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EXISTENCE OF SOLUTIONS FOR THE SEMILINEAR CORNER DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In this paper, we are concerned with the following elliptic equations:

$$\begin{cases} -\Delta_{\mathbb{M}} u = \lambda f & \text{in } z := (r,x,t) \in \mathbb{M}_0, \\ u = 0 & \text{on } \partial\mathbb{M}. \end{cases}$$

Here, $\lambda > 0$ and $M = [0,1) \times X \times [0,1)$ as a local model of stretched corner-manifolds, that is, the manifolds with corner singularities with dimension $N = n + 2 \geq 3$. Here X is a closed compact submanifold of dimension n embedded in the unit sphere of $\mathbb{R}^{n+1}$. We study the existence of nontrivial weak solutions for the semilinear corner degenerate elliptic equations without the Ambrosetti and Rabinowitz condition via the mountain pass theorem and fountain theorem.

1. Introduction

In this paper, we are concerned with some results about the existence and multiplicity of weak solutions for elliptic equations in a domain $\mathbb{M}$:

$$\begin{cases} -\Delta_{\mathbb{M}} u = \lambda f & \text{in } z := (r,x,s) \in \mathbb{M}_0, \\ u = 0 & \text{on } \partial\mathbb{M}. \end{cases} \tag{1.1}$$

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Write $\mathbb{M} = [0,1) \times X \times [0,1)$ as a local model of stretched corner-manifolds, that is, the manifolds with corner singularities with dimension $N = n+2 \geq 3$. Here X is a closed compact submanifold of dimension n embedded in the unit sphere of $\mathbb{R}^{n+1}$. Let $\mathbb{M}_0$ denote the interior of $\mathbb{M}$ and $\partial\mathbb{M} = \{0\} \times X \times \{0\}$ denote the boundary of $\mathbb{M}$. The so-called called corner Laplacian is defined as

$$\Delta_{\mathbb{M}} = (r\partial r)^2 + (\partial_{x_1})^2 + \ldots + (\partial_{x_n})^2 + (rs\partial_s)^2.$$

The corner-Laplacian is a degenerate elliptic operator on the boundary $\partial\mathbb{M}$. Such kinds of degenerate operators have been studied by many authors; see e.g. [5], [11], [12]. In [3], [4], Chen et. al. introduced the corner type weighted p-Sobolev spaces and discussed the various properties of this space.

On the other hand, the critical point theory, originally introduced in [2], plays a decisive role in finding solutions to elliptic equations of variational type. It is well known that one of crucial ingredients for ensuring the boundedness of Palais–Smale sequence of the Euler–Lagrange functional and to apply the critical point theory, is the Ambrosetti–Rabinowitz condition ((AR)-condition for short) in [2]:

(AR) There exist positive constants C and ζ such that $\zeta > p$ and

$$0 < \zeta F(x,t) \leq f(x,t)t \quad \text{for } x \in \Omega \text{ and } |t| \geq C,$$

where $F(x,t) = \int_0^t f(x,s)\,ds$ and Ω is a bounded domain in $\mathbb{R}^N$.

The (AR)-condition being natural to guarantee the boundedness of Palais–Smale sequence is very restrictive. Many people have tried to drop the (AR)-condition for elliptic type problem associated with the p-Laplacian; see [1], [8]–[10], [13]. In this regard, we are to show the existence of multiple solutions for problem (1.1) without the (AR)-condition. In particular, following in [8, Remark 1.8], there are many examples of problems where this condition on the nonlinear term f is not satisfied; see [1], [9], [10].

Thus, motivated by these examples and references, the main aim of this paper is to show the existence of weak solutions to the problem above without the (AR)-condition using the mountain pass theorem and fountain theorem. Novelty of this paper is to obtain existence results to the semilinear corner degenerate elliptic equations provided f has mild assumptions different from those of [1], [9], [10]. To the best of our knowledge, here are very few existence results in this situation.

Now following [3], [4], we define the weighted $L_p^{\gamma_1,\gamma_2}$-space on $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$ as follows.

Definition 1.1. Let $(r,x,s) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$, weight data $\gamma_1, \gamma_2 \in \mathbb{R}$ and $1 \leq p < \infty$. Then

$$L_p^{\gamma_1,\gamma_1}\left(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+, \frac{dr}{r}\,dx\,\frac{ds}{rs}\right)$$

denotes the space of all $u(r,x,s) \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+)$ such that

$$\|u\|_{L_p^{\gamma_1,\gamma_1}(\mathbb{R}_+\times\mathbb{R}^N\times\mathbb{R}_+)} \le \left(\int_{\mathbb{R}_+\times\mathbb{R}^N\times\mathbb{R}_+} \left|r^{N/p-\gamma_1} t^{N/p-\gamma_2} u(r,x,t)\right|^p \frac{dr}{r}\, dx\, \frac{ds}{rs} \right)^{1/p} < \infty.$$

By the above weighted $L_p^{\gamma_1,\gamma_2}$ space, we can define the following weighted $p-$Sobolev spaces on $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$ with natural scale for all $1 \le p < \infty$.

DEFINITION 1.2. Let $m \in \mathbb{N}, \gamma_1, \gamma_2 \in \mathbb{R}$, and set $N = n+2$,the weighted Sobolev space

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+) = \Big\{ u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+) : (r\partial_r)^l \partial_x^\alpha (rs\partial_t)^k u(r,x,s) \in L_p^{\gamma_1,\gamma_1}\left(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+, \frac{dr}{r}\, dx\, \frac{ds}{rs}\right) \Big\}$$

for $k,l \in \mathbb{N}$ and the multiindex $\alpha \in \mathbb{N}^n$, with $k+|\alpha|+l \le m$. Moreover, the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+)$ is denoted by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+)$.

Similarly, we can define the following weighted p-Sobolev spaces on an open stretched corner $\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+$,

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) = \Big\{ u \in \mathcal{D}'(\mathbb{R}_+ \times X \times \mathbb{R}_+) : (r\partial_r)^l \partial_x^\alpha (rs\partial_s)^k u(r,x,s) \in L_p^{\gamma_1,\gamma_1}\left(\mathbb{R}_+ \times X \times \mathbb{R}_+, \frac{dr}{r}\, dx\, \frac{ds}{rs}\right) \Big\}$$

for $k,l \in \mathbb{N}$ and the multiindex $\alpha \in \mathbb{N}^n$, with $k+|\alpha|+l \le m$, which is a Banach space with the following norm

$$\|u\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times X\times\mathbb{R}_+)} = \Big\{ \sum_{l+|\alpha|+k\le m} \int_{\mathbb{R}_+\times X\times\mathbb{R}_+} \left| r^{N/p-\gamma_1} s^{N/p-\gamma_2} (r\partial_r)^l \partial_x^\alpha (rs\partial_s)^k u(r,x,s) \right|^p \frac{dr}{r}\, dx\, \frac{ds}{rs} \Big\}^{1/p}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$ denotes as the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$. Now we can introduce the following weighted p-Sobolev space on the finite stretched corner $\mathbb{M}$.

DEFINITION 1.3. Let $m \in \mathbb{N}$, $1 \le p < \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. $W_{\mathrm{loc}}^{m,p}(\mathbb{M}_0)$ is the classical local Sobolev space. Then we define

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) = \big\{ u(r,x,t) \in W_{\mathrm{loc}}^{m,p}(\mathbb{M}_0) : (w\sigma)u \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \big\}$$

for any cutoff functions $w = w(r,x)$ and $\sigma = \sigma(t,x)$, supported by a collar neighbourhoods of $(0,1) \times \partial\mathbb{M}$ and $\partial\mathbb{M} \times (0,1)$, respectively.

The following compact embedding is given in [3, Proposition 3.3].

Lemma 1.4. *The embedding*

$$\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M}) \hookrightarrow \mathcal{H}_{l,0}^{1,((N-1)/l,Nl)}(\mathbb{M})$$

is compact for $1 < l < 2^*$.

2. Existence of weak solutions

In this section, we briefly recall definitions and some elementary properties of the weighted Sobolev spaces (see [3], [4] for a detailed description).

Let $X \subset S^n$ be a bounded open set in the unit sphere of $\mathbb{R}_x^{n+1}$, then the finite corner is defined as $M = (E \times [0,1))/(E \times \{0\})$, where the base E is a finite cone defined as $E = ([0,1) \times X)/(\{0\} \times X)$. Thus, the finite stretched corner is

$$\mathbb{M} \subset E \times [0,1) = [0,1) \times X \times [0,1)$$

with the smooth boundary $\partial\mathbb{M} = \{0\} \times X \times \{0\}$, and here we denote $\mathbb{M}_0$ as the interior of $\mathbb{M}$. In this paper, we shall use the coordinates $(r,x,t) \in \mathbb{M}$.

Next, we consider appropriate assumptions for the nonlinear term f. Let us denote $F(x,t) = \int_0^t f(x,s)\,ds$ and $2 < p < 2^*$.

(F1) $f\colon \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\,\cdot\,,t)$ is measurable for all $t \in \mathbb{R}$ and $f(x,\,\cdot\,)$ is continuous for almost all $x \in \mathbb{M}$.

(F2) There exist nonnegative functions $\rho, \sigma \in L^\infty(\mathbb{M})$ such that

$$|f(x,t)| \le \rho(x) + \sigma(x)\,|t|^{p-1}$$

for all $(x,t) \in M \times \mathbb{R}$ and for all $(x,t) \in \mathbb{M} \times \mathbb{R}$.

(F3) There exists $\delta > 0$ such that

$$F(x,t) \le 0, \quad \text{for } x \in M,\ |t| < \delta.$$

(F4) $\lim\limits_{|t|\to\infty} F(x,t)/|t|^q = \infty$ uniformly for almost all $x \in \mathbb{R}^N$ and $q > 1$.

(F5) There exist real numbers $c_0 > 0$, $r_0 \ge 0$, and $\kappa > N$ such that

$$|F(x,t)|^\kappa \le c_0\,|t|^\kappa\,\mathfrak{F}(x,t)$$

for all $(x,t) \in \mathbb{M} \times \mathbb{R}$ and $|t| \ge r_0$, where $\mathfrak{F}(x,t) = f(x,t)t - qF(x,t) \ge 0$ with $q > 2$.

Lemma 2.1. *For* $u := u(r,x,t) \in H_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{M})$, $1 \le p < \infty$, *the following estimate holds*

$$\|u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})} \le C\|\nabla_{\mathbb{M}} u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})},$$

where d is the diameter of $\mathbb{M}$.

For $u \in \mathcal{H}_{2,0}^{1,(N-1)/2,N/2}(\mathbb{M})$, the Euler–Lagrange functional

$$I_\lambda \colon \mathcal{H}_{2,0}^{1,(N-1)/2,N/2}(\mathbb{M}) \to \mathbb{R}$$

is defined by

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\, d\sigma - \int_{\mathbb{M}} rF(z,u)\, d\sigma, \quad d\sigma = \frac{dr}{r}\, dx\, \frac{ds}{rs}$$

DEFINITION 2.2. Let $N \geq 3$. We say that $u \in \mathcal{H}_{2,0}^{1,(N-1)/2,N/2}(\mathbb{M})$ is a weak solution of the problem (1.1) if

$$\langle I'(u), u\rangle = \int_{\mathbb{M}} r(\nabla_{\mathbb{M}} u)(\nabla_{\mathbb{M}}\varphi) d\sigma - \int_{\mathbb{M}} rf(z,u)\varphi\, d\sigma,$$

for all $\varphi \in \mathcal{H}_{2,0}^{1,(N-1)/2,N/2}(\mathbb{M})$. Here, $I'(\,\cdot\,)$ denotes the Fréchet differentiation.

REMARK 2.3. In [4], the critical points of $I_\lambda(u)$in $\mathcal{H}_{2,1}^{1,(N-1)/2,N/2}(\mathbb{M})$ are the weak solutions of Dirichlet problem.

In the following result we are to show that the energy functional I_λ satisfies the geometric conditions of the mountain pass theorem.

LEMMA 2.4. *Assume that* (F1)–(F4) *hold. Then the geometric conditions of the mountain pass theorem hold, i.e.*

(a) $u = 0$ *is a strict local minimum for* $I_\lambda(u)$.
(b) $I_\lambda(u)$ *is unbounded from below on* $\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$.

PROOF. By (F3), $u = 0$ is a strict local minimum for $I_\lambda(u)$. Next we show condition (b). It is obvious that I_λ is bounded from below and $I_\lambda(0) = -I_\lambda(0) = 0$. By (F4), for any $M > 0$, there exists a constant $\delta > 0$ such that

$$F(x,t) \geq M\,|t|^q, \quad q > 2 \tag{2.1}$$

for $|t| > \delta$ and for almost all $x \in \mathbb{R}^N$. Take $v \in \mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M}) \setminus \{0\}$ with $\|v\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = 1$. Then (2.1) implies that

$$I_\lambda(\widetilde{t}v) = \frac{\widetilde{t}^2}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} v|^2 d\sigma - \lambda\int_{\mathbb{M}} rF(x,\widetilde{t}v)\, d\sigma \leq \widetilde{t}\left(1 - \widetilde{t}^{q-2}\lambda\int_{\mathbb{M}} rF(x,\widetilde{t}v)\, d\sigma\right)$$

for sufficiently large $\widetilde{t} > 1$. If M is large enough, then we assert that $I_\lambda(\widetilde{t}v) \to -\infty$ as $\widetilde{t} \to \infty$. Hence the functional I_λ is unbounded from below. $\square$

Using similar arguments as in [7, Theorem 4.1], the following lemma is easily checked. Also, we can easily see that the functional I_λ as well as its derivative I'_λ are weakly-strongly continuous on X by the analogous arguments in [6, Lemma 3.2]: we omit the proof. That is, the operator I'_λ is a mapping of type (S_+).

Lemma 2.5. *Assume that* (F1)–(F4) *hold. Then the functional* $I_\lambda\colon X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X*. Moreover, the operator* I'_λ *is a mapping of type* (S_+)*, i.e. if*

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \limsup_{n\to\infty} \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \le 0,$$

then $u_n \to u$ *in* X *as* $n \to \infty$.

Lemma 2.6. *Let* $3 \le N$*. Assume that* (F1)–(F5) *hold. Then the functional* I_λ *satisfies the* $(\mathrm{C})_c$*-condition for any* $\lambda > 0$.

Proof. For $c \in \mathbb{R}$, let $\{u_n\}$ be a $(\mathrm{C})_c$-sequence in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$, that is,

$$\begin{aligned} &I_\lambda(u_n) \to c, \\ &\|I'_\lambda(u_n)\|_{\left(\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathcal{M})\right)^*}\left(1 + \|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}\right) \to 0 \end{aligned} \tag{2.2}$$

as $n \to \infty$, which implies that

$$c = I_\lambda(u_n) + o(1) \quad \text{and} \quad \langle I'_\lambda(u_n), u_n\rangle = o(1), \tag{2.3}$$

where $o(1) \to 0$ as $n \to \infty$. If $\{u_n\}$ is bounded in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$, it follows that $\{u_n\}$ converges strongly to u in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$. In fact, this statement is similar of that Proposition 2.6 in [3]. Hence, it suffices to verify that the sequence $\{u_n\}$ is bounded in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$. We argue by contradiction. Suppose that the sequence $\{u_n\}$ is unbounded in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$. So then we may assume that

$$\begin{aligned} &\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} > 1, \\ &\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} \to \infty, \quad \text{as } n \to \infty. \end{aligned} \tag{2.4}$$

Define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}$. Then it is obvious that $\{w_n\} \subset \mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$ and $\|w_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} = 1$. Hence, up to a subsequence, still denoted by $\{w_n\}$, we obtain $w_n \rightharpoonup w$ in $\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})$ as $n \to \infty$ and by Lemma 1.4, we have

$$\begin{aligned} w_n(x) &\to w(x) && \text{a.e. in } \mathbb{M}, \\ w_n &\to w && \text{in } \mathcal{H}^{1,((N-1)/p,N/p)}_{p,0}(\mathbb{M}) \end{aligned} \tag{2.5}$$

as $n \to \infty$ for $1 < p < 2^*$. Due to the condition (2.3), we have that

$$c = I_\lambda(u_n) + o(1) = \frac{1}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma - \lambda\int_{\mathbb{R}^N} rF(x,u_n)\,dx + o(1). \tag{2.6}$$

Note that, due to Lemma 2.1,

$$\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma \ge d\|u\|^2_{\mathcal{H}^{1,(N-1)/2,N/2}_{2,0}(\mathbb{M})}. \tag{2.7}$$

Since $\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} \to \infty$ as $n \to \infty$, we assert that

$$\int_{\mathbb{R}^N} rF(x,u_n)\,dx = \frac{1}{2\lambda}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma - \frac{c}{\lambda} + \frac{o(1)}{\lambda} \to \infty \quad \text{as } n \to \infty. \tag{2.8}$$

The assumption (F3) implies that there exists $t_0 > 1$ such that $F(x,t) > |t|$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. From (F1) and (F2), we have that there exists a positive constant $\mathcal{C}$ such that $|F(x,t)| \le \mathcal{C}$ for all $(x,t) \in \mathbb{R}^N \times [-t_0, t_0]$. Therefore we can choose a real number $\mathcal{C}_0$ such that $F(x,t) \ge \mathcal{C}_0$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{F(x,u_n) - \mathcal{C}_0}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma} \ge 0, \quad \text{for all } x \in \mathbb{R}^N \text{ and for all } n \in \mathbb{N}.$$

By (2.7), we get

$$\frac{F(x,u_n) - \mathcal{C}_0}{\|u\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}} \ge 0. \tag{2.9}$$

Set $\Omega = \{x \in \mathbb{R}^N : w(x) \ne 0\}$. By the convergence (2.5), we know that $|u_n(x)| = |w_n(x)|\,\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} \to \infty$ as $n \to \infty$ for all $x \in \Omega$. So then, it follows from the assumption (F3), and the relation (2.4) that for all $x \in \Omega$,

$$\begin{aligned} \lim_{n\to\infty} \frac{F(x,u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma} &\le \frac{1}{d}\lim_{n\to\infty} \frac{F(x,u_n)}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}} \\ &= \lim_{n\to\infty} \frac{F(x,u_n)}{|u_n(x)|^2}\,|w_n(x)|^2 = \infty. \end{aligned} \tag{2.10}$$

Hence we have $\mathrm{meas}(\Omega) = 0$. Indeed, if $\mathrm{meas}(\Omega) \ne 0$, then according to (2.6)–(2.10), and Fatou's lemma, we deduce that

$$\begin{aligned} \frac{1}{\lambda} &= \liminf_{n\to\infty} \frac{\displaystyle\int_{\mathbb{R}^N} rF(x,u_n)\,dx}{\lambda\displaystyle\int_{\mathbb{R}^N} rF(x,u_n)\,dx + c - o(1)} \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{2rF(x,u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u_n|^2\,d\sigma}\,dx \\ &\ge \liminf_{n\to\infty} \int_{\Omega} \frac{2F(x,u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma}\,d\sigma - \limsup_{n\to\infty} \int_{\Omega} \frac{2rM_0}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\,d\sigma}\,d\sigma \\ &= \liminf_{n\to\infty} \int_{\Omega} \frac{2r(F(x,u_n) - M_0)}{\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma}\,d\sigma \ge \int_{\Omega} \liminf_{n\to\infty} \frac{2r(F(x,u_n) - M_0)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma}\,d\sigma \\ &= \int_{\Omega} \liminf_{n\to\infty} \frac{2rF(x,u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma}\,d\sigma - \int_{\Omega} \limsup_{n\to\infty} \frac{2rM_0}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma}\,d\sigma = \infty, \end{aligned}$$

which is a contradiction. Thus $w(x) = 0$ for almost all $x \in \mathbb{M}$.

Observe that

$$
\begin{aligned}
(2.11)\quad c+1 &\geq I_\lambda(u_n) - \frac{1}{q}\langle I'_\lambda(u_n), u_n\rangle \\
&= \frac{1}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\, d\sigma - \lambda \int_{\mathbb{R}^N} rF(x,u_n)\, d\sigma \\
&\quad - \frac{1}{q}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\, d\sigma + \frac{\lambda}{q}\int_{\mathbb{R}^N} rf(x,u_n)u_n\, d\sigma \\
&= \frac{q-2}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 d\sigma + \lambda \int_{\mathbb{R}^N} \mathfrak{F}(x,u_n)\, d\sigma \geq \lambda \int_{\mathbb{R}^N} \mathfrak{F}(x,u_n)\, d\sigma
\end{aligned}
$$

for n large enough and $\mathfrak{F}$ is defined in (F5). Let us define $\Omega_n(a,b) := \{x \in \mathbb{M} : a \leq |u_n(x)| < b\}$ for $a \geq 0$. By (2.5), we note that

$$
(2.12)\quad
\begin{aligned}
w_n(x) &\to 0 \quad \text{a.e. in } \mathbb{M}, \\
w_n &\to 0 \quad \text{in } \mathcal{H}^{1,((N-1)/p,N/p)}_{p,0}(\mathbb{M})
\end{aligned}
$$

as $n \to \infty$ for $1 < p < 2^*$. Hence from the relations (2.8) and (2.7), we get

$$
\begin{aligned}
(2.13)\quad 0 < \frac{1}{2\lambda} &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{r\,|F(x,u_n)|}{\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2\, d\sigma}\, d\sigma \\
&\leq \frac{1}{d}\limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{r\,|F(x,u_n)|}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}}\, d\sigma.
\end{aligned}
$$

From (F2) and (2.12), we have

$$
\begin{aligned}
(2.14)\quad &\int_{\Omega_n(0,\widetilde{d})} \frac{rF(x,u_n)}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}}\, d\sigma \\
&\leq \int_{\Omega_n(0,\widetilde{d})} \frac{\rho(x)r\,|u_n(x)| + \sigma(x)r\,|u_n(x)|^q/q}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}}\, d\sigma \\
&\leq \frac{C\|\rho\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}} \\
&\quad + \frac{\|\sigma\|_{L^\infty(\mathbb{M})}}{q}\int_{\Omega_n(0,\widetilde{d})} |u_n(x)|^{q-p}\, r\,|w_n(x)|^p\, d\sigma \\
&\leq \frac{C\|\rho\|_{L^\infty(\mathbb{R}^N)}\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}}{\|u_n\|^2_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}} \\
&\quad + \frac{\|\sigma\|_{L^\infty(\mathbb{M})}}{q}\widetilde{d}^{q-p}\int_{\mathbb{R}^N} r\,|w_n(x)|^p\, d\sigma \\
&\leq \frac{C}{\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}} + \frac{\|\sigma\|_{L^\infty(\mathbb{M})}}{q}\,\widetilde{d}^{q-p}\int_{\mathbb{R}^N} r\,|w_n(x)|^p\, d\sigma \to 0
\end{aligned}
$$

as $n \to \infty$, where C is a positive and generic constant. Set $\tau' = \tau/(\tau - 1) > 1$. Since $\tau > N/2$, we see that $2 < 2\tau' < 2^*$. Hence, it follows from (F5), (2.11), and (2.12) that

$$\begin{aligned}
(2.15)\quad \int_{\Omega_n(\widetilde{d},\infty)} \frac{r\,|F(x,u_n)|}{\|u_n\|^2_{\mathcal{H}^{1,(N-1)/2,N/2}_{2,0}(\mathbb{M})}}\,d\sigma &= \int_{\Omega_n(\widetilde{d},\infty)} \frac{r\,|F(x,u_n)|}{|u_n(x)|^2}\,|w_n(x)|^2\,d\sigma \\
&\le \bigg\{\int_{\Omega_n(\widetilde{d},\infty)} \bigg(\frac{|F(x,u_n)|}{|u_n(x)|^2}\bigg)^{\tau} d\sigma\bigg\}^{1/\tau} \\
&\quad\cdot \bigg\{\int_{\Omega_n(\widetilde{d},\infty)} (r\,|w_n(x)|^2)^{\tau'}\,d\sigma\bigg\}^{1/\tau'} \\
&\le c_0^{1/\tau}\bigg\{\int_{\Omega_n(\widetilde{d},\infty)} \mathfrak{F}(x,u_n)\,d\sigma\bigg\}^{1/\tau}\bigg\{\int_{\mathbb{M}} r^{\tau'}\,|w_n(x)|^{2\tau'}\,d\sigma\bigg\}^{1/(2\tau')} \\
&\le c_0^{1/\tau}\bigg(\frac{c+1}{\lambda}\bigg)^{1/\tau}\bigg\{\int_{\mathbb{M}} r^{\tau'-1} r\,|w_n(x)|^{2\tau'}\bigg\}^{1/(2\tau')} \\
&\le C c_0^{1/\tau}\bigg(\frac{c+1}{\lambda}\bigg)^{1/\tau}\bigg\{\int_{\mathbb{M}} r\,|w_n(x)|^{2\tau'}\bigg\}^{1/(2\tau')},
\end{aligned}$$

as $n \to \infty$. Here, $C := \|r^{\tau'-1}\|_{L^\infty(d\sigma)} < \infty$ with $0 < r < 1$. Combining (2.14) with (2.15), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{r\,|F(x,u_n)|}{\|u_n\|^2_{\mathcal{H}^{1,(N-1)/2,N/2}_{2,0}(\mathbb{M})}}\,d\sigma &= \int_{\Omega_n(0,\widetilde{d})} \frac{r\,|F(x,u_n)|}{\|u_n\|^2_{\mathcal{H}^{1,(N-1)/2,N/2}_{2,0}(\mathbb{M})}}\,d\sigma \\
&\quad + \int_{\Omega_n(\widetilde{d},\infty)} \frac{r\,|F(x,u_n)|}{\|u_n\|^2_{\mathcal{H}^{1,(N-1)/2,N/2}_{2,0}(\mathbb{M})}}\,d\sigma \to 0
\end{aligned}$$

as $n \to \infty$, which contradicts the inequality (2.13). □

THEOREM 2.7. *Let $N \ge 3$. Assume that* (F1)–(F4) *and* (F5) *hold. Then problem* (1.1) *has a nontrivial weak solution for all $\lambda > 0$.*

PROOF. Note that $I_\lambda(0) = 0$. In view of Lemma 2.4, the geometric conditions of the mountain pass theorem are fulfilled. By Lemma 2.6, I_λ satisfies the $(\mathrm{C})_c$-condition for any $\lambda > 0$, and hence we see that the energy functional I_λ satisfies all conditions of the mountain pass theorem. Consequently, problem (1.1) has a nontrivial weak solution for all $\lambda > 0$. □

Next, using the oddity on f and applying the fountain theorem in [14, Theorem 3.6], we demonstrate infinitely many weak solutions for problem (1.1). To do this, let X be a separable and reflexive Banach space. It is well known that there are $\{e_n\} \subseteq X$ and $\{f_n^*\} \subseteq X^*$ such that

$$X = \overline{\mathrm{span}\{e_n : n = 1, 2, \dots\}}, \qquad X^* = \overline{\mathrm{span}\{f_n^* : n = 1, 2, \dots\}},$$

and

$$\langle f_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let us denote $X_n = \operatorname{span}\{e_n\}$, $Y_k = \bigoplus_{n=1}^{k} X_n$, and $Z_k = \overline{\bigoplus_{n=k}^{\infty} X_n}$. Then we have the following fountain theorem.

Lemma 2.8 ([14]). *Let X be a real reflexive Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the* $(\mathrm{C})_{\mathrm{c}}$*-condition for any $c > 0$ and I is even. If, for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > \delta_k > 0$ such that the following conditions hold*:

(a) $b_k := \inf\{I(u) : u \in Z_k, \|u\|_X = \delta_k\} \to \infty$ *as* $k \to \infty$;

(b) $a_k := \max\{I(u) : u \in Y_k, \|u\|_X = \rho_k\} \leq 0$.

Then the functional I has an unbounded sequence of critical values, i.e. there exists a sequence $\{u_n\} \subset X$ such that $I'(u_n) = 0$ and $I(u_n) \to \infty$ as $n \to \infty$.

Using Lemma 2.8, we show the existence of infinitely many nontrivial weak solutions for our problem.

Theorem 2.9. *Let $3 < N$. Assume that* (F1)–(F5) *hold. If $f(x, -t) = -f(x, t)$ holds for all $(x, t) \in \mathbb{M} \times \mathbb{R}$, then for any $\lambda > 0$, the functional I_λ has an unbounded sequence of nontrivial weak solutions $\{u_n\}$ in $\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$ such that $I_\lambda(u_n) \to \infty$ as $n \to \infty$.*

Proof. Obviously, I_λ is an even functional and satisfies $(C)_c$-condition. Note that $\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$ is a reflexive Banach space. According to Lemma 2.8, it suffices to show that there exist $\rho_k > \delta_k > 0$ such that

(a) $b_k := \inf\left\{I_\lambda(u) : u \in Z_k, \|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = \delta_k\right\} \to \infty$ as $n \to \infty$;

(b) $a_k := \max\left\{I_\lambda(u) : u \in Y_k, \|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = \rho_k\right\} \leq 0$,

for k large enough. Denote

$$\alpha_k := \sup_{u \in Z_k, \|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = 1} \left(\int_{\mathbb{M}} \frac{1}{q} |u(x)|^q \, d\sigma \right), \quad 1 < q < 2^*.$$

Then we have $\alpha_k \to 0$ as $k \to \infty$. In fact, suppose to the contrary that there exist $\varepsilon_0 > 0$ and the sequence $\{u_k\}$ in Z_k such that

$$\|u_k\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = 1, \qquad \int_{\mathbb{M}} \frac{1}{q} |u_k(x)|^q \, d\sigma \geq \varepsilon_0,$$

for all $k \geq k_0$. Since the sequence $\{u_k\}$ is bounded in $\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$, there exists $u \in \mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$ such that $u_k \rightharpoonup u$ in $\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})$ as $k \to \infty$ and

$$\langle f_j^*, u \rangle = \lim_{k \to \infty} \langle f_j^*, u_k \rangle = 0$$

for $j = 1, 2, \ldots$ Hence we get $u = 0$. However, we obtain

$$\varepsilon_0 \le \lim_{k\to\infty} \int_{\mathbb{M}} \frac{1}{q} |u_k(x)|^q \, d\sigma = \int_{\mathbb{M}} \frac{1}{q} |u(x)|^q \, d\sigma = 0,$$

which provides a contradiction.

Note that

$$\begin{aligned} \int_{\mathbb{M}} r|u|^p &\le \left(\int_{\mathbb{M}} r|u|^2 \, d\sigma \right)^{\sigma/2} \left(\int_{\mathbb{M}} r|u|^{2^*} \, d\sigma \right)^{(p-\sigma)/2^*} \\ &\le \|u\|^{\sigma}_{L_2^{(N-1)/2,1/2}} \|u\|^{p-\sigma}_{L_{2^*}^{(N-1)/2,1/2}}. \end{aligned}$$

By Lemma 2.1 and the corner type Sobolev inequality (see [4]), we get

$$\begin{aligned} \|u\|^{\sigma}_{L_2^{(N-1)/2,1/2}} &\le \lambda_{k_0}^{-\sigma/2} \|\nabla_{\mathbb{M}} u\|^{\sigma}_{L_2^{(N-1)/2,N/2}} \le \lambda_{k_0}^{-\sigma/2} \|u\|^{\sigma}_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}}, \\ \|u\|^{p-\sigma}_{L_{2^*}^{(N-1)/2^*,N/2^*}} &\le \widetilde{C} \|u\|^{p-\sigma}_{\mathcal{H}_{2,0}^{1,((N-1)/2,1/2)}}. \end{aligned} \tag{2.16}$$

For any $u \in Z_k$, it follows from condition (F2), and the Hölder inequality that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma - \int_{\mathbb{M}} rF(z,u) \, d\sigma \\ &\ge \frac{1}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma - \lambda \int_{\mathbb{M}} r \, |\rho(x)| \, |u(x)| \, d\sigma \\ &\quad - \lambda \int_{\mathbb{M}} r \frac{|\sigma(x)|}{q} \, |u(x)|^q \, d\sigma \\ &\ge \frac{1}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma - \lambda \|\rho\|_{L^\infty(\mathbb{M})} \int_{\mathbb{M}} |u(x)| \, d\sigma \\ &\quad - \frac{\lambda}{q} \|\sigma\|_{L^\infty(\mathbb{M})} \int_{\mathbb{M}} |u(x)|^q \, d\sigma \\ &\ge C_2 \|u\|^2_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}} - C_3 \lambda_{k_0}^{-\sigma/2} \|u\|^p_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}} \\ &\quad - C_4 \|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}} \\ &\ge \left(C_2 - C_3 \lambda_{k_0}^{-\sigma/2} \|u\|^{p-2}_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}} \right) \|u\|^2_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}} \\ &\quad - C_4 \|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}}, \end{aligned} \tag{2.17}$$

where $C_2 > 0$ depends on d in Lemma 2.1, $C_3 > 0$ depends on $\|\rho\|_{L^\infty(\mathbb{M})}$, p and $\widetilde{C}$ in (2.16), and $C_4 > 0$ depends on $\|\rho\|_{L^\infty(\mathbb{M})}$ and the measure $|M|$. Let $k_0$be large enough such that $C_2 - C_3 \lambda_{k_0}^{-\sigma/2} \delta_k^{p-2} > 0$. Hence, if $u \in Z_k$ and $\|u\|_{\mathcal{H}_{2,0}^{1,((N-1)/2,N/2)}(\mathbb{M})} = \delta_k$, then we deduce that $I_\lambda(u) \to \infty$ as $k \to \infty$, which implies (a).

Assume that condition (b) does not hold for some k. Then there exists a sequence $\{u_n\}$ in Y_k such that

$$
\begin{aligned}
&\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} > 1, \\
&\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} \to \infty, \quad \text{as } n \to \infty, \\
&I_\lambda(u_n) \geq 0.
\end{aligned}
\tag{2.18}
$$

Let $w_n = u_n/\|u_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})}$. Then it is obvious that

$$
\|w_n\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} = 1.
$$

Since $\dim Y_k < \infty$, there exists $w \in Y_k \setminus \{0\}$ such that up to a subsequence,

$$
\|w_n - w\|_{\mathcal{H}^{1,((N-1)/2,N/2)}_{2,0}(\mathbb{M})} \to 0 \quad \text{and} \quad w_n(x) \to w(x)
$$

for almost all $x \in \mathbb{M}$ as $n \to \infty$. For $x \in \Omega := \{x \in \mathbb{M} : w(x) \neq 0\}$, we get $|u_n(x)| \to \infty$ as $n \to \infty$. As seen in the proof of Lemma 2.6, we can choose a real number $\mathcal{C}_1$ such that

$$
\frac{F(x, u_n) - \mathcal{C}_1}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma} \geq 0 \tag{2.19}
$$

for $x \in \Omega$ and for all $n \in \mathbb{N}$. Taking into account (2.19) and the Fatou lemma, we assert by a similar argument to (2.10) that

$$
\begin{aligned}
\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma} \, d\sigma &\geq \liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n) - \mathcal{C}_1}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma} \, d\sigma \\
&\geq \int_\Omega \liminf_{n\to\infty} \frac{F(x, u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma} \, d\sigma = \infty.
\end{aligned}
\tag{2.20}
$$

Therefore, using the relation (2.20), we have

$$
\begin{aligned}
I_\lambda(u_n) &= \frac{1}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma - \int_{\mathbb{M}} rF(z, u) \, d\sigma \\
&\leq \frac{1}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma \left(1 - 2\lambda \int_\Omega \frac{F(x, u_n)}{\displaystyle\int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u|^2 \, d\sigma} \, d\sigma \right) \to -\infty
\end{aligned}
$$

as $n \to \infty$, which is contradiction to (2.18). $\square$

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