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CONVENIENT MAPS FROM ONE-RELATOR MODEL TWO-COMPLEXES INTO THE REAL PROJECTIVE PLANE

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Abstract. Let f be a map from a one-relator model two-complex $K_{\mathcal{P}}$ into the real projective plane. The composition $\varrho \circ f_{\#}$ of the homomorphism $f_{\#}$ induced by f on fundamental groups with the action ϱ of $\pi_1(\mathbb{R}\mathrm{P}^2)$ over $\pi_2(\mathbb{R}\mathrm{P}^2)$ provides a local integer coefficient system $f_{\#}^{\varrho}$ over $K_{\mathcal{P}}$. We prove that if the twisted integer cohomology group $H^2(K_{\mathcal{P}};{}_{f_{\#}^{\varrho}}\mathbb{Z}) = 0$, then f is homotopic to a non-surjective map. As an intermediary step for the proof, we show that if $H^2(K_{\mathcal{P}};{}_{\beta}\mathbb{Z}) = 0$ for some local integer coefficient system β over $K_{\mathcal{P}}$, then $K_{\mathcal{P}}$ is aspherical.

1. Introduction

The existence of strong surjections from a finite and connected n-dimensional CW complex K (a n-complex, to shorten) into a closed n-manifold Y has been investigated for at least a decade, specially from the viewpoint of the topological root theory.

For a *strong surjection* from K into Y we mean a (continuous) map $f\colon K \to Y$ whose free homotopy class $[f] \in [K;Y]$ has just surjective maps. In this case, we say also that f is *strongly surjective*. In the context of topological root theory, a map $f\colon K \to Y$ which is not strongly surjective is said to be *root free.*

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The Hopf–Whitney Classification Theorem [11, Corollary 6.19, p. 244] implies that the set $[K; S^n]$ of the free homotopy classes of maps from K into the n-sphere S^n is in one-to-one correspondence with the integer cohomology group $H^n(K;\mathbb{Z})$. Thus, there exists a strong surjection from K onto S^n if and only if $H^n(K;\mathbb{Z}) \neq 0$. Now, if $\widetilde{f}\colon K \to S^n$ is a strong surjection, its composition with the double-covering map $\mathfrak{p}\colon S^n \to \mathbb{R}\mathrm{P}^n$ produces a strong surjection $\mathfrak{p} \circ \widetilde{f}\colon K \to \mathbb{R}\mathrm{P}^n$. Therefore, the condition $H^n(K;\mathbb{Z}) \neq 0$ implies the existence of a strong surjection from K onto $\mathbb{R}\mathrm{P}^n$. The converse is the central problem of this article.

In order to contextualize, we highlight that this problem is part of a more general question: is the top integer cohomology group $H^n(K;\mathbb{Z})$ able to detect the existence of a strong surjection from the n-complex K into a closed n-manifold?

In the 2000's, D.L. Gonçalves and C. Aniz approached this problem in dimension three. In [1], C. Aniz proved that every map from a (finite and connected) three-complex K, with the top cohomology group $H^3(K;\mathbb{Z}) = 0$, into $S^1 \times S^2$ is homotopic to a non-surjective map, but for Y the non-orientable S^1-bundle over S^2, there exists a strong surjection $f\colon K \to Y$ from such a three-complex. In [2], C. Aniz proved that there is no strong surjection from such a three-complex into the orbit space of the three-sphere S^3 with respect to the action of the quaternion group Q_8 determined by the inclusion $Q_8 \subset S^3$.

Following the line of our works [4]–[7], we focus the problem in low-dimension, specifically in dimension two, which is often left out, since it does not permit the use of special techniques as obstruction theory and others.

In our latest work [4], we built a countable collection of two-complexes with trivial second integer cohomology group and, from each of them, a strong surjection onto the torus $S^1 \times S^1$. Therefore, the integer cohomology group $H^2(K;\mathbb{Z})$ is not able to detect the existence of strong surjection from a two-complex K onto the torus.

In [5] we studied the problem from a more restricted viewpoint: given a two-complex K with the top integer cohomology group $H^2(K;\mathbb{Z}) = 0$, we found necessary and sufficient conditions in order to get that all maps from K into $\mathbb{R}\mathrm{P}^2$ are homotopic to a constant map, i.e. $[K; \mathbb{R}\mathrm{P}^2] = 0$. The main theorem of [5] presents five of such conditions. We highlight one of them: given a two-complex K with $H^2(K;\mathbb{Z}) = 0$, we have $[K; \mathbb{R}\mathrm{P}^2] = 0$ if and only if the number of two-cells of K is equal to the first Betti number of its one-skeleton K^1. For a model two-complex $K_{\mathcal{P}}$ induced by a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ (see Section 2), this condition may be simplified by $\#\mathbf{x} = \#\mathbf{r}$.

At the other extreme, in this article we consider two-complexes with just one two-cell (with $\#\mathbf{r} = 1$ for model two-complexes). Furthermore, instead of the usual second integer cohomology group $H^2(K;\mathbb{Z})$, we consider the second cohomology groups $H^2(K;_\beta\mathbb{Z})$ with local integer coefficient systems $\beta\colon \pi_1(K) \to \mathrm{Aut}(\mathbb{Z})$.

We anticipate some details of our approach: the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^2)$ acts on $\pi_2(\mathbb{R}\mathrm{P}^2)$ exactly as the group of covering transformation $\mathfrak{C}(S^2;\mathbb{R}\mathrm{P}^2) = \{\mathrm{id}, -\mathrm{id}\}$ acts on the integer homology group $H_2(S^2;\mathbb{Z})$. Thus, the action of $\pi_1(\mathbb{R}\mathrm{P}^2)$ on $\pi_2(\mathbb{R}\mathrm{P}^2)$ provides, over $\mathbb{R}\mathrm{P}^2$, via the natural isomorphism$\pi_2(\mathbb{R}\mathrm{P}^2) \approx \mathbb{Z}$, the local integer coefficient system

$$\varrho\colon \pi_1(\mathbb{R}\mathrm{P}^2) \to \mathrm{Aut}(\mathbb{Z}) \quad \text{given by } \varrho(1) = \mathrm{id} \text{ and } \varrho(-1) = -\mathrm{id}.$$

Let K be a (finite and connected) two-complex with fundamental group Π and let $f\colon K \to \mathbb{R}\mathrm{P}^2$ be a cellular map. Consider the homomorphism $f_\#\colon \pi_1(K) \to \pi_1(\mathbb{R}\mathrm{P}^2)$ induced by f on fundamental groups and, over K, the local integer coefficient system

$$f_\#^\varrho\colon \Pi \to \mathrm{Aut}(\mathbb{Z}) \quad \text{given by } f_\#^\varrho = \varrho \circ f_\#.$$

The cohomology groups of K with local integer coefficient system $f_\#^\varrho$ are also called the *twisted integer cohomology groups* of K according $f_\#^\varrho$ and denoted by $H^*(K;_{f_\#^\varrho}\mathbb{Z})$. Of course, if $f_\#$ is the trivial homomorphism, then $f_\#^\varrho$ is the trivial local integer coefficient system ($f_\#^\varrho(\pi) = \mathrm{id}$ for all $\pi \in \Pi$) and $H^*(K;_{f_\#^\varrho}\mathbb{Z})$ is the integer cohomology group $H^*(K;\mathbb{Z})$.

In [6], a cellular map $f\colon K \to \mathbb{R}\mathrm{P}^2$ for which $H^2(K;_{f_\#^\varrho}\mathbb{Z}) = 0$ is called a *convenient map*. That is the reason for the title of this article, whose main theorem is about the nonexistence of convenient strong surjections from a one-relator model two-complex (or more general a finite and connected two-complex with just one two-cell) into the real projective plane.

THEOREM 1.1 (Main Theorem). *Let K be a finite and connected two-complex with just a two-cell and let $f\colon K \to \mathbb{R}\mathrm{P}^2$ be a cellular map. If the twisted integer cohomology group $H^2(K;_{f_\#^\varrho}\mathbb{Z}) = 0$, then f is homotopic to a non-surjective map.*

The proof of Theorem 1.1 is presented in Section 6 as a consequence of several results proved in the previous sections. Among them, Propositions 4.1 and 5.1 are the more interesting. Proposition 4.1 states that a one-relator model two-complex $K_{\mathcal{P}}$ for which $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = 0$, for some local integer coefficient system β, is aspherical (has contractible universal covering). Proposition 5.1 is more technical and treats the solubility of certain diophantine linear equations induced by maps from a model two-complex $K_{\mathcal{P}}$ into $\mathbb{R}\mathrm{P}^2$.

Sections 2 and 3 are surveys on model two-complexes and cohomology with local integer coefficient systems, respectively.

In the final section, we present a version of the Main Theorem for a special kind of one-relator model two-complexes, and we present examples to show that some results of the article does not hold for model two-complexes that are not one-relator.

Throughout the text, we simplify finite and connected two-dimensional CW complex by two-complex. We also simplify f is a continuous map by f is a map.

2. Model two-complexes

We remember that a (finite) *group presentation* $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ consists of a finite set $\mathbf{x} = \{x_1, \ldots, x_n\}$ of elements called *generators*, together with a finite set $\mathbf{r} = \{r_1, \ldots, r_m\}$ of elements called *relators* that are not necessarily reduced words with letters in the alphabet $\mathbf{x}$. Let $F(\mathbf{x})$ be the free group generated by the alphabet $\mathbf{x}$ and let $N(\mathbf{r})$ denote the smallest normal subgroup of $F(\mathbf{x})$ containing the words obtained by reducing the relation words of the set $\mathbf{r}$. The quotient group $\Pi = F(\mathbf{x})/N(\mathbf{r})$ is called the *group presented by* $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$. We take $\Omega\colon F(\mathbf{x}) \to \Pi$ to be the quotient homomorphism.

The *model two-complex* $K_{\mathcal{P}}$ of the finite group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ is the finite and connected two-complex whose one-skeleton

$$K^1_{\mathcal{P}} = \bigvee_j^n S^1_j = e^0 \cup e^1_1 \cup \ldots \cup e^1_n$$

is the bouquet of n circles (with minimal cellular decomposition) and whose m two-dimensional cells, we say $e^2_1, \ldots, e^2_m$, are attached on the one-skeleton according to the relators $r_1, \ldots, r_m$.

The fundamental groups $\pi_1(K^1_{\mathcal{P}})$ and $\pi_1(K_{\mathcal{P}})$ are naturally identified with the groups $F(\mathbf{x})$ and Π in such a way that the quotient homomorphism $\Omega\colon F(\mathbf{x}) \to \Pi$ corresponds to the homomorphism $\iota_\#\colon \pi_1(K^1_{\mathcal{P}}) \to \pi_1(K_{\mathcal{P}})$ induced on fundamental groups by the skeleton inclusion $\iota\colon K^1_{\mathcal{P}} \hookrightarrow K_{\mathcal{P}}$. All theses constructions and identifications are used in the text.

The following result corresponds to Theorem 1.9 of [10].

THEOREM 2.1. *The skeleton pair (K, K^1) of a finite and connected two-complex is homotopy equivalent to that of the model two-complex $K_{\mathcal{P}}$ of a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$.*

In views of this theorem, the study of strong surjections from two-complexes may be developed, without loss of generality, considering just model two-complexes. In fact, if K is a two-complex, there exists a homotopy equivalence $\varphi\colon K \to K_{\mathcal{P}}$ between K and a model two-complex $K_{\mathcal{P}}$, and a given map $f\colon K_{\mathcal{P}} \to Y$ is strong surjective if and only if so is the composed map $f \circ \varphi\colon K \to Y$.

Because of this, we develop all the results of the article for model two-complexes. Furthermore, we use the Cellular Approximation Theorem to consider that all maps given *a priori* are cellular. The identifications, results and notation introduced in this section are used throughout the article.

3. Computing top twisted (co)homology

Let $K_{\mathcal{P}}$ be the model two-complex induced by a group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r}\rangle$, with set of generators $\mathbf{x} = \{x_1, \dots, x_n\}$ and relator set $\mathbf{r} = \{r_1, \dots, r_m\}$. Consider the quotient homomorphism

$$\Omega\colon \pi_1(K^1_{\mathcal{P}}) \approx F(\mathbf{x}) \longrightarrow \Pi = \frac{F(\mathbf{x})}{N(\mathbf{r})} \approx \pi_1(K_{\mathcal{P}}).$$

For later use, let consider the integer $m \times n$ matrix $\Delta_{\mathcal{P}} = (\delta_{ij})$ in which each integer δ_{ij} is the sum of all powers of the letter x_j in the relator word r_i. In what follows, we identify each automorphism $\tau \in \mathrm{Aut}(\mathbb{Z})$ with its value $\tau(1)$.

For each local integer coefficient system $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$, the homomorphism $\beta' = \beta \circ \Omega\colon F(\mathbf{x}) \to \mathrm{Aut}(\mathbb{Z})$ satisfiyes $\beta'(r_i) = 1$ for all $r_i \in \mathbf{r}$, since $\ker \Omega = N(\mathbf{r})$. More precisely, a given homomorphism $\alpha'\colon F(\mathbf{x}) \to \mathrm{Aut}(\mathbb{Z})$ induces a local integer coefficient system $\alpha\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$ verifying $\alpha' = \alpha \circ \Omega$ if and only if $\alpha'(r_i) = 1$ for all $r_i \in \mathbf{r}$.

Since the group Π is generated by the elements $\overline{x}_j = \Omega(x_j)$, for $x_j \in \mathbf{x}$, a local integer coefficient system $\beta : \Pi \to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$ is uniquely defined by its values $\beta(\overline{x}_j) = \beta'(x_j)$, for $1 \le j \le n$. If $\beta(\overline{x}_j) = -1$, we say that β *twists* x_j; else we say that β *does not twist* x_j. If β does not twist neither of $x_1, \dots, x_n$, then β is the *trivial integer coefficient system.*

To shorten, the second cohomology group $H^2(K_{\mathcal{P}};_\beta \mathbb{Z})$ of $K_{\mathcal{P}}$ with the (trivial or not trivial) local integer coefficient system $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ will be called the *top twisted cohomology group of* $K_{\mathcal{P}}$ according to β. Of course, if β_0 is the trivial integer coefficient system, then $H^2(K_{\mathcal{P}};_{\beta_0}\mathbb{Z})$ is simply the integer cohomology group $H^2(K;\mathbb{Z})$.

Let $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ be a (trivial or not trivial) local integer coefficient system over $K_{\mathcal{P}}$. We define the β*-augmentation function* $\xi_\beta\colon \mathbb{Z}[\Pi] \to \mathbb{Z}$ by setting

$$\xi_\beta\Big(\sum_k n_k \pi_k\Big) = \sum_k n_k \beta(\pi_k).$$

In particular, if $\beta_0\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ is the trivial integer coefficient system, then the β_0-augmentation function is the canonical *augmentation function*

$$\xi_{\beta_0}\Big(\sum n_k \pi_k\Big) = \sum_k n_k.$$

It may be convenient to consider the β-augmentation function from $\mathbb{Z}[F(\mathbf{x})]$ into $\mathbb{Z}$, namely, the function $\xi'_\beta\colon \mathbb{Z}[F(\mathbf{x})] \to \mathbb{Z}$ given by

$$\xi'_\beta\Big(\sum_k n_k w_k\Big) = \sum_k n_k \beta'(w_k).$$

Thus, for each z in $\mathbb{Z}[F(\mathbf{x})]$, we have

$$\xi'_\beta(z) = \xi_\beta(\|z\|),$$

in which $\|\cdot\|\colon \mathbb{Z}[F(\mathbf{x})] \to \mathbb{Z}[\Pi]$ is the natural extension on group rings of the quotient homomorphism $\Omega\colon F(\mathbf{x}) \to \Pi$.

The *Reidmeister–Fox derivative*

$$\frac{\partial}{\partial x_j}\colon F(\mathbf{x}) \to \mathbb{Z}[F(\mathbf{x})]$$

associated with each generator x_j of $F(\mathbf{x})$ is the unique derivation, i.e. function satisfying

$$\frac{\partial w_1 w_2}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial w_2}{\partial x_j}, \quad \text{for all } w_1, w_2 \in F(\mathbf{x}),$$

whose value on the generator x_j is 1 and the values on the others generators are 0.

Consider the cellular decomposition of the model two-complex $K_{\mathcal{P}}$, namely,

$$K_{\mathcal{P}} = e^0 \cup e^1_1 \cup \ldots \cup e^1_n \cup e^2_1 \cup \ldots \cup e^2_m,$$

in such a way that each 1-cell e^1_j corresponds to the letter x_j, for $1 \le j \le n$, and each 2-cell e^2_i corresponds to the relator word r_i, for $1 \le i \le m$.

Consider the cellular chains of $K_{\mathcal{P}}$ with its natural identifications, namely,

$$\begin{aligned}
C_0(K_{\mathcal{P}}) &= H_0\big(K^0_{\mathcal{P}}\big) \approx \mathbb{Z}\big\langle e^0\big\rangle,\\
C_1(K_{\mathcal{P}}) &= H_1\big(K^1_{\mathcal{P}}, K^0_{\mathcal{P}}\big) \approx \mathbb{Z}^n\big\langle e^1_1, \ldots, e^1_n\big\rangle,\\
C_2(K_{\mathcal{P}}) &= H_2\big(K_{\mathcal{P}}, K^1_{\mathcal{P}}\big) \approx \mathbb{Z}^m\big\langle e^2_1, \ldots, e^2_m\big\rangle.
\end{aligned}$$

Let $\widetilde{K}_{\mathcal{P}}$ be the universal covering space of $K_{\mathcal{P}}$, endowed with its natural cellular structure. Select a 0-cell $\widetilde{e}^0$ over e^0, a 1-cell $\widetilde{e}^1_j$ over e^1_j for each $1 \le j \le n$, and a 2-cell $\widetilde{e}^2_i$ over e^2_i for each $1 \le i \le m$. The group Π acts on the left (via covering transformation) on the cellular chain complex $C_q\big(\widetilde{K}_{\mathcal{P}}\big) = H_q\big(\widetilde{K}^q_{\mathcal{P}}, \widetilde{K}^{q-1}_{\mathcal{P}}\big)$ making it into a left $\mathbb{Z}[\Pi]$-module, so that we have identifications

$$\begin{aligned}
C_0\big(\widetilde{K}_{\mathcal{P}}\big) &= \mathbb{Z}[\Pi]\big\langle \widetilde{e}^0\big\rangle,\\
C_1\big(\widetilde{K}_{\mathcal{P}}\big) &= \mathbb{Z}[\Pi]^n\big\langle \widetilde{e}^1_1, \ldots, \widetilde{e}^1_n\big\rangle,\\
C_2\big(\widetilde{K}_{\mathcal{P}}\big) &= \mathbb{Z}[\Pi]^m\big\langle \widetilde{e}^2_1 \ldots, \widetilde{e}^2_m\big\rangle.
\end{aligned}$$

Via this identifications and considering the action $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$, we have the corresponding (twisted) cellular chain complex of left $\mathbb{Z}[\Pi]$-modules

$$C^\beta_*(\widetilde{K}_{\mathcal{P}})\colon 0 \to C_2(\widetilde{K}_{\mathcal{P}}) \xrightarrow{\widetilde{\partial}^\beta_2} C_1(\widetilde{K}_{\mathcal{P}}) \xrightarrow{\widetilde{\partial}^\beta_1} C_0(\widetilde{K}_{\mathcal{P}}) \to 0,$$

in which the boundaries operators are given by

$$\widetilde{\partial}^{\beta}_1(\widetilde{e}^1_j) = \xi_\beta(1-\overline{x}_j)\widetilde{e}^0,$$
$$\widetilde{\partial}^{\beta}_2(\widetilde{e}^2_i) = \xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_1}\right\|\right)\widetilde{e}^1_1 + \ldots + \xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_n}\right\|\right)\widetilde{e}^1_n.$$

Now, consider the corresponding cellular co-chain complex

$$C^*_\beta\big(\widetilde{K}_{\mathcal{P}}\big)\colon 0 \to \mathrm{Hom}^{\Pi}\big(C_0\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big) \xrightarrow{\widetilde{\delta}^{\beta}_1}$$
$$\mathrm{Hom}^{\Pi}\big(C_1\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big) \xrightarrow{\widetilde{\delta}^{\beta}_2} \mathrm{Hom}^{\Pi}\big(C_2\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big) \to 0.$$

In each $\mathrm{Hom}^{\Pi}\big(C_j\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big)$, the integers $\mathbb{Z}$ is seen as a left $\mathbb{Z}[\Pi]$-module via the action $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$. The co-boundaries operators $\widetilde{\delta}^{\beta}_*$ are defined by the usual dual form $\widetilde{\delta}^{\beta}_*(\phi) = \phi \circ \widetilde{\partial}^{\beta}_*$. Explicitly, a given co-chain $\phi \in \mathrm{Hom}^{\Pi}\big(C_1\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big)$ is defined by its values $\phi\big(\widetilde{e}^1_j\big)$, for $1 \le j \le n$, and, for each $1 \le i \le m$, we have

$$\widetilde{\delta}^{\beta}_2(\phi)\big(\widetilde{e}^2_i\big) = \xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_1}\right\|\right)\phi\big(\widetilde{e}^1_1\big) + \ldots + \xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_n}\right\|\right)\phi\big(\widetilde{e}^1_n\big).$$

By definition, the second cohomology group of $K_{\mathcal{P}}$ with local integer coefficient system β is given by

$$H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z}) = \frac{\mathrm{Hom}^{\Pi}\big(C_2\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big)}{\mathrm{Im}\big(\widetilde{\delta}^{\beta}_2\big)}. \tag{3.1}$$

Now, for each $1 \le i \le m$ and each $1 \le j \le n$, put

$$\lambda^{\beta}_{ij} = \xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_j}\right\|\right), \tag{3.2}$$

and take the integer $m \times n$ matrix $\Lambda^{\beta}_{\mathcal{P}} = (\lambda^{\beta}_{ij})$. Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}^{\Pi}\big(C_1\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big) & \xrightarrow{\widetilde{\delta}^{\beta}_2} & \mathrm{Hom}^{\Pi}\big(C_2\big(\widetilde{K}_{\mathcal{P}}\big);\mathbb{Z}\big) \\
\approx\big\downarrow & & \big\downarrow\approx \\
\mathbb{Z}^n & \xrightarrow[\Lambda^{\beta}_{\mathcal{P}}]{} & \mathbb{Z}^m
\end{array}$$

It follows by equation (3.1) that

$$H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z}) \approx \frac{\mathbb{Z}^m}{\mathrm{Im}\big(\Lambda^{\beta}_{\mathcal{P}}\big)}. \tag{3.3}$$

The next result, which is used in [1] without a proof, provides a relationship between the integer $m \times n$ matrices $\Delta_{\mathcal{P}}$ and $\Lambda^{\beta}_{\mathcal{P}}$. Here we present a detailed proof.

LEMMA 3.1. *For each $1 \le i \le m$ and each $1 \le j \le n$ one has $\delta_{ij} \equiv \lambda^{\beta}_{ij}\,(\mathrm{mod}\,2)$.*

PROOF. Given a word $w \in F(\mathbf{x})$, for each $1 \leq j \leq n$, we define $s_j(w)$ to be the sum of all powers of the letter x_j in w. In order to prove the lemma, it is sufficient to prove that

$$s_j(w) \equiv \xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right) \pmod 2.$$

The word w may be written as $w = w_1 \dots w_k$ in which each word w_l is a *block* of the form $w_l = x_1^{d_{l1}} \dots x_n^{d_{ln}}$. Thus $s_j(w) = d_{1j} + \dots + d_{kj}$. We prove the result by induction on the number k of blocks.

If $k = 1$, then w may be write simply as $w = x_1^{d_1} \dots x_n^{d_n}$. In this case, $s_j(w) = d_j$ and

$$\frac{\partial w}{\partial x_j} = x_1^{d_1} \dots x_{j-1}^{d_{j-1}} \langle x_j, d_j\rangle^{\mathrm{sgn}(d_j)},$$

where $\mathrm{sgn}(d_j)$ is the signal of the integer d_j (which may be 0 or $+$ or $-$) and

$$\begin{aligned}
\langle x_j, d_j\rangle^0 &= 0,\\
\langle x_j, d_j\rangle^+ &= 1 + x_j + \dots + x_j^{d_j-1},\\
\langle x_j, d_j\rangle^- &= -x_j^{-1} - \dots - x_j^{d_j}.
\end{aligned}$$

Since $\xi_\beta(\|x_1^{d_1} \dots x_{j-1}^{d_{j-1}}\|) = 1$, it follows that

$$\xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right) = \xi_\beta(\|\langle x_j, d_j\rangle^{\mathrm{sgn}(d_j)}\|)$$

and so

$$\xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right) = \begin{cases} \pm d_j & \text{if } \beta \text{ does not twist } x_j,\\ 0 & \text{if } \beta \text{ twists } x_j \text{ and } d_j \text{ is even},\\ 1 & \text{if } \beta \text{ twists } x_j \text{ and } d_j \text{ is odd}.\end{cases}$$

In all the cases, it is clear that

$$\xi_\beta\left(\left\|\frac{\partial r_i}{\partial x_j}\right\|\right) \equiv s_j(w) \pmod 2.$$

Now, suppose that the result holds true for words with p blocks. Let us prove that the result holds true for a word $w = w_1 \dots w_p w_{p+1}$ with $p + 1$ blocks. We have

$$\frac{\partial w}{\partial x_j} = \frac{\partial w_1 \dots w_p}{\partial x_j} + w_1 \dots w_p \frac{\partial w_{p+1}}{\partial x_j}.$$

Applying the β-augmentation function $\xi_\beta(\| \cdot \|)$ in both sides, we obtain

$$\xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right) = \xi_\beta\left(\left\|\frac{\partial w_1 \dots w_p}{\partial x_j}\right\|\right) + \xi_\beta\left(\left\|\frac{\partial w_{p+1}}{\partial x_j}\right\|\right).$$

By the case $k = 1$ and by the induction hypothesis, we have

$$\xi_\beta\left(\left\|\frac{\partial w_{p+1}}{\partial x_j}\right\|\right) \equiv s_j(w_{p+1}) \pmod 2$$

and

$$\xi_\beta\left(\left\|\frac{\partial w_1 \dots w_p}{\partial x_j}\right\|\right) \equiv s_j(w_1 \dots w_p) \pmod 2.$$

Since $s_j(w) = s_j(w_1 \dots w_p) + s_j(w_{p+1})$, it follows that

$$\xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right) \equiv s_j(w) \pmod 2.$$

The induction step is complete and, therefore, we have proved the lemma. □

The statement of Lemma 3.1 may be simplified by writing the matrix congruence

$$\Delta_{\mathcal{P}} \equiv \Lambda^{\beta}_{\mathcal{P}} \pmod 2. \tag{3.4}$$

Of course, for the trivial integer coefficient system β_0 one has $\Lambda^{\beta_0}_{\mathcal{P}} = \Delta_{\mathcal{P}}$.

It follows from equation (3.3) that the twisted cohomology group $H^2(K_{\mathcal{P}};_\beta\mathbb{Z})$ is not trivial if $m > n$. If, on the other hand, we assume $m \leq n$, then the nullity of $H^2(K_{\mathcal{P}};_\beta\mathbb{Z})$ depends on the *minors* of the $m \times n$ matrix $\Lambda^{\beta}_{\mathcal{P}}$. Here, by a minor of a $m\times n$ matrix A, with $m \leq n$, we mean the determinant of a $m\times m$ sub-matrix of A. We have the following result:

Proposition 3.2. *If $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = 0$ for some (trivial or not trivial) local integer coefficient system β over $K_{\mathcal{P}}$, then at least one of the minors of the matrix $\Delta_{\mathcal{P}}$ is odd.*

Proof. As we have seen, we have an isomorphism $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) \approx \mathbb{Z}^m/\mathrm{Im}(\Lambda^{\beta}_{\mathcal{P}})$, in which the $m \times n$ matrix $\Lambda^{\beta}_{\mathcal{P}}$ is seen as a homomorphism of abelian free groups $\Lambda^{\beta}_{\mathcal{P}} \colon \mathbb{Z}^n \to \mathbb{Z}^m$.

It follows from well known results of linear algebra over commutative rings (over $\mathbb{Z}$), that $H^2(K_{\mathcal{P}};_\beta\mathbb{Z})$ is infinite if $n < m$, and, on the other hand, if $m \leq n$, then the index

$$[\mathbb{Z}^m : \mathrm{Im}(\Lambda^{\beta}_{\mathcal{P}})] = \gcd(d_1, \dots, d_s),$$

in which $d_1, \dots, d_s$ are all the minors of the matrix $\Lambda^{\beta}_{\mathcal{P}}$.

Therefore, if $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = 0$, then $m \leq n$ and $\gcd(d_1, \dots, d_s) = 1$, with means that at least one of the minors $d_1, \dots, d_s$ is odd. Since, by equation (3.4), the matrix $\Delta_{\mathcal{P}} \equiv \Lambda^{\beta}_{\mathcal{P}} \pmod 2$, it follows that the corresponding minor of the matrix $\Delta_{\mathcal{P}}$ is also odd. □

4. Asphericity of one-relator model two-complexes

A (finite) group presentation $\mathcal{P} = \langle \mathbf{x} \,|\, \mathbf{r} \rangle$ is called a *one-relator group presentation* if the relator set $\mathbf{r}$ is single, that is, $\mathcal{P}$ is of the form $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r \rangle$, in which r is a not necessarily reduced word in the alphabet $\mathbf{x} = \{x_1, \dots, x_n\}$.

A model two-complex $K_{\mathcal{P}}$ of a one-relator group presentation is called a *one-relator model two-complex*; it has jus one cell of dimension two.

Next, we prove that for a one-relator model two-complex $K_{\mathcal{P}}$, the nullity of the twisted cohomology group $H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z})$ for some (trivial or not trivial) local integer coefficient system β over $K_{\mathcal{P}}$ implies that $K_{\mathcal{P}}$ is aspherical. Before that, we remember some things.

Following [3] we remember that a (not necessarily one-relator) model two-complex $K_{\mathcal{P}}$ is *aspherical* (has contractible universal covering) if and only if $\pi_2(K_{\mathcal{P}}) = 0$. More generally, $K_{\mathcal{P}}$ (or rather $\mathcal{P}$) is said to be *combinatorially aspherical* if $\pi_2(K_{\mathcal{P}})$ is generated by a set of based aspherical pictures over $\mathcal{P}$ that contain exactly two discs. Further, we say that the presentation $\mathcal{P}$ satisfies the *Relator Hypothesis* if no relator of $\mathcal{P}$ is freely trivial nor is a conjugate of any other relator or its inverse. It follows by [8, Proposition 5] that $K_{\mathcal{P}}$ is aspherical if and only if (i) $\mathcal{P}$ satisfies the Relator Hypothesis, (ii) each relator of $\mathcal{P}$ has period one (is not a proper power of another word) and (iii) $K_{\mathcal{P}}$ is combinatorially aspherical. See also [3, Subsection 2.1]. As a fundamental example, the Simple Identity Theorem of R.C. Lyndon [9] implies that if the relator of a one-relator presentation $\mathcal{P}$ is not freely trivial, then $K_{\mathcal{P}}$ is combinatorially aspherical. See again [3, Subsection 2.1]).

Therefore, *a one-relator model two-complex $K_{\mathcal{P}}$ is aspherical if and only if the single relator of $\mathcal{P}$ is not freely trivial and has period one.*

With this characterization of asphericity for one-relator model two-complexes we prove:

Proposition 4.1. *Let $K_{\mathcal{P}}$ be a one-relator model two-complex. If $H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z}) = 0$ for some (trivial or non-trivial) local integer coefficient system β over $K_{\mathcal{P}}$, then $K_{\mathcal{P}}$ is aspherical.*

Proof. Let $\mathcal{P} = \langle x_1, \ldots, x_n \,|\, r\rangle$ be the one-relator group presentation inducing the one-relator model two-complex $K_{\mathcal{P}}$ and consider the integer line-matrix $\Delta_{\mathcal{P}} = [\delta_1 \ \ldots \ \delta_n]$, in which each δ_j is the sum of all powers of the letter x_j in the relator word r.

Suppose $K_{\mathcal{P}}$ is not aspherical. Then either r is freely trivial or r has period greater than one. We consider the two cases separately:

Firstly suppose r is freely trivial. Then the matrix $\Delta_{\mathcal{P}} = 0$. Let $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ be an arbitrary local coefficient system over $K_{\mathcal{P}}$. By equation (3.4), the matrix $\Lambda^\beta_{\mathcal{P}} \equiv 0 \,(\mathrm{mod}\, 2)$, which means that all the entries of $\Lambda^\beta_{\mathcal{P}}$ are even. It follows that $H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z}) \approx \mathbb{Z}/\mathrm{Im}(\Lambda^\beta_{\mathcal{P}})$ is either infinite cyclic or cyclic of even order. In any of the cases $H^2(K_{\mathcal{P}};{}_\beta\mathbb{Z}) \neq 0$.

Now suppose that r has period $k \geq 2$, so that $r = w^k$ for some word $w \in F(\mathbf{x})$. Then, for each $1 \leq j \leq n$, we have

$$\frac{\partial r}{\partial x_j} = \frac{\partial w}{\partial x_j} + w\,\frac{\partial w^{k-1}}{\partial x_j} = \ldots = \frac{\partial w}{\partial x_j} + w\,\frac{\partial w}{\partial x_j} + \ldots + w^{k-1}\,\frac{\partial w}{\partial x_j}.$$

Thus, given an arbitrary local integer coefficient system $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$ we have

$$\lambda_j^\beta = \xi_\beta\left(\left\|\frac{\partial r}{\partial x_j}\right\|\right) = \xi_\beta\left(\left\|(1+w+\ldots+w^{k-1})\frac{\partial w}{\partial x_j}\right\|\right) = k\cdot\xi_\beta\left(\left\|\frac{\partial w}{\partial x_j}\right\|\right).$$

It follows that the integer $k \geq 2$ divides each entry of the matrix $\Lambda_{\mathcal{P}}^\beta$, which implies that $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) \approx \mathbb{Z}/\mathrm{Im}(\Lambda_{\mathcal{P}}^\beta) \approx \mathbb{Z}/ck\mathbb{Z}$ for some positive integer c. Again $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) \neq 0$. □

Proposition 4.1 does not hold true for a model two-complex whose corresponding presentation group has more than one relator. For instance, consider the group presentation $\mathcal{P} = \langle x, y \,|\, x^3y^{-5}, (xy)^2y^{-5}\rangle$ of the Poincaré's binary icosahedral group 2I, which has order 120. Since $\det(\Delta_{\mathcal{P}}) = 1$, we have $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$. However, since $K_{\mathcal{P}}$ is a finite complex and $\pi_1(K_{\mathcal{P}}) \approx$ 2I is a finite group, the complex $K_{\mathcal{P}}$ is not an Eilenberg–Macline complex K(2I,1), with means that $\pi_2(K_{\mathcal{P}}) \neq 0$. Additionally, we remark that there is not a local integer coefficient system $\beta\colon$ 2I $\to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$ other than the trivial one.

5. On certain diophantine linear equations

Let $K_{\mathcal{P}}$ be a one-relator model two-complex and take $\mathcal{P} = \langle x_1, \ldots, x_n \,|\, r\rangle$ to be the corresponding one-relator group presentation. Consider the quotient homomorphism $\Omega\colon F(\mathbf{x}) \to \Pi = F(\mathbf{x})/N(r)$ and the integer line-matrix $\Delta_{\mathcal{P}} = [\delta_1 \ldots \delta_n]$ in which, for each index $1 \leq j \leq n$, the integer δ_j is the sum of all powers of the letter x_j in the word r.

In what follows, we consider diophantine linear equations of the form $\Delta_{\mathcal{P}}Y{=}b$, where $Y = [y_1 \,\ldots\, y_n]^{\mathrm{T}}$ is the unknown vector and b is an integer. Thus

$$\Delta_{\mathcal{P}}Y = b \quad \text{means} \quad \delta_1y_1 + \ldots + \delta_ny_n = b.$$

Let $f\colon K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ be a cellular map. Then f restricts itself on one-skeletons to a cellular map $f^1\colon K_{\mathcal{P}}^1 \to S^1$ making commutative the diagram below, in which the vertical arrows are the skeleton inclusions:

$$\begin{array}{ccc} K_{\mathcal{P}}^1 & \xrightarrow{f^1} & S^1 \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle l} \\ K_{\mathcal{P}} & \xrightarrow[f]{} & \mathbb{R}\mathrm{P}^2 \end{array}$$

The homomorphism $\iota_\#\colon \pi_1(K_{\mathcal{P}}^1) \to \pi_1(K_{\mathcal{P}})$ corresponds to $\Omega\colon F(\mathbf{x}) \to \Pi$ via natural identifications. Furthermore, the homomorphism $l_\#\colon \pi_1(S^1) \approx F(a) \to \pi_1(\mathbb{R}\mathrm{P}^2) \approx \mathbb{Z}_2$ corresponds to the natural quotient homomorphism $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, so that $\ker(l_\#)$ is the subgroup of $F(a)$ generated by a^2. From the commutativity

$l_\# \circ f^1_\# = f_\# \circ \Omega$, it follows that $f^1_\#(r) \in \ker(l_\#)$. Thus, there exists an integer d_f, so called the *relator-degree* of f, such that

$$f^1_\#(r) = a^{2d_f}. \tag{5.1}$$

PROPOSITION 5.1. *Let $K_\mathcal{P}$ be a one-relator model two-complex and let $f\colon K_\mathcal{P} \to \mathbb{RP}^2$ be a cellular map of relator-degree d_f. If at least one of the entries of the line-matrix $\Delta_\mathcal{P}$ is odd, then the linear diophantine equation $\Delta_\mathcal{P} Y = d_f$ has an integer solution.*

PROOF. Put $\mathcal{P} = \langle x_1, \dots, x_n \,|\, r\rangle$ and $\Delta_\mathcal{P} = [\delta_1 \dots \delta_n]$ as above. The induced homomorphism

$$f^1_\# : \pi_1(K^1_\mathcal{P}) = F(x_1, \dots, x_n) \to F(a) \approx \pi_1(S^1)$$

is defined by its values on the generators $x_1, \dots, x_n$. If it is necessary, we may reindex the generators in such a way that, for a certain $k \in \{1, \dots, n\}$, the generators $x_1, \dots, x_k$ are mapped by $f^1_\#$ to odd powers of a and the generators $x_{k+1}, \dots, x_n$ are mapped by $f^1_\#$ to even powers of a, we say

$$f^1_\#(x_j) = a^{2q_j+1} \quad \text{for } 1 \le j \le k \quad \text{and} \quad f^1_\#(x_j) = a^{2q_j} \quad \text{for } k+1 \le j \le n.$$

Thus, we have

$$f^1_\#(r) = a^{(2q_1+1)\delta_1 + \dots + (2q_k+1)\delta_k + 2q_{k+1}\delta_{k+1} + \dots + 2q_n\delta_n},$$

which implies, according to equation (5.1), that

$$(2q_1+1)\delta_1 + \dots + (2q_k+1)\delta_k + 2q_{k+1}\delta_{k+1} + \dots + 2q_n\delta_n = 2d_f.$$

Since both, the integer $2d_f$ and the parcel $2q_{k+1}\delta_{k+1} + \dots + 2q_n\delta_n$ are even, also the parcel $(2q_1+1)\delta_1 + \dots + (2q_k+1)\delta_k$ is even. Hence there exist integers d_1 and d_2 such that

$$(2q_1+1)\delta_1 + \dots + (2q_k+1)\delta_k = 2d_1, \tag{5.2}$$

$$2q_{k+1}\delta_{k+1} + \dots + 2q_n\delta_n = 2d_2, \tag{5.3}$$

$$d_f = d_1 + d_2. \tag{5.4}$$

Suppose that at least one of the entries of $\Delta_\mathcal{P}$ is odd. We consider two cases. Firstly, suppose that some of the integers $\delta_1, \dots, \delta_k$ is odd. Then $\gcd(\delta_1, \dots, \delta_k)$ is odd. On the other hand, equation (5.2) implies that $\gcd(\delta_1, \dots, \delta_k)$ divides $2d_1$. It follows that $\gcd(\delta_1, \dots, \delta_k)$ divides d_1. By Bézout's Lemma, there exist integers $b, b_1, \dots, b_k$ such that

$$d_1 = b \cdot \gcd(\delta_1, \dots, \delta_k) = b(b_1\delta_1 + \dots + b_k\delta_k).$$

It follows by equations (5.3) and (5.4) that

$$d_f = bb_1\delta_1 + \dots + bb_k\delta_k + q_{k+1}\delta_{k+1} + \dots + q_n\delta_n.$$

Therefore, the linear diophantine equation $\Delta_\mathcal{P} Y = d_f$ has an integer solution.

Secondly, suppose that the integers $\delta_1, \dots, \delta_k$ are all even and put $\delta_j = 2\varepsilon_j$ for $1 \le j \le k$. By equations (5.2), (5.3) and (5.4) one has

$$d_f = (2q_1 + 1)\varepsilon_1 + \dots + (2q_k + 1)\varepsilon_k + q_{k+1}\delta_{k+1} + \dots + q_n\delta_n.$$

Thus $\gcd(\varepsilon_1, \dots, \varepsilon_k, \delta_{k+1}, \dots, \delta_n)$ divides d_f, that is, there exists an integer c such that

$$d_f = c \cdot \gcd(\varepsilon_1, \dots, \varepsilon_k, \delta_{k+1}, \dots, \delta_n). \tag{5.5}$$

Now, since $\delta_1, \dots, \delta_k$ are all even and $\Delta_{\mathcal{P}}$ has at least one odd entry, it follows that some of integers $\delta_{k+1}, \dots, \delta_n$ is odd. Thus implies that

$$\gcd(\delta_1, \dots, \delta_n) = \gcd(\varepsilon_1, \dots, \varepsilon_k, \delta_{k+1}, \dots, \delta_n). \tag{5.6}$$

By Bézout's Lemma and equations (5.5) and (5.6), there exist integers $c_1, \dots, c_n$ such that $d_f = c(c_1\delta_1 + \dots + c_n\delta_n)$. Therefore, the linear diophantine equation $\Delta_{\mathcal{P}}Y = d_f$ has an integer solution. □

6. Proof of the Main Theorem

This section is devoted to the proof of the Main Theorem of the article, namely, Theorem 1.1. The proof is based on the central results of Sections 4 and 5, besides the version of Proposition 8.6 of [6] for maps from one-relator model two-complexes into the projective plane, which we present now:

PROPOSITION 6.1. *Let $K_{\mathcal{P}}$ be a one-relator model two-complex and let $f\colon K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ be a cellular map of relator-degree d_f. If f is homotopic to a non-surjective map, then the induced homomorphism $f_{\#_2}\colon \pi_2(K_{\mathcal{P}}) \to \pi_2(\mathbb{R}\mathrm{P}^2)$ is trivial and the linear diophantine equation $\Delta_{\mathcal{P}}Y = d_f$ has an integer solution. The converse is true if $H^2(K_{\mathcal{P}};_{f^{\varrho}_{\#}}\mathbb{Z}) = 0$.*

Finally, we prove the Main Theorem of the article:

PROOF OF THEOREM 1.1. According to Section 2, it is sufficient to prove the result for model two-complexes. Therefore, let $K_{\mathcal{P}}$ be a one-relator model two-complex and let $f\colon K_{\mathcal{P}} \to \mathbb{R}\mathrm{P}^2$ be a cellular map for which $H^2(K_{\mathcal{P}};_{f^{\varrho}_{\#}}\mathbb{Z}) = 0$. Proposition 3.2 tells us that at least one of the entries of the line-matrix $\Delta_{\mathcal{P}}$ is odd, which implies, by Proposition 5.1, that the diophantine linear equation $\Delta_{\mathcal{P}}Y = d_f$ has an integer solution. Furthermore, $K_{\mathcal{P}}$ is aspherical, by Proposition 4.1. Therefore, the result follows by Proposition 6.1. □

7. Maps from one-block-relator model two-complexes

It is proved in [1, Lemma 3.3] that for a finite and connected two-complex K, if $H^2(K;\mathbb{Z}) = 0$, then $H^2(K;_{\beta}\mathbb{Z})$ is finite of odd order for every local integer coefficient system β over K. Next we present a family of one-relator model two-complexes for which the nullity of the group $H^2(\,\cdot\,;\mathbb{Z})$ implies the nullity of all the groups $H^2(\,\cdot\,;_\beta\mathbb{Z})$. Therefore, in views of Theorem 1.1, for such a one-relator model two-complex $K_{\mathcal{P}}$, the nullity of $H^2(K_{\mathcal{P}};\mathbb{Z})$ implies the nonexistence of strong surjection from $K_{\mathcal{P}}$ into $\mathbb{RP}^2$.

A model two-complex $K_{\mathcal{P}}$ is called a *one-block-relator model two-complex* if it is a one-relator model two-complex whose unique relator r is such that each generator of the presentation $\mathcal{P}$ appears just once in r, case in which we say that r has just a block.

LEMMA 7.1. *Let $K_{\mathcal{P}}$ be a one-block-relator model two-complex. If $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$, then $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = 0$ for all local integer coefficient system β over $K_{\mathcal{P}}$.*

PROOF. Assume $\mathcal{P} = \langle x_1, \ldots, x_n \,|\, r\rangle$ with $r = x_1^{\delta_1} \ldots x_n^{\delta_n}$. We may suppose that each $\delta_j \neq 0$. In fact, if $\delta_1 = 0$, then $K_{\mathcal{P}} = S^1 \vee L$, in which L is the model two-complex induced by the group presentation $\langle x_2, \ldots, x_n \,|\, x_2^{\delta_2} \ldots x_n^{\delta_n}\rangle$, and we have $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) \approx H^2(L;_\beta\mathbb{Z})$ for all β. The same happens for any δ_j instead of δ_1. Furthermore, we may suppose that each $\delta_j > 0$. In fact, if $\delta_j < 0$, then we replace the generator x_j by x_j^{-1}.

Let $\Omega\colon F(\mathbf{x}) \to \Pi = F(\mathbf{x})/N(r)$ be the quotient homomorphism and write $\overline{x}_j = \Omega(x_j)$ for each $1 \leq j \leq n$. Computing the Reidmeister–Fox Derivatives of r we obtain:

$$\begin{aligned}
\left\|\frac{\partial r}{\partial x_1}\right\| &= \left\|1 + x_1 + \ldots + x_1^{\delta_1 - 1}\right\| = 1 + \overline{x}_1 + \ldots + \overline{x}_1^{\delta_1 - 1},\\
\left\|\frac{\partial r}{\partial x_2}\right\| &= \left\|x_1^{\delta_1}\big(1 + x_2 + \ldots + x_2^{\delta_2 - 1}\big)\right\| = \overline{x}_1^{\delta_1}\big(1 + \overline{x}_2 + \ldots + \overline{x}_2^{\delta_2 - 1}\big),\\
&\;\vdots\\
\left\|\frac{\partial r}{\partial x_n}\right\| &= \left\|x_1^{\delta_1} \ldots x_{n-1}^{\delta_{n-1}}\big(1 + x_n + \ldots + x_n^{\delta_n - 1}\big)\right\|\\
&= \overline{x}_1^{\delta_1} \ldots \overline{x}_{n-1}^{\delta_{n-1}}\big(1 + \overline{x}_n + \ldots + \overline{x}_n^{\delta_n - 1}\big).
\end{aligned}$$

Let $\beta\colon \Pi \to \mathrm{Aut}(\mathbb{Z})$ by an arbitrary (possible trivial) local integer coefficient system over $K_{\mathcal{P}}$. Applying the ξ_β-augmentation function we obtain

$$\begin{aligned}
\xi_\beta\left(\left\|\frac{\partial r}{\partial x_1}\right\|\right) &= 1 + \beta(\overline{x}_1) + \ldots + \beta(\overline{x}_1)^{\delta_1 - 1},\\
\xi_\beta\left(\left\|\frac{\partial r}{\partial x_2}\right\|\right) &= \beta(\overline{x}_1)^{\delta_1}\big(1 + \beta(\overline{x}_2) + \ldots + \beta(\overline{x}_2)^{\delta_2 - 1}\big),\\
&\;\vdots\\
\xi_\beta\left(\left\|\frac{\partial r}{\partial x_n}\right\|\right) &= \beta(\overline{x}_1)^{\delta_1} \ldots \beta(\overline{x}_{n-1})^{\delta_{n-1}}\big(1 + \beta(\overline{x}_n) + \ldots + \beta(\overline{x}_n)^{\delta_n - 1}\big).
\end{aligned}$$

But we have $\beta(\overline{x}_j)^{\delta_j} = \pm 1$ for all $1 \leq j \leq n$. Thus, for each $1 \leq j \leq n$, we have

$$\lambda_j^\beta = \pm\big(1 + \beta(\overline{x}_j) + \ldots + \beta(\overline{x}_j)^{\delta_i - 1}\big).$$

This implies that

$$\lambda_j^\beta = \begin{cases} \pm\delta_j & \text{if } \beta(\overline{x}_j) = 1, \\ \pm 1 & \text{if } \beta(\overline{x}_j) = -1 \text{ and } \delta_j \text{ is odd}, \\ 0 & \text{if } \beta(\overline{x}_j) = -1 \text{ and } \delta_j \text{ is even}. \end{cases}$$

Therefore, we find:

- If β twists all the letters $x_1, \ldots, x_n$ and all the integers $\delta_1, \ldots, \delta_n$ are even, then $\Lambda_{\mathcal{P}}^\beta = 0$ and so $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) \approx \mathbb{Z}$.
- If β twists some letter x_j and the corresponding δ_j is odd, then $\Lambda_{\mathcal{P}}^\beta \colon \mathbb{Z}^n \to \mathbb{Z}$ is onto and so $H^2(K_{\mathcal{P}};_\beta \mathbb{Z}) = 0$.
- If β is the trivial coefficient system, then $\Lambda_{\mathcal{P}}^\beta = \Delta_{\mathcal{P}}$ and do $H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = H^2(K_{\mathcal{P}};\mathbb{Z}) \approx \mathbb{Z}/\delta\mathbb{Z}$, with $\delta = \gcd(\delta_1, \ldots, \delta_n)$.

Therefore,

$$H^2(K_{\mathcal{P}};_\beta\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \beta \text{ twists all } x_1, \ldots, x_n \text{ and all } \delta_1, \ldots, \delta_n \text{ are even}, \\ \mathbb{Z}/\delta\mathbb{Z} & \text{if } \beta \text{ is the trivial system and } \delta = \gcd(\delta_1, \ldots, \delta_n), \\ 0 & \text{otherwise}. \end{cases}$$

Since $H^2(K_{\mathcal{P}};\mathbb{Z}) = \mathbb{Z}/\delta\mathbb{Z}$, the condition $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$ implies $\delta = 1$ and so the integers δ_i are not all even. The result follows. □

As we anticipated at the beginning of the section, we have the following corollary of the Main Theorem:

COROLLARY 7.2 (of Main Theorem). *Let $K_{\mathcal{P}}$ be a one-block-relator model two-complex. If $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$, then there is non strong surjection from $K_{\mathcal{P}}$ into $\mathbb{R}\mathrm{P}^2$.*

PROOF. It follows from Lemma 7.1 and Theorem 1.1. □

We still do not know if the result of this corollary holds true for one-relator model two-complex whose relator has more than one block, but we known that Lemma 7.1 does not hold, in general, for such two-complexes. We present an example:

EXAMPLE 7.3. Let $K_{\mathcal{P}}$ be the one-relator model two-complex induced by the group presentation $\mathcal{P} = \langle x_1, x_2 \,|\, x_1^{k_1} x_2^{k_2} x_1^{l_1} x_2^{l_2} \rangle$, in which all the powers k_1, l_1, k_2, l_2 are positive integers, so that the relator $s = x_1^{k_1} x_2^{k_2} x_1^{l_1} x_2^{l_2}$ has two blocks. The line-matrix $\Delta_{\mathcal{P}}$ is given by $\Delta_{\mathcal{P}} = [k_1 + l_1 \;\; k_2 + l_2]$ and we have

$$H^2(K_{\mathcal{P}};\mathbb{Z}) = 0 \;\Leftrightarrow\; \gcd(k_1{+}l_1, k_2{+}l_2) = 1.$$

We have three "possible" local integer coefficient systems $\beta_1, \beta_2, \beta_{12} \colon \pi_1(K_{\mathcal{P}}) \to \mathrm{Aut}(\mathbb{Z})$ over $K_{\mathcal{P}}$, other than the trivial one, namely:

$$\beta_1(\overline{x}_1) = -\beta_1(\overline{x}_2) = -1, \quad -\beta_2(\overline{x}_1) = \beta_2(\overline{x}_2) = -1, \quad \beta_{12}(\overline{x}_1) = \beta_{12}(\overline{x}_2) = -1.$$

We spare the reader the necessary calculations to conclude that:

(1) The local coefficient system β_1 is defined if and only if $k_1 + l_1$ is even, and in this case

$$H^2(K_{\mathcal{P}};_{\beta_1}\mathbb{Z}) \approx \begin{cases} \mathbb{Z}/|l_2 + k_2|\mathbb{Z} & \text{if } k_1 \text{ is even,} \\ \mathbb{Z}/|l_2 - k_2|\mathbb{Z} & \text{if } k_1 \text{ is odd.} \end{cases}$$

(2) The local coefficient system β_2 is defined if and only if $k_2 + l_2$ is even, and in this case

$$H^2(K_{\mathcal{P}};_{\beta_2}\mathbb{Z}) \approx \begin{cases} \mathbb{Z}/|k_1 + l_1|\mathbb{Z} & \text{if } k_2 \text{ is even,} \\ \mathbb{Z}/|k_1 - l_1|\mathbb{Z} & \text{if } k_2 \text{ is odd.} \end{cases}$$

(3) The local coefficient system β_{12} is defined if and only if $k_1 + l_1 + k_2 + l_2$ is even, and in this case

$$H^2(K_{\mathcal{P}};_{\beta_{12}}\mathbb{Z}) \approx \begin{cases} 0 & \text{if either } k_1,\, l_1 \text{ or } k_2,\, l_2 \text{ have oposite parity,} \\ \mathbb{Z}_2 & \text{if } k_1,\, l_1,\, k_2,\, l_2 \text{ are odd,} \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Thus, for instance, for $\mathcal{P} = \langle x_1, x_2 \,|\, x_1 x_2^4 x_1 x_2 \rangle$, the unique local coefficient system over $K_{\mathcal{P}}$, other than the trivial one, is β_1, and we have $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$ and $H^2(K_{\mathcal{P}};_{\beta_1}\mathbb{Z}) \approx \mathbb{Z}_3$.

Lemma 7.1 also does not hold, in general, for model two-complexes with more than one relator, even if each relator has just a block. We present an example: Consider the model two-complex $K_{\mathcal{P}}$ of the group presentation $\mathcal{P} = \langle x_1, x_2, x_3 \,|\, x_1^2 x_2 x_3, x_1^2 x_2 x_3^2 \rangle$. The unique local integer coefficient system over $K_{\mathcal{P}}$, other than the trivial one, is β_1, which twists just the generator $\overline{x}_1$ of $\pi_1(K_{\mathcal{P}})$. It is not hard to compute $H^2(K_{\mathcal{P}};\mathbb{Z}) = 0$ and $H^2(K_{\mathcal{P}};_{\beta_1}\mathbb{Z}) \approx \mathbb{Z}_2$.

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