Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 1, 2018, 259–283
DOI: 10.12775/TMNA.2018.023

© 2018 Juliusz Schauder Centre for Nonlinear Studies
Nicolaus Copernicus University

STEADY SOLUTIONS TO THE NAVIER–STOKES–FOURIER SYSTEM FOR DENSE COMPRESSIBLE FLUID

Šimon Axmann — Piotr B. Mucha — Milan Pokorný

In memory of Marek Burnat

Abstract. We establish existence of strong solutions to the stationary Navier–Stokes–Fourier system for compressible flows with density dependent viscosities in regime of heat conducting fluids with very high densities. In comparison to the known results considering the low Mach number case, we work in the L^p-setting combining the methods for the weak solutions with the method of decomposition. Moreover, the magnitude of gradient of the density as well as other data are not limited, our only assumption is the given total mass must be sufficiently large.

2010 *Mathematics Subject Classification.* 35Q35, 76N10.

Key words and phrases. Steady compressible Navier–Stokes–Fourier system; low Mach number limit; strong solution; denisty dependent viscosities; large data; existence via Schauder typefixed point theorem.

The first and the third author were supported by Czech Science Foundation grant No. 16-03230S.

The second author has been partly supported by Polish NCN grant No. 2014/13/B/ST1/03094.

1. Introduction and the main result

Let us consider the following steady version of the Navier–Stokes–Fourier system for the compressible heat-conducting fluid in a bounded domain $\Omega \subset \mathbb{R}^3$

$$\operatorname{div}(\varrho\mathbf{u}) = 0, \tag{1.1}$$

$$\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}\mathbb{S} + \nabla p(\varrho,\theta) = \varrho\mathbf{F}, \tag{1.2}$$

$$\operatorname{div}(\varrho e\mathbf{u}) + \operatorname{div}\mathbf{q} = \mathbb{S} : \mathbb{D}(\mathbf{u}) - p(\varrho,\theta)\operatorname{div}\mathbf{u}, \tag{1.3}$$

with unknowns: the density $\varrho\colon \Omega \to (0,\infty)$, velocity field $\mathbf{u}\colon \Omega \to \mathbb{R}^3$, and the absolute temperature $\theta\colon \Omega \to (0,\infty)$ of the fluid. The viscous part of the stress tensor $\mathbb{S}$, pressure p, internal energy e as well as the heat flux $\mathbf{q}$ are assumed to be given functions of unknowns. The fluid is assumed to be Newtonian with density dependent viscosity

$$\mathbb{S}(\varrho,\nabla\mathbf{u}) = \varrho\big(\nabla\mathbf{u} + \nabla^T\mathbf{u}\big) = 2\varrho\mathbb{D}(\mathbf{u}), \tag{1.4}$$

where we denoted by $\mathbb{D}(\mathbf{u}) = \big(\nabla\mathbf{u} + \nabla^T\mathbf{u}\big)/2$ the symmetric part of the velocity gradient. It is possible to consider more general dependency of the viscosity on the density than (1.4), but it would lead to unnecessary technicalities. In fact, the only important assumption concerning the viscosity is the linear growth. The required form of the tensor $\mathbb{S}$ is connected to shallow water models (see [5], [6] and references therein), where viscosity coefficients are linear functions of the density.

The pressure is assumed to be the sum of the so-called cold pressure and the ideal gas law

$$p(\varrho,\theta) = p_c(\varrho) + \varrho\theta = \varrho^\gamma + \varrho\theta, \tag{1.5}$$

$$e(\varrho,\theta) = \frac{1}{\gamma-1}\varrho^{\gamma-1} + c_v\theta \quad \text{with } \gamma > 1. \tag{1.6}$$

For the general form of pressure we refer to [11], [12]. Here we choose a simple form of the constitutive equations in order to avoid unnecessary technical complications and in the considerations we put just $c_v = 1$. The system is considered in a bounded domain $\Omega \subset \mathbb{R}^3$ with a C^2- smooth boundary. We impose the partial slip boundary condition for the velocity field and we allow the heat flux through the boundary

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{at } \partial\Omega, \tag{1.7}$$

$$\mathbf{n}\cdot\mathbb{S}(\varrho,\nabla\mathbf{u})\cdot\boldsymbol{\tau}^k + f\mathbf{u}\cdot\boldsymbol{\tau}^k = 0 \quad \text{at } \partial\Omega, \tag{1.8}$$

$$\mathbf{q}\cdot\mathbf{n} = L(\varrho,\theta)(\theta - \Theta_0) \qquad \text{at } \partial\Omega, \tag{1.9}$$

where $\boldsymbol{\tau}^k$, $k = 1, 2$ are two linearly independent tangent vectors to $\partial\Omega$, $\mathbf{n}$ denotes the normal vector. The friction is given by

$$f = f_0\varrho(x) \quad \text{with } f_0 > 0, \quad \text{while } \Theta_0(x) \geq \Theta_* > 0 \tag{1.10}$$

stands for the temperature of the boundary from outside.

The heat flux satisfies the Fourier law

$$\mathbf{q} = -\kappa(\varrho,\theta)\nabla\theta.$$

The transport coefficients are assumed in the form

$$\kappa(\varrho,\theta) = \varrho k_1(\theta), \qquad L(\varrho,\theta) = \varrho k_2(\theta), \tag{1.11}$$

with k_i being bounded continuously differentiable functions bounded away from zero. Note that we consider the viscosities independent of the temperature, hence the only influence of the thermal effects on the momentum equation is coming from the pressure. We could consider also some sublinear growth in temperature, but it would lead to unnecessary technicalities (see [11], [16]).

Our main result is stated in the following way.

Theorem 1.1. *Let $\gamma > 1$. Let $\Omega \subset \mathbb{R}^3$ be a C^2 bounded domain, let $p > 3$, $\mathbf{F} \in L^p(\Omega)$, $\Theta_0 \in W^{1-1/p,p}(\partial\Omega)$. Then there exists m_0 sufficiently large with respect to $\|\mathbf{F}\|_p$ and $\|\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}$ in the sense of condition* (3.7) *such that for any $m \geq m_0$, where $m = (1/\Omega)\int_\Omega \varrho\,dx$ is the average of the density, there exists a strong stationary solution to the Navier–Stokes–Fourier system* (1.1)–(1.9). *Moreover, it possesses the regularity $(\varrho,\mathbf{u},\theta) \in W^{1,p}(\Omega)\times W^{2,p}(\Omega;\mathbb{R}^3)\times W^{2,p}(\Omega)$ and there exists a constant $C_{\mathbf{F}}$ depending on $\|\mathbf{F}\|_p$ and $\|\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}$ such that*

$$m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} + \|\theta\|_{2,p} + m^{\gamma-1}\|\operatorname{div}\mathbf{u}\|_p \leq C_{\mathbf{F}}.$$

The estimate in Theorem 1.1 determines, in particular, the smallness of $\operatorname{div}\mathbf{u}$, making the constructed flow slightly compressible. However for $\gamma \in (1,2)$ the magnitude of the perturbation of the density r is of order $m^{2-\gamma}$, what does not allow to look at the system as a perturbation of a constant density flow. On the other hand, the condition imposed on $C_{\mathbf{F}}$ and m excludes the possibility of creation of vacuum regions.

Remark 1.2. If we replace assumption (1.10) by $f = f_0 \geq 0$ we get the same result as in Theorem 1.1 assuming that domain Ω is not axially symmetric. This is connected with the validity of the Korn inequality, i.e. for $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ we have either

$$\|\mathbf{u}\|_{1,p} \leq C(\|\mathbb{D}(\mathbf{u})\|_p + \|\mathbf{u}\|_{L^2(\partial\Omega)}) \tag{1.12}$$

or, if Ω is additionally not axially symmetric,

$$\|\mathbf{u}\|_{1,p} \leq C\|\mathbb{D}(\mathbf{u})\|_p.$$

The result is a generalization of our previous work [3], where the isothermal case was treated. Similar results were obtained by Choe and Jin [7] in the isothermal case and by Dou et al. [8] in the heat-conducting case. The authors study there the low Mach number problem for the steady system with Dirichlet boundary conditions. Working in the H^m framework, they obtain also large solutions as a perturbation of corresponding incompressible flows. The connection to the low Mach number limit problem is through parameter m, which can be compared to the inverse of the Mach number — see [3] for details. In our case the main troublemaker is the equation of energy. The problem is not located in the pressure, where we find an explicit dependence on the temperature, but in energy production term $\mathbb{S} : \mathbb{D}(\mathbf{u})$. We shall underline that dependence on order of parameter m does not neglect this term from main considerations. The only way to treat it comes from the theory of weak solutions [16]. Namely, we have to employ here the entropy type estimate. Roughly speaking, this type of bound is obtained by testing of the energy equation by $-\theta^{-\delta}$ (the physical entropy is related to $\delta = 1$). Note that the magnitude of the velocity field is of order of $\|\mathbf{F}\|_p$ and $\|\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}$, hence it can be arbitrary large. Therefore, we can not expect to obtain the uniqueness property for our system.

The strong solutions for the compressible Navier–Stokes equations were intensively studied in eighties, in the papers by M. Padula [19], [20] or A. Valli [24], [25] in the context of energy method and in the second half of eighties in the papers by H. Beirão da Veiga [4] and M. Padula [21] in the L^p-setting. Later, in the first half of nineties, many further results appeared, considering different situations as bounded or unbounded domains, different methods of proof (energy method, L^p-estimates or method of decomposition) and/or different boundary conditions, see e.g. [2], [7], [9], [10], [14], [18] or [22].

The rest of the article is devoted to the proof of Theorem 1.1. In the next section we derive available *a priori* estimates for the solutions of the system, provided we work within the class (2.2). Next, inspired by the *a priori* bounds, we define suitable function class in which we look for the solution. We also introduce a linearized problem, and show its solvability within this class. We finish the proof by a series of fixed point arguments. Throughout the article we will denote generic constants by C, their value can change from line to line or even in the same formula. Nevertheless, we will be more precise when necessary, especially for the formulation of our assumption for the mean density. We will use the following abbreviate notation for the norms of the Lebesgue and Sobolev spaces over Ω: $\|f\|_p = \|f\|_{L^p(\Omega)}$, $\|f\|_{k,p} = \|f\|_{W^{k,p}(\Omega)}$.

2. *A priori* estimates

Here we derive the *a priori* estimate for sought solutions. First we introduce notation which allows to keep the dependence from parameter m. Assume

$$\varrho = m + r, \quad \text{with} \int_\Omega r \, dx = 0 \quad \text{and} \quad \frac{1}{|\Omega|}\int_\Omega \varrho \, dx = m, \quad \text{with } m \text{ large enough.}$$

Define for $p > d = 3$

$$\Xi = m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} + \|\theta\|_{2,p} + m^{\gamma-1}\|\operatorname{div}\mathbf{u}\|_p \tag{2.1}$$

and consider solutions satisfying the following structure conditions

$$m \gg \Xi, \quad m > 2\|r\|_\infty \quad \text{and} \quad \|\operatorname{div}\mathbf{u}\|_p \ll 1. \tag{2.2}$$

Now we state the expected bounds for the solutions.

LEMMA 2.1. *Let* $(\rho, \mathbf{u}, \theta)$ *be sufficiently smooth solution to* (1.1)–(1.3). *Let* m *be so big that*(2.2) *is fulfilled, then there exists constants* H, E, $C_{\mathbf{F}}$ *depending only on* $\|\mathbf{F}\|_p$ *and* $\|\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}$ *such that*

$$\begin{aligned} \|\theta\|_{3(1-\delta)} + \|\mathbf{u}\|^2_{1,6(1-\delta)/(3-2\delta)} + \|\nabla\theta\|^{1/2}_{3(1-\delta)/(2-\delta)} &\le H \quad \textit{for } \delta \in (0,1/3), \\ \|\mathbf{u}\|_{1,2} &\le E, \\ m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} + \|\theta\|_{2,p} + m^{\gamma-1}\|\operatorname{div}\mathbf{u}\|_p &\le C_{\mathbf{F}}. \end{aligned} \tag{2.3}$$

The rest of this section is devoted to the proof of this lemma. First, we test the internal energy equation by $-\theta^{-\delta}$. Suitable $\delta \in (0, 1/3)$ will be chosen later. In further considerations we will require to fix δ small, but the choice will depend only on the power of integrability of solutions p – see the end of the proof of Lemma 3.2, indeed in the construction of solutions we will require that $\delta \in (0, \delta_0(p))$ by relation (3.31). Hence one can treat δ as given in comparison to the magnitude of $\|\operatorname{div}\mathbf{u}\|_p$. The choice $\delta = 1$ would correspond to the usual entropy estimate. However, this leads only to certain logarithmic estimate for temperature, which is not sufficient. In addition, thanks to (2.2) one can estimate ϱ by $m/2$ from below and $3m/2$ from above. Hence, we obtain after a straightforward computation

$$\begin{aligned} &\int_\Omega \Big(-\kappa(\varrho,\theta)\nabla\theta\cdot\nabla(\theta^{-\delta}) + \frac{\mathbb{S}:\nabla\mathbf{u}}{\theta^\delta}\Big)\,dx + \int_{\partial\Omega} \frac{L(\varrho,\theta)\Theta_0}{\theta^\delta}\,dS \\ &\qquad = \int_{\partial\Omega} L(\varrho,\theta)\theta^{1-\delta}\,dS + \int_\Omega \varrho\mathbf{u}\cdot\nabla\theta\,\theta^{-\delta}\,dx + \int_\Omega \varrho\theta^{1-\delta}\operatorname{div}\mathbf{u}\,dx, \end{aligned} \tag{2.4}$$

whence using the assumptions on $\mathbb{S}$, Θ_0, κ, and L

$$\begin{aligned} &m\delta\int_\Omega \frac{|\nabla\theta|^2}{\theta^{1+\delta}}\,dx + m\int_\Omega \frac{|\mathbb{D}(\mathbf{u})|^2}{\theta^\delta}\,dx + m\int_{\partial\Omega}\frac{1}{\theta^\delta}\,dS \\ &\qquad \le C\bigg(m\int_{\partial\Omega}\theta^{1-\delta}\,dS + \int_\Omega \varrho\mathbf{u}\cdot\frac{\nabla(\theta^{1-\delta})}{1-\delta}\,dx + m\int_\Omega \theta^{1-\delta}|\operatorname{div}\mathbf{u}|\,dx\bigg). \end{aligned}$$

Therefore, using the continuity equation in the second term on the right-hand side we have

$$
\begin{aligned}
(2.5)\qquad m\delta\|\nabla(\theta^{(1-\delta)/2})\|_2^2 + m\bigg\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\bigg\|_2^2
&\leq Cm\bigg(\int_{\partial\Omega}\theta^{1-\delta}\,dS + \int_\Omega \theta^{1-\delta}|\operatorname{div}\mathbf{u}|\,dx\bigg)\\
&\leq Cm\bigg(\int_{\partial\Omega}\theta^{1-\delta}\,dS + \|\theta\|_{3(1-\delta)}^{1-\delta}\|\operatorname{div}\mathbf{u}\|_p\bigg).
\end{aligned}
$$

Besides,

$$
\begin{aligned}
(2.6)\qquad \|\theta\|_{3(1-\delta)}^{1-\delta} = \big\|\theta^{(1-\delta)/2}\big\|_6^2 &\leq C\big\|\theta^{(1-\delta)/2}\big\|_{1,2}^2\\
&\leq C\big(\big\|\theta^{(1-\delta)/2}\big\|_{L^2(\partial\Omega)}^2 + \|\nabla(\theta^{(1-\delta)/2})\|_2^2\big)\\
&= C\bigg(\int_{\partial\Omega}\theta^{1-\delta}\,dS + \big\|\nabla(\theta^{(1-\delta)/2})\big\|_2^2\bigg),
\end{aligned}
$$

and since $\|\operatorname{div}\mathbf{u}\|_p \ll 1$, we put the second term on the right-hand side of (2.5) to the left-hand side. It is possible to do so, since the magnitude of δ depends only on p (see (3.31)). Further,

$$
\begin{aligned}
(2.7)\qquad \|\mathbf{u}\|_{6(1-\delta)} &\leq C\big(\|\mathbb{D}(\mathbf{u})\|_{6(1-\delta)/(3-2\delta)} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big)\\
&\leq C\bigg(\bigg\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\bigg\|_2\|\theta^{\delta/2}\|_{6(1-\delta)/\delta} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\bigg)\\
&= C\bigg(\bigg\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\bigg\|_2\|\theta\|_{3(1-\delta)}^{\delta/2} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\bigg).
\end{aligned}
$$

Thus, combining (2.7) with (2.5) and (2.6) we get

$$
\begin{aligned}
(2.8)\qquad \|\mathbf{u}\|_{6(1-\delta)} &\leq C\big(\big(\|\theta\|_{L^1(\partial\Omega)}^{1-\delta}\big)^{1/2+\delta/(2(1-\delta))} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big)\\
&= C\big(\|\theta\|_{L^1(\partial\Omega)}^{1/2} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big).
\end{aligned}
$$

Further, integrating the internal energy over Ω and testing the momentum equation by $\mathbf{u}$ we obtain for $\delta < 1/2$ the following total energy equality

$$
\begin{aligned}
\int_{\partial\Omega} f|\mathbf{u}\times\mathbf{n}|^2 + L(\varrho,\theta)\theta\,dS &= \int_\Omega \varrho\mathbf{u}\cdot\mathbf{F}\,dx + \int_{\partial\Omega} L(\varrho,\theta)\Theta_0\,dS\\
&\leq Cm\big(\|\mathbf{u}\|_{6(1-\delta)}\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} + \|\Theta_0\|_{L^1(\partial\Omega)}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(2.9)\qquad & mf_0\|\mathbf{u}\|_{L^2(\partial\Omega)}^2 + m\|\theta\|_{L^1(\partial\Omega)}\\
&\qquad \leq Cm\big(\|\mathbf{u}\|_{6(1-\delta)}\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} + \|\Theta_0\|_{L^1(\partial\Omega)}\big).
\end{aligned}
$$

Now, as $\|\operatorname{div}\mathbf{u}\|_p \ll 1$, using the Korn inequality (1.12), we combine (2.4) with (2.9) in order to get

$$\begin{aligned}\|\theta\|_{3(1-\delta)} + \|\mathbf{u}\|^2_{1,6(1-\delta)/(3-2\delta)} &\leq C\big(\|\theta\|_{L^1(\partial\Omega)} + \|\mathbf{u}\|^2_{L^2(\partial\Omega)} + 1\big)\\ &\leq C\big(\|\mathbf{u}\|_{6(1-\delta)}\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} + \|\Theta_0\|_{L^1(\partial\Omega)} + 1\big),\\ \|\theta\|_{3(1-\delta)} + \|\mathbf{u}\|^2_{1,6(1-\delta)/(3-2\delta)} &\leq C\big(\|\mathbf{F}\|^2_{(6-6\delta)/(5-6\delta)} + \|\Theta_0\|_{L^1(\partial\Omega)} + 1\big)\\ &= C(\mathbf{F}, \Theta_0).\end{aligned}$$

Thus, we obtained the first a priori bound. Further we estimate from (2.5) and (2.6)

$$\|\nabla\theta\|_{3(1-\delta)/(2-\delta)} \leq \left\|\frac{\nabla\theta}{\theta^{(1+\delta)/2}}\right\|_2 \|\theta^{(1+\delta)/2}\|_{6(1-\delta)/(1+\delta)} \leq C.$$

Thus, there exists constant H depending only on the given data (independent from m) such that

$$\|\theta\|_{3(1-\delta)} + \|\mathbf{u}\|^2_{1,6(1-\delta)/(3-2\delta)} + \|\nabla\theta\|^{1/2}_{3(1-\delta)/(2-\delta)} \leq H. \tag{2.10}$$

Furthermore, we go back to the momentum equation, and test it by the velocity field, and get

$$\begin{aligned}m\|\nabla\mathbf{u}\|_2^2 + m\|\mathbf{u}\|^2_{L^2(\partial\Omega)} &\leq C\left|\int_\Omega (\nabla(\varrho\theta)\cdot\mathbf{u} + \varrho\mathbf{F}\cdot\mathbf{u})\, dx\right|\\ &\leq C\|\varrho\|_\infty(\|\theta\|_2\|\operatorname{div}\mathbf{u}\|_2 + \|\mathbf{F}\|_{6/5}\|\mathbf{u}\|_6),\end{aligned}$$

hence due to the previous estimate, note that $3(1-\delta) > 2$ for $\delta < 1/3$, we have

$$\|\mathbf{u}\|_{1,2} \leq C(\|\theta\|_2 + \|\mathbf{F}\|_{6/5}) \leq C(\mathbf{F}, \Theta_0) =: E. \tag{2.11}$$

Here we underline that E and H are independent of m.

The above information is not sufficient, we are required to improve the regularity. Further, we use the ideas of the method of decomposition from [18], in the same spirit as in [3]. More precisely, we use the Helmholtz decomposition for functions in $L^p(\Omega)$ with values in $\mathbb{R}^3$. Let us decompose $\mathbf{g} = P_H(\mathbf{g}) + \nabla P_\triangledown(\mathbf{g})$ and denote the corresponding linear operators by

$$P_\triangledown\colon L^p(\Omega) \to W^{1,p}(\Omega) \quad \text{and} \quad P_H\colon L^p(\Omega) \to L^p_{\operatorname{div}}(\Omega). \tag{2.12}$$

They possess the following properties $\int_\Omega P_\triangledown \mathbf{g}\, dx = 0$, $\operatorname{div}\mathbf{g} = \Delta P_\triangledown(\mathbf{g})$, and $\mathbf{n}\cdot P_H(\mathbf{g}) = 0$ on $\partial\Omega$. The main idea of the method of decomposition is to estimate the solenoidal and the gradient part of the momentum equation separately. We are allowed to follow this procedure thanks to properties of slip boundary conditions put on the velocity field. In what follows, we denote

$$\mathcal{G} = -\varrho\mathbf{u}\cdot\nabla\mathbf{u} + 2\operatorname{div}(r\mathbb{D}(\mathbf{u})) + \varrho\mathbf{F}$$

for the sake of clarity. First, we apply curl-operator on (1.2) yielding, for $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$,

$$-m\Delta\boldsymbol{\omega} = \operatorname{curl}\mathcal{G} \quad \text{in } \Omega$$

with the boundary conditions

$$\operatorname{div}\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega}\cdot\boldsymbol{\tau}_1 = (f/(m+r) - 2\chi_2)\mathbf{u}\cdot\boldsymbol{\tau}_2, \quad \boldsymbol{\omega}\cdot\boldsymbol{\tau}_2 = (2\chi_1 - f/(m+r))\mathbf{u}\cdot\boldsymbol{\tau}_2$$

on $\partial\Omega$, where χ_i stand for the curvatures corresponding to the vectors $\boldsymbol{\tau}_i$, tangent to $\partial\Omega$. Note that $f/(m+r) = f_0$ is constant. The form of boundary conditions in the system above can be deduced by differentiation of (1.7), see [17], [13]. Thus, due to Lemma A.2 (see Appendix)

$$m\|\boldsymbol{\omega}\|_{1,p} \le C\big(\|\operatorname{curl}\mathcal{G}\|_{(W^{1,p'}(\Omega))^*} + m\|\mathbf{u}\|_{W^{1-1/p,p}(\partial\Omega)}\big),$$

where $(W^{1,p'}(\Omega))^*$ denotes the dual space to $W_0^{1,p'}(\Omega)$. Note that

$$\|\mathbf{u}\|_{W^{1-1/p,p}(\partial\Omega)} \le C\|\mathbf{u}\|_{1,p}.$$

Further, $P_H\mathbf{u}$ satisfies the overdetermined system

$$\begin{aligned} \operatorname{curl} P_H\mathbf{u} &= \boldsymbol{\omega} && \text{in } \Omega,\\ \operatorname{div} P_H\mathbf{u} &= 0 && \text{in } \Omega,\\ P_H\mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.13}$$

so by Theorem A.5 (in the Appendix)

$$\|\nabla^2 P_H\mathbf{u}\|_p \le C\|\boldsymbol{\omega}\|_{1,p}, \quad \text{thus} \quad \|\nabla^2 P_H\mathbf{u}\|_p \le \frac{C}{m}\big(\|\mathcal{G}\|_p + m\|\nabla\mathbf{u}\|_p\big). \tag{2.14}$$

Similarly, the potential part of the momentum equation (1.2) reads

$$p(\varrho) - \{p(\varrho)\}_\Omega - 2m\operatorname{div}\mathbf{u} = P_\nabla\big(\mathcal{G} + m\Delta P_H\mathbf{u}\big). \tag{2.15}$$

Here we introduce the notation: $\{g\}_\Omega = (1/|\Omega|)\int_\Omega g\,dx$ for any g integrable.

In our considerations we keep in mind that $P_\nabla(\Delta\mathbf{u}) = \Delta P_\nabla\mathbf{u} + P_\nabla(\Delta P_H\mathbf{u})$ and $\Delta P_\nabla\mathbf{u} = \operatorname{div}\mathbf{u}$. Observe (Taylor's expansion) that

$$p_c(\varrho) = \varrho^\gamma = (m+r)^\gamma = m^\gamma + \gamma m^{\gamma-1} r + R_m(r), \tag{2.16}$$

where $R_m(r) = p_c''(\xi)r^2/2$ and ξ is between m and $m+r$, whence $|R_m(r)| = |p_c''(\xi)r^2|/2 \le Cm^{\gamma-2}r^2$. Subtracting the average from (2.16) yields

$$p(\varrho) - \{p(\varrho)\}_\Omega = \gamma m^{\gamma-1} r + R_m(r) - \{R_m(r)\}_\Omega + \varrho\theta - \{\varrho\theta\}_\Omega. \tag{2.17}$$

Then we combine

$$\gamma m^{\gamma-1} r - 2m\operatorname{div}\mathbf{u} + R_m(r) - \{R_m(r)\}_\Omega + \varrho\theta - \{\varrho\theta\}_\Omega = P_\nabla(\mathcal{G} + m\Delta P_H\mathbf{u}) \tag{2.18}$$

with the continuity equation

$$m\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla r + r\operatorname{div}\mathbf{u} = 0, \tag{2.19}$$

in order to get

$$\begin{aligned} \gamma m^{\gamma-1} r + 2\mathbf{u}\cdot\nabla r = {}& -2r\operatorname{div}\mathbf{u} + P_\nabla(\mathcal{G} + m\Delta P_H\mathbf{u})\\ &- R_m(r) + \{R_m(r)\}_\Omega - \varrho\theta + \{\varrho\theta\}_\Omega. \end{aligned} \tag{2.20}$$

Further, by differentiating (2.20), we obtain

$$\gamma m^{\gamma-1}\nabla r + 2\mathbf{u}\cdot\nabla\nabla r = -2\nabla r \operatorname{div}\mathbf{u} - 2r\nabla\operatorname{div}\mathbf{u} - 2\nabla\mathbf{u}\nabla r - \nabla R_m(r) - \theta\nabla r - \varrho\nabla\theta + \nabla P_\triangledown(\mathcal{G} + m\Delta P_H\mathbf{u}). \tag{2.21}$$

Note that $P_\triangledown$ is continuous from L^p to $W^{1,p}$, so $\nabla P_\triangledown$ is actually a zero order operator. Thus, to obtain from (2.21) the required information about ∇r, we test the k-th component of (2.21) by $\partial_k r|\partial_k r|^{p-2}$. The second term on the left hand side can be then rewritten using integration by parts as

$$\int_\Omega \mathbf{u}\cdot\nabla\partial_k r|\partial_k r|^{p-2}\partial_k r\ dx = -\frac{1}{p}\int_\Omega \operatorname{div}\mathbf{u}|\partial_k r|^p\ dx;$$

$|\nabla R_m(r)| \le Cm^{\gamma-2}|r||\nabla r|$. Thus, we get due to the Poincaré inequality and the fact that $\|\theta\|_\infty + \|\nabla\mathbf{u}\|_{1,p} \ll m^{\gamma-1}$ following from (2.2)

$$\begin{aligned} m^{\gamma-1}\|r\|_{1,p} &\le C\big(\|\nabla r\|_p\|\nabla\mathbf{u}\|_{1,p} + \|\theta\nabla r\|_p + m\|\nabla\theta\|_p \\ &\quad + \|\mathcal{G}\|_p + m\|\nabla^2 P_H\mathbf{u}\|_p\big) \\ &\le C\big(m\|\nabla\theta\|_p + \|\mathcal{G}\|_p + m\|\nabla^2 P_H\mathbf{u}\|_p\big). \end{aligned} \tag{2.22}$$

Furthermore, using (2.18), we bound the potential part of the velocity. Since

$$2m\nabla\operatorname{div}\mathbf{u} = \gamma m^{\gamma-1}\nabla r + \nabla\big(R_m(r) - \{R_m(r)\}_\Omega + \varrho\theta - \{\varrho\theta\}_\Omega\big) - \nabla P_\triangledown\big(\mathcal{G} + m\Delta P_H\mathbf{u}\big),$$

we obtain for the quantity $\nabla\operatorname{div}\mathbf{u}$ a similar estimate, namely

$$m\|\nabla\operatorname{div}\mathbf{u}\|_p \le C\big(m^{\gamma-1}\|\nabla r\|_p + \|\mathcal{G}\|_p + \|\theta\nabla r\|_p + m\|\nabla\theta\|_p + m\|\nabla^2 P_H\mathbf{u}\|_p\big). \tag{2.23}$$

Putting together (2.14), (2.22) and (2.23) yields

$$m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} \le C\bigg(\frac{1}{m}\|\mathcal{G}\|_p + \|\nabla\mathbf{u}\|_p + \|\nabla\theta\|_p\bigg).$$

Recalling that

$$\mathcal{G} = -\varrho\mathbf{u}\cdot\nabla\mathbf{u} + 2\mathbb{D}(\mathbf{u})\nabla r + r\Delta\mathbf{u} + r\nabla\operatorname{div}\mathbf{u} + \varrho\mathbf{F},$$

it is easy to see that the most restrictive term is, except the external force, the convective term. The rest can be put to the left-hand side by means of the Young inequality due to the fact $\|\nabla r\|_p \ll m$, like $\|\mathbb{D}(\mathbf{u})\nabla r/m\|_p \ll \|\mathbf{u}\|_{1,\infty} \le \|\mathbf{u}\|_{2,p}$.

We estimate the convective term by means of the Gagliardo–Nirenberg interpolation inequality (A.1) (see Appendix) and energy inequality (2.11) as follows

$$\begin{aligned}\|\varrho\mathbf{u}\cdot\nabla\mathbf{u}\|_p &\le \|\varrho\|_\infty\|\mathbf{u}\|_\infty\|\nabla\mathbf{u}\|_p,\\ \|\mathbf{u}\|_\infty &\le C\|\mathbf{u}\|_6^{2/3}\|\mathbf{u}\|_{1,\infty}^{1/3},\\ \|\nabla\mathbf{u}\|_p &\le C\|\nabla\mathbf{u}\|_2^{2p/(5p-6)}\|\nabla\mathbf{u}\|_{1,p}^{(3p-6)/(5p-6)},\\ \|\varrho\mathbf{u}\cdot\nabla\mathbf{u}\|_p &\le C(m+\|r\|_\infty)\|\mathbf{u}\|_6^{2/3}\|\nabla\mathbf{u}\|_2^{2p/(5p-6)}\|\mathbf{u}\|_{1,\infty}^{1/3}\|\nabla\mathbf{u}\|_{1,p}^{(3p-6)/(5p-6)}\\ &\le CmE^{(16p-12)/(15p-18)}\|\mathbf{u}\|_{2,p}^{(14p-24)/(15p-18)}.\end{aligned}$$

Note that $(14p-24)/(15p-18)<1$ for $p>18/15$, and E is defined by (2.11). Hence the decomposition method applied on the momentum equation eventually yields

$$\|\mathbf{u}\|_{2,p}+m^{\gamma-2}\|r\|_{1,p}\le C(\|\mathbf{F}\|_p+\|\nabla\theta\|_p).$$

Interpolation (A.1) allows us to estimate the term with the gradient of the temperature

$$\|\nabla\theta\|_p\le C\|\nabla\theta\|_{3(1-\delta)/(2-\delta)}^{1-\alpha}\|\nabla\theta\|_{1,p}^{\alpha}\quad\text{with } \alpha=\frac{2p-3+\delta(3-p)}{3p-3+\delta(3-2p)}<1. \tag{2.24}$$

Thus, we have from (2.24) with α

$$\|\mathbf{u}\|_{2,p}+m^{\gamma-2}\|r\|_{1,p}\le C(\|\nabla\theta\|_{1,p}^{\alpha}+1), \tag{2.25}$$

where C depends only on given data.

Therefore, it remains to bound the second gradient of the temperature. State the energy equation in the following form

$$\begin{aligned}-\operatorname{div}(\kappa(\varrho,\theta)\nabla\theta) &= \varrho|\nabla\mathbf{u}|^2-\varrho\mathbf{u}\cdot\nabla\theta-\varrho\theta\operatorname{div}\mathbf{u} &&\text{in } \Omega,\\ -\kappa(\varrho,\theta)\nabla\theta\cdot\mathbf{n} &= L(\varrho,\theta)(\theta-\Theta_0) &&\text{on } \partial\Omega.\end{aligned}$$

By Lemma A.3 (after dividing by m) we conclude

$$\begin{aligned}\|\theta\|_{2,p}\le C\big(&\|\nabla\mathbf{u}\|_{2p}^2+\|\theta\|_\infty\|\operatorname{div}\mathbf{u}\|_p\\ &+\|\mathbf{u}\cdot\nabla\theta\|_p+\|\theta-\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}\big)=C(I_1+I_2+I_3+I_4).\end{aligned} \tag{2.26}$$

Let us estimate the right-hand side of (2.26) term by term. To start, I_2 can be directly put to the left-hand side due to smallness of $\operatorname{div}\mathbf{u}$. Next, in order to bound $\nabla\mathbf{u}$ in L^q for $q>2$, we interpolate between the energy norm and higher order estimate of $\nabla^2\mathbf{u}$, namely

$$\|\nabla\mathbf{u}\|_{2p}\le C\|\nabla\mathbf{u}\|_2^{1-\beta}\|\nabla\mathbf{u}\|_{1,p}^{\beta}\quad\text{with } \beta=\frac{3p-3}{5p-6},$$

so $I_1\le CE^{2-2\beta}\|\nabla\mathbf{u}\|_{1,p}^{2\beta}$, where

$$2\alpha\beta=2\cdot\frac{2p-3+\delta(3-p)}{3p-3+\delta(3-2p)}\cdot\frac{3p-3}{5p-6}<1\quad\text{for any } p>3,\ \delta<\frac{1}{2}.$$

Hence we put the term to the left-hand side of (2.25) by means of the Young inequality. For term I_3 we again interpolate, see (A.1)

$$\|\mathbf{u}\|_\infty \le C\|\mathbf{u}\|_6^{1-\lambda}\|\mathbf{u}\|_{2,p}^\lambda \tag{2.27}$$

with $\lambda = p/(5p-6)$. Thus, using (2.24) and (2.27)

$$I_3 \le \|\mathbf{u}\|_\infty\|\nabla\theta\|_p \le CE^{1-\lambda}(\|\nabla^2\mathbf{u}\|_p^\lambda + E^\lambda)H^{1-\alpha}(\|\nabla^2\theta\|_p^\alpha + H^\alpha).$$

In order to move the highest terms to the left-hand side of (2.25) and (2.26), we need $\lambda\alpha + \alpha < 1$, so

$$\frac{\lambda\alpha}{1-\alpha} = \frac{p}{5p-6}\cdot\frac{2p-3+\delta(3-p)}{p(1-\delta)} < 1,$$

which is definitely satisfied for any $p > 3$ and $\delta < 1/3$. Finally, note

$$\begin{aligned}\|\theta-\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)} &\le C(1+\|\theta\|_{1,p})\\ &\le C(1+\|\nabla\theta\|_{3(1-\delta)/(2-\delta)}^{1-\alpha}\|\nabla\theta\|_{1,p}^\alpha) \le C(1+H^{1-\alpha}\|\nabla\theta\|_{1,p}^\alpha).\end{aligned}$$

Therefore, from (2.10), (2.11), (2.25), and analysis for (2.26) we conclude

$$\Xi \le C_{\mathbf{F}}.$$

We set $m \gg C_{\mathbf{F}}$ and it guarantees that we will work in the announced regularity class. Finally, looking at the continuity equation which can be stated as

$$m\operatorname{div}\mathbf{u} = -r\operatorname{div}\mathbf{u} - \mathbf{u}\cdot\nabla r,$$

we obtain

$$\begin{gathered}(m-\|r\|_\infty)\|\operatorname{div}\mathbf{u}\|_p \le \|\mathbf{u}\cdot\nabla r\|_p,\\ m\|\operatorname{div}\mathbf{u}\|_p \le 2\|\mathbf{u}\cdot\nabla r\|_p \le 2C_{\mathbf{F}}^2 m^{2-\gamma}.\end{gathered} \tag{2.28}$$

3. Approximation, construction of solutions

In this part we prove the main theorem. The construction of solutions is done in spaces described by Section 2. Let us denote classes of functions, where solutions to (1.1)–(1.3) are constructed.

$$\begin{aligned}M_r(m) &= \Big\{f\in W^{1,p}(\Omega),\ \int_\Omega f\,dx = 0,\ m^{\gamma-2}(\|f\|_\infty + \|\nabla f\|_p) \le C_{\mathbf{F}}\Big\},\\ M_{\mathbf{u}}(m) &= \big\{\mathbf{f}\in W^{2,p}(\Omega;\mathbb{R}^3),\ \mathbf{f}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega,\ \|\nabla\mathbf{f}\|_2 \le E,\\ &\qquad \|\nabla\mathbf{f}\|_\infty + \|\mathbf{f}\|_\infty + \|\nabla^2\mathbf{f}\|_p \le C_{\mathbf{F}},\ m^{\gamma-1}\|\operatorname{div}\mathbf{f}\|_p \le 2C_{\mathbf{F}}^2\big\},\\ M_\theta(m) &= \big\{f\in W^{1,\infty}(\Omega),\ f>0 \text{ in } \Omega,\\ &\qquad \|f\|_{3(1-\delta)} + \|\nabla f\|_{3(1-\delta)/(2-\delta)}^{1/2} \le H,\ \|f\|_\infty + \|\nabla f\|_\infty \le C_{\mathbf{F}}\big\},\end{aligned}$$

where $C_{\mathbf{F}}$, E and H depend only on given data Ω, p, $\mathbf{F}$, Θ_0 and $\delta \in (0,\delta_0(p))$ — see (3.31).

Note that $M_{\mathbf{u}}(m)$ is not a compact subset of $W^{2,p}(\Omega)$. Therefore, to perform a fixed point argument, there is a need to introduce additionally another set, which is a closed subset of $W^{1,\infty}(\Omega)$, $M_{\mathbf{u}}(m) \subset M_{\operatorname{div}\mathbf{u}}$, namely

$$M_{\operatorname{div}\mathbf{u}} = \big\{\mathbf{f} \in W^{1,\infty}(\Omega;\mathbb{R}^3),\ \mathbf{f}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega, \\ \big\|\nabla\mathbf{f}\big\|_2 \le E,\ \|\nabla\mathbf{f}\|_\infty + \|\mathbf{f}\|_\infty \le C_{\mathbf{F}},\ m^{\gamma-1}\|\operatorname{div}\mathbf{f}\|_p \le 2C_{\mathbf{F}}^2\big\}.$$

In order to explain the definition of $M_\theta(m)$ let us note that the part bounded by H comes from the entropy type estimate (one with δ) and the second one is coming from the full regularity obtained for the solution to the heat law.

Our general strategy, which heavily depends on the fact that the temperature occurs in the momentum equation only through the pressure, is the following. First, we solve for fixed $\mathbf{U} \in M_{\operatorname{div}\mathbf{u}}$ and $\widetilde{\theta} \in M_\theta(m)$ the system

$$m\operatorname{div}\mathbf{u} + \operatorname{div}(r\mathbf{u}) = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$(m+r)\,\mathbf{U}\cdot\nabla\mathbf{u} - \operatorname{div}(2(m+r)\mathbb{D}(\mathbf{u})) + \nabla p\big(m+r,\widetilde{\theta}\big) = (m+r)\mathbf{F} \quad \text{in } \Omega, \tag{3.2}$$

with boundary conditions (1.7)–(1.8). Then, for the resulting $\varrho, \mathbf{u}$ we look at he problem for the temperature

$$-\operatorname{div}(\kappa(\varrho,\widetilde{\theta})\nabla\theta) = -\operatorname{div}(\varrho e\mathbf{u}) + \mathbb{S}:\mathbb{D}(\mathbf{u}) - p(\varrho,\theta)\operatorname{div}\mathbf{u} \quad \text{in } \Omega, \tag{3.3}$$

$$-\kappa(\varrho,\widetilde{\theta})\nabla\theta\cdot\mathbf{n} = L(\varrho,\widetilde{\theta})(\theta-\Theta_0) \quad \text{on } \partial\Omega. \tag{3.4}$$

This linearization enables to show easily the uniqueness as well as the comparison principle for the resulting temperature. Finally, we verify that the mapping

$$(\mathbf{U},\widetilde{\theta}) \mapsto (\mathbf{u},\theta)$$

maps continuously $M_{\operatorname{div}\mathbf{u}} \times M_\theta$ into its compact subset $M_{\mathbf{u}} \times (M_\theta \cap W^{2,p}(\Omega))$ thus we will be able to find a fixed point of this mapping via the Schauder fixed point theorem.

Lemma 3.1. *Suppose* $\mathbf{U} \in M_{\operatorname{div}\mathbf{u}}(m)$, *and* $\widetilde{\theta} \in M_\theta(m)$ *for* m *sufficiently large, then problem* (3.1)–(3.2) *with boundary conditions* (1.7)–(1.9) *admits a unique solution* $(r,\mathbf{u})$ *in the class* $M_r(m) \times M_{\mathbf{u}}(m)$.

Proof. The proof follows ideas from [3, Proposition 4.1]. From that reason we point here only the main estimate. The only difference is hidden in the pressure which is temperature dependent. Additional term $\nabla(\varrho\widetilde{\theta})$ is bounded in $L^\infty(\Omega)$, hence it is treated in the same way as the right-hand side $\mathbf{F}$. Let us briefly explain the construction of solutions to (3.1)–(3.2) done in [3]. The main troublemaker is the hyperbolic equation and so lack of compactness for the perturbation of density r. Here we apply the well-known approach from the theory of weak solutions to the compressible Navier–Stokes equations [16]. It is based on the regularization of the continuity equation by adding $-\varepsilon\Delta r$. Then for given $\varepsilon > 0$, there is no problem to construct solutions. Then, having a sequence depending on ε with suitable estimates, we pass to limit $\varepsilon \to 0$ by standard arguments. For details we refer to [3]. The proof is based on several fixed point arguments within the regularity classes $M_r(m)$ and $M_{\mathbf{u}}(m)$.

Crucial are the precise estimates. The energy estimate reads in our case

$$m\|\nabla \mathbf{u}\|_2^2 \le \left| \int_\Omega \varrho \mathbf{F} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla(\varrho\widetilde{\theta})\, dx \right| \le Cm\big(\|\mathbf{F}\|_{6/5}\|\mathbf{u}\|_6 + \|\operatorname{div} \mathbf{u}\|_2\|\widetilde{\theta}\|_2\big)$$
$$\le Cm\big(\|\mathbf{F}\|_{6/5}^2 + \|\widetilde{\theta}\|_2^2\big) + \frac{m}{2}\|\nabla \mathbf{u}\|_2^2.$$

Therefore

$$\|\nabla \mathbf{u}\|_2^2 \le C\big(\|\mathbf{F}\|_{6/5}^2 + H^2\big) =: E^2.$$

Further, following the *a priori* approach, we get

$$m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} \le C(\|\mathbf{F}\|_p + \|\nabla\widetilde{\theta}\|_p).$$

We interpolate

$$\|\nabla\widetilde{\theta}\|_p \le \|\nabla\widetilde{\theta}\|_{3(1-\delta)/(2-\delta)}^{1-\alpha'}\|\nabla\widetilde{\theta}\|_\infty^{\alpha'} \quad \text{with} \quad \alpha' = 1 - \frac{3(1-\delta)}{p(2-\delta)} = \frac{2p-3+\delta(3-p)}{2p-\delta p} < 1. \tag{3.5}$$

Therefore,

$$m^{\gamma-2}\|r\|_{1,p} + \|\mathbf{u}\|_{2,p} \le C\big(\|\mathbf{F}\|_p + H^{1-\alpha'}C_{\mathbf{F}}^{\alpha'}\big) =: C_{\mathbf{F}} \tag{3.6}$$

for $C_{\mathbf{F}}$ sufficiently large. At this stage we choose m sufficiently larger than $C_{\mathbf{F}}$. Namely, we obtain the restriction on the magnitude of parameter m in terms of E and $C_{\mathbf{F}}$ defined in Section 2 as follows

$$\min\big(m, m^{(\gamma-1)/4}\big) > \max\big(C_{\mathbf{F}}(1+E^2), C_{\mathbf{F}}^2(1+E^2)\big)C_\Omega, \tag{3.7}$$

where C_Ω represents constant depending purely on Ω by means of the constants from the Korn, Poincaré, and embedding inequalities. Note that we can later on redefine $C_{\mathbf{F}}$ and consequently m, as inequality (3.6) is satisfied for any larger $C_{\mathbf{F}}$. We also recall (2.28). □

LEMMA 3.2. *Suppose* $\mathbf{u} \in M_{\mathbf{u}}(m)$ *and* $r \in M_r(\Omega)$ *are from Lemma* 3.1*, then for* m *sufficiently large there exists a unique solution* θ *to problem* (3.3) *in the class* $M_\theta(m) \cap W^{2,p}(\Omega)$.

PROOF. To begin, we rewrite the equation for the internal energy using constitutive relations (1.5)–(1.6), the continuity equation in the following way

$$-\operatorname{div}(\kappa(\varrho,\widetilde{\theta})\nabla\theta) = \mathbb{S} : \mathbb{D}(\mathbf{u}) - \varrho\mathbf{u}\cdot\nabla\theta - \varrho\theta \operatorname{div}\mathbf{u} \quad \text{in } \Omega. \tag{3.8}$$

Let us first consider for fixed $\overline{\theta} \in W^{1,\infty}(\Omega)$ the problem

$$-\operatorname{div}(\kappa(\varrho,\widetilde{\theta})\nabla\theta) = \mathbb{S} : \mathbb{D}(\mathbf{u}) - \varrho\mathbf{u}\cdot\nabla\overline{\theta} - \varrho\overline{\theta} \operatorname{div}\mathbf{u} \quad \text{in } \Omega, \tag{3.9}$$

$$-\kappa(\varrho,\widetilde{\theta})\nabla\theta\cdot\mathbf{n} = L(\varrho,\widetilde{\theta})(\overline{\theta} - \Theta_0) \quad \text{at } \partial\Omega, \tag{3.10}$$

it can be uniquely solved in $W^{2,p}(\Omega)$. Indeed, the coefficient of the elliptic operator is in $W^{1,p}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$ with some $\alpha > 0$ and the right-hand side of (3.9) is bounded in $L^p(\Omega)$, see (3.29). Moreover, the mapping $\overline{\theta} \mapsto \theta$ is for fixed $\mathbf{u}$, ϱ continuous in $W^{1,2}(\Omega)$ and since $W^{2,p}(\Omega)$ is compactly embedded into $W^{1,\infty}(\Omega)$, also compact. Therefore, the map has a fixed point. Let us show that it is positive everywhere and unique. A rough idea is that the right-hand side of (3.8) is non-negative plus some perturbation which is in some sense small, because we are close to the incompressibility.

For the comparison principle, we use the following standard approach. For a solution θ we denote $\theta^- = \min(0, \theta - \Theta_*)$ with Θ_* from (1.10) and test equation (3.8) by θ^-. We obtain

$$\begin{aligned}(3.11)\quad \int_\Omega -\operatorname{div}\big(\kappa(\varrho,\widetilde{\theta})\nabla\theta\big)\theta^- \, dx &= \int_\Omega \mathbb{S}:\mathbb{D}(\mathbf{u})\theta^- \, dx \\ &\quad - \int_\Omega \varrho\,\mathbf{u}\cdot\nabla\theta\,\theta^- \, dx - \int_\Omega \varrho\,\theta \operatorname{div}\mathbf{u}\,\theta^- \, dx,\end{aligned}$$

$$\begin{aligned}(3.12)\quad &\int_\Omega \kappa\big(\varrho,\widetilde{\theta}\big)\nabla\theta\cdot\nabla\theta^- \, dx + \int_{\partial\Omega} L\big(\varrho,\widetilde{\theta}\big)(\theta-\Theta_0)\theta^- \, dS \\ &= \int_\Omega \mathbb{S}:\mathbb{D}(\mathbf{u})\theta^- \, dx - \int_\Omega \varrho\mathbf{u}\cdot\nabla\frac{|\theta^-|^2}{2} \, dx - \int_\Omega \varrho\operatorname{div}\mathbf{u}|\theta^-|^2 \, dx,\end{aligned}$$

thus using the constitutive assumptions on $\mathbb{S}$, L, κ

$$\begin{aligned}(3.13)\quad &\int_\Omega m|\nabla\theta^-|^2 \, dx + \int_{\partial\Omega} m|\theta^-|^2 \, dS \le C\bigg(\int_{\partial\Omega} m\Theta_0\theta^- \, dS \\ &+ \int_\Omega m|\mathbb{D}(\mathbf{u})|^2\theta^- \, dx + \int_\Omega |\operatorname{div}(\varrho\mathbf{u})|\frac{|\theta^-|^2}{2} \, dx + \int_\Omega |\varrho\operatorname{div}\mathbf{u}||\theta^-|^2 \, dx\bigg).\end{aligned}$$

The first two terms on the right-hand side of (3.13) are non-positive by definition of θ^-, the third one vanishes due to the continuity equation, while the last one can be estimated

$$m\|\theta^-\|_{1,2}^2 \le Cm\|\operatorname{div}\mathbf{u}\|_p\|\theta^-\|_6^2,$$

but since $m\|\operatorname{div}\mathbf{u}\|_p \ll m$, we conclude $\|\theta^-\|_{1,2} \le 0$, $\theta^- = 0$, and thus $\theta \ge \Theta_*$ in Ω.

The proof of the uniqueness follows similar lines as before. We consider two solutions θ_1, θ_2 satisfying for $i = 1, 2$

$$(3.14)\quad -\operatorname{div}(\kappa(\varrho,\widetilde{\theta})\nabla\theta_i) = \mathbb{S}:\mathbb{D}(\mathbf{u}) - \varrho\mathbf{u}\cdot\nabla\theta_i - \varrho\theta_i\operatorname{div}\mathbf{u} \quad \text{in } \Omega,$$

$$(3.15)\quad -\kappa(\varrho,\widetilde{\theta})\nabla\theta_i\cdot\mathbf{n} = L(\varrho,\widetilde{\theta})(\theta_i - \Theta_0) \quad \text{on } \partial\Omega,$$

hence taking the difference

$$-\operatorname{div}\big(\kappa(\varrho,\widetilde{\theta})\nabla(\theta_1-\theta_2)\big) = -\varrho\mathbf{u}\cdot\nabla(\theta_1-\theta_2) - \varrho(\theta_1-\theta_2)\operatorname{div}\mathbf{u} \quad \text{in } \Omega, \tag{3.16}$$

$$-\kappa(\varrho,\widetilde{\theta})\nabla(\theta_1-\theta_2)\cdot\mathbf{n} = L(\varrho,\widetilde{\theta})(\theta_1-\theta_2) \quad \text{on } \partial\Omega. \tag{3.17}$$

Testing (3.16) by $(\theta_1-\theta_2)$ yields as before

$$m\|\nabla(\theta_1-\theta_2)\|_2^2 + m\|\theta_1-\theta_2\|_{L^2(\partial\Omega)}^2 \le C\int_\Omega \operatorname{div}(\varrho\mathbf{u})\frac{|\theta_1-\theta_2|^2}{2}\,dx + Cm\|\operatorname{div}\mathbf{u}\|_p\|\theta_1-\theta_2\|_6^2,$$

since $\|\operatorname{div}\mathbf{u}\|_p \ll 1$, $\|\theta_1-\theta_2\|_{1,2}=0$ and the uniqueness is proved.

Let us now show that the resulting θ is in $M_\theta(m)$. It is a solution to

$$-\operatorname{div}(\kappa(\varrho,\widetilde{\theta})\nabla\theta) = \mathbb{S}:\mathbb{D}(\mathbf{u}) - \varrho\mathbf{u}\cdot\nabla\theta - \varrho\theta\operatorname{div}\mathbf{u}, \tag{3.18}$$

$$-\kappa(\varrho,\widetilde{\theta})\nabla\theta\cdot\mathbf{n} = L(\varrho,\widetilde{\theta})(\theta-\Theta_0) \quad \text{at } \partial\Omega, \tag{3.19}$$

where r and $\mathbf{u}$ satisfy (3.1)–(3.2) with (1.7)–(1.8). We follow the heuristic approach; we test (3.18) by $-\theta^{-\delta}$. This yields

$$\int_\Omega\Big(-\kappa(\varrho,\widetilde{\theta})\nabla\theta\cdot\nabla(\theta^{-\delta}) + \frac{\mathbb{S}:\nabla\mathbf{u}}{\theta^\delta}\Big)\,dx + \int_{\partial\Omega}\frac{L(\varrho,\widetilde{\theta})\Theta_0}{\theta^\delta}\,dS = \int_{\partial\Omega}L(\varrho,\widetilde{\theta})\theta^{1-\delta}\,dS + \int_\Omega \varrho\mathbf{u}\cdot\nabla\theta\,\theta^{-\delta}\,dx + \int_\Omega\varrho\theta^{1-\delta}\operatorname{div}\mathbf{u}\,dx, \tag{3.20}$$

$$m\delta\int_\Omega\frac{|\nabla\theta|^2}{\theta^{1+\delta}}\,dx + m\int_\Omega\frac{|\mathbb{D}(\mathbf{u})|^2}{\theta^\delta}\,dx \le C\Big(\int_{\partial\Omega}L(\varrho,\widetilde{\theta})\theta^{1-\delta}\,dS + \int_\Omega\varrho\mathbf{u}\cdot\frac{\nabla(\theta^{1-\delta})}{1-\delta}\,dx + m\int_\Omega\theta^{1-\delta}|\operatorname{div}\mathbf{u}|\,dx\Big). \tag{3.21}$$

Hence

$$\begin{aligned} m\delta\big\|\nabla(\theta^{(1-\delta)/2})\big\|_2^2 + m\Big\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\Big\|_2^2 &\le C\Big(\int_{\partial\Omega}L(\varrho,\widetilde{\theta})\theta^{1-\delta}\,dS + m\int_\Omega\theta^{1-\delta}|\operatorname{div}\mathbf{u}|\,dx\Big) \\ &\le Cm\Big(\int_{\partial\Omega}\theta^{1-\delta}\,dS + \|\theta\|_{3(1-\delta)}^{1-\delta}\|\operatorname{div}\mathbf{u}\|_p\Big). \end{aligned} \tag{3.22}$$

Besides,

$$\begin{aligned} \|\theta\|_{3(1-\delta)}^{1-\delta} &= \|\theta^{(1-\delta)/2}\|_6^2 \le C\|\theta^{(1-\delta)/2}\|_{1,2}^2 \\ &\le C\big(\big\|\theta^{(1-\delta)/2}\big\|_{L^2(\partial\Omega)}^2 + \|\nabla(\theta^{(1-\delta/2)})\|_2^2\big) \\ &= C\Big(\int_{\partial\Omega}\theta^{1-\delta}\,dS + \|\nabla(\theta^{(1-\delta)/2})\|_2^2\Big), \end{aligned}$$

and since $\|\operatorname{div}\mathbf{u}\|_p \ll 1$, we put the second term on the right-hand side of (3.22) to the left-hand side. Further,

$$
\begin{aligned}
\|\mathbf{u}\|_{6(1-\delta)} &\le C\big(\|\mathbb{D}(\mathbf{u})\|_{6(1-\delta)/(3-2\delta)} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big) \\
&\le C\bigg(\bigg\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\bigg\|_2 \|\theta^{\delta/2}\|_{6(1-\delta)/\delta} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\bigg) \\
&= C\bigg(\bigg\|\frac{\mathbb{D}(\mathbf{u})}{\theta^{\delta/2}}\bigg\|_2 \|\theta\|^{\delta/2}_{3(1-\delta)} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\bigg).
\end{aligned}
\tag{3.23}
$$

Thus, combining (3.23) with (3.22) we get

$$
\begin{aligned}
\|\mathbf{u}\|_{6(1-\delta)} &\le C\big(\big(\|\theta\|^{1-\delta}_{L^1(\partial\Omega)}\big)^{1/2+\delta/(2(1-\delta))} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big) \\
&= C\big(\|\theta\|^{1/2}_{L^1(\partial\Omega)} + \|\mathbf{u}\|_{L^2(\partial\Omega)}\big).
\end{aligned}
\tag{3.24}
$$

Further, we obtain for $\delta < 1/2$ the following total energy inequality

$$
\begin{aligned}
\int_{\partial\Omega} (f|\mathbf{u}\times\mathbf{n}|^2 + L(\varrho,\widetilde{\theta})\theta)\,dS &= -\int_\Omega \varrho(\mathbf{U}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,dx \\
&\quad + \int_\Omega \varrho\mathbf{u}\cdot\mathbf{F} + \big(p(\varrho,\widetilde{\theta}) - p(\varrho,\theta)\big)\operatorname{div}\mathbf{u}\,dx + \int_{\partial\Omega} L(\varrho,\widetilde{\theta})\Theta_0\,dS \\
&\le C\int_\Omega m|\mathbf{u}|^2|\operatorname{div}\mathbf{U}| + |\nabla r||\mathbf{U}||\mathbf{u}|^2 + m|\mathbf{u}\cdot\mathbf{F}| + m|\widetilde{\theta}-\theta||\operatorname{div}\mathbf{u}|\,dx \\
&\quad + \int_{\partial\Omega} L(\varrho,\widetilde{\theta})\Theta_0\,dS \\
&\le C\bigg(m\|\mathbf{u}\|^2_{6(1-\delta)}\|\operatorname{div}\mathbf{U}\|_p + \|\nabla r\|_p\|\mathbf{U}\|_\infty\|\mathbf{u}\|^2_{6(1-\delta)} \\
&\quad + m\|\mathbf{u}\|_{6(1-\delta)}\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} \\
&\quad + m\|\widetilde{\theta}-\theta\|_{3(1-\delta)}\|\operatorname{div}\mathbf{u}\|_p + \int_{\partial\Omega} L(\varrho,\widetilde{\theta})\Theta_0\,dS\bigg) \\
&\le Cm\big(\|\mathbf{u}\|_{6(1-\delta)}(\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} + 1) \\
&\quad + \|\Theta_0\|_{L^1(\partial\Omega)} + \|\widetilde{\theta}-\theta\|_{3(1-\delta)}\|\operatorname{div}\mathbf{u}\|_p\big),
\end{aligned}
\tag{3.25}
$$

where we have assumed that $E\|\operatorname{div}\mathbf{U}\|_p < 1$, $C^2_{\mathbf{F}} < m$, which is guaranteed by the choice of m. Therefore,

$$
\begin{aligned}
\|\mathbf{u}\|^2_{L^2(\partial\Omega)} + \|\theta\|_{L^1(\partial\Omega)} &\le C\big(\|\mathbf{u}\|_{6(1-\delta)}(\|\mathbf{F}\|_{(6-6\delta)/(5-6\delta)} + 1) \\
&\quad + \|\Theta_0\|_{L^1(\partial\Omega)} + \|\widetilde{\theta}-\theta\|_{3(1-\delta)}\|\operatorname{div}\mathbf{u}\|_p\big).
\end{aligned}
\tag{3.26}
$$

Now, as $\|\operatorname{div}\mathbf{u}\|_p \ll 1$, we combine (3.20) and (3.26) to get

$$
\begin{aligned}
&\|\theta_{3(1-\delta)}\| + \|\mathbf{u}\|^2_{1,6(1-\delta)/(3-2\delta)} \le C\|\theta\|_{L^1(\partial\Omega)} \\
&\quad\le C\big(\|\mathbf{u}\|_{6(1-\delta)}(\|\mathbf{F}\|_{6-6\delta/(5-6\delta)} + 1) + \|\Theta_0\|_{L^1(\partial\Omega)} + \|\widetilde{\theta}\|_{3(1-\delta)}\|\operatorname{div}\mathbf{u}\|_p\big) \\
&\quad\le C\big(\|\mathbf{F}\|^2_{(6-6\delta)/(5-6\delta)} + 1 + \|\Theta_0\|_{L^1(\partial\Omega)} + H\|\operatorname{div}\mathbf{u}\|_p\big) \le H(\mathbf{F},\Theta_0)
\end{aligned}
\tag{3.27}
$$

for H sufficiently large. Further we estimate

$$\|\nabla\theta\|_{3(1-\delta)/(2-\delta)} \le \left\| \frac{\nabla\theta}{\theta^{(1+\delta)/2}} \right\|_2 \|\theta^{(1+\delta)/2}\|_{6(1-\delta)/(1+\delta)} \le H^2.$$

According to results of Lemma 3.1, we have with α' from (3.5)

$$\|\nabla^2\mathbf{u}\|_p + m^{\gamma-2}\|\nabla r\|_p \le C\big(\|\nabla^2\widetilde{\theta}\|_p^{\alpha'} + 1\big) \le C\big(C_{\mathbf{F}}^{\alpha'} + 1\big), \tag{3.28}$$

where C depends only on the given data.

It remains to bound the second gradient of the temperature. We have

$$\begin{aligned} -\operatorname{div}\big(\kappa(\varrho,\widetilde{\theta})\nabla\theta\big) &= \varrho|\nabla\mathbf{u}|^2 - \varrho\mathbf{u}\cdot\nabla\theta - \varrho\theta\operatorname{div}\mathbf{u} \quad &&\text{in } \Omega,\\ -\kappa(\varrho,\widetilde{\theta})\nabla\theta\cdot\mathbf{n} &= L(\varrho,\widetilde{\theta})(\theta-\Theta_0) &&\text{on } \partial\Omega, \end{aligned}$$

so, by Lemma A.2,

$$\begin{aligned} \|\theta\|_{2,p} \le C\big(&\|\nabla\mathbf{u}\|_{2p}^2 + \|\theta\|_\infty\|\operatorname{div}\mathbf{u}\|_p \\ &+ \|\mathbf{u}\cdot\nabla\theta\|_p + \|\theta-\Theta_0\|_{W^{1-1/p,p}(\partial\Omega)}\big) = C(I_1+I_2+I_3+I_4). \end{aligned} \tag{3.29}$$

We estimate the right-hand side of (3.29) term by term, analogously as in the *a priori* approach. The only difference is that in (3.28) we shall interpolate only with $W^{1,\infty}(\Omega)$ in order to reach compactness of the resulting mapping. Term I_2 is again put to the left-hand side according to smallness of divergence. For I_1 we have

$$I_1 \le CE^{2-2\beta}\|\nabla\mathbf{u}\|_{1,p}^{2\beta} \quad \text{with } \beta = \frac{3p-3}{5p-6},$$

and we need

$$2\alpha'\beta = 2\cdot\frac{2p-3+\delta(3-p)}{2p-\delta p}\cdot\frac{3p-3}{5p-6} < 1.$$

This inequality can be satisfied for some $\delta > 0$ sufficiently small only for

$$p < \frac{9+3\sqrt{5}}{2}. \tag{3.30}$$

In the case of $\mathbf{F} \in L^p(\Omega)$ with greater p, we have to perform the construction first with some p satisfying (3.30) and then we study the regularity a posteriori. Condition (3.30) looks artificial, however we shall remember that in order to estimate I_1 we use only $W^{1,\infty}$, not $W^{2,p}$. And α' given by (3.5) is essentially different than α from (2.24).

For the term I_3 we obtain

$$I_3 \le \|\mathbf{u}\|_\infty\|\nabla\theta\|_p \le CE^{1-\lambda}\|\nabla^2\mathbf{u}\|_p^\lambda H^{1-\alpha}\|\nabla^2\theta\|_p^\alpha$$

with $\lambda = p/(5p-6)$ and α from (2.24). Again, we need

$$\frac{\lambda\alpha'}{1-\alpha} = \frac{p}{5p-6}\cdot\frac{2p-3+\delta(3-p)}{p(2-\delta)}\cdot\frac{3p-3+\delta(3-2p)}{p(1-\delta)} < 1, \tag{3.31}$$

which can be satisfied for any $p > 3$ provided δ is chosen sufficiently small, relation (3.31) determines $\delta_0(p)$.

To summarize, we conclude $\|\theta\|_{2,p} \leq C_{\mathbf{F}}$, which completes the proof. $\square$

In order to apply the Schauder fixed point theorem on the mapping $(\mathbf{U}, \widetilde{\theta}) \mapsto (\mathbf{u}, \theta)$, it remains to prove that the mapping is continuous. This is a consequence of the following two lemmas.

Lemma 3.3. *The solution operator* $\big(\mathbf{U}, \widetilde{\theta}\big) \mapsto (\mathbf{u}, r)$ *of problem* (3.1)–(3.2) *is continuous in* $M_{\operatorname{div}\mathbf{u}}(m) \times M_\theta(m)$ *as a mapping from* $W^{1,2} \times L^2$ *to* $W^{1,2} \times L^2$.

Proof. Let us consider a solution to (3.1)–(3.2) $\mathbf{u}_1, r_1$ corresponding to $\widetilde{\theta}_1, \mathbf{U}_1$. Denoting $\varrho_1 = m + r_1$ we have

$$m \operatorname{div} \mathbf{u}_1 + \operatorname{div}(r_1 \mathbf{u}_1) = 0, \tag{3.32}$$

$$\begin{aligned}(m + r_1)\mathbf{U}_1 \cdot \nabla \mathbf{u}_1 - \operatorname{div}\big(2(m + r_1)\mathbb{D}(\mathbf{u}_1)\big) + \gamma m^{\gamma-1}\nabla r_1 + \nabla R_m(r_1)& \\ = \varrho_1 \mathbf{F} - \nabla(\varrho_1 \widetilde{\theta}_1)&,\end{aligned} \tag{3.33}$$

and similarly, for $\mathbf{u}_2, r_2$ corresponding to $\widetilde{\theta}_2, \mathbf{U}_2$,

$$m \operatorname{div} \mathbf{u}_2 + \operatorname{div}(r_2 \mathbf{u}_2) = 0, \tag{3.34}$$

$$\begin{aligned}(m + r_2)\mathbf{U}_2 \cdot \nabla \mathbf{u}_2 - \operatorname{div}\big(2(m + r_2)\mathbb{D}(\mathbf{u}_2)\big) + \gamma m^{\gamma-1}\nabla r_2 + \nabla R_m(r_2)& \\ = \varrho_2 \mathbf{F} - \nabla(\varrho_2 \widetilde{\theta}_2)&.\end{aligned} \tag{3.35}$$

Subtraction yields

$$m \operatorname{div}(\mathbf{u}_1 - \mathbf{u}_2) + \operatorname{div}((r_1 - r_2)\mathbf{u}_1) + \operatorname{div}(r_2(\mathbf{u}_1 - \mathbf{u}_2)) = 0, \tag{3.36}$$

$$\begin{aligned}&\varrho_1(\mathbf{U}_1 - \mathbf{U}_2) \cdot \nabla \mathbf{u}_1 + \varrho_1 \mathbf{U}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2) \\ &\quad + (r_1 - r_2)\mathbf{U}_2 \cdot \nabla \mathbf{u}_2 - \operatorname{div}\big(2\varrho_1 \mathbb{D}(\mathbf{u}_1 - \mathbf{u}_2) + 2(r_1 - r_2)\mathbb{D}(\mathbf{u}_2)\big) \\ &\qquad + \gamma m^{\gamma-1}\nabla(r_1 - r_2) + \nabla\big(R_m(r_1) - R_m(r_2)\big) \\ &\qquad\qquad = (r_1 - r_2)\mathbf{F} - \nabla\big((r_1 - r_2)\widetilde{\theta}_1 + \varrho_2(\widetilde{\theta}_1 - \widetilde{\theta}_2)\big).\end{aligned} \tag{3.37}$$

Let us test equation (3.36) by difference $\gamma m^{\gamma-2}(r_1 - r_2)$, this yields after some integration by parts

$$\begin{aligned}&\int_\Omega \gamma m^{\gamma-1}(\mathbf{u}_1 - \mathbf{u}_2) \cdot \nabla(r_1 - r_2)\, dx \\ &\qquad = \gamma m^{\gamma-2} \int_\Omega \frac{|r_1 - r_2|^2}{2} \operatorname{div} \mathbf{u}_1 + (r_1 - r_2)(\mathbf{u}_1 - \mathbf{u}_2) \\ &\qquad\qquad \cdot \nabla r_2 + r_2(r_1 - r_2)\operatorname{div}(\mathbf{u}_1 - \mathbf{u}_2)\, dx.\end{aligned} \tag{3.38}$$

Further, testing equation (3.37) by $\mathbf{u}_1 - \mathbf{u}_2$ we obtain the kinetic energy estimate, the second resulting term can be transformed as follows

$$\begin{aligned}\int_\Omega (m + r_1)\mathbf{U}_2 \cdot \nabla(\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{u}_1 - \mathbf{u}_2)\, dx &= \frac{1}{2}\int_\Omega (m + r_1)\mathbf{U}_2 \cdot \nabla|\mathbf{u}_1 - \mathbf{u}_2|^2\, dx \\ = -\int_\Omega (m + r_1)\operatorname{div}\mathbf{U}_2 \frac{|\mathbf{u}_1 - \mathbf{u}_2|^2}{2}\, dx &- \int_\Omega \frac{|\mathbf{u}_1 - \mathbf{u}_2|^2}{2}\mathbf{U}_2 \cdot \nabla r_1\, dx,\end{aligned}$$

in order to be pushed to the left-hand side of the resulting inequality, while for the leading term of the pressure we can use (3.38). Therefore, as in [3] we can conclude

$$
\begin{aligned}
(3.39)\quad m\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 \le{}& Cm^{\gamma-2}\|r_1-r_2\|_2^2\|\operatorname{div}\mathbf{u}_1\|_\infty \\
&+C\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\big(m\|\mathbf{U}_1-\mathbf{U}_2\|_3\|\nabla\mathbf{u}_1\|_2+m^{\gamma-2}\|r_1-r_2\|_2\|\nabla r_2\|_3 \\
&+m^{\gamma-2}\|r_1-r_2\|_2\|r_2\|_\infty+m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2 \\
&+\|r_1-r_2\|_2\big(\|\mathbf{U}_2\cdot\nabla\mathbf{u}_2\|_3+\|\nabla\mathbf{u}_2\|_\infty+\|\mathbf{F}\|_3+\big\|\widetilde{\theta}_1\big\|_\infty\big)\big),
\end{aligned}
$$

and with the help of the Young inequality

$$
\begin{aligned}
m\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 \le{}& Cm^{\gamma-2}\|r_1-r_2\|_2^2\|\operatorname{div}\mathbf{u}_1\|_\infty \\
&+C\big(m\|\mathbf{U}_1-\mathbf{U}_2\|_3^2\|\nabla\mathbf{u}_1\|_2^2+m^{2\gamma-5}\|r_1-r_2\|_2^2\|\nabla r_2\|_3^2 \\
&+m^{2\gamma-5}\|r_1-r_2\|_2^2\|r_2\|_\infty^2+m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2^2 \\
&+m^{-1}\|r_1-r_2\|_2^2\big(\|\mathbf{U}_2\cdot\nabla\mathbf{u}_2\|_3^2+\|\nabla\mathbf{u}_2\|_\infty^2+\|\mathbf{F}\|_3^2+\big\|\widetilde{\theta}_1\big\|_\infty^2\big)\big),
\end{aligned}
$$

and

$$
\begin{aligned}
(3.40)\quad m\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 \le{}& C\big(m^{\gamma-2}\|r_1-r_2\|_2^2\|\operatorname{div}\mathbf{u}_1\|_\infty+m\|\mathbf{U}_1-\mathbf{U}_2\|_3^2E^2 \\
&+m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2^2+\|r_1-r_2\|_2^2\big(m^{-1}C_{\mathbf{F}}^2(E^2+1)+m^{-1}\|\mathbf{F}\|_3^2\big)\big).
\end{aligned}
$$

To finish the estimates we need to obtain L^2 estimate of the difference r_1-r_2 by means of the Bogovskiĭ operator from Lemma A.4. We test (3.37) by $\mathbf{\Phi}=\mathcal{B}[\varrho_2-\varrho_1]$ in order to obtain

$$
\begin{aligned}
m^{\gamma-1}\|r_1-r_2\|_2^2 \le{}& C\big(m\|\mathbf{U}_2\|_3\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\|\mathbf{\Phi}\|_6+m\|\mathbf{U}_1-\mathbf{U}_2\|_3\|\nabla\mathbf{u}_2\|_2\|\mathbf{\Phi}\|_6 \\
&+\|r_1-r_2\|_2\|\mathbf{F}\|_3\|\mathbf{\Phi}\|_6+\|r_1-r_2\|_2\|\mathbf{U}_2\|_\infty\|\nabla\mathbf{u}_2\|_3\|\mathbf{\Phi}\|_6 \\
&+2m\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\|\nabla\mathbf{\Phi}\|_2+\|r_1-r_2\|_2\|\nabla\mathbf{u}_2\|_\infty\|\nabla\mathbf{\Phi}\|_2 \\
&+\|r_1-r_2\|_2\big\|\widetilde{\theta}_1\big\|_\infty\|\nabla\mathbf{\Phi}\|_2+m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2\|\nabla\mathbf{\Phi}\|_2\big),
\end{aligned}
$$

and using again the Young inequality and the fact that $C_{\mathbf{F}}^2, \|\mathbf{F}\|_3 \ll m^{\gamma-1}$

$$
\begin{aligned}
(3.41)\quad m^{\gamma-1}\|r_1-r_2\|_2^2 \le{}& Cm^{3-\gamma}\big(\|\mathbf{U}_2\|_3^2\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 \\
&+\|\mathbf{U}_1-\mathbf{U}_2\|_3^2\|\nabla\mathbf{u}_2\|_2^2+\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2+\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2^2\big) \\
\le{}& Cm^{3-\gamma}\big(\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2(E^2+1) \\
&+\|\mathbf{U}_1-\mathbf{U}_2\|_3^2\|\nabla\mathbf{u}_2\|_2^2+\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2^2\big).
\end{aligned}
$$

Then we combine (3.41) with estimate (3.40) and conclude

$$
(3.42)\quad m^{\gamma-2}\|r_1-r_2\|_2^2+\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 \le C(\mathbf{F},m)\big(\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2^2+\|\mathbf{U}_1-\mathbf{U}_2\|_{1,2}^2\big). \square
$$

Further, we write estimates of higher order for systems for (3.32) and (3.34), based on analysis from Section 2,

$$m^{\gamma-1}\|r_i\|_{1,p}+\|\mathbf{u}_i\|_{2,p}\le C(\mathbf{F},m)\big(\|\mathbf{U}_i\|_{1,\infty}+\big\|\widetilde{\theta}_i\big\|_{1,\infty}+1\big). \tag{3.43}$$

Hence, interpolation yields

$$\begin{aligned}\|\mathbf{u}_1-\mathbf{u}_2\|_{1,\infty}&\le C\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^{\beta}\|\mathbf{u}_1-\mathbf{u}_2\|_{2,p}^{1-\beta}\\&\le C(\mathbf{F},m)\big(\|\mathbf{U}_1-\mathbf{U}_2\|_{1,2}+\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2\big)^{\beta}\big(\|\mathbf{U}_1\|_{1,\infty}^{1-\beta}+\|\widetilde{\theta}_1\|_{1,\infty}^{1-\beta}+1\big)\end{aligned}$$

for some $\beta\in(0,1)$, and similarly for the density since

$$\|r_1-r_2\|_{1,2}\le C\|r_1-r_2\|_2^{1-\beta'}\big(\|r_1\|_{1,p}^{\beta'}+\|r_2\|_{1,p}^{\beta'}\big)$$

with $\beta'\in(0,1)$. This implies that the mapping from Lemma 3.3 is in fact continuous in norms

$$W^{1,\infty}(\Omega)\times W^{1,\infty}(\Omega)\to W^{1,\infty}(\Omega)\times W^{1,2}(\Omega). \tag{3.44}$$

Lemma 3.4. *The mapping* $\big(\mathbf{u},r,\widetilde{\theta}\big)\mapsto\theta$ *of the solution operator to* (3.3) *is continuous on* $M_{\operatorname{div}\mathbf{u}}(m)\times M_r(m)\times M_\theta(m)$ *as a mapping from* $W^{1,2}\times W^{1,2}\times W^{1,2}$ *to* $W^{1,2}$.

Proof. Consider two solutions θ_i, $i=1,2$ corresponding to $\mathbf{u}_i$, r_i, $\widetilde{\theta}_i$

$$\begin{aligned}-\operatorname{div}(\kappa(\varrho_i,\widetilde{\theta}_i)\nabla\theta_i)&=\mathbb{S}(\varrho_i,\mathbf{u}_i):\mathbb{D}(\mathbf{u}_i)-\varrho_i\mathbf{u}_i\cdot\nabla\theta_i-\varrho\theta_i\operatorname{div}\mathbf{u}_i&&\text{in }\Omega,\\-\kappa(\varrho_i,\widetilde{\theta}_i)\nabla\theta_i\cdot\mathbf{n}&=L(\varrho_i,\widetilde{\theta}_i)(\theta_i-\Theta_0)&&\text{on }\partial\Omega.\end{aligned}$$

Taking the difference of those two equations, multiplying by $\theta_1-\theta_2$ and integrating over Ω yields

$$\begin{aligned}&\int_\Omega\big(\kappa(\varrho_1,\widetilde{\theta}_1)\nabla\theta_1-\kappa(\varrho_2,\widetilde{\theta}_2)\nabla\theta_2\big)\cdot\nabla(\theta_1-\theta_2)dx\\&\quad+\int_{\partial\Omega}\big(L(\varrho_1,\widetilde{\theta}_1)(\theta_1-\Theta_0)-L(\varrho_2,\widetilde{\theta}_2)(\theta_2-\Theta_0)\big)(\theta_1-\theta_2)\,dS\\&=-\int_\Omega\big(\varrho_1\theta_1\operatorname{div}\mathbf{u}_1-\varrho_2\theta_2\operatorname{div}\mathbf{u}_2+\varrho_1\mathbf{u}_1\cdot\nabla\theta_1-\varrho_2\mathbf{u}_2\cdot\nabla\theta_2\big)(\theta_1-\theta_2)\,dx\\&\quad+\int_\Omega\big(\varrho_1|\mathbb{D}(\mathbf{u}_1)|^2-\varrho_2|\mathbb{D}(\mathbf{u}_2)|^2\big)(\theta_1-\theta_2)\,dx.\end{aligned} \tag{3.45}$$

We rearrange the terms in order to see the differences. The left-hand side of (3.45) can be written

$$\begin{aligned}\text{l.h.s.}=\int_\Omega&\kappa(\varrho_1,\widetilde{\theta}_1)|\nabla(\theta_1-\theta_2)|^2+\big((r_1-r_2)k_1(\widetilde{\theta}_1)\\&+\varrho_2\big(k_1(\widetilde{\theta}_1)-k_1(\widetilde{\theta}_2)\big)\big)\nabla\theta_2\cdot\nabla(\theta_1-\theta_2)\,dx\\+\int_{\partial\Omega}&L(\varrho_1,\widetilde{\theta}_1)|\theta_1-\theta_2|^2+\big((r_1-r_2)k_2(\widetilde{\theta}_1)\\&+\varrho_2\big(k_2(\widetilde{\theta}_1)-k_2(\widetilde{\theta}_2)\big)\big)(\theta_2-\Theta_0)(\theta_1-\theta_2)\,dS\end{aligned}$$

and the right-hand side

$$
\begin{aligned}
\text{r.h.s} = \int_\Omega & \varrho_1\big(|\mathbb{D}(\mathbf{u}_1)|^2 - |\mathbb{D}(\mathbf{u}_2)|^2\big)(\theta_1-\theta_2) + (r_1-r_2)|\mathbb{D}(\mathbf{u}_2)|^2(\theta_1-\theta_2) \\
& - \int_\Omega \varrho_1 \operatorname{div}\mathbf{u}_1(\theta_1-\theta_2)^2 \\
& \qquad + (r_1-r_2)\theta_2 \operatorname{div}\mathbf{u}_1(\theta_1-\theta_2) + \varrho_2\theta_2\operatorname{div}(\mathbf{u}_1-\mathbf{u}_2)(\theta_1-\theta_2)\,dx \\
& - \int_\Omega \varrho_1\mathbf{u}_1\cdot\nabla\frac{(\theta_1-\theta_2)^2}{2} \\
& \qquad + (r_1-r_2)\mathbf{u}_1\nabla\theta_2(\theta_1-\theta_2) + \varrho_2(\mathbf{u}_1-\mathbf{u}_2)\nabla\theta_2(\theta_1-\theta_2)\,dx = \sum_{n=1}^{8} J_n.
\end{aligned}
$$

The leading terms on the left-hand side give us estimate on $m\|\theta_1-\theta_2\|_{1,2}^2$, while the rest will be estimated. Using the Lipschitz continuity of k_1 and k_2

$$
\begin{aligned}
\int_\Omega & \big((r_1-r_2)k_1(\widetilde{\theta}_1) + \varrho_2\big(k_1(\widetilde{\theta}_1)-k_1(\widetilde{\theta}_2)\big)\big)\nabla\theta_2\cdot\nabla(\theta_1-\theta_2)\ dx \\
& \le C\big(\|r_1-r_2\|_2 + m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_2\big)\|\nabla\theta_2\|_\infty\|\theta_1-\theta_2\|_{1,2},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\int_{\partial\Omega} & \big((r_1-r_2)k_2(\widetilde{\theta}_1) + \varrho_2\big(k_2(\widetilde{\theta}_1)-k_2(\widetilde{\theta}_2)\big)\big)(\theta_2-\Theta_0)(\theta_1-\theta_2)\,dS \\
& \le C\big(\|r_1-r_2\|_{1,2} + m\big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_{1,2}\big)\big(\|\theta_2\|_{1,p} + \|\Theta\|_{L^\infty(\partial\Omega)}\big)\|\theta_1-\theta_2\|_{1,2}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
|J_1| &\le Cm\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\|\mathbf{u}_1+\mathbf{u}_2\|_{1,\infty}\|\theta_1-\theta_2\|_2, \\
|J_2| &\le C\|r_1-r_2\|_2\|\mathbf{u}_2\|_{1,\infty}^2\|\theta_1-\theta_2\|_2, \\
|J_3| &\le Cm\|\operatorname{div}\mathbf{u}_1\|_p\|\theta_1-\theta_2\|_{1,2}^2, \\
|J_4| &\le \|r_1-r_2\|_2\|\theta_2\|_\infty\|\operatorname{div}\mathbf{u}_1\|_\infty\|\theta_1-\theta_2\|_2, \\
|J_5| &\le Cm\|\theta_2\|_\infty\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\|\theta_1-\theta_2\|_2, \\
J_6 &= \int_\Omega \operatorname{div}(\varrho_1\mathbf{u}_1)\frac{(\theta_1-\theta_2)^2}{2}\ dx = 0, \\
|J_7| &\le C\|r_1-r_2\|_2\|\mathbf{u}_1\|_\infty\|\theta_2\|_{1,\infty}\|\theta_1-\theta_2\|_2, \\
|J_8| &\le m\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}\|\theta_2\|_{1,\infty}\|\theta_1-\theta_2\|_2.
\end{aligned}
$$

Therefore, the term J_3 can be put directly to the left-hand side, since $\|\operatorname{div}\mathbf{u}_1\|_p \ll 1$, while for all the others we use the Young inequality. Thus, we conclude

$$
\|\theta_1-\theta_2\|_{1,2}^2 \le C(\mathbf{F},\Theta_0,m)\big(\|\mathbf{u}_1-\mathbf{u}_2\|_{1,2}^2 + \|r_1-r_2\|_{1,2}^2 + \big\|\widetilde{\theta}_1-\widetilde{\theta}_2\big\|_{1,2}^2\big). \quad \square
$$

In order to finish the proof of the main theorem, we again use the higher order estimate for θ with (3.43)

$$\|\theta_i\|_{2,p} \leq C(m)\big(\|\mathbf{u}_i\|_{2,p} + \|r_i\|_{2,p} + \big\|\widetilde{\theta}_i\big\|_{1,\infty} + 1\big) \leq C(m)\big(\|\mathbf{U}_i\|_{1,\infty} + \big\|\widetilde{\theta}_i\big\|_{1,\infty} + 1\big)$$

and interpolate with some $\beta \in (0,1)$

$$\|\theta_1 - \theta_2\|_{1,\infty} \leq C\|\theta_1 - \theta_2\|_{1,2}^{\beta}\big(\|\theta_1\|_{2,p} + \|\theta_2\|_{2,p}\big)^{1-\beta}$$

to conclude that the mapping from the previous lemma is in fact continuous in norms

$$W^{1,\infty} \times W^{1,2} \times W^{1,\infty} \mapsto W^{1,\infty}.$$

Therefore, combining with (3.44) $(\mathbf{U}, \widetilde{\theta}) \mapsto (\mathbf{u}, \theta)$ continuously maps closed subset $M_{\operatorname{div}\mathbf{u}}(m) \times M_\theta(m) \subset W^{1,\infty} \times W^{1,\infty}$ into its compact subset $M_{\mathbf{u}}(m) \times (M_\theta(m) \cap W^{2,p}(\Omega))$. Thus, according to the Schauder theorem it possesses a fixed point. This concludes the proof of the main theorem.

Remark 3.5. Note that we could replace the assumption that the viscosity and the heat conductivity depend on the density by an assumption that these quantities are constant and sufficiently large.

Remark 3.6. We proved in detail the existence of a solution only for $p < (9+3\sqrt{5})/2$. Note however, that if p is larger than this value (or equal to it), we get existence of a solution in the corresponding spaces for some $p_0 < (9+3\sqrt{5})/2$. Remember, the domain is bounded, so the imbeddings in a lower space are trivial. Since the temperature is bounded strictly away from zero (see the proof of Lemma 3.2), we can repeat the arguments from Section 2 (a priori estimates) to show that our solution belongs not only to spaces corresponding to the integrability exponent p_0, but also to the spaces corresponding to the integrability exponent $p \geq (9+3\sqrt{5})/2$.

Appendix A

At this place we want to state few technical lemmas, which are used throughout the proof. We frequently use the well-known Gagliardo–Nirenberg interpolation inequality

Lemma A.1. *Let $k \in \mathbb{N}$, $p > 3$, $\lambda \in [0,1]$ then we have*

$$\|f\|_r \leq C\|f\|_q^{1-\lambda}\|f\|_{k,p}^{\lambda} \quad \textit{provided} \quad \frac{1}{r} = \lambda\left(\frac{1}{p} - \frac{k}{3}\right) + (1-\lambda)\frac{1}{q}. \tag{A.1}$$

We recall the L^p-regularity properties of an elliptic equation in a divergence form with Neumann boundary condition

$$\begin{aligned} -\operatorname{div}(A(x)\nabla u) &= g \quad \text{in } \Omega,\\ A\nabla u\cdot\mathbf{n} &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{A.2}$$

where we assume A to be elliptic $\Big(\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$, for all $\xi \in \mathbb{R}^3\Big)$ with some $\alpha > 0$ and symmetric ($a_{ij} = a_{ji}$). The following result due to Agmon, Douglis, Nirenberg (see [1]) holds true.

Lemma A.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary, $1 < p < \infty$, assume that $A \in C^1(\overline{\Omega})$, $g \in L^p(\Omega)$, $h \in W^{1-1/p,p}(\partial\Omega)$. Then there exists $C > 0$ such that any solution to* (A.2) *satisfies*

$$\|u\|_{2,p} \leq C\big(\|g\|_p + \|h\|_{W^{1-1/p,p}(\partial\Omega)} + \|u\|_p\big). \tag{A.3}$$

Further, we need the regularity properties of the semilinear heat equation

$$\begin{aligned} -\operatorname{div}\big(\varrho(x)k(u)\nabla u\big) &= g \quad \text{in } \Omega,\\ -\varrho(x)k(u)\nabla u\cdot\mathbf{n} &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{A.4}$$

with $\varrho \geq m/2 > 0$, and k as in (1.11).

Lemma A.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2 boundary, $3 < p < \infty$, assume that $\varrho \in W^{1,p}(\Omega)$, $g \in L^p(\Omega)$, $h \in W^{1-1/p,p}(\partial\Omega)$. Then there exists $C > 0$ such that any solution to* (A.4) *satisfies*

$$\|u\|_{2,p} \leq C\big(\|g\|_p + \|h\|_{W^{1-1/p,p}(\partial\Omega)} + \|u\|_p\big). \tag{A.5}$$

Proof. The proof is a consequence of previous Lemma A.2 and the Kirchhoff transform. Namely, we define

$$T = \widetilde{k}(u) = \int_0^u k(s)\,ds,$$

then we have $\operatorname{div}(\varrho k(u)\nabla u) = \operatorname{div}(\varrho\nabla T)$ and thus (A.4) transforms to linear equation

$$\begin{aligned} -\operatorname{div}(\varrho(x)\nabla T) &= g \quad \text{in } \Omega,\\ -\varrho(x)\nabla T\cdot\mathbf{n} &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{A.6}$$

as $W^{1,p}(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$ the equation is elliptic with regular enough coefficients hence by previous lemma we have

$$m\|T\|_{2,p} \leq C\big(\|g\|_p + \|h\|_{W^{1-1/p,p}(\partial\Omega)} + \|T\|_p\big).$$

Finally, as $\widetilde{k}^{-1}$ is Lipschitz continuous, we have $\theta = \widetilde{k}^{-1}(T) \in W^{1,\infty}(\Omega)$. Thus, the whole coefficient $\varrho(x)k(u)$ in (A.4) possesses the regularity $W^{1,p}(\Omega)$, and we can apply Lemma A.2 directly on (A.4), which finishes the proof. □

Lemma A.4 (Bogovskiĭ). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, then there exists a bounded linear operator $\mathcal{B}$*

$$\mathcal{B} : \left\{f, f \in L^p(\Omega), \int_\Omega f(x)\;dx = 0\right\} \to W^{1,p}(\Omega), \quad 1 < p < \infty$$

such that

(a) $\operatorname{div}(\mathcal{B}[f]) = f$ *almost everywhere in* Ω, *id est* $\mathbf{u} = \mathcal{B}[f]$ *solves the equation* $\operatorname{div}\mathbf{u} = f$ *with the boundary condition* $\mathbf{u} = \mathbf{0}$ *on* $\partial\Omega$

(b) *there exists a constant* $c = c(d, p, \Omega)$ *such that*

$$\|\mathcal{B}[f]\|_{1,p} \le c\|f\|_p, \quad \text{for all } 1 < p < \infty.$$

We will also need the regularity properties of the solutions to the following (overdetermined) system

$$\begin{aligned} \operatorname{curl}\mathbf{u} &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{A.7}$$

with the compatibility conditions $\operatorname{div}\mathbf{f} = 0$ in Ω, $\mathbf{f}\cdot\mathbf{n} = 0$ on $\partial\Omega$. The following result holds true, see Solonnikov [23]; the assumption concerning the regularity of the domain can be relaxed, see Mucha, Pokorný [15].

Theorem A.5. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz boundary, let* $\mathbf{f} \in W^{1,p}(\Omega)$, $1 < p < +\infty$, $\operatorname{div}\mathbf{f} = 0$ *in* Ω, $\mathbf{f}\cdot\mathbf{n} = 0$ *on* $\partial\Omega$. *Then there exists a constant* $c = c(\Omega, p)$ *such that the unique solution* $\mathbf{u}$ *to system* (A.7) *satisfies*

$$\|\nabla\mathbf{u}\|_p \le c\|\mathbf{f}\|_p, \qquad \|\nabla^2\mathbf{u}\|_p \le c\|\mathbf{f}\|_{1,p}.$$

References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions.* I, Comm. Pure Appl. Math. **12** (1959), 623–727.

[2] T. Alazard, *Low Mach number limit of the full Navier–Stokes equations*, Arch. Ration. Mech. Anal. **180** (2006), no. 1, 1–73.

[3] Š. Axmann, P.B. Mucha and M. Pokorný, *Steady solutions to viscous shallow water equations. The case of heavy water*, Commun. Math. Sci. **15** (2017), no. 5, 1385–1402.

[4] H. Beirão Da Veiga, *An* L^p*-theory for the* n*-dimensional, stationary, compressible Navier–Stokes equations, and the incompressible limit for compressible fluids. The equilibrium solutions*, Comm. Math. Phys. **109** (1987), no. 2, 229–248.

[5] D. Bresch and B. Desjardins, *Existence of global weak solutions for a* 2D *viscous shallow water equations and convergence to the quasi-geostrophic model*, Comm. Math. Phys. **238** (2003), 211–223.

[6] D. Bresch, B. Desjardins and D. Gerard-Varet, *On compressible Navier–Stokes equations with density dependent viscosities in bounded domains*, J. Math. Pures Appl. (9) **87** (2007), 227–235.

[7] H.J. Choe and B.J. Jin, *Existence of solutions of stationary compressible Navier–Stokes equations with large force*, J. Funct. Anal. **177** (2000), no. 1, 54–88.

[8] Ch. Dou, F. Jiang, S. Jiang and Y.-F. Yang, *Existence of strong solutions to the steady Navier–Stokes equations for a compressible heat-conductive fluid with large forces*, J. Math. Pures Appl. **103**, no. 5, 1163–1197.

[9] R. Farwig, *Stationary solutions of the Navier–Stokes equations for a compressible, viscous and heat-conductive fluid*, SFB 256, University Bonn, 1988, preprint.

[10] R. Farwig, *Stationary solutions of compressible Navier–Stokes equations with slip boundary conditions*, Comm. Partial Differential Equations **14** (1989), no. 11, 1579–1606.

[11] E. FEIREISL, *Dynamics of Viscous Compressible Fluids*, Oxford University Press, Oxford (2004).

[12] E. FEIREISL, P.B. MUCHA, A. NOVOTNÝ AND M. POKORNÝ, *Time-periodic solutions to the full Navier–Stokes–Fourier system*, Arch. Ration. Mech. Anal. **204** (2012), 745–786.

[13] P.B. MUCHA, *On cylindrical symmetric flows through pipe-like domains*, J. Differential Equations **201** (2004), no. 2, 304–323.

[14] P.B. MUCHA AND T. PIASECKI, *Compressible perturbation of Poiseuille type flow*, J. Math. Pures Appl. (9) **102** (2014), no. 2, 338–363.

[15] P.B. MUCHA AND M. POKORNÝ, *The rot-div system in exterior domains*, J. Math. Fluid Mech. **16** (2014), no. 4, 701–720.

[16] P.B. MUCHA AND M. POKORNÝ, *On the steady compressible Navier–Stokes–Fourier system*, Comm. Math. Phys. **288** (2009), no. 1, 349–377.

[17] P.B. MUCHA AND R. RAUTMANN, *Convergence of Rothe's scheme for the Navier–Stokes equations with slip conditions in* 2D *domains*, ZAMM Z. Angew. Math. Mech. **86** (2006), no. 9, 691–701.

[18] A. NOVOTNÝ AND M. PADULA, *L^p-approach to steady flows of viscous compressible fluids in exterior domains*, Arch. Ration. Mech. Anal. **126** (1994), no. 3, 243–297.

[19] M. PADULA, *Existence and continuous dependence for solutions to the equations of a one-dimensional model in gas dynamics*, Meccanica **17** (1981), 128–135.

[20] M. PADULA, *Existence and uniqueness for viscous steady compressible motions*, Proc. Sem. Fis. Mat., Dinamica dei Fluidi e dei gaz ionizzati, Trieste (1982).

[21] M. PADULA, *Existence and uniqueness for viscous steady compressible motions*, Arch. Ration. Mech. Anal. **97** (1987), no. 2, 89–102.

[22] T. PIASECKI AND M. POKORNÝ, *Strong solutions to the Navier–Stokes–Fourier system with slip–inflow boundary conditions*, ZAMM Z. Angew. Math. Mech. **94** (2014), no. 12, 1035–1057.

[23] V.A. SOLONNIKOV, *Overdetermined elliptic boundary-value problems*, J. Sov. Math. **1** (1973), no. 4, 477–512.

[24] A. VALLI, *Periodic and stationary solutions for compressible Navier–Stokes equations via a stability method*, Ann. Sc. Norm. Sup. Pisa (1) **4** (1983), 607–646.

[25] A. VALLI, *On the existence of stationary solutions to compressible Navier–Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire (1) **4** (1987), 99–113.

Manuscript received June 27, 2017

accepted February 9, 2018

ŠIMON AXMANN AND MILAN POKORNÝ
Charles University
Faculty of Mathematics and Physics
Sokolovska 83
186 75 Praha 8, CZECH REPUBLIC
E-mail address: axmanns@vscht.cz, pokorny@karlin.mff.cuni.cz

PIOTR B. MUCHA
Institute of Applied Mathematics and Mechanics
University of Warsaw
02-097 Warsaw, POLAND
E-mail address: p.mucha@mimuw.edu.pl