & 0 & 0 & \mu\beta'\xi_2 & \mu\beta''\xi_2 & \overline{B}_1\xi_2 & -\overline{B}_1\xi_1-\overline{B}_3\xi_3 & \overline{B}_3\xi_2 \\ \mu\alpha'\xi_3 & 0 & 0 & 0 & \mu\beta'\xi_3 & \mu\beta''\xi_3 & \overline{B}_1\xi_3 & \overline{B}_2\xi_3 & -\overline{B}_1\xi_1-\overline{B}_2\xi_2 \\ 0 & \mu\beta'\xi_1 & \mu\beta'\xi_2 & \mu\beta'\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu\beta''\xi_1 & \mu\beta''\xi_2 & \mu\beta''\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\overline{B}_2\xi_2-\overline{B}_3\xi_3 & \overline{B}_1\xi_2 & \overline{B}_1\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_2\xi_1 & -\overline{B}_1\xi_1-\overline{B}_3\xi_3 & \overline{B}_2\xi_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_3\xi_1 & \overline{B}_3\xi_2 & -\overline{B}_1\xi_1-\overline{B}_2\xi_2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(3.15)\quad \mathcal{B}(\xi) := \widetilde{\mathcal{B}} + |\xi|^2\widetilde{\mathcal{D}} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\mu\beta'\zeta}{\gamma'} + \frac{\mu\beta'\delta'}{\gamma'}|\xi|^2 & -\frac{\mu\beta'\eta}{\gamma'} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\mu\beta''\pi}{\gamma''} & \frac{\mu\beta''\sigma_a}{\gamma''} + \frac{\mu\beta''\delta''}{\gamma''}|\xi|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda|\xi|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda|\xi|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda|\xi|^2 \end{pmatrix}.$$

Solving this equation with initial condition $\widehat{U}_0(\xi)$ we get

$$(3.16)\qquad \widehat{U}(t,\xi) = \exp\big[t\widetilde{\mathcal{A}}_0^{-1}E(\xi)\big]\widehat{U}_0(\xi).$$

In the strictly hyperbolic case $\widetilde{\mathcal{D}} = 0$, under the Kalman rank condition [12] for the pair $(\mathcal{A}(\xi), \mathcal{B})$, it can be proved [1] that

$$\exists\, C > 0,\ \lambda(\xi) > 0 : \exp\big[t\widetilde{\mathcal{A}}_0^{-1}E(\xi)\big] \leq Ce^{-\lambda(\xi)t}.$$

Due to the partially parabolic nature of the system, one can expect a similar result when $\widetilde{\mathcal{D}} \neq 0$ with a parabolic smoothing effect at low frequencies and an extra damping in the high frequency regime.

Taking benefit of the damping, we can use the Shizuta–Kawashima condition (SK) [22] which applies to the previous system. Following the arguments of Beauchard and Zuazua [1], we have

Lemma 3.1. *For any $\xi \in S^2$, a necessary and sufficient condition for matrices $\mathcal{B}(\xi)$ and $\mathcal{A}(\xi)$ defined by* (3.14) *and* (3.15) *to satisfy the Shizuta–Kawashima condition* (SK):

$$(3.17)\qquad \big\{\textit{eigenvectors of } \big(\widetilde{\mathcal{A}}_0\big)^{-1}\mathcal{A}(\xi)\big\} \cap \ker\mathcal{B}(\xi) = \{0\},$$

is that $\nu > 0$.

Proof. (1) We first consider the case $\nu \neq 0$. One checks that $\ker\mathcal{B}(\xi)$ is the 1-dimensional subspace spanned by the vector $(1,0,0,0,0,0,0,0,0)$. Therefore, if $X \in \ker\mathcal{B}(\xi) \setminus \{0\}$ is an eigenvector of $\big(\widetilde{\mathcal{A}}_0\big)^{-1}\mathcal{A}(\xi)$, we have $X = (x_1,0,0,0,0,0,0,0,0)$, $x_1 \neq 0$, and $\mathcal{A}(\xi)X = \lambda\widetilde{\mathcal{A}}_0X$, for some $\lambda \in \mathbb{R}$. This, together with the definition of $\widetilde{\mathcal{A}}_0$ and $\mathcal{A}(\xi)$, implies that $\lambda = 0$, $\xi_1 = \xi_2 = \xi_3 = 0$, which is in contradiction with the hypothesis $\xi \in S^2$.

(2) Next, we assume that $\nu = 0$. One checks that $\ker\mathcal{B}(\xi)$ is the 4-dimensional subspace spanned by the vectors $(x_1,x_2,x_3,x_4,0,0,0,0,0)$.

Let us denote by (λ, X) an eigenpair of $\mathcal{A}(\xi)$, with non zero eigenvector $X \in \ker \mathcal{B}(\xi)$. X satisfies the system

$$\begin{gathered}
\mu\alpha'\xi_1 x_2 + \mu\alpha'\xi_2 x_3 + \mu\alpha'\xi_3 x_4 = \lambda x_1, \\
\mu\alpha'\xi_1 x_1 = \lambda x_2, \qquad \mu\alpha'\xi_2 x_1 = \lambda x_3, \qquad \mu\alpha'\xi_3 x_1 = \lambda x_4, \\
\mu\beta'\xi_1 x_2 + \mu\beta'\xi_2 x_3 + \mu\beta'\xi_3 x_4 = 0, \\
\mu\beta''\xi_1 x_2 + \mu\beta''\xi_2 x_3 + \mu\beta''\xi_3 x_4 = 0, \\
-(\overline{B}_2\xi_2 + \overline{B}_3\xi_3)x_2 + \overline{B}_1\xi_2 x_3 + \overline{B}_1\xi_3 x_4 = 0, \\
\overline{B}_2\xi_1 x_2 - (\overline{B}_1\xi_1 + \overline{B}_3\xi_3)x_3 + \overline{B}_2\xi_3 x_4 = 0, \\
\overline{B}_3\xi_1 x_2 + \overline{B}_3\xi_2 x_3 - (\overline{B}_1\xi_1 + \overline{B}_2\xi_2)x_4 = 0.
\end{gathered}$$

Denoting $\overrightarrow{B} = (\overline{B}_1, \overline{B}_2, \overline{B}_3)$, $\overrightarrow{\xi} = (\xi_1, \xi_2, \xi_3)$, and $\overrightarrow{x} = (x_2, x_3, x_4)$, the system rewrites

$$\begin{gathered}
\mu\alpha' \, \overrightarrow{x} \cdot \overrightarrow{\xi} = \lambda x_1, \qquad \mu\alpha' x_1 \overrightarrow{\xi} = \lambda \overrightarrow{x}, \qquad \mu\beta' \, \overrightarrow{x} \cdot \overrightarrow{\xi} = 0, \qquad \mu\beta'' \, \overrightarrow{x} \cdot \overrightarrow{\xi} = 0, \\
-(\overrightarrow{B} \cdot \overrightarrow{\xi})\overrightarrow{x} + (\overrightarrow{\xi} \cdot \overrightarrow{x})\overrightarrow{B} = 0.
\end{gathered}$$

In particular, this implies $\overrightarrow{x} \cdot \overrightarrow{\xi} = 0$, which in turn implies $\lambda x_1 = 0$. As a consequence, we have

$$\lambda x_1 = 0, \qquad \overrightarrow{x} \cdot \overrightarrow{\xi} = 0, \qquad \mu\alpha' x_1 \overrightarrow{\xi} = \lambda \overrightarrow{x}, \qquad (\overrightarrow{\xi} \cdot \overrightarrow{B})\overrightarrow{x} = 0.$$

Choosing $\lambda = 0$, we see that any $\overrightarrow{x} \in \overrightarrow{\xi}^{\perp}$ for $\xi \in \overrightarrow{B}^{\perp}$ gives a nontrivial eigenpair (λ, X) with $\lambda = 0$ and $X = (0, \overrightarrow{x}, 0, 0, 0, 0, 0)$. Hence the SK condition is not satisfied. □

As in the equilibrium case, (3.17) is equivalent to the existence of a compensating matrix:

PROPOSITION 3.2. *For any* $\xi \in S^2$*, the matrices* $\widetilde{\mathcal{A}}_0$*,* $\mathcal{B}(\xi)$ *and* $\mathcal{A}(\xi)$ *being defined by* (3.8), (3.14) *and* (3.15)*, there exists a matrix-valued function*

$$K\colon S^2 \to \mathbb{R}^{6\times 6}, \qquad \omega \mapsto K(\omega) \tag{3.18}$$

such that

(a) $\omega \mapsto K(\omega)$ *is a* C^∞ *function, and satisfies* $K(-\omega) = -K(\omega)$ *for any* $\omega \in S^2$*.*
(b) $K(\omega)\tilde{\mathcal{A}}_0$ *is a skew-symmetric matrix for any* $\omega \in S^2$*.*
(c) *Denoting by* $[A] = (A + A^T)/2$ *the symmetric part of* A*, the matrix* $[K(\omega)\mathcal{A}(\omega)] + \mathcal{B}(\omega)$ *is symmetric positive definite for any* $\omega \in S^2$*.*

3.2. Entropy properties. Adding equations (1.9), (1.12) and (1.13) we get

$$
(3.19)\quad \partial_t\left(\frac{1}{2}\varrho\big|\overrightarrow{u}\big|^2+\varrho e+E_r+\frac{1}{2\mu}\big|\overrightarrow{B}\big|^2\right)
+\operatorname{div}_x\left((\varrho E+E_r)\overrightarrow{u}+(p+p_r)\overrightarrow{u}+\frac{1}{\mu}\,\overrightarrow{E}\times\overrightarrow{B}\right)
=\operatorname{div}_x\big(\kappa\nabla_x\vartheta\big)+\operatorname{div}_x\left(\frac{1}{3\sigma_s}\,\nabla_x E_r\right).
$$

We define $S_r := 4aT_r^3/3$ the radiative entropy. With these definitions, equation (1.13) rewrites

$$
\partial_t S_r+\operatorname{div}_x\big(S_r\overrightarrow{u}\big)=\frac{1}{T_r}\operatorname{div}_x\left(\frac{1}{3\sigma_s}\,\nabla_x E_r\right)-\sigma_a\frac{E_r-a\vartheta^4}{T_r},
$$

that is,

$$
(3.20)\quad \partial_t S_r+\operatorname{div}_x(S_r\overrightarrow{u})=\operatorname{div}_x\left(\frac{1}{3\sigma_s T_r}\,\nabla_x E_r\right)
+\frac{4a}{3\sigma_s}\,T_r|\nabla_x T_r|^2-\sigma_a\frac{E_r-a\vartheta^4}{T_r}.
$$

Replacing equation (1.12) by the internal energy equation

$$
(3.21)\quad \partial_t(\varrho e)+\operatorname{div}_x\big(\varrho e\overrightarrow{u}\big)+p\operatorname{div}_x\overrightarrow{u}-\nu\big|\overrightarrow{u}\big|^2
=\operatorname{div}_x(\kappa\nabla_x\vartheta)-\sigma_a(a\vartheta^4-E_r)+\frac{1}{\sigma\mu^2}\,|\operatorname{curl}_x\overrightarrow{B}|^2.
$$

The entropy s of the fluid is defined by the Gibbs law $\vartheta ds = de + pd(1/\varrho)$. Hence, dividing (3.21) by ϑ, we find

$$
(3.22)\quad \partial_t(\varrho s)+\operatorname{div}_x(\varrho s\overrightarrow{u})-\frac{\nu}{\vartheta}\big|\overrightarrow{u}\big|^2
=\operatorname{div}_x\left(\frac{\kappa\nabla_x\vartheta}{\vartheta}\right)+\frac{\kappa|\nabla_x\vartheta|^2}{\vartheta^2}-\sigma_a\frac{a\vartheta^4-E_r}{\vartheta}+\frac{1}{\sigma\mu^2\vartheta}\,\big|\operatorname{curl}_x\overrightarrow{B}\big|^2.
$$

So adding (3.22) and (3.20) we obtain

$$
(3.23)\quad \partial_t(\varrho s+S_r)+\operatorname{div}_x\big((\varrho s+S_r)\overrightarrow{u}\big)-\operatorname{div}_x\left(\frac{\kappa\nabla_x\vartheta}{\vartheta}+\frac{1}{3\sigma_s T_r}\,\nabla_x E_r\right)
=\frac{\kappa|\nabla_x\vartheta|^2}{\vartheta^2}+\frac{4a}{3\sigma_s}\,T_r|\nabla_x E_r|^2+\frac{a\sigma_a}{\vartheta T_r}(\vartheta-T_r)^2(\vartheta+T_r)(\vartheta^2+T_r^2)
+\frac{1}{\sigma\mu^2\vartheta}\,|\operatorname{curl}_x\overrightarrow{B}|^2+\frac{\nu}{\vartheta}\,|\overrightarrow{u}|^2.
$$

Introducing the Helmholtz functions $H_{\overline{\vartheta}}(\varrho,\vartheta) := \varrho(e-\overline{\vartheta}s)$ and $H_{r,\overline{\vartheta}}(T_r) := E_r-\overline{\vartheta}S_r$, we check that the quantities $H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$

and $H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r)$ are non-negative and strictly coercive functions reaching zero minima at the equilibrium state $(\overline{\varrho}, \overline{\vartheta}, \overline{E}_r)$.

Lemma 3.3. *Let $\overline{\varrho}$ and $\overline{\vartheta} = \overline{T}_r$ be given positive constants. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the sets defined by*

$$\mathcal{O}_1 := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 : \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta}, \right\}, \tag{3.24}$$

$$\mathcal{O}_2 := \left\{ T_r \in \mathbb{R} : \frac{\overline{T}_r}{2} < T_r < 2\overline{T}_r \right\}. \tag{3.25}$$

There exist positive constants $C_{1,2}(\overline{\varrho}, \overline{\vartheta})$ and $C_{3,4}(\overline{T}_r)$ such that

(a) *for all $(\varrho, \vartheta) \in \mathcal{O}_1$*

$$\begin{aligned} C_1(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \\ \leq C_2(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2), \end{aligned} \tag{3.26}$$

(b) *for all $T_r \in \mathcal{O}_2$*

$$C_3 |T_r - \overline{T}_r|^2 \leq H_{r,\overline{\vartheta}}(T_r) - H_{r,\overline{\vartheta}}(\overline{T}_r) \leq C_4 |T_r - \overline{T}_r|^2. \tag{3.27}$$

Proof. (a) is proved in [9] and we only sketch the proof for convenience. We have the decomposition

$$\varrho \mapsto H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) = \mathcal{F}(\varrho) + \mathcal{G}(\varrho),$$

where $\mathcal{F}(\varrho) = H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$ and $\mathcal{G}(\varrho) = H_{\overline{\vartheta}}(\varrho, \vartheta) - H_{\overline{\vartheta}}(\varrho, \overline{\vartheta})$. Using the Gibbs law $\vartheta ds = de + pd(1/\varrho)$, one easily proves that $\partial^2_\varrho H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) = (\varrho\overline{\vartheta}/(\varrho\overline{\vartheta}))\partial_\varrho p(\varrho, \overline{\vartheta})$, which is positive according to (2.1). Hence, $\mathcal{F}$ is strictly convex and reaches a zero minimum at $\overline{\varrho}$. Turning to $\mathcal{G}$, we have, still using Gibbs law, $\partial_\vartheta H_{\overline{\vartheta}}(\varrho, \overline{\vartheta}) = \varrho((\vartheta - \overline{\vartheta})/\vartheta)\partial_\vartheta e(\varrho, \overline{\vartheta})$. Thus, using (2.1) again, we infer that $\mathcal{G}$ is strictly decreasing for $\vartheta < \overline{\vartheta}$ and strictly increasing for $\vartheta > \overline{\vartheta}$. Computing the derivatives of $H_{\overline{\vartheta}}$ leads directly to the estimate (3.26).

(b) follows from the properties of the function

$$x \mapsto H_{r,\overline{\vartheta}}(x) - H_{r,\overline{\vartheta}}(\overline{T}_r) = ax^3 \left(x - \frac{4}{3}\overline{\vartheta} \right) + \frac{a}{3}\overline{\vartheta}^4 \qquad \square$$

From this simple result, we can obtain an identity leading to energy estimates. Multiplying (3.23) by $\overline{\vartheta}$, subtracting the result to (3.19) and using the conservation of mass, we get

$$\begin{aligned}(3.28)\quad &\partial_t\bigg(\frac{1}{2}\,\varrho|\overrightarrow{u}|^2+H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\\&\qquad-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})+H_{r,\overline{\vartheta}}(T_r)-H_{r,\overline{\vartheta}}(\overline{T}_r)+\frac{1}{2\mu}|\overrightarrow{B}|^2\bigg)\\&\qquad+\operatorname{div}_x\bigg((\varrho(E-\overline{e})+E_r)\overrightarrow{u}+(p+p_r)\overrightarrow{u}\\&\qquad-\overline{\vartheta}(\varrho(s-\overline{s})+S_r)\overrightarrow{u}+\frac{1}{\mu}\,\overrightarrow{E}\times\overrightarrow{B}\bigg)\\&=\operatorname{div}_x\bigg(\kappa\nabla_x\vartheta+\frac{1}{3\sigma_s}\,\nabla_xE_r\bigg)-\overline{\vartheta}\operatorname{div}_x\bigg(\frac{\kappa\nabla_x\vartheta}{\vartheta}+\frac{1}{3\sigma_sT_r}\,\nabla_xE_r\bigg)\\&\qquad-\overline{\vartheta}\,\frac{\kappa|\nabla_x\vartheta|^2}{\vartheta^2}-\overline{\vartheta}\,\frac{4a}{3\sigma_s}\,T_r|\nabla_xE_r|^2\\&\qquad-\overline{\vartheta}\,\frac{a\sigma_a}{\vartheta T_r}(\vartheta-T_r)^2(\vartheta+T_r)(\vartheta^2+T_r^2)-\frac{\nu}{\vartheta}|\overrightarrow{u}|^2-\frac{1}{\sigma\mu^2\vartheta}\big|\operatorname{curl}_x\overrightarrow{B}\big|^2.\end{aligned}$$

In the sequel, we define

$$V=\big(\rho,\overrightarrow{u},\vartheta,E_r,\overrightarrow{B}\big)^T,\qquad\overline{V}=\big(\overline{\rho},0,\overline{\vartheta},\overline{E_r},\overline{\overrightarrow{B}}\big)^T,$$

and

$$\begin{aligned}(3.29)\qquad N(t)^2=&\sup_{0\le s\le t}\|V(s)-\overline{V}\|^2_{H^d(\mathbb{R}^3)}\\&+\int_0^t\big(\|\nabla_xV(s)\|^2_{H^{d-1}(\mathbb{R}^3)}+\|\nabla_x\vartheta(s)\|^2_{H^d(\mathbb{R}^3)}\\&+\|\nabla_xE_r(s)\|^2_{H^d(\mathbb{R}^3)}+\big\|\nabla_x\overrightarrow{B}(s)\big\|^2_{H^d(\mathbb{R}^3)}\big)\,ds\\&+\int_0^t\big(\|\vartheta(s)-T_r(s)\|^2_{H^{d-1}(\mathbb{R}^3)}+\|\overrightarrow{u}(s)\|^2_{H^{d-1}(\mathbb{R}^3)}\big)\,ds.\end{aligned}$$

Recall that $T_r=E_r^{1/4}a^{-1/4}$. Note also that, since $\operatorname{div}_x\big(\overrightarrow{B}\big)=0$, we have

$$(3.30)\qquad\int_{\mathbb{R}^3}\big|\operatorname{curl}_x\overrightarrow{B}\big|^2=\int_{\mathbb{R}^3}\big|\nabla_x\overrightarrow{B}\big|^2,$$

as far as $\overrightarrow{B}\in H^1(\mathbb{R}^3)$, and similarly for any H^s norm. This allows, in the sequel, to replace $\operatorname{curl}_x\overrightarrow{B}$ by $\nabla_x\overrightarrow{B}$ in all bounds.

3.2.1. $L^\infty(H^d)$ estimates. Using the entropy properties, we are going to prove the following result:

Proposition 3.4. *Let the assumptions of Theorem* 2.1 *be satisfied. Consider a solution* $(\varrho,\overrightarrow{u},\vartheta,E_r)$ *of system* (1.10)–(1.14) *on* $[0,t]$, *for some* $t>0$. *Then,* *the energy defined by* (3.29) *satisfies*

$$
\begin{aligned}
(3.31)\quad \|V(t)-\overline{V}\|^2_{L^2(\mathbb{R}^3)} &+ \int_0^t \big(\|\nabla_x\vartheta(s)\|^2_{L^2(\mathbb{R})} + \|\nabla_x E_r(s)\|^2_{L^2(\mathbb{R}^3)} \\
&+ \|\vartheta(s)-T_r(s)\|^2_{L^2(\mathbb{R}^3)} + \|\overrightarrow{u}(s)\|^2_{L^2(\mathbb{R}^3)} + \|\nabla_x\overrightarrow{B}(s)\|^2_{L^2(\mathbb{R}^3)}\big)\,ds \\
&\le C(N(t))N(0)^2,
\end{aligned}
$$

where the function C is non-decreasing.

Proof. Following the proof of [13, Lemma 3.1] we define

$$
(3.32)\quad \eta(t,x)=H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}\big(\overline{\varrho},\overline{\vartheta}\big)-H_{\overline{\vartheta}}\big(\overline{\varrho},\overline{\vartheta}\big)+H_{r,\overline{\vartheta}}(T_r)-H_{r,\overline{\vartheta}}(\overline{T}_r).
$$

We multiply (3.23) by $\overline{\vartheta}$, and subtract the result from (3.19). Integrating over $[0,t]\times\mathbb{R}^3$, we find

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(\frac{1}{2}\varrho(t)\big|\overrightarrow{u}\big|^2(t)+\eta(t,x)+\frac{1}{2\mu}\big|\overrightarrow{B}\big|^2\Big)\,dx+\int_0^t\int_{\mathbb{R}^3}\kappa\,\frac{\overline{\vartheta}}{\vartheta^2}|\nabla_x\vartheta|^2+\frac{4a}{3\sigma_s}\,T_r|\nabla_x E_r|^2\overline{\vartheta} \\
&+\int_0^t\int_{\mathbb{R}^3}\overline{\vartheta}\,\frac{a\sigma_a}{\vartheta T_r}(\vartheta+T_r)(\vartheta-T_r)^2(\vartheta^2+T_r^2)+\frac{\overline{\vartheta}}{-}\vartheta\nu\big|\overrightarrow{u}\big|^2+\frac{\overline{\vartheta}}{\sigma\mu^2\vartheta}\,\big|\mathrm{curl}_x\overrightarrow{B}\big|^2 \\
&\qquad\le\int_{\mathbb{R}^3}\eta(0,x)\,dx+\int_{\mathbb{R}^3}\varrho_0\big|\overrightarrow{u}_0\big|^2+\frac{1}{2\mu}\int_{\mathbb{R}^3}\big|\overrightarrow{B}_0\big|^2.
\end{aligned}
$$

Defining

$$
\begin{aligned}
(3.33)\quad M(t)=\sup_{0\le s\le t}\sup_{x\in\mathbb{R}^3}\big[&\max(|\varrho(s,x)-\overline{\varrho}|,|\overrightarrow{u}(s,x)|, \\
&|\vartheta(s,x)-\overline{\vartheta}|,|E_r(s,x)-\overline{E_r}|,\big|\overrightarrow{B}-\overline{\overrightarrow{B}}\big|)\big],
\end{aligned}
$$

and applying Lemma 3.3, we find that

$$
\begin{aligned}
\|V(t)-\overline{V}\|^2_{L^2(\mathbb{R}^3)}&+\int_0^t\big(\|\nabla_x\vartheta(s)\|^2_{L^2(\mathbb{R}^3)}+\|\nabla_x E_r(s)\|^2_{L^2(\mathbb{R}^3)}+\|\vartheta(s)-T_r(s)\|^2_{L^2(\mathbb{R}^3)} \\
&+\|\overrightarrow{u}(s)\|^2_{L^2(\mathbb{R}^3)}+\|\mathrm{curl}_x\overrightarrow{B}(s)\|^2_{L^2(\mathbb{R}^3)}\big)\,ds\le C(M(t))N(0),
\end{aligned}
$$

where $C\colon\mathbb{R}^+\to\mathbb{R}^+$ is non-decreasing. Equation (3.30) allows to replace $\mathrm{curl}_x\overrightarrow{B}$ by $\nabla_x\overrightarrow{B}$ in the above estimate. Finally, we point out that, since $d>7/2>3/2$, due to Sobolev embeddings, there exists a universal constante C_0 such that $M(t)\le C_0N(t)$. Since C is non-decreasing, this proves (3.31). □

Proposition 3.5. *Setting $V=\big(\varrho,\overrightarrow{u},\vartheta,E_r,\overrightarrow{B}\big)^T$, under the same assumptions as in Theorem* 2.1, *we have the following estimate*:

$$
\begin{aligned}
(3.34)\quad \|\partial_tV(t)\|_{H^{d-1}(\mathbb{R}^3)}&\le C(N(t))\big(\|\nabla_xV\|_{H^{d-1}(\mathbb{R}^3)}+\|\nabla_x\vartheta\|_{H^d(\mathbb{R}^3)} \\
&+\|\nabla_xE_r\|_{H^d(\mathbb{R}^3)}+\|\vartheta-T_r\|_{H^{d-1}(\mathbb{R}^3)}+\|\overrightarrow{u}\|_{H^{d-1}(\mathbb{R}^3)}+\|\nabla_x\overrightarrow{B}\|_{H^d(\mathbb{R}^3)}\big).
\end{aligned}
$$

Proof. The system satisfied by V may be written formally as

$$\partial_t V + \sum_{j=1}^{3} \widehat{\mathcal{A}}_j(V)\partial_{x_j} V = \widehat{\mathcal{D}}(V)\Delta V - \widehat{\mathcal{B}}(V) = 0, \tag{3.35}$$

where

$$\widehat{\mathcal{A}}_1 = \begin{pmatrix} 0 & \varrho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \widehat{\alpha}' & 0 & 0 & 0 & \widehat{\beta}' & \widehat{\beta}'' & 0 & B_2/\varrho\mu & B_3/\varrho\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_1/\varrho\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_1/\varrho\mu \\ 0 & \widehat{\gamma}' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\gamma}'' & 0 & 0 & 0 & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & 0 & 0 & u_1 \end{pmatrix},$$

$$\widehat{\mathcal{A}}_2 = \begin{pmatrix} 0 & 0 & \varrho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -B_2/\varrho\mu & 0 & 0 \\ \widehat{\alpha}' & 0 & 0 & 0 & \widehat{\beta}' & \widehat{\beta}'' & B_1/\varrho\mu & 0 & B_3/\varrho\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_2/\varrho\mu \\ 0 & 0 & \widehat{\gamma}' & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widehat{\gamma}'' & 0 & 0 & u_2 & 0 & 0 & 0 \\ 0 & -B_2 & B_1 & 0 & 0 & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & B_3 & -B_2 & 0 & 0 & 0 & 0 & u_2 \end{pmatrix},$$

$$\widehat{\mathcal{A}}_3 = \begin{pmatrix} 0 & 0 & 0 & \varrho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -B_3/\varrho\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_3/\varrho\mu & 0 \\ \widehat{\alpha}' & 0 & 0 & 0 & \widehat{\beta}' & \widehat{\beta}'' & B_1/\varrho\mu & B_2/\varrho\mu & 0 \\ 0 & 0 & 0 & \widehat{\gamma}' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\gamma}'' & 0 & u_3 & 0 & 0 & 0 \\ 0 & -B_3 & 0 & B_1 & 0 & 0 & u_3 & 0 & 0 \\ 0 & 0 & -B_3 & B_2 & 0 & 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_3 \end{pmatrix},$$

and

$$\widehat{\mathcal{D}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \widehat{\delta}' & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\delta}'' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$\widehat{\mathcal{B}} := \begin{pmatrix} 0 \\ -\nu \frac{\overrightarrow{u}}{\varrho} \\ -\frac{\sigma_a}{\varrho C_v}(a\vartheta^4 - E_r) + \frac{1}{\varrho C_v}\frac{\lambda}{\mu}|\mathrm{curl}_x \overrightarrow{B}|^2 + \frac{\nu\left|\overrightarrow{u}\right|^2}{\varrho C_v} \\ \sigma_a(a\vartheta^4 - E_r) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$\widehat{\alpha}' = \frac{p_\varrho}{\varrho}, \qquad \widehat{\beta}' = \frac{p_\vartheta}{\varrho}, \qquad \widehat{\beta}'' = \frac{1}{3\varrho}, \qquad \widehat{\gamma}' = \frac{3\varrho p_\vartheta}{3\varrho C_v},$$
$$\widehat{\delta}' = \frac{\kappa}{\varrho C_v}, \qquad \widehat{\gamma}'' = \frac{4}{3}E_r, \qquad \widehat{\delta}'' = \frac{1}{3\sigma_s}.$$

It is possible to symmetrize this nonlinear system in the same spirit as what we have done for the linearized system (3.7). However, we do not need to do so here. So we write

$$\partial_t V = -\sum_{j=1}^{3}\big[\widehat{\mathcal{A}}_j(V) - \widehat{\mathcal{A}}_j(\overline{V})\big]\partial_{x_j} V - \sum_{j=1}^{n}\widehat{\mathcal{A}}_j(\overline{V})\partial_{x_j} V$$
$$+ \big[\widehat{\mathcal{D}}(V) - \widehat{\mathcal{D}}(\overline{V})\big]\Delta V + \widehat{\mathcal{D}}(\overline{V})\Delta V - \widehat{\mathcal{B}}(V).$$

We first observe that these matrices are Lipschitz continuous with respect to V, away from $\varrho = 0$ and $\vartheta = 0$ and also that the matrices $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{D}}$ have, respectively, the same structure as those defined in (3.10). Note also that, since $d - 1 > 5/2 = 3/2 + 1$, Sobolev embeddings imply that $H^{d-1}(\mathbb{R}^3)$ is an algebra. Therefore, we have

$$\|\partial_t V\|_{H^{d-1}(\mathbb{R}^3)} \le C_0\Big(1 + \sum_{j=1}^{3}\big\|\widehat{\mathcal{A}}_j(V) - \widehat{\mathcal{A}}_j(\overline{V})\big\|_{H^{d-1}(\mathbb{R}^3)}\Big)\|\nabla_x V\|_{H^{d-1}(\mathbb{R}^3)}$$
$$+ C_0\big(1 + \big\|\widehat{\mathcal{D}}(V) - \widehat{\mathcal{D}}(\overline{V})\big\|_{H^{d-1}(\mathbb{R}^3)}\big)$$

$$
\begin{aligned}
&\cdot \big(\|\Delta\vartheta\|_{H^{d-1}(\mathbb{R}^3)} + \|\Delta E_r\|_{H^{d-1}(\mathbb{R}^3)} + \|\mathrm{curl}_x(\mathrm{curl}_x \overrightarrow{B})\|_{H^{d-1}(\mathbb{R}^3)}\big) \\
&+ C_0\big(1 + \big\|\widehat{\mathcal{B}}(V) - \widehat{\mathcal{B}}(\overline{V})\big\|_{H^{d-1}(\mathbb{R}^3)}\big) \\
&\cdot \big(\|\vartheta - T_r\|_{H^{d-1}(\mathbb{R}^3)} + \|\overrightarrow{u}\|_{H^{d-1}(\mathbb{R}^3)}\big),
\end{aligned}
$$

whence,

$$
\begin{aligned}
\|\partial_t V\|_{H^{d-1}(\mathbb{R}^3)} \le C_0\big(1 + \|V - \overline{V}\|_{H^{d-1}(\mathbb{R}^3)}\big)& \\
\cdot \big(\|\nabla_x V\|_{H^{d-1}(\mathbb{R}^3)} + \|\Delta\vartheta\|_{H^{d-1}(\mathbb{R}^3)} + \|\Delta E_r\|_{H^{d-1}(\mathbb{R}^3)}& \\
+ \|\Delta \overrightarrow{B}\|_{H^{d-1}(\mathbb{R}^3)} + \|\vartheta - T_r\|_{H^{d-1}(\mathbb{R}^3)} + \|\overrightarrow{u}\|_{H^{d-1}(\mathbb{R}^3)}\big)&,
\end{aligned}
$$

which proves (3.34). □

Next, we bound the spatial derivatives as follows:

Proposition 3.6. *Assume that the hypotheses of Theorem* 2.1 *are satisfied. Let* $k \in \mathbb{N}^3$ *be such that* $1 \le |k| \le d$*, where* $d > 7/2$*. Then, we have*

$$
\begin{aligned}
\|\partial_x^k V(t)\|_{L^2(\mathbb{R}^3)}^2 &+ \int_0^t \big(\|\partial_x^k \nabla_x \vartheta(s)\|_{L^2(\mathbb{R}^3)}^2 \\
&+ \|\partial_x^k \nabla_x E_r(s)\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_x^k(\vartheta - T_r)(s)\|_{L^2(\mathbb{R}^3)}^2 \\
&+ \big\|\partial_x^k \nabla_x \overrightarrow{B}(s)\big\|_{L^2(\mathbb{R}^3)}^2 + \big\|\partial_x^k \overrightarrow{u}(s)\big\|_{L^2(\mathbb{R}^3)}^2\big)\, ds \\
\le C_0 N(0)^2 &+ C_0 N(t) \int_0^t \big(\|\nabla_x V(s)\|_{H^{d-1}(\mathbb{R}^3)}^2 \\
&+ \|\nabla_x \vartheta(s)\|_{H^d(\mathbb{R}^3)}^2 + \|\nabla_x E_r\|_{H^d(\mathbb{R}^3)}^2 + \|\vartheta(s) - T_r(s)\|_{H^{d-1}(\mathbb{R}^3)}^2 \\
&+ \big\|\overrightarrow{u}(s)\big\|_{H^{d-1}(\mathbb{R}^3)}^2 + \big\|\nabla_x \overrightarrow{B}(s)\big\|_{H^d(\mathbb{R}^3)}^2\big)\, ds.
\end{aligned}
\tag{3.36}
$$

Proof. Here, we need to symmetrize the nonlinear system. For this purpose, we multiply (3.35) on the right by the matrix

$$
\widehat{\mathcal{A}}_0(V) = \begin{pmatrix}
\widehat{\mu/\alpha'\varrho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \widehat{\mu/\beta'}\widehat{\gamma}' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \widehat{\mu/\beta''}\widehat{\gamma}'' & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\tag{3.37}
$$

This gives

$$
\widehat{\mathcal{A}}_0(V)\partial_t V = -\sum_{j=1}^3 \breve{\mathcal{A}}_j(V)\partial_{x_j} V + \breve{\mathcal{D}}(V)\Delta V - \breve{\mathcal{B}}(V) = 0,
\tag{3.38}
$$

where $\breve{\mathcal{A}}_j(V) = \widehat{\mathcal{A}}_0(V)\widehat{\mathcal{A}}_j(V)$, $\breve{\mathcal{B}}(V) = \widehat{\mathcal{A}}_0(V)\widehat{\mathcal{B}}(V)$ and $\breve{\mathcal{D}}(V) = \widehat{\mathcal{A}}_0(V)\widehat{\mathcal{D}}(V)$ are all symmetric matrices. Applying ∂_x^k to (3.38) then taking the scalar product with the vector $\partial_x^k V$, and integrating over $[0,t]\times\mathbb{R}^3$, we find

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}\big[\big(\widehat{\mathcal{A}}_0\partial_x^k V\big)\cdot\partial_x^k V\big]_0^t\,dx\\
&\quad+\int_0^t\int_{\mathbb{R}^3}\big(\breve{\mathcal{D}}\nabla_x\big(\partial_x^k V\big)\big)\cdot\nabla_x(\partial_x^k V)\,dx\,dt+\int_0^t\int_{\mathbb{R}^3}\big(\breve{\mathcal{B}}(V)\partial_x^k V\big)\cdot(\partial_x^k V)\,dx\,dt\\
&\qquad=\int_0^t\int_{\mathbb{R}^3}\left(\frac{1}{2}(I_1+I_2)-I_3-I_4-I_5\right)dx\,dt,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \partial_t\big(\widehat{\mathcal{A}}_0(V)\big)\partial_x^k V\cdot\partial_x^k V, &\qquad I_2 &= \sum_{j=1}^3\partial_{x_j}\big(\breve{\mathcal{A}}_j(V)\big)\partial_x^k V\cdot\partial_x^k V,\\
I_3 &= \big[\partial_x^k,\widehat{\mathcal{A}}_0(V)\big]\partial_t V\cdot\partial_x^k V, & I_4 &= \sum_{j=1}^3\big[\partial_x^k,\breve{\mathcal{A}}_j(V)\big]\partial_{x_j}V\cdot\partial_x^k V,\\
I_5 &= \partial_x^k\big(\breve{\mathcal{B}}(V)\big)\cdot\partial_x^k V.
\end{aligned}$$

We estimate separately each term of the right-hand side. First, we have

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}|I_1| &\le C\int_0^t\int_{\mathbb{R}^3}\big|\partial_x^k V\big|^2|\partial_t V|\\
&\le C\int_0^t\int_{\mathbb{R}^3}\big|\partial_x^k V\big|^2(|\nabla_x V|+|\mathcal{B}(V)|+|\mathcal{D}\Delta V|)\\
&\le CN(t)\int_0^t\big\|\partial_x^k V(s)\big\|_{L^2(\mathbb{R}^3)}^2\,ds,
\end{aligned}$$

where we have used the Sobolev embeddings and the fact that $d>7/2$. A similar computation gives

$$\int_0^t\int_{\mathbb{R}^3}|I_2|\le CN(t)\int_0^t\big\|\partial_x^k V(s)\big\|_{L^2(\mathbb{R}^3)}^2\,ds.$$

We estimate I_3 by applying the Cauchy–Schwarz inequality:

$$\int_0^t\int_{\mathbb{R}^3}|I_3\le\int_0^t\big\|\partial_x^k V\big\|_{L^2(\mathbb{R}^3)}\big\|\big[\partial_x^k,\widehat{\mathcal{A}}_0(V)\big]\partial_t V\big\|_{L^2(\mathbb{R}^3)}.$$

Next, we apply the same estimate for commutators and the composition of functions (see [15, Proposition 2.1]), and $|k|\le d$:

$$\begin{aligned}
&\big\|\big[\partial_x^k,\widehat{\mathcal{A}}_0(V)\big]\partial_t V\big\|_{L^2(\mathbb{R}^3)}=\big\|\big[\partial_x^k,\widehat{\mathcal{A}}_0(V)-\widehat{\mathcal{A}}_0(\overline{V})\big]\partial_t V\big\|_{L^2(\mathbb{R}^3)}\\
&\le C\big(\|\partial_t V\|_{L^\infty(\mathbb{R}^3)}\big\|\nabla_x\widehat{\mathcal{A}}_0(V)\big\|_{H^{d-1}(\mathbb{R}^3)}+\|\partial_t V\|_{H^{d-1}(\mathbb{R}^3)}\big\|\nabla_x\widehat{\mathcal{A}}_0(V)\big\|_{L^\infty(\mathbb{R}^3)}\big).
\end{aligned}$$

Moreover, we have

$$\big\|\nabla_x\widehat{\mathcal{A}}_0(V)\big\|_{H^{d-1}(\mathbb{R}^3)} \le C\|V-\overline{V}\|_{H^d(\mathbb{R}^3)} \le CN(t), \tag{3.39}$$

$$\big\|\nabla_x\widehat{\mathcal{A}}_0(V)\big\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla_x V\|_{H^{d-1}(\mathbb{R}^3)} \le CN(t). \tag{3.40}$$

Hence, I_3 satisfies

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^3}|I_3| \le{}& C(N(t))N(t)\\ &\cdot\int_0^t \big(\|\nabla_x V(s)\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\nabla_x\vartheta(s)\|^2_{H^d(\mathbb{R}^3)} + \|\nabla_x E_r(s)\|^2_{H^d(\mathbb{R}^3)}\\ &+\|\vartheta(s)-T_r(s)\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\overrightarrow{u}(s)\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\nabla_x\overrightarrow{B}(s)\|^2_{H^d(\mathbb{R}^3)}\big)\,ds.\end{aligned}$$

Here, we have used (3.34).

The integral of I_4 is dealt with by using similar computations.

Turning to I_5, we use the particular form of $\partial_x^k\breve{\mathcal{B}}(V)$. More precisely, we have

$$\begin{aligned}\partial_x^k\big(\breve{\mathcal{B}}(V)\big)\cdot\partial_x^k V ={}& \partial_x^k\bigg(\frac{\overrightarrow{f}_m}{\varrho} - \nu\,\frac{\overrightarrow{u}}{\varrho}\bigg)\cdot\partial_x^k\overrightarrow{u}\\ &-\partial_x^k\bigg(\frac{\sigma_a}{\varrho}(a\vartheta^4-E_r) + \frac{E_m - \overrightarrow{f}_m\cdot\overrightarrow{u}}{\varrho}\bigg)\cdot\partial_x^k\vartheta + \partial_x^k(\sigma_a(a\vartheta^4-E_r))\cdot\partial_x^k E_r,\end{aligned}$$

from which, using estimates for the composition of functions (see Proposition 2.1 in [15]) we infer

$$\int_0^t\int_{\mathbb{R}^3}|I_5| \le CN(t)\int_0^t\|\partial_x^k V(s)\|^2_{L^2(\mathbb{R}^3)}\,ds.$$

Collecting the estimates on I_1, I_2, I_3, I_4 and I_5, we have proved (3.36). □

The above results allow to derive the following bound:

Proposition 3.7. *Assume that the assumptions of Theorem* 2.1 *are satisfied. Then, there exists a non-decreasing function* $C\colon\mathbb{R}^+\to\mathbb{R}^+$ *such that*

$$\begin{aligned}&\big\|V-\overline{V}\big\|^2_{H^d(\mathbb{R}^3)} + \int_0^t\big(\|\nabla_x\vartheta(s)\|^2_{H^d(\mathbb{R}^3)} + \|\nabla_x E_r(s)\|^2_{H^d(\mathbb{R}^3)}\\ &\qquad+\|\vartheta(s)-T_r(s)\|^2_{H^d(\mathbb{R}^3)} + \big\|\overrightarrow{u}(s)\big\|^2_{H^d(\mathbb{R}^3)} + \big\|\nabla_x\overrightarrow{B}(s)\big\|^2_{H^d(\mathbb{R}^3)}\big)\,ds\\ &\le C(N(t))\bigg[N(0)^2 + N(t)\int_0^t\big(\|\nabla_x V(s)\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\nabla_x\vartheta(s)\|^2_{H^d(\mathbb{R}^3)}\\ &\qquad+\|\nabla_x E_r(s)\|^2_{H^d(\mathbb{R}^3)} + \|\vartheta(s)-T_r(s)\|^2_{H^{d-1}(\mathbb{R}^3)}\\ &\qquad+\|\overrightarrow{u}(s)\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\nabla_x\overrightarrow{B}(s)\|^2_{H^d(\mathbb{R}^3)}\bigg)\,ds\bigg].\end{aligned}\tag{3.41}$$

Proof. We sum up estimates (3.36) over all multi-indices k such that $|k|\le d$, and add this to (3.31). This leads to (3.41). □

3.2.2. $L^2(H^{d-1})$ estimates. In this section, we derive bounds on the right-hand side of (3.41). For this purpose, we adapt the strategy of [22], which was further developed in [10]. We apply the Fourier transform to the linearized system and use the compensating matrix K to prove estimates on the space derivatives of V.

Proposition 3.8. *Assume that the assumptions of Theorem* 2.1 *are satisfied. Then there exists a non-decreasing function* $C\colon \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\int_0^t \|\nabla_x V(s)\|_{H^{d-1}(\mathbb{R}^3)}\,ds \le C(N(t))\big(N(t) + \|V_0 - \overline{V}\|_{H^d(\mathbb{R}^3)}\big). \tag{3.42}$$

Proof. As a first step, we apply the symmetrizer of the linearized system (3.7) (which leads to (3.9)) to the nonlinear system (1.10)–(1.12), which then reads

$$\widetilde{\mathcal{A}}_0(V)\partial_t V + \sum_{j=1}^3 \widetilde{\mathcal{A}}_j(V)\partial_{x_j} V = \widetilde{\mathcal{D}}\Delta V - \widetilde{\mathcal{B}}(V)V.$$

Of course, this system is not symmetric. However, the corresponding linearized system (3.9) is symmetric. Next, we rewrite the nonlinear system by setting $U = V - \overline{V}$:

$$\widetilde{\mathcal{A}}_0(V)\partial_t U + \sum_{j=1}^3 \widetilde{\mathcal{A}}_j(V)\partial_{x_j} U = \widetilde{\mathcal{D}}\Delta U - \widetilde{\mathcal{B}}(V)U - \widetilde{\mathcal{B}}(V)\overline{V}.$$

Therefore, multiplying this system on the left by $\widetilde{\mathcal{A}}_0(\overline{V})\big(\widetilde{\mathcal{A}}_0(V)\big)^{-1}$, we find

$$\widetilde{\mathcal{A}}_0(\overline{V})\partial_t U + \sum_{j=1}^3 \widetilde{\mathcal{A}}_j(\overline{V})\partial_{x_j} U = H, \tag{3.43}$$

where

$$\begin{aligned} H = -\widetilde{\mathcal{A}}_0(\overline{V}) \sum_{j=1}^3 \big[\big(\widetilde{\mathcal{A}}_0(V)\big)^{-1}\widetilde{\mathcal{A}}_j(V) - \big(\widetilde{\mathcal{A}}_0(\overline{V})\big)^{-1}\widetilde{\mathcal{A}}_j(\overline{V})\big]\partial_{x_j} V \\ + \widetilde{\mathcal{A}}_0(\overline{V})\big(\widetilde{\mathcal{A}}_0(V)\big)^{-1}\widetilde{\mathcal{D}}\Delta U - \widetilde{\mathcal{A}}_0(\overline{V})\big(\widetilde{\mathcal{A}}_0(V)\big)^{-1}\widetilde{\mathcal{B}}(V)U \\ - \widetilde{\mathcal{A}}_0(\overline{V})\big(\widetilde{\mathcal{A}}_0(V)\big)^{-1}\widetilde{\mathcal{B}}(V)\overline{V}. \end{aligned}$$

We apply the Fourier transform to (3.43), and then multiply on the left by $-i(\widehat{U})^* K(\xi/|\xi|)$, where * denotes the transpose of the complex conjugate, and K is the compensating matrix (see Proposition 3.2). Taking the real part of the result, we infer

$$\begin{aligned} \operatorname{Im}\bigg((\widehat{U})^* K\Big(\frac{\xi}{|\xi|}\Big)\mathcal{A}_0(\overline{V})\partial_t \widehat{U}\bigg) + |\xi|(\widehat{U})^* K\Big(\frac{\xi}{|\xi|}\Big)\mathcal{A}\Big(\frac{\xi}{|\xi|}\Big)\widehat{U} \\ = \operatorname{Im}\bigg((\widehat{U})^* K\Big(\frac{\xi}{|\xi|}\Big)\widehat{H}\bigg), \end{aligned} \tag{3.44}$$

where the matrix $\mathcal{A}(\xi/|\xi|)$ is defined by (3.14). According to Proposition 3.2, $K\mathcal{A}_0(\overline{V})$ is skew-symmetric, hence

$$\operatorname{Im}\bigg((\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}_0(\overline{V})\partial_t\widehat{U}\bigg)=\frac{1}{2}\frac{d}{dt}\operatorname{Im}\bigg((\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}_0(\overline{V})\,\widehat{U}\bigg).$$

Next, we also have

$$\begin{aligned}(3.45)\quad |\xi|(\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}\bigg(\frac{\xi}{|\xi|}\bigg)\widehat{U}=|\xi|(\widehat{U})^*\bigg[K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}\bigg(\frac{\xi}{|\xi|}\bigg)+\mathcal{B}\bigg(\frac{\xi}{|\xi|}\bigg)\bigg]\widehat{U}\\ -|\xi|(\widehat{U})^*\widetilde{\mathcal{B}}\widehat{U}-|\xi|(\widehat{U})^*\widetilde{\mathcal{D}}\widehat{U}.\end{aligned}$$

Hence, still applying Proposition 3.2, there exists $\alpha_1>0$ and $\alpha_2>0$ such that

$$\begin{aligned}(3.46)\quad |\xi|(\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}\bigg(\frac{\xi}{|\xi|}\bigg)\widehat{U}\geq\alpha_1|\xi||\widehat{U}|^2-\alpha_2\frac{1}{|\xi|}\big(\big|\xi\widehat{(\vartheta-\overline{\vartheta})}\big|^2\\ +\big|\xi\widehat{(E_r-\overline{E_r})}\big|^2+|\xi|^2\big|\widehat{\vec{B}-\overline{\vec{B}}}\big|^2+|\xi|^2\big|\widehat{\vec{u}}\big|^2+|\xi|^2\big|\widehat{\vartheta-T_r}\big|^2\big).\end{aligned}$$

Finally, we estimate the right-hand side of (3.44) using the Cauchy–Schwarz inequality and the Young inequality:

$$(3.47)\qquad \bigg|\operatorname{Im}\bigg((\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\widehat{H}\bigg)\bigg|\leq\varepsilon|\xi||\widehat{U}|^2+C_\varepsilon\frac{1}{|\xi|}|\widehat{H}|^2,$$

for any $\varepsilon>0$. We choose ε small enough, insert (3.45)-(3.46)-(3.47) into (3.44), and find

$$\begin{aligned}|\xi||\widehat{U}|^2\leq C\bigg[\frac{1}{|\xi|}\big(\big|\xi\widehat{(\vartheta-\overline{\vartheta})}\big|^2+\big|\xi\widehat{(E_r-\overline{E_r})}\big|^2+|\xi|^2\big|\widehat{\vec{B}-\overline{\vec{B}}}\big|^2+|\xi|^2\big|\widehat{\vec{u}}\big|^2\\ +|\xi|^2\big|\widehat{\vartheta-T_r}\big|^2\big)+\frac{1}{|\xi|}|\widehat{H}|^2-\frac{d}{dt}\operatorname{Im}\bigg((\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}_0(\overline{V})\,\widehat{U}\bigg)\bigg].\end{aligned}$$

We multiply this inequality by $|\xi|^{2l-1}$, for some $1\leq l\leq d$, and get

$$\begin{aligned}(3.48)\quad |\xi|^{2l}|\widehat{U}|^2\leq C\bigg[|\xi|^{2l-2}\big(\big|\xi\widehat{(\vartheta-\overline{\vartheta})}\big|^2+\big|\xi\widehat{(E_r-\overline{E_r})}\big|^2\\ +|\xi|^2\big|\widehat{\vec{B}-\overline{\vec{B}}}\big|^2+|\xi|^2\big|\widehat{\vec{u}}\big|^2+|\xi|^2\big|\widehat{\vartheta-T_r}\big|^2\big)\\ +|\xi|^{2l-2}|\widehat{H}|^2-|\xi|^{2l-1}\frac{d}{dt}\operatorname{Im}\bigg((\widehat{U})^*K\bigg(\frac{\xi}{|\xi|}\bigg)\mathcal{A}_0(\overline{V})\widehat{U}\bigg)\bigg].\end{aligned}$$

We integrate this inequality over $[0,t]\times\mathbb{R}^3$, and use Plancherel's theorem:

$$\tag{3.49} \int_0^t \int_{\mathbb{R}^3} \sum_{|k|=l-1} \left|\partial_x^k \nabla_x V\right|^2$$

$$\leq C\int_0^t \int_{\mathbb{R}^3} \sum_{|k|=l-1} \left(\left|\partial_x^k\nabla_x\vartheta\right|^2 + \left|\partial_x^k\nabla_x E_r\right|^2 + \left|\partial_x^k\nabla_x \overrightarrow{B}\right|^2 + \left|\partial_x^k\nabla_x\overrightarrow{u}\right|^2 + \left|\partial_x^k H\right|^2\right)$$

$$+ C\,\mathrm{Im}\int_{\mathbb{R}^3} |\xi|^{2l-1}\left[(\widehat{U})^* K\left(\frac{\xi}{|\xi|}\right)\mathcal{A}_0(\overline{V})\widehat{U}\right]_0^t.$$

The matrix $K(\xi/|\xi|)$ is uniformly bounded for $\xi\in\mathbb{R}^3\setminus\{0\}$, so we have

$$\mathrm{Im}\int_{\mathbb{R}^3} |\xi|^{2l-1}\left[(\widehat{U})^* K\left(\frac{\xi}{|\xi|}\right)\mathcal{A}_0(\overline{V})\widehat{U}\right]_0^t$$

$$\leq C\left(\int_{\mathbb{R}^3}(1+|\xi|^2)^l\left|\widehat{U}(t)\right|^2 + \int_{\mathbb{R}^3}(1+|\xi|^2)^l\left|\widehat{U}_0\right|^2\right)$$

$$\leq C\left(\|V-\overline{V}\|^2_{H^l(\mathbb{R}^3)} + \|V_0-\overline{V}\|^2_{H^l(\mathbb{R}^3)}\right).$$

We insert this estimate into (3.49), sum the result over $1\leq l\leq d$, which leads to

$$\tag{3.50} \int_0^t \|\nabla_x V\|_{H^{d-1}(\mathbb{R}^3)} \leq C\bigg(\|V-\overline{V}\|^2_{H^d(\mathbb{R}^3)} + \|V_0-\overline{V}\|^2_{H^d(\mathbb{R}^3)}$$

$$+\int_0^t\left(\|\nabla_x\vartheta\|^2_{H^{d-1}(\mathbb{R}^3)} + \|\nabla_x E_r\|^2_{H^{d-1}(\mathbb{R}^3)} + \left\|\nabla_x\overrightarrow{u}\right\|^2_{H^{d-1}(\mathbb{R}^3)}\right.$$

$$\left. + \left\|\nabla_x\overrightarrow{B}\right\|^2_{H^{d-1}(\mathbb{R}^3)} + \|H\|^2_{H^{d-1}(\mathbb{R}^3)}\right)\bigg).$$

In order to conclude, we need to estimate the perturbation H. For this purpose, we use that $H^{d-1}(\mathbb{R}^3)$ is an algebra: for any $s\leq t$,

$$\|H(s)\|^2_{H^{d-1}(\mathbb{R}^3)} \leq CN(t)\|\nabla_x V\|_{H^{d-1}(\mathbb{R}^3)}.$$

Inserting this into (3.50), we prove (3.42). □

We are now in position to conclude with the

PROOF OF THEOREM 2.1. We first point out that local existence for system (1.10)–(1.12) may be proved using standard fix-point methods. We refer to [15] for the proof. The existence is proved in the following functional space:

$$X(0,T) = \big\{V,\ V-\overline{V}\in C([0,T];H^d(\mathbb{R}^3)),\ \nabla_x V\in L^2([0,T];H^{d-1}(\mathbb{R}^3)),$$
$$\nabla_x\vartheta, \nabla_x E_r,\ \nabla_x\overrightarrow{B}\in L^2([0,T];H^d(\mathbb{R}^3))\big\}.$$

In order to prove global existence, we argue by contradiction, and assume that $T_c>0$ is the maximum time existence. Then, we necessarily have

$$\tag{3.51} \lim_{t\to T_c} N(t) = +\infty,$$

where $N(t)$ is defined by (3.29). We are thus reduced to prove that N is bounded. For this purpose, we use the method of [13], which was also used in [18]. First note that, due to Proposition 3.7 on the one hand, and to Proposition 3.8 on the other hand, we know that there exists a non-decreasing continuous function $C\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$N(t)^2 \leq C(N(t))\big(N(0)^2 + N(t)^3\big) \quad \text{for all } T \in [0, T_c],. \tag{3.52}$$

Hence, setting $N(0) = \varepsilon$, we have

$$\frac{N(t)^2}{\varepsilon^2 + N(t)^3} \leq C(N(t)). \tag{3.53}$$

Studying the variation of $\phi(N) = N^2/(\varepsilon^2 + N^3)$, we see that $\phi'(0) = 0$, that ϕ is increasing on the interval $[0, (2\varepsilon^2)^{1/3}]$ and decreasing on the interval $[(2\varepsilon^2)^{1/3}, +\infty)$. Hence,

$$\max \phi = \phi\big((2\varepsilon^2)^{1/3}\big) = \frac{1}{3}\left(\frac{2}{\varepsilon}\right)^{2/3}.$$

Hence, the function C being independent of ε, we can choose ε small enough to have $\phi(N) \leq C(N)$ for all $N \in [0, N^*]$, where $N^* > 0$. Since C is continuous, (3.53) implies that $N \leq N^*$. This is clearly in contradiction with (3.51). □

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Xavier Blanc
Univ. Paris Diderot, Sorbonne Paris Cité
Laboratoire Jacques-Louis Lions
UMR 7598, UPMC, CNRS
F-75205 Paris, FRANCE

E-mail address: blanc@ann.jussieu.fr

Bernard Ducomet
Université Paris-Est
LAMA (UMR 8050)
UPEMLV, UPEC, CNRS
61 avenue du Général de Gaulle
94010 Créteil Cedex 10, FRANCE

E-mail address: bernard.ducomet@cea.fr

Šárka Nečasová
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žitná 25
115 67 Praha 1, CZECH REPUBLIC

E-mail address: matus@math.cas.cz