

**GEODESICS ON $SO(N)$
AND A CLASS OF SPHERICALLY SYMMETRIC MAPS
AS SOLUTIONS TO A NONLINEAR GENERALISED
HARMONIC MAP PROBLEM**

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ABSTRACT. We address questions on existence, multiplicity as well as qualitative features including rotational symmetry for certain classes of geometrically motivated maps serving as solutions to the nonlinear system

$$\begin{cases} -\operatorname{div}[F'(|x|, |\nabla u|^2)\nabla u] = F'(|x|, |\nabla u|^2)|\nabla u|^2 u & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial\mathbb{X}^n. \end{cases}$$

Here $\varphi \in \mathcal{C}^\infty(\partial\mathbb{X}^n, \mathbb{S}^{n-1})$ is a suitable boundary map, F' is the derivative of F with respect to the second argument, $u \in W^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1})$ for a fixed $1 < p < \infty$ and $\mathbb{X}^n = \{x \in \mathbb{R}^n : a < |x| < b\}$ is a generalised annulus. Of particular interest are spherical twists and whirls, where following [26], a spherical twist refers to a rotationally symmetric map of the form $u: x \mapsto Q(|x|x|x|^{-1})$ with Q some suitable path in $\mathcal{C}([a, b], SO(n))$ and a whirl has a similar but more complex structure with only 2-plane symmetries. We establish the existence of an infinite family of such solutions and illustrate an interesting discrepancy between odd and even dimensions.

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1. Introduction

Let $\mathbb{X}^n = \mathbb{X}^n[a, b] = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$ be a generalised annulus and consider the energy functional

$$(1.1) \quad \mathbb{F}[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} F(|x|, |\nabla u|^2) dx.$$

Here $F \in \mathcal{C}^{1,2}([a, b] \times \mathbb{R})$, that is, \mathcal{C}^1 with respect to the first variable and \mathcal{C}^2 with respect to the second, is bounded below, suitably convex and monotone in the second argument with a p -growth at infinity (see Section 2 for details). In this paper we aim to extremise \mathbb{F} over the space of admissible maps given by

$$(1.2) \quad \mathcal{A}_\varphi^p(\mathbb{X}^n) := \{u \in W^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) : u = \varphi \text{ on } \partial\mathbb{X}^n\},$$

for some fixed $1 < p < \infty$ and suitable boundary map $\varphi \in \mathcal{C}^\infty(\partial\mathbb{X}^n, \mathbb{S}^{n-1})$. Note that here $\partial\mathbb{X}^n = \partial\mathbb{X}_a^n \cup \partial\mathbb{X}_b^n$ and as customary ⁽¹⁾

$$W^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1}) := \{u \in W^{1,p}(\mathbb{X}^n, \mathbb{R}^n) : |u| = 1 \text{ a.e. in } \mathbb{X}^n\}.$$

One motivation for studying such problems comes from liquid crystal theory and in particular the well-known Oseen–Frank model where the aim is to describe and classify the director fields u arising as extremisers and minimisers of the energy functional

$$\mathbb{E}_{\text{OF}}[u; \Omega] = \int_{\Omega} \mathcal{W}(u, \nabla u) dx, \quad u \in W^{1,2}(\Omega, \mathbb{S}^{n-1}),$$

subject to suitable boundary conditions. In this setting $\Omega \subset \mathbb{R}^3$ is a bounded domain representing the body, u is a unit vector-field on Ω (the director field) with ∇u denoting its gradient and the energy density $\mathcal{W} = \mathcal{W}(u, \nabla u)$ is given by

$$\begin{aligned} \mathcal{W}(u, \nabla u) &= k_1 |\nabla \cdot u|^2 + k_2 |u \cdot (\nabla \times u)|^2 \\ &\quad + k_3 |u \times (\nabla \times u)|^2 + (k_2 + k_4)(\text{tr}(\nabla u)^2 - (\nabla \cdot u)^2). \end{aligned}$$

Here k_j ($1 \leq j \leq 4$) are the Frank constants that are assumed to satisfy the strict form of Ericksen inequalities [9]: $k_1, k_2, k_3 > 0$, $k_2 > |k_4|$ and $2k_1 > k_2 + k_4$, which result in the coercivity inequality $\mathcal{W}(u, \nabla u) \geq \alpha |\nabla u|^2$ for all vector fields u and some $\alpha > 0$. Using the identity

$$\text{tr}(\nabla u)^2 + |u \cdot (\nabla \times u)|^2 + |u \times (\nabla \times u)|^2 = |\nabla u|^2,$$

it is seen that in the case of “equal elastic constants”, that is, $k_1 = k_2 = k_3 =: k$, ($k_4 = 0$) the Oseen–Frank energy reduces to a constant multiple of the Dirichlet energy, here, \mathbb{F} with $F(r, t) = kt$. Note also that $\text{tr}(\nabla u)^2 - (\nabla \cdot u)^2$ is a null-Lagrangian, that is, its integral depends only on the boundary values of u (cf. e.g. [1], [10], [28]).

⁽¹⁾ For more on the structure of Sobolev spaces of maps between Riemannian manifolds see [2], [4], [10], [12], [23] and the references therein.

Moving on by considering the first order condition $d/d\varepsilon(\mathbb{F}[u_\varepsilon; \mathbb{X}^n])|_{\varepsilon=0} = 0$ where $u_\varepsilon = (u + \varepsilon\psi)/|u + \varepsilon\psi|$ for $\psi \in \mathcal{C}_0^\infty(\mathbb{X}^n, \mathbb{R}^n)$ and $\varepsilon \in \mathbb{R}$ is sufficiently small one can formulate the Euler–Lagrange equation associated with \mathbb{F} on $\mathcal{A}_\varphi^p(\mathbb{X}^n)$ as the nonlinear system

$$(1.3) \quad \begin{cases} \mathcal{L}[u] = \operatorname{div}[F'(|x|, |\nabla u|^2)\nabla u] + F'(|x|, |\nabla u|^2)|\nabla u|^2 u = 0 & \text{in } \mathbb{X}^n, \\ |u| = 1 & \text{in } \mathbb{X}^n, \\ u = \varphi & \text{on } \partial\mathbb{X}^n. \end{cases}$$

Note that here F' denotes the derivative of F with respect to the second variable and the divergence operator on the first line acts row-wise. ⁽²⁾

We will look at the particular example $F(r, t) = h(r)t^{p/2}$ with $1 < p < \infty$ and $h \in \mathcal{C}^1([a, b])$ satisfying $h > 0$ on $[a, b]$. (Here in view of the explicit form of the integrand we can slightly relax the regularity assumptions as stated earlier.) This leads to a generalisation of the usual Dirichlet energy called the *weighted p-energy* taking the form

$$(1.4) \quad \mathbb{E}_p^h[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} h(|x|)|\nabla u|^p dx = \int_a^b \int_{\mathbb{S}^{n-1}} h(r)|\nabla u|^p r^{n-1} dr d\mathcal{H}^{n-1}(\theta),$$

where the Euler–Lagrange equation in this case is as formulated by the system (1.3) with the differential operator \mathcal{L} being

$$(1.5) \quad \mathcal{L}[u] = h\Delta_p u + |\nabla u|^{p-2}\nabla u \nabla h + h|\nabla u|^p u = 0.$$

Here $\nabla h = \dot{h}x|x|^{-1}$, $\dot{h} = dh/dr$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian. For $h \equiv 1$ and $p = 2$, as stated, (1.4) is the Dirichlet energy while for the given range of p , (1.5) is the celebrated p -harmonic map equation. In analogy, we refer to solutions to (1.5) for arbitrary h as *weighted p-harmonic maps*. (For more on p -harmonic maps — the case $h \equiv 1$ — the reader is referred to [10], [13], [17], [22] for regularity results and [3], [11], [15], [16] for further results in particular on the structure of minimisers. See also [8], [10], [14] and the references therein.)

The first class of maps we examine in this paper as solutions to the nonlinear system (1.3) are the so-called spherical twists as introduced in [26] (see also [20], [21], [24], [25]). Recall that a *spherical twist* by definition is a map $u \in \mathcal{C}(\overline{\mathbb{X}^n}, \mathbb{S}^{n-1})$ of the form

$$(1.6) \quad u : x = r\theta \mapsto Q(r)\theta = Q(|x|x|x|^{-1}), \quad x \in \overline{\mathbb{X}^n},$$

where $a \leq r = |x| \leq b$ and $Q \in \mathcal{C}([a, b], SO(n))$. For obvious geometric reasons the continuous curve $[a, b] \ni r \mapsto Q(r) \in SO(n)$ will be referred to as the *twist path* associated with the spherical twist $u = Q(r)\theta$

⁽²⁾ A particular solution to this system is the radial projection $u(x) = x/|x|$ where $\varphi = x/|x|$ on $\partial\mathbb{X}^n$. For reasons that will be clear later we call this the *trivial* solution (see Section 2).

to be admissible, that is, to lie in $\mathcal{A}_\varphi^P(\mathbb{X}^n)$, it is evident that the boundary map φ must have the form

$$(1.7) \quad \varphi(r\theta) = \begin{cases} R_a\theta, & \partial\mathbb{X}_a^n = \{x : |x| = a\}, \\ R_b\theta, & \partial\mathbb{X}_b^n = \{x : |x| = b\}, \end{cases}$$

with R_a, R_b in $SO(n)$; or on the level of the twist path, $Q(a) = R_a, Q(b) = R_b$. Indeed since for any rotation $v = Pu$ of u , upon referring to (1.3), we have

$$\mathcal{L}[v] = \mathcal{L}[Pu] = P\mathcal{L}[u], \quad \mathcal{L}[v] = 0 \iff P\mathcal{L}[u] = 0 \iff \mathcal{L}[u] = 0,$$

there is no loss of generality in specialising to $R_a = I_n, R_b = R, R \in SO(n)$. We henceforth aim to describe those twist paths $Q \in \mathcal{C}^2([a, b], SO(n))$ such that the corresponding spherical twist $u = Q(r)\theta$ gives a (*classical*) solution to the system of Euler–Lagrange equations (1.3). This as will be seen involves a study of the geodesics of the compact lie group $SO(n)$ and their links with geodesics on the sphere.

The approach here is to restrict the energy to the space of spherical twists. Existence of multiple solutions to the resulting Euler–Lagrange equation follows by using variational methods. A more refined analysis then fully characterises the spherical twists associated with such extremising twist paths that grant solutions to the original system (1.3). We will see that here there is a curious discrepancy between odd and even dimensions in that in even dimensions there is an infinite family of spherical twist solutions to the system (1.3) whereas in odd dimensions there is either one or none depending as to whether $R = I_n$ or not! More specifically the following statement is proved in Section 2. Note that in the formulation below we are taking advantage of the diagonalisation $R = G_R D_R G_R^t$ with G_R, D_R orthogonal and D_R block diagonal, specifically, $D_R = \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta])$ for n even and $D_R = \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta], 1)$ for n odd where $\mathcal{R}[\eta]$ is the usual 2×2 rotation matrix by angle $\eta \in \mathbb{R}$. (See (2.7)–(2.9) for more details.)

THEOREM 1.1. *The spherical twist $u = Q(r)\theta$ is a solution of the Euler–Lagrange system (1.3) provided that depending on n being even or odd the twist path $Q = Q(r)$ has the explicit form below:*

(a) (*n even*) For $m \in \mathbb{Z}$ and P in the centraliser of D_R in $O(n)$ we have

$$(1.8) \quad Q(r; m) = G_R P \text{diag}(\mathcal{R}[\mathcal{G}](r), \dots, \mathcal{R}[\mathcal{G}](r)) P^t G_R^t,$$

where $\mathcal{G} = \mathcal{G}(r; m) \in \mathcal{C}^2([a, b], \mathbb{R})$ is the unique solution to the boundary value problem

$$(1.9) \quad \begin{cases} \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + \mathcal{G}^2 \right) r^{n-1} \mathcal{G} \right] = 0, \\ \mathcal{G}(a) = 0, \\ \mathcal{G}(b) = 2\pi m + \eta. \end{cases}$$

(b) (n odd) $R = I_n$ and $Q(r) \equiv I_n$.

We remark that in the case n odd, that is, (b) above, the problem admits a solution in the form of a spherical twist *if and only if* $R = I_n$, in which case this solution is the radial projection $u = x/|x|$. Hence unlike the n even case here there are no solutions for other choices of R . We also introduce and establish conditions that result in a converse to this theorem (see Remark 2.3 and Lemma 2.5 for details).

The second class of maps we examine in this paper as solutions to the system (1.3) are the spherical whirls (or whirls for simplicity). These by definition are maps $u \in \mathcal{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ of the form

$$u: x \mapsto Q(\rho_1, \dots, \rho_N)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}^n,$$

where $Q = Q(\rho_1, \dots, \rho_N)$ is a continuous $SO(n)$ -valued map depending on the spatial variable $x = (x_1, \dots, x_n)$ through the 2-plane variables $\rho = (\rho_1, \dots, \rho_N)$, that, depending on the dimension n being even or odd, we have the description:

(a) (n even) writing $n = 2N$ we set $k = N$ and then

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2} \quad \text{for } 1 \leq j \leq N.$$

(b) (n odd) writing $n = 2N - 1$ we set $k = N - 1$ and then

$$\rho_j := \begin{cases} \sqrt{x_{2j-1}^2 + x_{2j}^2} & \text{for } 1 \leq j \leq N - 1, \\ x_n & \text{for } j = N. \end{cases}$$

It is seen that the 2-plane variables $\rho = (\rho_1, \dots, \rho_N)$ lie in $\overline{\mathbb{A}}_N \subset \mathbb{R}^N$ where $\mathbb{A}_N = \{\rho \in \mathbb{R}_+^N : a < |\rho| < b\}$ ($n = 2N$) and $\mathbb{A}_N = \{\rho \in \mathbb{R}_+^{N-1} \times \mathbb{R} : a < |\rho| < b\}$ ($n = 2N - 1$). On account of this and as a result of the earlier discussion we hereafter suppose $Q \in \mathcal{C}(\overline{\mathbb{A}}_N, SO(n))$. Indeed prompted by the commutativity and rotational symmetry of whirls as will become clear later (see also [7], [18], [19]) we assume Q to take values on a fixed maximal torus of $SO(n)$, here, and for spherical whirls as ⁽³⁾

$$(1.10) \quad Q(\rho) = \begin{cases} \text{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k]) & \text{for } n = 2N, \\ \text{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k], 1) & \text{for } n = 2N - 1, \end{cases}$$

where for each $1 \leq l \leq k$, $f_l \in \mathcal{C}(\overline{\mathbb{A}}_N, \mathbb{R})$ satisfies $f_l \equiv 0$ when $|\rho| = a$ and $f_l \equiv 2\pi m + \eta$ for some $m \in \mathbb{Z}$ when $|\rho| = b$. For a spherical whirl $u = Q(\rho_1, \dots, \rho_N)\theta$

⁽³⁾ As any pair of maximal tori on a compact Lie group are *conjugate* the general form of (1.10) will be $PQ(\rho)P^t$ with $P \in SO(n)$. Here without loss of generality we have set $P = I_n$ and taken the maximal torus the one just described. Note that as a result in the description of the boundary map φ we have $R = \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta])$ ($n = 2N$) and $R = \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta], 1)$ ($n = 2N - 1$).

the weighted p -energy (1.4) can be seen to reduce for $p = 2$ to

$$\mathbb{E}_2^h[u; \mathbb{X}^n] = \int_{\mathbb{X}^n} h(r) \left[\frac{n-1}{r^2} + \frac{1}{r^2} \sum_{l=1}^k |\nabla f_l|^2 \rho_l^2 \right] dx.$$

As a result extremisers of \mathbb{E}_2^h within the class of whirls should satisfy for each $f_l = f_l(\rho_1, \dots, \rho_N)$ the equation (see Section 3 for details and notation)

$$\begin{cases} \operatorname{div} \left(h(r) \frac{\rho_l^2}{r^2} \prod_{j=1}^k \rho_j \nabla f \right) = 0 & \text{in } \mathbb{A}_N, \\ f \equiv 0 & \text{on } (\partial \mathbb{A}_N)_a, \\ f \equiv 2m\pi + \eta & \text{on } (\partial \mathbb{A}_N)_b, \\ \rho_l^2 \prod_{j=1}^k \rho_j \partial_\nu f = 0 & \text{on } \Gamma_N. \end{cases}$$

We see that the above has a unique solution for each $m \in \mathbb{Z}$ given explicitly by the formulation

$$(1.11) \quad f(\rho; m) = \frac{2\pi m + \eta}{\beta(b)} \beta(|\rho|), \quad \beta(t) := \int_a^t \frac{s^{1-n}}{h(s)} ds, \quad \rho \in \overline{\mathbb{A}}_N,$$

where $|\rho|^2 = \sum_{l=1}^N \rho_l^2$ and for the integral on the right $a \leq t \leq b$. Using the techniques developed and used in Section 2 we then prove the following statement.

THEOREM 1.2. *Consider the weighted Dirichlet energy \mathbb{E}_2^h as given by (1.4) with $h \in \mathcal{C}^1([a, b])$ satisfying $h > 0$ on $[a, b]$, the system (1.3) with $F(r, t) = h(r)t$ and φ as above. Then, depending on n being even or odd, the following hold.*

- (a) (*n even*) *There is a countably infinite family of spherical whirls serving as solutions to (1.3). Specifically, for each $m \in \mathbb{Z}$ the map $u(x) = Q(\rho; m)x|x|^{-1}$ with $Q(\rho; m) = \operatorname{diag}(\mathcal{R}[f](\rho; m), \dots, \mathcal{R}[f](\rho; m))$ and $f = f(\rho, m)$ as in (1.11) is a solution to (1.3).*
- (b) (*n odd*) *The only solution to (1.3) in the form of a spherical whirl is the radial projection $u \equiv x|x|^{-1}$ and only when $R = I_n$ (respectively, $\varphi = x|x|^{-1}$).*

2. Spherical twists as extremisers of the \mathbb{F} -energy and the nonlinear system (1.3)

Let $\mathbb{X}^n = \mathbb{X}^n[a, b]$ be as before and consider the \mathbb{F} -energy functional as given by the integral

$$(2.1) \quad \mathbb{F}[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} F(|x|, |\nabla u|^2) dx.$$

We assume that $F \in \mathcal{C}^{1,2}([a, b] \times \mathbb{R})$, that is, \mathcal{C}^1 with respect to the first variable and \mathcal{C}^2 with respect to the second. Furthermore, we assume there exist $c_1, c_2 > 0$ and $c_0 \in \mathbb{R}$ such that

$$\begin{aligned} |F'(r, \zeta^2)\zeta| &\leq c_2|\zeta|^{p-1} && \text{for all } a \leq r \leq b, \text{ for all } \zeta \in \mathbb{R}, \\ c_0 + c_1|\zeta|^p &\leq F(r, \zeta^2) \leq c_2|\zeta|^p && \text{for all } a \leq r \leq b, \text{ for all } \zeta \in \mathbb{R}, \end{aligned}$$

with $1 < p < \infty$ fixed. As a result \mathbb{F} is well-defined and finite on $W^{1,p}(\mathbb{X}^n, \mathbb{S}^{n-1})$. We further assume that $F' > 0$, $F'' \geq 0$ on $[a, b] \times (0, \infty)$ (recall that $'$ denotes derivative in the second argument) and that the function $\zeta \mapsto F(r, \zeta^2)$ is uniformly convex in ζ for all $a \leq r \leq b$ and $\zeta \in \mathbb{R}$.

We seek to extremise \mathbb{F} over the space $\mathcal{A}_\varphi^p(\mathbb{X}^n)$ defined by (1.2) with φ given by (1.7) and establish the existence of multiple spherical twist solutions to the associated Euler-Lagrange system given by (1.3). We start by calculating some useful quantities associated with spherical twists.

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PROPOSITION 2.1 (Key identities). *Suppose u is a spherical twist with a twice continuously differentiable twist path Q . Then with $r = |x|$, $\theta = x/|x|$ we have*

$$\begin{aligned} \text{(a)} \quad \nabla u &= \frac{Q + (r\dot{Q} - Q)\theta \otimes \theta}{r}. \\ \text{(b)} \quad |\nabla u|^2 &= \frac{n-1}{r^2} + |\dot{Q}\theta|^2. \\ \text{(c)} \quad \Delta u &= \frac{1}{r^2} ((n-1)(r\dot{Q} - Q) + r^2\ddot{Q})\theta. \\ \text{(d)} \quad \Delta_p u &= |\nabla u|^{p-4} \left\{ \frac{p-2}{2r} (Q + (r\dot{Q} - Q)\theta \otimes \theta) \left(\nabla |\dot{Q}\theta|^2 - \frac{2(n-1)}{r^3} \theta \right) \right. \\ &\quad \left. + \frac{1}{r^2} \left(\frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) ((n-1)(r\dot{Q} - Q) + r^2\ddot{Q}) \theta \right\}. \\ \text{(e)} \quad \operatorname{div} [F'(r, |\nabla u|^2)\nabla u] &= \frac{1}{r} F''(r, |\nabla u|^2) (Q + (r\dot{Q} - Q)\theta \otimes \theta) \left(\nabla |\dot{Q}\theta|^2 - \frac{2(n-1)}{r^3} \theta \right) \\ &\quad + \partial_r F'(r, |\nabla u|^2) \dot{Q}\theta + \frac{1}{r^2} F'(r, |\nabla u|^2) ((n-1)(r\dot{Q} - Q) + r^2\ddot{Q})\theta. \end{aligned}$$

PROOF. Let $u = Q(r)\theta$ be a spherical twist with a differentiable twist path Q . A straightforward differentiation then yields $\nabla u = (Q + (r\dot{Q} - Q)\theta \otimes \theta)/r$. Hence upon calculating the Hilbert-Schmidt norm of this gradient we have

$$|\nabla u|^2 = \operatorname{tr} \{ [\nabla u][\nabla u]^t \} = \frac{n-1}{r^2} + |\dot{Q}\theta|^2.$$

These are the first two identities in the list. To pass on to the remaining ones, assuming a further differentiability on Q , we can write, by using the chain rule:

$$(2.2) \quad \begin{aligned} \operatorname{div} [F'(r, |\nabla u|^2) \nabla u] &= \nabla u \nabla [F'(r, |\nabla u|^2)] + F'(r, |\nabla u|^2) \Delta u \\ &= F''(r, |\nabla u|^2) \nabla u \nabla (|\nabla u|^2) + \partial_r F'(r, |\nabla u|^2) \nabla u \theta + F'(r, |\nabla u|^2) \Delta u. \end{aligned}$$

Now differentiating the Hilbert–Schmidt norm squared $|\nabla u|^2$ using the second identity in the list we can write

$$(2.3) \quad \nabla |\nabla u|^2 = \nabla [(n-1)r^{-2} + |\dot{Q}\theta|^2] = \nabla |\dot{Q}\theta|^2 - 2(n-1)r^{-3}\theta,$$

and likewise taking the divergence of ∇u gives Δu as formulated in the third identity. Substituting these two into (2.2) gives the fifth identity in the list. Finally the fourth identity describing $\Delta_p u$ follows from the fifth one by specialising to $F(r, t) = 2p^{-1}t^{p/2}$. \square

Using the calculations in Proposition 2.1, for a given spherical twist $u = Q(r)\theta$ and exponent $1 \leq p < \infty$, we can express the $W^{1,p}$ -Sobolev norm by writing

$$\begin{aligned} \|u\|_{W^{1,p}(\mathbb{X}^n)}^p &= \int_{\mathbb{X}^n} (|u|^p + |\nabla u|^p) dx \\ &= \int_a^b \int_{\mathbb{S}^{n-1}} \left\{ |Q\theta|^p + \left(\frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right)^{p/2} \right\} r^{n-1} dr d\mathcal{H}^{n-1}(\theta), \end{aligned}$$

and its straightforward to see that this is finite if and only if

$$Q \in W^{1,p}((a, b), \operatorname{SO}(n)).$$

With this in mind we introduce the space of admissible twist paths

$$(2.4) \quad \mathcal{B}_R^p = \mathcal{B}_R^p([a, b]) := \{Q \in W^{1,p}((a, b), \operatorname{SO}(n)) : Q(a) = I_n, Q(b) = R\}.$$

Then, for the boundary map φ defined by (1.7) with $R_a = I_n$ and $R_b = R \in \operatorname{SO}(n)$, we have

$$u = Q(r)\theta \in \mathcal{A}_\varphi^p(\mathbb{X}^n) \iff Q \in \mathcal{B}_R^p([a, b]).$$

Again, by referring to the calculations in Proposition 2.1, we can write the \mathbb{F} -energy of a spherical twist $u = Q(r)\theta$ as

$$(2.5) \quad \begin{aligned} \mathbb{F}[Q(r)\theta; \mathbb{X}^n] &= \int_{\mathbb{X}^n} F\left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2\right) dx \\ &= \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2\right) r^{n-1} dr d\mathcal{H}^{n-1}(\theta) \\ &= \int_a^b G(r, \dot{Q}) r^{n-1} dr =: \mathbb{G}[Q; (a, b)] \end{aligned}$$

hence, introducing the energy functional $Q \mapsto \mathbb{G}[Q; (a, b)]$ on the space of twist paths \mathcal{B}_R^p , where in the last line the integrand $G = G(r, H)$ with $a \leq r \leq b$ and

H first in the space of $n \times n$ skew-symmetric matrices (i.e. satisfying $H^t = -H$) and then extended to all $n \times n$ matrices H (see Lemma 2.2 below) is given by

$$(2.6) \quad G(r, H) := \int_{\mathbb{S}^{n-1}} F\left(r, \frac{n-1}{r^2} + |H\theta|^2\right) d\mathcal{H}^{n-1}(\theta).$$

In what follows for the sake of convenience and future reference we assume that the orthogonal matrix R , describing the boundary map φ , is expressed in an orthogonally block diagonalised form as:

(1) (n even) with $n = 2k$,

$$(2.7) \quad R = G_R D_R G_R^t := G_R \operatorname{tr}(\nabla u)^2 - (\nabla \cdot u)^2 \operatorname{diag}(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \dots, \mathcal{R}[\eta_k]) G_R^t,$$

(2) (n odd) with $n = 2k + 1$,

$$(2.8) \quad R = G_R D_R G_R^t := G_R \operatorname{diag}(\mathcal{R}[\eta_1], \mathcal{R}[\eta_2], \dots, \mathcal{R}[\eta_k], 1) G_R^t.$$

Here G_R and D_R are both $n \times n$ orthogonal matrices and $\mathcal{R}[s]$ (with real s) refers to the usual 2×2 rotation matrix (counter clockwise rotation by angle s) as given by (cf. also (2.15) below)

$$(2.9) \quad \mathcal{R}[s] := \exp(sJ) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}.$$

Note that there is no uniqueness in the choice of G_R , however, in what follows we pick one such G_R and then fix it throughout. We begin by computing the Euler–Lagrange equation associated with \mathbb{G} over the space \mathcal{B}_R^p .

LEMMA 2.2. *The Euler–Lagrange equation associated with the energy \mathbb{G} defined by (2.5) over the space of admissible twist paths \mathcal{B}_R^p is given by*

$$(2.10) \quad \int_{\mathbb{S}^{n-1}} \frac{d}{dr} \left\{ r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) \cdot [\dot{Q}\theta \otimes Q\theta - Q\theta \otimes \dot{Q}\theta] \right\} d\mathcal{H}^{n-1}(\theta) = 0.$$

PROOF. First fix Q and for $\varepsilon > 0$ define the variation $Q_\varepsilon = Q + \varepsilon(F - F^t)Q$ where $F \in \mathcal{C}_0^\infty((a, b), \mathbb{M}^{n \times n})$ is arbitrary. Then to the first order in ε one can compute that Q_ε takes values in $SO(n)$. We differentiate with respect to ε and

set equal to zero and with a slight abuse of notation write

$$\begin{aligned}
 (2.11) \quad 0 &= \frac{d}{d\varepsilon} \mathbb{G}[\mathbb{Q}_\varepsilon; (a, b)] \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_a^b G(r, \dot{\mathbb{Q}}_\varepsilon) r^{n-1} dr \Big|_{\varepsilon=0} \\
 &= \int_a^b \int_{\mathbb{S}^{n-1}} 2F' \left(r, \frac{n-1}{r^2} + |\dot{\mathbb{Q}}\theta|^2 \right) \langle \dot{\mathbb{Q}}\theta, (\dot{\mathbb{F}} - \dot{\mathbb{F}}^t) \mathbb{Q}\theta \rangle r^{n-1} d\mathcal{H}^{n-1} dr \\
 &= \int_a^b \left\langle \int_{\mathbb{S}^{n-1}} -2 \frac{d}{dr} \left[r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{\mathbb{Q}}\theta|^2 \right) \dot{\mathbb{Q}}\theta \otimes \mathbb{Q}\theta \right] d\mathcal{H}^{n-1}, \right. \\
 &\qquad \qquad \qquad \left. \mathbb{F} - \mathbb{F}^t \right\rangle dr.
 \end{aligned}$$

Since this holds for all $\mathbb{F} \in \mathcal{C}_0^\infty((a, b), \mathbb{M}^{n \times n})$ the skew-symmetric part of the matrix represented by the spherical integral must be zero. \square

REMARK 2.3. In view of Lemma 2.2 a sufficient condition for a twist path $\mathbb{Q} = \mathbb{Q}(r)$ to be an extremiser of the energy functional \mathbb{G} is for it to satisfy the stronger condition

$$(2.12) \quad \frac{d}{dr} \left\{ r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{\mathbb{Q}}\theta|^2 \right) [\dot{\mathbb{Q}}\theta \otimes \mathbb{Q}\theta - \mathbb{Q}\theta \otimes \dot{\mathbb{Q}}\theta] \right\} = 0,$$

for $a < r < b$, $\theta \in \mathbb{S}^{n-1}$. In general, however, this is not a two way implication. As part of the analysis below we exploit some consequences of this *stronger* equation, particularly, in virtue of the relation it bears to the original system (1.3) as well as the nature of the geodesics on $\text{SO}(n)$.

Now we do not intend to solve (2.12) directly but instead show that under sufficient regularity, if $\mathbb{Q} = \mathbb{Q}(r)$ solves (2.12) for all $\theta \in \mathbb{S}^{n-1}$, then it must have the form $\mathbb{Q}(r) = \exp(\mathcal{G}(r)\mathbb{H})$ for suitable $\mathcal{G} = \mathcal{G}(r) \in \mathcal{C}^1([a, b])$ and constant $n \times n$ skew-symmetric \mathbb{H} ; thus relating to the formulation of \mathbb{Q} as given in parts (a) and (b) in Theorem 1.1. We first establish the following lemma that provides a connection between geodesics on \mathbb{S}^{n-1} and geodesics on $\text{SO}(n)$. This lemma will also enforce restrictions on the matrix \mathbb{R} associated with the boundary map φ . For notational convenience let us also write \mathbb{J}_n for the $n \times n$ skew-symmetric matrix $\mathbb{J}_n = \text{diag}(\mathbb{J}, \dots, \mathbb{J})$ when n is even and $\mathbb{J}_n = \text{diag}(\mathbb{J}, \dots, \mathbb{J}, 0)$ when n is odd. Here of course $\mathbb{J} = \mathbb{J}_2 = \mathcal{R}(\pi/2)$ (see (2.15) below).

LEMMA 2.4. *Let $0 < \ell < \infty$ and $\mathbb{K} \in \mathcal{C}^2((0, \ell), \text{SO}(n)) \cap \mathcal{C}([0, \ell], \text{SO}(n))$ satisfy $\mathbb{K}(0) = \mathbb{I}_n$ and $\mathbb{K}(\ell) = \mathbb{R}$. Then the following are equivalent:*

- (a) *The curve $s \mapsto \mathbb{K}(s)\theta$ with $0 \leq s \leq \ell$ is a geodesic on \mathbb{S}^{n-1} for every $\theta \in \mathbb{S}^{n-1}$.*
- (b) *Depending on n being even or odd one of (i) or (ii) described below holds:*
 - (i) *(n even) $\mathbb{R} = \mathbb{G}_\mathbb{R} \text{diag}(\mathcal{R}[\eta], \dots, \mathcal{R}[\eta]) \mathbb{G}_\mathbb{R}^t$ for some $\mathbb{G}_\mathbb{R}$ in $\text{O}(n)$ and $\eta \in \mathbb{R}$ and that there exists $m \in \mathbb{Z}$ and \mathbb{P} in the centraliser of $\mathbb{D}_\mathbb{R}$*

in $O(n)$ such that, with $\mathcal{H}(s; m; \ell) := (2\pi m + \eta)s/\ell$, K admits the factorisation

$$(2.13) \quad \begin{aligned} K(s; m) &= \exp(\mathcal{H}(s; m; \ell)H) \quad H = G_R P J_n P^t G_R^t, \quad 0 \leq s \leq \ell, \\ &= G_R P \operatorname{diag}(\mathcal{R}[\mathcal{H}](s; m; \ell), \dots, \mathcal{R}[\mathcal{H}](s; m; \ell)) P^t G_R^t, \end{aligned}$$

(ii) (n odd) $R = I_n$ and $K(s) \equiv I_n$.

PROOF. It is well known that on a round sphere geodesics $\gamma: [0, \ell] \rightarrow \mathbb{S}^{n-1}$ are great circles and satisfy the geodesic equation $\ddot{\gamma} + |\dot{\gamma}|^2\gamma = 0$. Hence, if $s \mapsto K(s)\theta$ is a geodesic for all θ then K satisfies the equation

$$(2.14) \quad [\ddot{K} + |\dot{K}\theta|^2 K]\theta = 0$$

for all $\theta \in \mathbb{S}^{n-1}$. We claim that if K satisfies (2.14) then $|\dot{K}\theta|^2$ is constant in θ and s . Indeed, differentiating with respect to s yields

$$\frac{d}{ds} |\dot{K}\theta|^2 = 2\langle \ddot{K}\theta, \dot{K}\theta \rangle = -2|\dot{K}\theta|^2 \langle K\theta, \dot{K}\theta \rangle = -2|\dot{K}\theta|^2 \langle \dot{K}^t K\theta, \theta \rangle = 0,$$

where we have used the fact that $\dot{K}^t K$ is skew-symmetric. It therefore follows that $|\dot{K}\theta|^2 = \psi(\theta)$ for some $\psi \in \mathcal{C}(\mathbb{S}^{n-1}, \mathbb{R})$. By rearranging (2.14) we obtain $K^t \ddot{K}\theta = -\psi(\theta)\theta$. For fixed s this implies that $-\psi(\theta)$ is an eigenvalue of the matrix $K^t \ddot{K}$. However, since $K^t \ddot{K}$ has at most n eigenvalues it follows, by the continuity of ψ , that ψ must be constant say $\psi(\theta) = |\dot{K}\theta|^2 = t^2$.

We now conclude that if $K(s)\theta$ is a geodesic for all θ then K must satisfy the linear ODE: $\ddot{K} + t^2 K = 0$. To solve this we make the ansatz $K(s) = \exp(sA)K_0$ for some skew-symmetric matrix A and some $K_0 \in SO(n)$. By differentiating K it is seen that A satisfies $[A^2 + t^2 I_n]K = 0$. This implies that $A^2 = -t^2 I_n$. Using spectral theorem we may write A in block diagonal form, that is,

$$(2.15) \quad A = \begin{cases} P \operatorname{diag}(t_1 J, \dots, t_k J) P^t & \text{if } n = 2k, \\ P \operatorname{diag}(t_1 J, \dots, t_k J, 0) P^t & \text{if } n = 2k + 1, \end{cases} \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

for some $P \in O(n)$ and $(t_j)_{j=1}^k \subset \mathbb{R}$. By writing A in block diagonal form as above we see that for $n = 2k$, we have

$$A^2 = -P \operatorname{diag}(t_1^2 I_2, \dots, t_k^2 I_2) P^t = -t^2 I_n \Rightarrow |t_1| = \dots = |t_k| = |t|.$$

Similarly for $n = 2k + 1$, we have

$$A^2 = -P \operatorname{diag}(t_1^2 I_2, \dots, t_k^2 I_2, 0) P^t = -t^2 I_n \Rightarrow |t_1| = \dots = |t_k| = |t| = 0.$$

We now chose A and K_0 so that $K(0) = I_n$ and $K(\ell) = R$. First, if A is chosen, then $K(0) = I_n$ implies $K_0 = \exp(-0 \cdot A) = I_n$ and hence $K(s) = \exp(sA)$. We now choose A such that $K(\ell) = \exp(\ell A) = R$. This is done by writing $\exp(\ell A)$ in block diagonal form and comparing with R (cf. [26, Theorem 2.1]). \square

Referring to the discussion in the previous lemma, for a $Q \in \mathcal{C}^1([a, b], SO(n))$, the integral

$$(2.16) \quad I(Q, \theta) = \int_a^b |\dot{Q}\theta| \, dr,$$

represents the length of the spherical curve $\gamma(r) = Q(r)\theta$. In the next lemma we show that for a given Q satisfying (2.12) for every θ , if the value of this integral is also independent of θ then Q has a factorisation similar to that given in Lemma 2.4.

LEMMA 2.5. *Let R be as given in Lemma 2.4 and for each $m \in \mathbb{Z}$ let $\mathcal{G} = \mathcal{G}(\cdot, m) \in \mathcal{C}^2([a, b])$ be a solution to the boundary value problem*

$$(2.17) \quad \begin{cases} \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + \mathcal{G}^2 \right) r^{n-1} \mathcal{G} \right] = 0 & \text{in } (a, b), \\ \mathcal{G}(a) = 0, \\ \mathcal{G}(b) = 2\pi m + \eta. \end{cases}$$

Let $Q \in \mathcal{C}^2((a, b), SO(n)) \cap \mathcal{C}^1([a, b], SO(n))$ satisfy $Q(a) = I_n$ and $Q(b) = R$. Then the following are equivalent:

- (a) Q is a solution to (2.12) for every $\theta \in \mathbb{S}^{n-1}$ and the integral in (2.16) is independent of θ .
- (b) Depending on n being even or odd, one of (i) or (ii) below holds.
 - (i) (n even) There exists $m \in \mathbb{Z}$ and P is in the centraliser of D_R in $O(n)$ such that $Q = Q(r; m)$ admits the factorisation

$$(2.18) \quad \begin{aligned} Q(r; m) &= \exp(\mathcal{G}(r; m)H) \quad H = G_R P J_n P^t G_R^t, \quad a \leq r \leq b, \\ &= G_R P \operatorname{diag}(\mathcal{R}[\mathcal{G}](r; m), \dots, \mathcal{R}[\mathcal{G}](r; m)) P^t G_R^t. \end{aligned}$$

- (ii) (n odd) $R = I_n$ and $Q(r) \equiv I_n$.

PROOF. We start by verifying that functions of the form (2.18) are solutions to (2.12) (the odd case is trivial). If Q is given by (2.18) then we have

$$\begin{aligned} \dot{Q}\theta \otimes Q\theta &= \dot{\mathcal{G}}HQ\theta \otimes Q\theta, \\ \ddot{Q}\theta \otimes Q\theta &= \ddot{\mathcal{G}}HQ\theta \otimes Q\theta - \mathcal{G}^2 Q\theta \otimes Q\theta, \end{aligned}$$

where in concluding the second identity we have used

$$(2.19) \quad H^2 = G_R P J_n^2 P^t G_R^t = -G_R P I_n P^t G_R^t = -I_n.$$

Therefore by substitution and straightforward differentiation starting from (2.12) we have

$$\begin{aligned}
 (2.20) \quad & \frac{d}{dr} \left\{ r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) [\dot{Q}\theta \otimes Q\theta - Q\theta \otimes \dot{Q}\theta] \right\} \\
 & = \left\{ \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + \mathcal{G}^2 \right) r^{n-1} \right] \mathcal{G} + F' \left(r, \frac{n-1}{r^2} + \mathcal{G}^2 \right) r^{n-1} \mathcal{G}' \right\} \\
 & \quad \times (HQ\theta \otimes Q\theta - Q\theta \otimes HQ\theta) = 0,
 \end{aligned}$$

as a result of \mathcal{G} being a solution to (2.17).

Now to prove the reverse implication, first note that if $I(Q, \theta) = 0$ then we have $|\dot{Q}\theta| = 0$ and hence $Q \equiv I_n$. We can therefore assume for the rest of the proof that $I(Q, \theta) > 0$.

Next, observe that if Q is a solution to (2.12) then multiplying (2.12) by $Q\theta$ and using the observation $[Q\theta \otimes \ddot{Q}\theta]Q\theta = -|\dot{Q}\theta|^2 Q\theta$ it follows that Q satisfies

$$\begin{aligned}
 (2.21) \quad & \frac{d}{dr} \left[r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) \dot{Q}\theta \right] \\
 & \quad + r^{n-1} F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) |\dot{Q}\theta|^2 Q\theta = 0.
 \end{aligned}$$

Now the quantity $p(r, \theta) = F'(r, (n-1)r^{-2} + |\dot{Q}\theta|^2)r^{n-1}|\dot{Q}\theta|$ is constant in r . To see this we differentiate with respect to r and use the fact that Q solves (2.21) to find

$$\begin{aligned}
 \frac{d}{dr} p(r, \theta) &= \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) r^{n-1} \right] |\dot{Q}\theta| \\
 & \quad + \left[F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) r^{n-1} \right] \frac{\langle \ddot{Q}\theta, \dot{Q}\theta \rangle}{|\dot{Q}\theta|} \\
 &= \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) r^{n-1} \right] |\dot{Q}\theta| \\
 & \quad - \frac{d}{dr} \left[F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) r^{n-1} \right] \frac{\langle \dot{Q}\theta, \dot{Q}\theta \rangle}{|\dot{Q}\theta|} \\
 & \quad - \left[F' \left(r, \frac{n-1}{r^2} + |\dot{Q}\theta|^2 \right) r^{n-1} \right] \frac{\langle \dot{Q}\theta, Q\theta \rangle |\dot{Q}\theta|^2}{|\dot{Q}\theta|} = 0,
 \end{aligned}$$

where in the last line we have used the fact that $Q^t \dot{Q}$ is skew-symmetric. Note that this argument shows that $p(r, \theta)$, as a function of r , is a positive constant on any interval on which $|\dot{Q}\theta|$ is non-zero and so a basic continuity argument implies that either $|\dot{Q}\theta| \equiv 0$ on $[a, b]$ or $|\dot{Q}\theta| > 0$ on $[a, b]$. It follows that for each θ the function

$$(2.22) \quad \mathcal{F}(r, \theta) := \int_a^r |\dot{Q}(\tau)\theta| d\tau, \quad a \leq r \leq b,$$

is a solution to (2.17) with $\mathcal{F}(b, \theta) = I(Q, \theta) > 0$. We now claim that $\mathcal{F} = \mathcal{F}(r, \theta)$ is independent of θ . Indeed solutions to (2.10) are extremals of the energy functional

$$\mathcal{G} \mapsto \int_a^b F\left(r, \frac{n-1}{r^2} + \dot{\mathcal{G}}^2\right) r^{n-1} dr.$$

It is straightforward to verify that this functional is strictly convex (due to the assumptions on F : $F' > 0$ and F being convex in t). Therefore, using standard convexity arguments, solutions to (2.10) are the unique minimisers of the energy functional with respect to their own boundary conditions. This implies that as $\mathcal{F}(r, \theta)$ is a solution to (2.10) for all θ and as the boundary conditions of \mathcal{F} are independent of θ it follows from uniqueness that the function $\mathcal{F}(r, \theta)$ must be independent of θ .

Next, since $F' > 0$ we have that non-trivial solutions to (2.17) are strictly monotone and hence invertible. Thus we can make use of the change of variables $r \mapsto s(r)$ with $s(r) = \mathcal{F}(r)$. Let $K(s) = Q(r(s))$ where $r = r(s)$ denotes the inverse of $s = s(r)$ for $a \leq r \leq b$. Then $K \in \mathcal{C}^2((0, \ell), \text{SO}(n)) \cap \mathcal{C}([0, \ell], \text{SO}(n))$, where $\ell = \mathcal{F}(b)$. Upon writing $Q(r) = K(s(r))$, using the chain rule and utilising (2.21), we have

$$(2.23) \quad \frac{d}{ds} \left[r^{n-1} F' \left(r, \frac{n-1}{r^2} + \dot{s}^2 |\dot{K}\theta|^2 \right) \dot{s} \dot{K}\theta \right] + r^{n-1} F' \left(r, \frac{n-1}{r^2} + \dot{s}^2 |\dot{K}\theta|^2 \right) |\dot{K}\theta|^2 \dot{s} K\theta = 0.$$

Since $|\dot{Q}\theta|^2 = \dot{\mathcal{F}}^2$ we have that $|\dot{K}\theta|^2 = |\dot{Q}\theta|^2 / \dot{\mathcal{F}}^2 = 1$. Hence, since \mathcal{F} satisfies (2.17), we have that

$$(2.24) \quad r^{n-1} F' \left(r, \frac{n-1}{r^2} + \dot{s}^2 |\dot{K}\theta|^2 \right) \dot{s} = r^{n-1} F' \left(r, \frac{n-1}{r^2} + \dot{\mathcal{F}}^2 \right) \dot{\mathcal{F}} = c,$$

for some constant $c \in \mathbb{R}$. Therefore, equation (2.23) reduces to

$$(2.25) \quad c \left(\frac{d}{ds} \dot{K}\theta + |\dot{K}\theta|^2 K \right) \theta = 0.$$

This is precisely the geodesic equation on spheres (2.14) from the proof of Lemma 2.4. In odd dimensions there are no K that solve this with $|\dot{K}\theta|^2 = 1$ and so there are no solutions if $I(Q, \theta) \neq 0$. On the other hand if n is even by Lemma 2.4 we must have $\mathcal{F}(b) = |2\pi m + \eta|$ for some $m \in \mathbb{Z}$ and K is given by

$$K(s) = G_{\mathbb{R}} P \text{diag}(\mathcal{R}[\mathcal{H}](s; m), \dots, \mathcal{R}[\mathcal{H}](s; m)) P^t G_{\mathbb{R}}^t$$

where $\mathcal{H}(s; m) = \pm s$ depending on $2\pi m + \eta$ being positive or negative. (Note that when $2\pi m + \eta \geq 0$ then \mathcal{F} is a solution to (2.17) and when $2\pi m + \eta < 0$ then $-\mathcal{F}$ is a solution to (2.17).) It now follows that when n is even

$$Q(r) = K(\mathcal{F}(r)) = G_{\mathbb{R}} P \text{diag}(\mathcal{R}[\mathcal{G}](r), \dots, \mathcal{R}[\mathcal{G}](r)) P^t G_{\mathbb{R}}^t.$$

This therefore completes the proof of the lemma. □

PROOF OF THEOREM 1.1. We start by proving the existence of solutions to (2.17). The equation (2.17) is the Euler–Lagrange equation associated with the energy $\mathcal{G} \mapsto \mathbb{F}[\exp(\mathcal{G}(r)\mathbf{H})]$. Due to the assumptions on the integrand F this energy is sequentially lower semi-continuous and coercive. One can then prove existence of solutions to (2.17) for each $m \in \mathbb{Z}$ using the direct method and regularity is obtained by standard methods (cf. [5]).

We now justify that the spherical twist $u(x) = Q(r)\theta$ with $Q = Q(r)$ defined by Lemma 2.5, satisfies the Euler–Lagrange equation (1.3). First, for Q of the form (2.18), we have $\dot{Q}(r) = \dot{\mathcal{G}}(r)\mathbf{H}Q$ where \mathbf{H} is the skew-symmetric matrix $\mathbf{H} = G_R P J_n P^t G_R^t$ satisfying (2.19), hence

$$|\dot{Q}\theta|^2 = \langle \dot{\mathcal{G}}(r)\mathbf{H}Q\theta, \dot{\mathcal{G}}(r)\mathbf{H}Q\theta \rangle = \dot{\mathcal{G}}(r)^2 \langle \theta, \theta \rangle = \dot{\mathcal{G}}(r)^2.$$

Anticipating (2.1), we can easily calculate that $\nabla|\dot{Q}\theta|^2 = 2\dot{\mathcal{G}}(r)\ddot{\mathcal{G}}(r)\theta$ thus

$$\nabla u \nabla(|\nabla u|^2) = 2 \left(\dot{\mathcal{G}}(r)\ddot{\mathcal{G}}(r) - \frac{n-1}{r^3} \right) \dot{\mathcal{G}}(r)\mathbf{H}Q(r)\theta.$$

We can now verify, using Proposition 2.1, that the spherical twist with twist path as defined in part (a) of the theorem is a solution to (1.3). Indeed referring to (1.3) we can write

$$\begin{aligned} (2.26) \quad \mathcal{L}[u] &= \operatorname{div} [F'(r, |\nabla u|^2)\nabla u] + F'(r, |\nabla u|^2)|\nabla u|^2 u \\ &= 2F''(r, |\nabla u|^2) \left(\dot{\mathcal{G}}\ddot{\mathcal{G}} - \frac{n-1}{r^3} \right) \dot{Q}\theta + \partial_r F'(r, |\nabla u|^2) \dot{Q}\theta \\ &\quad + F'(r, |\nabla u|^2) \times \frac{1}{r^2} \left((n-1)(r\dot{Q} - Q) + r^2\ddot{Q} \right) \theta \\ &\quad + F'(r, |\nabla u|^2) \left(\dot{\mathcal{G}}^2 + \frac{n-1}{r^2} \right) Q\theta \\ &= \left\{ 2F''(r, |\nabla u|^2) \left(\dot{\mathcal{G}}\ddot{\mathcal{G}} - \frac{n-1}{r^3} \right) \dot{\mathcal{G}} \right. \\ &\quad \left. + \partial_r F'(r, |\nabla u|^2) \dot{\mathcal{G}} + F'(r, |\nabla u|^2) \left(\frac{n-1}{r} \dot{\mathcal{G}} + \ddot{\mathcal{G}} \right) \right\} \mathbf{H}Q\theta = 0. \end{aligned}$$

Note that the term in the curly brackets in the last equation in (2.26) is zero as a consequence of \mathcal{G} satisfying the first equation in (2.17) from Lemma 2.5. This finishes part (a).

Now for part (b) we note that for $u = x/|x|$ referring to the key identities in Proposition 2.1 with $Q \equiv I_n$ we can write

$$\begin{aligned} (2.27) \quad \mathcal{L}[u] &= \operatorname{div} [F'(r, |\nabla u|^2)\nabla u] + F'(r, |\nabla u|^2)|\nabla u|^2 u \\ &= \nabla u \nabla F'(r, |\nabla u|^2) + F'(r, |\nabla u|^2)[\Delta u + |\nabla u|^2 u] = 0 \end{aligned}$$

as a result of $\Delta u + |\nabla u|^2 u = 0$ and $F'(r, |\nabla u|^2)$ being solely a function of r and therefore its gradient being a (scalar function) multiple of $\theta = x|x|^{-1}$. \square

2.1. Weighted p -harmonic maps for a continuous h . In this section we take a closer look at the integrand $F(r, \zeta^2) = h(r)\zeta^p$ for some weight $h \in \mathcal{C}^1([a, b])$ satisfying the strict inequality $h > 0$ on $[a, b]$ and consider the resulting weighted p -energy (with $p > 1$) given by the integral

$$(2.28) \quad \mathbb{E}_p^h[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} h(|x|)|\nabla u|^p dx = \int_a^b \int_{\mathbb{S}^{n-1}} h(r)|\nabla u|^p r^{n-1} dr d\mathcal{H}^{n-1}.$$

Recall that extremisers of \mathbb{E}_p^h hereafter called weighted p -harmonic maps satisfy the associated Euler–Lagrange system (1.5). It is the aim here to characterise the spherical twists u granting solutions to this system. Indeed using the discussion and arguments in the previous section it is plain that here we have the following conclusion serving as the counterpart of Theorem 1.1 in the introduction.

THEOREM 2.6. *Let $h \in \mathcal{C}^1([a, b])$ satisfy $h > 0$ on $[a, b]$ and suppose R is as in Lemma 2.4. Then the spherical twist $u = Q(r)\theta$ is a solution to the system (1.3)–(1.5) provided that depending on n being even or odd, the twist path $Q = Q(r)$ has the explicit form.*

- (a) (n even) For any $m \in \mathbb{Z}$ and P is in the centraliser of D_R in $O(n)$, we have

$$(2.29) \quad Q(r; m) = G_R P \operatorname{diag}(\mathcal{R}[\mathcal{G}](r), \dots, \mathcal{R}[\mathcal{G}](r)) P^t G_R^t,$$

where $\mathcal{G} = \mathcal{G}(r; m) \in \mathcal{C}^2([a, b], \mathbb{R})$ is a solution to the boundary value problem

$$(2.30) \quad \begin{cases} \ddot{\mathcal{G}} \left[(p-1)\mathcal{G}^2 + \frac{n-1}{r^2} \right] \\ + \frac{n-1}{r} \dot{\mathcal{G}} \left[\mathcal{G}^2 + \frac{n+1-p}{r^2} \right] + \frac{\dot{h}}{h} \mathcal{G} \left[\mathcal{G}^2 + \frac{n-1}{r^2} \right] = 0, \\ \mathcal{G}(a) = 0, \\ \mathcal{G}(b) = 2\pi m + \eta. \end{cases}$$

- (b) (n odd) $R = I_n$ and $Q(r) \equiv I_n$.

Note that again in the n odd case there is spherical twist solution to the system (1.3)–(1.5) only when $R = I_n$ in which case u must be the radial projection $x/|x|$. We next specialise to the case $p = 2$ and explicitly compute spherical twists that are weighted 2-harmonic maps for a given weight function h as above. First

the ODE in Theorem 2.6 in this case reduces to

$$\begin{cases} \ddot{\mathcal{G}} + \frac{n-1}{r} \dot{\mathcal{G}} + \frac{\dot{h}}{h} \mathcal{G} = 0 & \text{in } (a, b), \\ \mathcal{G}(a) = 0, \\ \mathcal{G}(b) = 2\pi m + \eta. \end{cases}$$

Interestingly the explicit solution $\mathcal{G} = \mathcal{G}(r; m)$ with $a \leq r \leq b$ can then be seen to be given by the formulation

$$(2.31) \quad \mathcal{G}(r) = \frac{2\pi m + \eta}{\beta(b)} \beta(r),$$

where β is the function given by the integral

$$(2.32) \quad \beta(r) = \int_a^r \frac{s^{1-n}}{h(s)} ds.$$

As a result in the n even case the required weighted 2-harmonic maps $u = Q(r)\theta$ are obtained via (2.29) by substituting for \mathcal{G} from (2.31). Note that the \mathbb{E}_2^h -energy of these extremising maps takes the explicit form

$$\begin{aligned} \mathbb{E}_2^h[Q(r)\theta; \mathbb{X}^n] &= \int_a^b \int_{\mathbb{S}^{n-1}} r^{n-1} h(r) |\nabla[Q(r)\theta]|^2 dr d\mathcal{H}^{n-1}(\theta) \\ &= \int_a^b n\omega_n r^{n-1} h(r) \left[\frac{n-1}{r^2} + \left(\frac{(2\pi m + \eta)r^{1-n}}{\beta(b)h(r)} \right)^2 \right] dr, \end{aligned}$$

where $\omega_n = |\mathbb{B}_1^n|$. By inspection this energy is seen to diverge quadratically as $|m| \nearrow \infty$. Note also that the condition $h > 0$ here ensures the coercivity of the energy and the ellipticity of the resulting Euler–Lagrange system whilst for the sole purpose of (2.31)–(2.32) all that is required of h is for $s^{1-n}/h(s)$ to be L^1 -integrable on (a, b) .

2.2. Weighted harmonic maps with discontinuous h . As a variation of the above theme in this subsection we consider the weighted Dirichlet energy where the weight function h has jump discontinuities. For the ease of exposition and clarity here we assume that there are only finitely many such jumps within the interval (a, b) but a similar analysis can also be carried out for infinitely many jumps. To start let us consider first the case of a single jump discontinuity at a point $c \in (a, b)$:

$$\mathbb{E}_2^h[u; \mathbb{X}^n] = \int_{\mathbb{X}^n} h(|x|) |\nabla u|^2 dx, \quad \text{where } h(r) = \begin{cases} h_1 & \text{if } r \in [a, c], \\ h_2 & \text{if } r \in (c, b], \end{cases}$$

and $h_1, h_2 \in \mathbb{R}$ are fixed (non-zero) constants. ⁽⁴⁾ If we restrict to spherical twists then we can write $\mathbb{E}_2^h[\mathbb{Q}\theta; \mathbb{X}^n] = \omega_n \mathbb{G}_2^h[\mathbb{Q}]$ where

$$\mathbb{G}_2^h[\mathbb{Q}] := h_1 \int_a^c \left[n \frac{n-1}{r^2} + |\dot{\mathbb{Q}}|^2 \right] r^{n-1} dr + h_2 \int_c^b \left[n \frac{n-1}{r^2} + |\dot{\mathbb{Q}}|^2 \right] r^{n-1} dr.$$

Similarly to Lemma 2.2 we can compute that the Euler–Lagrange equation associated with \mathbb{G}_2^h is the system

$$\begin{cases} \frac{d}{dr} [r^{n-1} \dot{\mathbb{Q}} \mathbb{Q}^t] = \mathbb{0} & \text{on } (a, c) \cup (c, b), \\ \mathbb{Q}(a) = \mathbb{I}_n, \quad \mathbb{Q}(b) = \mathbb{R}, \\ h_1 \partial_- \mathbb{Q}(c) \mathbb{Q}(c)^t - h_2 \partial_+ \mathbb{Q}(c) \mathbb{Q}(c)^t = \mathbb{0}, \end{cases}$$

where ∂_-, ∂_+ denote the left and right derivative, respectively. We point out that the $W^{1,2}$ integrability of the twist path $\mathbb{Q} = \mathbb{Q}(r)$ implies that \mathbb{Q} is continuous on the interval $[a, b]$. Integrating the ODE on $(a, c) \cup (c, b)$ we observe that for suitable $n \times n$ skew-symmetric A_1, A_2 and orthogonal $\mathbb{Q}_1, \mathbb{Q}_2$:

$$(2.33) \quad \mathbb{Q}(r) = \begin{cases} \exp(A_1/r^{n-2}) \times \mathbb{Q}_1 & \text{for } a \leq r \leq c, \\ \exp(A_2/r^{n-2}) \times \mathbb{Q}_2 & \text{for } c \leq r \leq b. \end{cases}$$

Now $\mathbb{Q}(a) = \mathbb{I}_n$ gives $\mathbb{Q}_1 = \exp(-A_1/a^{n-2})$ and the continuity of \mathbb{Q} at $r = c$ gives $\mathbb{Q}_2 = \exp((c^{2-n} - a^{2-n})A_1 - c^{2-n}A_2)$. Next computing the left and right derivatives at c results in $h_1 A_1 = h_2 A_2$. Finally to choose A_1 so that $\mathbb{Q}(b) = \mathbb{R}$ we write A_1 in block diagonal form, that is,

$$A_1 = \begin{cases} P_A \text{diag}(s_1 J, \dots, s_k J) P_A^t & \text{if } n = 2k, \\ P_A \text{diag}(s_1 J, \dots, s_k J, 0) P_A^t & \text{if } n = 2k + 1. \end{cases}$$

for suitable $P_A \in \text{SO}(n)$ and $(s_i)_{i=1}^k \subset \mathbb{R}$. Next upon writing $\mathbb{R} = G_R D_R G_R^t$, with \mathbb{R} given by (2.7) or (2.8), it follows that the scalars s_i must be given by

$$(2.34) \quad s_i = \frac{(2\pi m_i + \eta_i)/h_1}{(c^{2-n} - a^{2-n})/h_1 + (b^{2-n} - c^{2-n})/h_2}, \quad 1 \leq i \leq k,$$

for suitable $(m_i)_{i=1}^k \subset \mathbb{Z}$ and $P_A = G_R P$ for some P in the centraliser of D_R . Finally in a fashion similar to what was done in Section 2 (cf. [26]) it can be shown that for n even and with $s_i = s_j$ for all i, j the spherical twist associated with the twist path (2.33) is a solution to the system of Euler–Lagrange equations associated with \mathbb{E}_2^h .

By iterating the above argument and invoking the matrix exponential map one can prove the following extension. Note that despite the discontinuity and

⁽⁴⁾ For technical reasons we additionally require $(c^{2-n} - a^{2-n})/h_1 + (b^{2-n} - c^{2-n})/h_2 \neq 0$ (see (2.34)). Note also that ruling out the case where either of h_1, h_2 is zero is due to the basic fact that here there would be an uncountable family of curves $\mathbb{Q} = \mathbb{Q}(r)$ in the corresponding sub-interval serving trivially as solutions.

sign condition on the weight function h these solutions agree with those given by the integral representation (2.31)–(2.32) in Theorem 2.6.

THEOREM 2.7. *Let $n = 2k$ and \mathbb{R} be as in Lemma 2.4. Furthermore, assume the finite sequence $(h_i : 1 \leq i \leq l) \subset \mathbb{R}$ satisfies $h_i \neq 0$, $(c_i : 1 \leq i \leq l) \subset \mathbb{R}$ satisfies $a = c_0 < c_1 < \dots < c_l = b$ and that the weighted sum in the formulation of ζ in (2.37) is non-zero. Let h be the discontinuous piecewise constant weight function*

$$h(r) = \sum_{i=1}^l h_i \chi_{[c_{i-1}, c_i)}(r), \quad a \leq r \leq b,$$

where $\chi_{[c_{i-1}, c_i)}$ is the characteristic function of the interval $[c_{i-1}, c_i)$. For fixed $m \in \mathbb{Z}$ and \mathbf{P} in the centraliser of $D_{\mathbb{R}}$ consider the twist path $\mathbf{Q} = \mathbf{Q}(r; m)$ given by the factorisation

$$(2.35) \quad \mathbf{Q}(r; m) = G_{\mathbb{R}} \mathbf{P} \operatorname{diag}(\mathcal{R}[\mathcal{G}](r), \dots, \mathcal{R}[\mathcal{G}](r)) \mathbf{P}^t G_{\mathbb{R}}^t$$

where $\mathcal{G} = \mathcal{G}(r; m)$ with $a \leq r \leq b$ is the continuous function given piecewisely, for $1 \leq j \leq l$, by

$$(2.36) \quad \mathcal{G}(r) = \left[\sum_{i=1}^{j-1} \left(\frac{c_i^{2-n} - c_{i-1}^{2-n}}{h_i} \right) + \frac{r^{2-n} - c_{j-1}^{2-n}}{h_j} \right] \zeta \quad \text{for } r \in [c_{j-1}, c_j]$$

and ζ is the quantity given by

$$(2.37) \quad \zeta = \frac{2\pi m + \eta}{\sum_{i=1}^l h_i^{-1} (c_i^{2-n} - c_{i-1}^{2-n})}.$$

Then the spherical twist $u = \mathbf{Q}(r; m)\theta$ is a solution to the Euler–Lagrange system associated with the energy \mathbb{E}_2^h , that is, u is a weighted 2-harmonic map.

3. Spherical whirls as extremisers of the weighted Dirichlet energy

The aim of this section is to consider and examine a second class of maps with less symmetry as solutions to the system of Euler–Lagrange equations associated with the weighted Dirichlet energy. These can be regarded as a generalisation of spherical twists where the twist path depends on the spatial variable in a more complex way whilst its range is confined to a fixed maximal torus. Indeed instead of a usual twist path $\mathbf{Q} = \mathbf{Q}(r)$ we consider an $SO(n)$ -valued map $\mathbf{Q} = \mathbf{Q}(\rho)$ depending on the 2-plane radial variables $\rho = (\rho_1, \dots, \rho_N)$ (see below) and we solely restrict to the energy functional \mathbb{E}_2^h for some fixed \mathcal{C}^1 function h satisfying $h > 0$ on $[a, b]$.

Let us proceed by formally introducing the *spherical whirls*. Towards this end we first define the 2-plane radial variables $\rho = (\rho_1, \dots, \rho_N)$ as functions of the spatial variable $x = (x_1, \dots, x_n)$ on \mathbb{X}^n given, depending on n being even or odd, by

(a) (n even) set $n = 2k$ and put

$$(3.1) \quad \rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2} \quad \text{for } 1 \leq j \leq k.$$

(b) (n odd) set $n = 2k + 1$ and put

$$(3.2) \quad \rho_j := \begin{cases} \sqrt{x_{2j-1}^2 + x_{2j}^2} & \text{for } 1 \leq j \leq k, \\ x_{2k+1} & \text{for } j = k + 1. \end{cases}$$

In order to ease notation we also introduce $N = N(n)$ by writing $N = k$ when $n = 2k$ and $N = k + 1$ when $n = 2k + 1$. Then $\rho = (\rho_1, \dots, \rho_N)$ is seen to lie in $\overline{\mathbb{A}}_N$ where the open domain $\mathbb{A}_N \subset \mathbb{R}^N$ is given by

$$\mathbb{A}_N = \begin{cases} \{\rho \in \mathbb{R}_+^N : a < |\rho| < b\} & \text{for } n = 2k, \\ \{\rho \in \mathbb{R}_+^{N-1} \times \mathbb{R} : a < |\rho| < b\} & \text{for } n = 2k + 1. \end{cases}$$

We will write $(\partial\mathbb{A}_N)_a = \{\rho \in \partial\mathbb{A}_N : |\rho| = a\}$, $(\partial\mathbb{A}_N)_b = \{\rho \in \partial\mathbb{A}_N : |\rho| = b\}$ and $\Gamma_N = \partial\mathbb{A}_N \setminus \{\rho \in \partial\mathbb{A}_N : |\rho| = a \text{ or } |\rho| = b\}$ to denote the three segments of the boundary of \mathbb{A}_N . Now we have

$$(3.3) \quad u : x \mapsto Q(\rho)\theta = Q(\rho_1, \dots, \rho_N)x|x|^{-1}, \quad x \in \mathbb{X}^n,$$

where $\theta = x|x|^{-1}$, $\rho = \rho(x) = (\rho_1, \dots, \rho_N)$ is the vector of 2-plane variables as above and $Q \in \mathcal{C}(\overline{\mathbb{A}}_N, \text{SO}(n))$. We further assume that Q takes values on some fixed maximal torus of $\text{SO}(n)$, specifically, we consider $\text{SO}(n)$ -valued maps Q of the form ⁽⁵⁾

$$(3.4) \quad Q(\rho_1, \dots, \rho_N) = \begin{cases} \text{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k]) & \text{for } n = 2k, \\ \text{diag}(\mathcal{R}[f_1], \dots, \mathcal{R}[f_k], 1) & \text{for } n = 2k + 1, \end{cases}$$

where, for $1 \leq l \leq k$, $f_l \in \mathcal{C}(\overline{\mathbb{A}}_N, \mathbb{R})$ satisfies $f_l \equiv 0$ on $(\partial\mathbb{A}_N)_a$ and $f_l \equiv 2\pi m + \eta$ on $(\partial\mathbb{A}_N)_b$. This ensures that $u = \varphi$ on $\partial\mathbb{X}_n$. We start by calculating some of the quantities associated with spherical whirls to facilitate future derivations.

LEMMA 3.1 (Key identities). *Let $u = Q(\rho_1, \dots, \rho_N)\theta$ be a spherical whirl on \mathbb{X}^n with $Q \in \mathcal{C}(\overline{\mathbb{A}}_N, \text{SO}(n)) \cap \mathcal{C}^2(\mathbb{A}_N, \text{SO}(n))$. Then we have the following:*

$$(a) \quad \nabla u = \frac{1}{r} (Q - Q\theta \otimes \theta) + \sum_{l=1}^N Q_{,l}\theta \otimes \nabla \rho_l,$$

$$(b) \quad |\nabla u|^2 = \frac{n-1}{r^2} + \sum_{l=1}^N |Q_{,l}\theta|^2,$$

⁽⁵⁾ In weakening the radial symmetry of twists we demand that spherical whirls should instead commute with only a subgroup of $\text{SO}(n)$ — here, the described maximal torus of block diagonal 2×2 planar rotations. By maximality reasons this then implies that any spherical whirl itself should take values on the same maximal torus. See [7], [18], [19].

$$(c) \quad \Delta u = \sum_{l=1}^N \left[Q_{,il}\theta + \frac{2}{r} Q_{,i}\nabla\rho_l + \left(\Delta\rho_l - \frac{2\rho_l}{r^2} \right) Q_{,i}\theta \right] - \frac{n-1}{r^2} Q\theta.$$

Here $Q_{,i}$ and $Q_{,il}$ denote the first and second order derivatives of Q with respect to ρ_l respectively and $\nabla\rho_l$ is the gradient of ρ_l with respect to the spatial variables $x = (x_1, \dots, x_n)$.

PROOF. A straightforward differentiation using the described representation of u as in (3.3) and with $r = \sqrt{x_1^2 + \dots + x_n^2} = \sqrt{\rho_1^2 + \dots + \rho_N^2}$ gives

$$\nabla u = Q\nabla\theta + \sum_{l=1}^N Q_{,i}\theta \otimes \nabla\rho_l = \frac{1}{r} (Q - Q\theta \otimes \theta) + \sum_{l=1}^N Q_{,i}\theta \otimes \nabla\rho_l,$$

where in deducing the second identity we have made use of the formulation $\nabla\theta = r^{-1}(\mathbf{I}_n - \theta \otimes \theta)$. With the aid of this we can now calculate the Hilbert-Schmidt norm of the gradient ∇u as

$$\begin{aligned} (3.5) \quad |\nabla u|^2 &= \text{tr} [\nabla u \nabla u^t] \\ &= \text{tr} \left\{ \frac{1}{r^2} (\mathbf{I}_n - Q\theta \otimes Q\theta) + \frac{1}{r} \sum_{l=1}^N (Q - Q\theta \otimes \theta) (\nabla\rho_l \otimes Q_{,i}\theta) \right. \\ &\quad \left. + \sum_{l=1}^N Q_{,i}\theta \otimes Q_{,i}\theta \right\} \\ &= \frac{n-1}{r^2} + \frac{1}{r} \sum_{l=1}^N \{ \langle Q\nabla\rho_l, Q_{,i}\theta \rangle - \langle Q\theta, Q_{,i}\theta \rangle \langle \theta, \nabla\rho_l \rangle + r|Q_{,i}\theta|^2 \} \\ &= \frac{n-1}{r^2} + \frac{1}{r} \sum_{l=1}^N \{ \langle Q\nabla\rho_l, Q_{,i}\theta \rangle + r|Q_{,i}\theta|^2 \}, \end{aligned}$$

where in the second to last line we have used the fact that the product $Q_{,i}^t Q$ is skew-symmetric. We now focus on the term $\langle Q\nabla\rho_l, Q_{,i}\theta \rangle$. First, recalling that Q is of the form (3.4), a straight forward differentiation gives

$$Q^t Q_{,i} = \begin{cases} \text{diag}(\partial_l f_1 J, \dots, \partial_l f_k J) & \text{if } n = 2k, \\ \text{diag}(\partial_l f_1 J, \dots, \partial_l f_k J, 0) & \text{if } n = 2k + 1, \end{cases}$$

where J is

$$Q^t Q_{,i}\theta = Q^t Q_{,i} x |x|^{-1} = \begin{cases} |x|^{-1} (\partial_l f_1 J y_1, \dots, \partial_l f_k J y_k) & \text{if } n = 2k, \\ |x|^{-1} (\partial_l f_1 J y_1, \dots, \partial_l f_k J y_k, 0) & \text{if } n = 2k + 1. \end{cases}$$

Next differentiating ρ_j using (3.2) and (3.1) we have $\nabla\rho_j = \rho_j^{-1}(0, \dots, y_j, \dots, 0)$ for $1 \leq j \leq N$. Hence by substitution (note that in the writing of the second equality below we are excluding the case $j = N$ with n odd in which the required

identity is trivially true) we have

$$\langle Q \nabla \rho_j, Q_{,l} \theta \rangle = \langle \nabla \rho_j, Q^t Q_{,l} \theta \rangle = |x|^{-1} \frac{\partial_l f_j}{\rho_j} \langle y_j, J y_j \rangle = 0, \quad 1 \leq j, l \leq N,$$

in virtue of J is skew-symmetric. It is now evident, by (3.5), that

$$(3.6) \quad |\nabla u|^2 = \frac{n-1}{r^2} + \sum_{l=1}^N |Q_{,l} \theta|^2.$$

Finally we obtain Δu by taking the divergence of ∇u and making note of the identities $\nabla \rho_l \cdot \nabla \rho_j = \delta_{lj}$, $\nabla \rho_j \cdot \theta = \rho_j / r$, $\Delta \rho_j = 1 / \rho_j$ except for n odd and $j = N$ where $\Delta \rho_N = 0$ and $[\nabla \rho]^t \rho = x^t$. This therefore completes the proof. \square

Using the above description of $|\nabla u|^2$ we can proceed by writing the weighted Dirichlet energy \mathbb{E}_2^h of a spherical whirl $u = Q(\rho)\theta$ as

$$\begin{aligned} \mathbb{E}_2^h[u; \mathbb{X}^n] &= \int_{\mathbb{X}^n} h(r) |\nabla [Q(\rho)\theta]|^2 \\ &= \int_{\mathbb{X}^n} h(r) \left[\frac{n-1}{r^2} + \sum_{l=1}^N |Q_{,l} \theta|^2 \right] dx \\ &= \int_{\mathbb{X}^n} h(r) \left[\frac{n-1}{r^2} + \frac{1}{r^2} \sum_{l=1}^k |\nabla f_l|^2 \rho_l^2 \right] dx, \end{aligned}$$

or after a basic changing of variables as

$$(3.7) \quad \mathbb{E}_2^h[u; \mathbb{X}^n] = \int_{\mathbb{X}^n} h(r) \frac{n-1}{r^2} dx + (2\pi)^k \sum_{l=1}^k \int_{\mathbb{A}_N} \frac{h(r)}{r^2} |\nabla f_l|^2 \rho_l^2 \prod_{j=1}^k \rho_j d\rho.$$

Prompted by the variational role of the integrals in the second term on the right in (3.7) we define, for $1 \leq l \leq k$, the energy functionals

$$\mathbb{H}_l[f; \mathbb{A}_N] = (2\pi)^k \int_{\mathbb{A}_N} \frac{h(r)}{r^2} |\nabla f|^2 \rho_l^2 \prod_{j=1}^k \rho_j d\rho.$$

Here the admissible functions f are assumed to lie in the space

$$\mathcal{B}(\mathbb{A}_N) := \bigcup_{m \in \mathbb{Z}} \mathcal{B}_m(\mathbb{A}_N)$$

where for each $m \in \mathbb{Z}$ we have

$$\mathcal{B}_m(\mathbb{A}_N) = \{f \in W^{1,2}(\mathbb{A}_N) : f|_{(\partial \mathbb{A}_N)_a} \equiv 0, f|_{(\partial \mathbb{A}_N)_b} \equiv 2m\pi + \eta\}.$$

Referring to (3.7) it is easily seen that we can rewrite this as

$$(3.8) \quad \mathbb{E}_2^h[u; \mathbb{X}^n] = \int_a^b n(n-1)\omega_n h(r) r^{n-3} dr + \sum_{l=1}^k \mathbb{H}_l[f; \mathbb{A}_N].$$

Now, since for a spherical whirl the energy \mathbb{E}_2^h is a finite sum of the energies \mathbb{H}_l ($1 \leq l \leq k$) where each \mathbb{H}_l depends only on $f = f_l$, we proceed by extremising

each \mathbb{H}_l separately. Indeed the Euler–Lagrange equations associated with \mathbb{H}_l over $\mathcal{B}_m(\mathbb{A}_N)$ is seen to be

$$(3.9) \quad \begin{cases} \operatorname{div} \left(h(r) \frac{\rho_l^2}{r^2} \prod_{j=1}^k \rho_j \nabla f \right) = 0 & \text{in } \mathbb{A}_N, \\ f \equiv 0 & \text{on } (\partial \mathbb{A}_N)_a, \\ f \equiv 2m\pi + \eta & \text{on } (\partial \mathbb{A}_N)_b, \\ \rho_l^2 \prod_{j=1}^k \rho_j \partial_\nu f = 0, & \text{on } \Gamma_N. \end{cases}$$

Note that ∂_ν is the partial derivative in the outward pointing normal direction. The following proposition leads to the conclusion formulated in Theorem 1.2.

PROPOSITION 3.2. *For each fixed $m \in \mathbb{Z}$ the Euler–Lagrange equation (3.9) admits a unique solution $f = f(\rho; m)$ in $\mathcal{B}(\mathbb{A}_N)$ given explicitly by*

$$f(\rho; m) = \frac{2\pi m + \eta}{\beta(b)} \beta \left(\sqrt{\sum_{l=1}^N \rho_l^2} \right), \quad \rho = (\rho_1, \dots, \rho_N) \in \overline{\mathbb{A}_N},$$

where $\beta = \beta(t)$ is the function defined on $[a, b]$ via the weight h through the integral

$$\beta(t) := \int_a^t \frac{s^{1-n}}{h(s)} ds.$$

PROOF. We first verify that f satisfies (3.9). Indeed f is easily seen to satisfy the boundary conditions. To ease notation we write $c = (2m\pi + \eta)/\beta(b)$ for the rest of this proof. Now basic differentiation yields

$$\frac{\partial f}{\partial \rho_i} = \dot{\beta} \left(\sqrt{\sum_{l=1}^N \rho_l^2} \right) \frac{c \rho_i}{\sqrt{\sum_{l=1}^N \rho_l^2}} = \frac{c \rho_i}{h(r) r^n}.$$

We approach the cases of odd and even n separately. First, if $n = 2k$, we have

$$(3.10) \quad \begin{aligned} \operatorname{div} \left(h(r) \frac{\rho_l^2}{r^2} \nabla f \prod_{j=1}^k \rho_j \right) &= \sum_{i=1}^k \frac{\partial}{\partial \rho_i} c \left(h(r) \rho_l^2 \frac{\rho_i}{h(r) r^{n+2}} \prod_{j=1}^k \rho_j \right) \\ &= \sum_{i=1}^k c \left\{ 2\rho_l \delta_i^l \frac{\rho_i}{r^{2+n}} \prod_{j=1}^k \rho_j + \rho_l^2 \frac{\rho_i}{r^{n+2}} \prod_{j=1, j \neq i}^k \rho_j \right. \\ &\quad \left. - \rho_l^2 \rho_i^2 (2+n) r^{-n-4} \prod_{j=1}^k \rho_j + \rho_l^2 \frac{1}{r^{n+2}} \prod_{j=1}^k \rho_j \right\} \\ &= \frac{c}{r^{n+2}} \rho_l \prod_{j=1}^k \rho_j (2\rho_l + k\rho_l - (2+2k)\rho_l + k\rho_l) = 0. \end{aligned}$$

On the other hand if $n = 2k + 1$ the calculations are the similar but we must observe the subtle difference of $\rho_{k+1} = x_{2k+1}$ and so we have

$$\begin{aligned}
 (3.11) \quad \operatorname{div} \left(h(r) \frac{\rho_l^2}{r^2} \nabla f \prod_{j=1}^k \rho_j \right) &= c \sum_{i=1}^k \frac{\partial}{\partial \rho_i} \left(h(r) \rho_l^2 \frac{\rho_i}{h(r) r^{n+2}} \prod_{j=1}^k \rho_j \right) \\
 &\quad + c \frac{\partial}{\partial \rho_{k+1}} \left(h(r) \rho_l^2 \frac{\rho_{k+1}}{h(r) r^{n+2}} \prod_{j=1}^k \rho_j \right) \\
 &= \frac{c}{r^{n+2}} \rho_l \prod_{j=1}^k \rho_j \left(2\rho_l + k\rho_l - \frac{2+n}{r^2} \rho_l \sum_{i=1}^k \rho_i^2 + k\rho_l \right) \\
 &\quad + c \left(\prod_{j=1}^k \rho_j \right) \rho_l \left(-\frac{2+n}{r^{4+n}} \rho_{k+1}^2 \rho_l + \frac{\rho_l}{r^{2+n}} \right) \\
 &= \frac{c\rho_l}{r^{n+2}} \prod_{j=1}^k \rho_j (2\rho_l + k\rho_l - (2k+3)\rho_l + (k+1)\rho_l) = 0.
 \end{aligned}$$

Finally to justify uniqueness suppose f^1, f^2 are two solutions to (3.9) and put $g = f^1 - f^2$. Then g solves (3.9) with zero boundary conditions. Using the divergence theorem and the asserted boundary conditions we then have

$$(3.12) \quad \int_{\mathbb{A}_N} \frac{1}{r^2} |\nabla g|^2 \rho_l^2 \prod_{j=1}^k \rho_j \, d\rho = \int_{\partial \mathbb{A}_N} \frac{1}{r^2} g \frac{\partial g}{\partial \nu} \rho_l^2 \prod_{j=1}^k \rho_j \, d\mathcal{H}^{n-1} = 0.$$

Since $\rho_1, \dots, \rho_k > 0$ in \mathbb{A}_N it then follows that $|\nabla g|^2 = 0$ in \mathbb{A}_N and so we must have $g \equiv 0$ in \mathbb{A}_N due to the boundary conditions. We can therefore conclude that $f^1 = f^2$ and so uniqueness follows. □

We note from the explicit description of the solution $f = f(\rho; m)$ in the above proposition that f is indeed a function of the radial variable r . In conclusion the associated spherical whirl has the form $u = Q(r)\theta$ where $Q \in \mathcal{C}^2([a, b], \text{SO}(n))$. Maps of this form are the spherical twists that were introduced and thoroughly discussed in Section 2. The proof of Theorem 1.2 now follows from the results in Section 2 for spherical twists. It is remarkable to note that, despite structural differences, the spherical whirl solutions to the system (1.3) in this case coincide exactly with the spherical twist solutions as given in Theorem 1.1.

REFERENCES

[1] F. ALOUGES AND J. GHIDAGLIA, *Minimizing Oseen–Frank energy for nematic liquid crystals: algorithms and numerical results*, *Annales de l’IHP Physique Théorique* **66** (1977), 411–447.
 [2] F. BETHUEL, *The approximation problem for Sobolev maps between two manifolds*, *Acta Math.* **167** (1991), 153–206.
 [3] J.C. BOURGOIN, *The minimality of the map $x/|x|$ for weighted energy*, *Calc. Var. Partial Differential Equations* **25** (2006), 469–489.

- [4] H. BREZIS AND Y. LI, *Topology and Sobolev spaces*, J. Funct. Anal. **183** (2001), 321–369.
- [5] L. CESARI, *Optimization Theory and Applications: Problems with Ordinary Differential Equations*, Applications of Mathematics, Vol. 17, Springer, 1983.
- [6] S. DAY AND A. TAHERI, *Stability of spherical harmonic twists, second variations and conjugate points on $SO(n)$* , (2018). (submitted)
- [7] S. DAY AND A. TAHERI, *A class of extremising sphere-valued maps with inherent maximal tori symmetries in $SO(n)$* , Bound. Value Probl. (2017).
- [8] J. EELLS AND L. LEMAIRE, *Two reports on harmonic maps*, Bull. Lond. Math. Soc. **10** (1978), 1–68; **20** (1988), 385–524.
- [9] J.L. ERICKSEN, *Inequalities in liquid crystal theory*, Physics of Fluids **9** (1966), 1205–1207.
- [10] M. GIAQUINTA, G. MODICA AND J. SOUCEK, *Cartesian Currents in the Calculus of Variations*, Vol. I, II, Springer, Berlin, 1998.
- [11] R. GULLIVER AND J.M. CORON, *Minimizing p -harmonic maps into spheres*, J. Reine Angew. Math. **401** (1989), 82–100.
- [12] F.B. HANG AND F.H. LIN, *Topology of Sobolev mappings*, Math. Res. Lett. **8** (2001), 321–330.
- [13] R. HARDT AND F.H. LIN, *Mappings minimizing the L^p norm of the gradient*, Comm. Pure Appl. Math. **40** (1987), 555–588.
- [14] F. HÉLEIN AND J.C. WOOD, *Harmonic maps*, Handbook of Global Analysis, Elsevier, Amsterdam, 2008, 417–491.
- [15] M.C. HONG, *On the minimality of the p -harmonic map $x/|x|: \mathbb{B}^n \rightarrow \mathbb{S}^{n-1}$* , Calc. Var. Partial Differential Equations **13** (2001), 459–468.
- [16] F.H. LIN, *A remark on the map $x/|x|$* , C.R. Acad. Sci. **305** (1987), 529–531.
- [17] S. LUCKHAUS, *Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold*, Indiana Univ. Math. J. **37** (1988), 349–367.
- [18] C. MORRIS AND A. TAHERI, *Annular rearrangements, incompressible axi-symmetric whirls and L^1 -local minimisers of the distortion energy*, NoDEA **25** (2018), Art. No. 2.
- [19] C. MORRIS AND A. TAHERI, *Whirl mappings on generalised annuli and incompressible symmetric equilibria of the Dirichlet energy*, J. Elasticity (2018). (to appear)
- [20] M.S. SHAHROKHI-DEHKORDI AND A. TAHERI, *Generalised twists, $SO(n)$ and the p -energy over a space of measure preserving maps*, Ann. Inst. H. Poincaré Anal. Non Linéaire (C) **26** (2009), 1897–1924.
- [21] M.S. SHAHROKHI-DEHKORDI AND A. TAHERI, *Generalised twists, stationary loops, and the Dirichlet energy over a space of measure preserving maps*, Calc. Var. Partial Differential Equations **35** (2009), 191–213.
- [22] L. SIMON, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel, 1996.
- [23] A. TAHERI, *Local minimizers and quasiconvexity – the impact of topology*, Arch. Rational Mech. Anal. **176** (2005), 363–414.
- [24] A. TAHERI, *Homotopy classes of self-maps of annuli, generalised twists and spin degree*, Arch. Rational Mech. Anal. **197** (2010), 239–270.
- [25] A. TAHERI, *Spherical twists, $SO(n)$ and the lifting of their twist paths to $Spin(n)$ in low dimensions*, Q. J. Math. **63** (2012), 723–751.
- [26] A. TAHERI, *Spherical twists, stationary paths and harmonic maps from generalised annuli into spheres*, NoDEA **19** (2012), 79–95.
- [27] A. TAHERI, *Function Spaces and Partial Differential Equations*, Vol. I, II, Oxford Lecture Series in Mathematics and its Applications, OUP, Oxford, 2015.

- [28] E.G. VIRGA, *Variational Theories for Liquid Crystals*, Applied Mathematics and Mathematical Computation, Vol. 8, Chapman & Hall, London, 1994.

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