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POSITIVE SOLUTIONS FOR SINGULAR IMPULSIVE DIRICHLET BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, a class of singular impulsive Dirichlet boundary value problems is considered. By using variational method and critical point theory, different parameter ranges are obtained to guarantee existence and multiplicity of positive classical solutions of the problem when nonlinearity exhibits different growths.

1. Introduction

The main purpose of this paper is to study positive classical solutions of the following singular impulsive Dirichlet boundary value problem

$$\begin{cases} -u''(t) - \dfrac{1}{u^{\alpha}(t)} = \lambda f(t,u(t)), & t \in \Omega, \quad \text{(1.1a)} \\ \Delta(u'(t_i)) := u'(t_i^+) - u'(t_i^-) = I_i(u(t_i)), & i = 1,\ldots,p, \quad \text{(1.1b)} \\ u(0) = u(1) = 0, & \quad \text{(1.1c)} \end{cases}$$

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where $\lambda \in (0,+\infty)$ is a parameter, $\alpha \in (0,1)$, $\Omega := (0,1) \setminus \{t_1,\dots,t_p\}$, $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = 1$, $u'(t_i^+)$ and $u'(t_i^-)$ denote the right and left limits of $u'(t)$ at $t = t_j$ respectively, the nonlinear term $f \in C([0,1] \times [0,+\infty); [0,+\infty))$ and the impulsive function $I_i \in C([0,+\infty); (-\infty,0])$ for each $i = 1,\dots,p$.

A function $u \in C[0,1]$ satisfying the boundary condition (1.1c) is said to be a classical solution of (1.1) if, for each $i = 0,\dots,p$, $u|_{(t_i,t_{i+1})} \in C^2(t_i,t_{i+1})$ satisfies the equation (1.1a) on (t_i,t_{i+1}), the limits $u'(t_i^-)$ and $u'(t_i^+)$ exist for each $i = 1,\dots,p$ and satisfy the impulsive condition (1.1b). By a positive solution u of (1.1) we mean a classical solution such that $u(t) > 0$ for $t \in (0,1)$.

The question of existence of solutions for singular problems has attracted much attention of many mathematicians and physicists over many years. Topological methods and variational approach have been widely applied to study such problems (see e.g. [2]–[5], [10]).

Impulsive effects arise from the real world and are used to describe sudden, discontinuous jumps. For some general and recent works on the theory of impulsive differential equations we refer the readers to [6], [11], [13], [14], [16]–[18], [22]–[24], [26].

For the study of impulsive singular problems, some classical tools have been used in the literature, such as the method of upper and lower solutions and the monotone iterative technique, fixed point theory and Leray–Schauder alternative principle (see e.g. [1], [9], [12]). Using variational method to study such problems is more recent, the number of references is small [19]–[21] and all these references are focused on periodic weak solutions. In [21] Sun and O'Regan established that the problem

$$u''(t) - \frac{b(t)}{u^\alpha(t)} = e(t); \qquad \Delta(u'(t_i)) = I_i(u(t_i)) \tag{1.2}$$

has at least one periodic weak solution by using the mountain pass theorem. After that, when $b(t) \equiv 1$, Sun and his coworkers studied the existence of one positive periodic weak solution generated by impulses for the problem (1.2) in [20] and obtained a necessary and sufficient condition for the existence of one positive periodic weak solution of the problem (1.2) in [19]. However, the study of solutions for singular impulsive Dirichlet boundary value problems via variational method has received considerably less attention.

Motivated mainly by [4], [16], in this paper we devote ourselves to studying existence and multiplicity of positive classical solutions of (1.1) via critical point theory. It is worth stressing that different parameter ranges are obtained to guarantee the solvability of (1.1) when the nonlinear term f exhibits different growths.

Choosing $\varepsilon \in (0, 2^{-1/(\alpha+1)})$, for $\lambda > 0$ define $f_{\varepsilon,\lambda}\colon (0,1) \times \mathbb{R} \to (0,+\infty)$ by

$$f_{\varepsilon,\lambda}(t,x) := \lambda f(t,(x-\varphi_\varepsilon(t))^+ + \varphi_\varepsilon(t)) + \big[(x-\varphi_\varepsilon(t))^+ + \varphi_\varepsilon(t)\big]^{-\alpha}$$

and for each $i = 1, \dots, p$, define $I_{i,\varepsilon}\colon \mathbb{R} \to (-\infty, 0]$ by

$$I_{i,\varepsilon}(x) := I_i\big((x - \varphi_\varepsilon(t_i))^+ + \varphi_\varepsilon(t_i)\big),$$

where $\varphi_\varepsilon(t) := \varepsilon t(1-t)$ and $u^\pm := \max\{\pm u, 0\}$. It could be verified that $\varphi_\varepsilon^{-\alpha} \in L^1(0,1)$. The continuity of f and I_i implies that $f_{\varepsilon,\lambda} \in C((0,1) \times \mathbb{R}; (0,+\infty))$ and $I_{i,\varepsilon} \in C(\mathbb{R}; (-\infty, 0])$.

In view of $\varepsilon \in (0, 2^{-1/(\alpha+1)})$, when $x \in (\varphi_\varepsilon(t), \varepsilon)$, we find

$$2\varepsilon \le \varepsilon^{-\alpha} \le x^{-\alpha} \le f_{\varepsilon,\lambda}(t,x) = \lambda f(t,x) + x^{-\alpha} \le \lambda C_\varepsilon + \varphi_\varepsilon^{-\alpha}(t). \tag{1.3}$$

where $C_\varepsilon := \max\limits_{[0,1]\times[0,\varepsilon]} f(t,x)$; when $x \in (-\infty, \varphi_\varepsilon(t)]$, we have

$$2\varepsilon \le \varepsilon^{-\alpha} \le \varphi_\varepsilon^{-\alpha}(t) \le f_{\varepsilon,\lambda}(t,x) = \lambda f(t, \varphi_\varepsilon(t)) + \varphi_\varepsilon^{-\alpha}(t) \le \lambda C_\varepsilon + \varphi_\varepsilon^{-\alpha}(t),$$

which combined with (1.3) yields to

$$2\varepsilon \le f_{\varepsilon,\lambda}(t,x) \le \varphi_\varepsilon^{-\alpha}(t) + \lambda C_\varepsilon, \quad \text{for } (t,x) \in (0,1) \times (-\infty, \varepsilon) \text{ and } \lambda > 0. \tag{1.4}$$

For the convenience, we introduce some assumptions:

(H1) There exist $0 < a < \pi^2$ and $C > 0$ such that

$$f(t,x) \le ax + C, \quad (t,x) \in (0,1) \times [\varepsilon, +\infty);$$

(H2) $$\lim_{x\to+\infty} \frac{f(t,x)}{x} = \pi^2 \quad \text{uniformly for } t \in (0,1);$$

(H3) There exist $b > \pi^2$ and $C > 0$ such that

$$C(x+1) \ge f(t,x) \ge bx - C, \quad (t,x) \in (0,1) \times [\varepsilon, +\infty);$$

(H4) There exist $\sigma > 2$, $\tau > 0$ and $C > 0$ such that

$$\liminf_{x\to+\infty} \frac{1}{x^{1-\alpha}} \bigg(f(t,x)x - \sigma \int_\varepsilon^x f(t,y)\,dy \bigg) \ge \tau \tag{1.5}$$

uniformly for $t \in (0,1)$, and for each $i = 1, \dots, p$,

$$\limsup_{x\to+\infty} \bigg(I_i(x)x - \sigma \int_\varepsilon^x I_i(y)\,dy \bigg) \le C; \tag{1.6}$$

(H5) There exist $0 \le \beta < 2$ and $C > 0$ such that

$$\liminf_{x\to+\infty} \frac{1}{x^\beta} \bigg(f(t,x)x - 2 \int_\varepsilon^x f(t,y)\,dy \bigg) > -C$$

uniformly for $t \in (0,1)$;

(H6) There exists $C > 0$ such that

$$\lim_{x\to+\infty} \frac{1}{x^{1-\alpha}} \bigg(f(t,x)x - 2 \int_\varepsilon^x f(t,y)\,dy \bigg) = +\infty \tag{1.7}$$

uniformly for $t \in (0,1)$, and for each $i = 1, \dots, p$,

$$\limsup_{x\to+\infty} \bigg(I_i(x)x - 2 \int_\varepsilon^x I_i(y)\,dy \bigg) \le C; \tag{1.8}$$

(H7) There exists $C > 0$ such that

$$\lim_{x\to+\infty} \frac{1}{x^{1-\alpha}} \bigg(f(t,x)x - 2\int_\varepsilon^x f(t,y)\,dy \bigg) = -\infty \tag{1.9}$$

uniformly for $t \in (0,1)$, and for each $i = 1, \dots, p$,

$$\liminf_{x\to+\infty} \bigg(I_i(x)x - 2\int_\varepsilon^x I_i(y)\,dy \bigg) \geq -C; \tag{1.10}$$

(H8) For each $i = 1, \dots, p$, there exist $a_i \geq 0$, $b_i \geq 0$ and $\gamma_i \in [0,1]$ (among which $\gamma_i = 1$ for $i \in \mathcal{A} \subseteq \{1, \dots, p\}$ and $\gamma_i \in [0,1)$ for $i \in \mathcal{B} := \{1, \dots, p\} \setminus \mathcal{A}$) such that $I_i(x) \geq -a_i x^{\gamma_i} - b_i$, for $x \in [\varepsilon, +\infty)$.

Let

$$h(r) := \frac{r^2 - r^{1-\alpha} + \sum_{i=1}^{p} \min_{[0,r]} I_i(x)x}{\max_{[0,1]\times[0,r]} f(t,x)x} \quad \text{and} \quad \sum_{i\in\mathcal{A}} a_i = 0 \quad \text{if } \mathcal{A} = \emptyset.$$

Main results of this paper are presented as follows.

Theorem 1.1. *The problem* (1.1) *is solvable in the following cases*:

(a) *If* (H1) *holds, then the problem* (1.1) *has a positive classical solution provided the assumption* (H8) *holds and*

$$0 < \lambda < \frac{\pi^2}{a}\bigg(1 - \sum_{i\in\mathcal{A}} a_i\bigg);$$

(b) *If* (H2) *holds, then*

(b1) *the problem* (1.1) *has a positive classical solution provided the assumptions* (H5) *and* (H8) *hold, and*

$$0 < \lambda < 1 - \sum_{i\in\mathcal{A}} a_i;$$

(b2) *the problem* (1.1) *has a positive classical solution provided the assumptions* (H6) *and* (H8) *hold with* $b_i \equiv 0$ *and* $\mathcal{A} = \{1, \dots, p\}$, *and*

$$0 < \lambda = 1 - \sum_{i\in\mathcal{A}} a_i;$$

(b3) *the problem* (1.1) *has two positive classical solutions provided the assumptions* (H7) *and* (H8) *hold, and* $1 \leq \lambda < \sup_{r>0} h(r)$;

(c) *If* (H3) *holds, then the problem* (1.1) *has two positive classical solutions provided the assumption* (H8) *holds and* $\pi^2/b < \lambda < \sup_{r>0} h(r)$;

(d) *If* (H4) *holds, then the problem* (1.1) *has two positive classical solutions provided* $(\sigma - 1 + \alpha)/(\tau(1-\alpha)) < \lambda < \sup_{r>0} h(r)$.

EXAMPLE 1.2. Consider the following singular impulsive problem

$$\begin{cases} -u''(t) - \dfrac{1}{u^{1/3}(t)} = \lambda f(t,u(t)), & t \in (0,t_1)\cup(t_1,1), \\ \Delta(u'(t_1)) = I_1(u(t_1)), \\ u(0) = u(1) = 0. \end{cases} \tag{1.11}$$

In view of Theorem 1.1, we have the following results:

(a) When $f(t,x) = x$ and $I_1(x) = -x/2$, the problem (1.11) has a positive classical solution provided $0 < \lambda < \pi^2/2$;

(b1) When $f(t,x) = \pi^2 x$ and $I_1(x) = -1/2x$, the problem (1.11) has a positive classical solution provided $0 < \lambda < 1/2$;

(b2) When $f(t,x) = \pi^2 x - x^{1/2} + 1$ and $I_1(x) = -x/2$, the problem (1.11) has a positive classical solution provided $\lambda = 1/2$;

(b3) When $f(t,x) = \pi^2 x + x^{2/3} - 60x^{1/3} + 55$ and $I_1(x) = -0.1x$, the problem (1.11) has two positive classical solutions provided $1 \le \lambda < \sup_{r>0} h(r)$. In fact,

$$\max_{[0,1]\times[0,4.2]} f(t,x)x = [f(t,x)x]|_{x=4.2} \approx 9.4490 \quad \text{and} \quad h(4.2) \approx 1.4047;$$

(c) When $f(t,x) = 10x - 400x^{1/10} + 420$ and $I_1(x) = -0.1x$, the problem (1.11) has two positive classical solutions provided $\pi^2/10 < \lambda < \sup_{r>0} h(r)$. In fact, (H3) holds with $b = 10 - \vartheta$ for any $\vartheta \in (0, 10-\pi^2)$,

$$\max_{[0,1]\times[0,7.2]} f(t,x)x = [f(t,x)x]|_{x=7.2} \approx 33.8656 \quad \text{and} \quad h(7.2) \approx 1.2676;$$

(d) When $f(t,x) = e^x$ and $I_1(x) = -x^{3/2} + 2x - 1.2$, the problem (1.11) has two positive classical solutions provided $0 < \lambda < \sup_{r>0} h(r)$. In fact, (H4) holds for any $\tau > 0$ and $\sigma = 2.4$. What is more,

$$\min_{[0,2.6]} I_1(x)x = [I_1(x)x]|_{x=2.6} \approx -0.5002 \quad \text{and} \quad h(2.6) \approx 0.1248.$$

The remaining part of this paper is organized as follows. In the next section, some fundamental facts are given. Proof of the main results are presented in Section 3.

Throughout this paper, by C we denote a positive constant whose value may vary from line to line.

2. Preliminaries

We recall some facts which will be used in the proof of our main result. Let $H_0^1(0,1)$ be the Sobolev space endowed with the norm

$$\|u\|_{H_0^1} := \left(\int_0^1 |u'(t)|^2 + |u(t)|^2 \, dt \right)^{1/2},$$

and $H_0^1(0,1)$ is a reflexive Banach space. It is a consequence of Poincaré's inequality that

$$\int_0^1 |u(t)|^2\,dt \leq \frac{1}{\lambda_1}\int_0^1 |u'(t)|^2\,dt, \tag{2.1}$$

where $\lambda_1 = \pi^2$ is the first eigenvalue of the Dirichlet problem

$$-u''(t) = \lambda u(t),\ t \in (0,1); \qquad u(0) = u(1) = 0,$$

and the associated eigenfunction of λ_1 is $\sin(\pi t)$. So

$$\int_0^1 \sin'(\pi t)u'(t)\,dt = \pi^2 \int_0^1 \sin(\pi t)u(t)\,dt, \quad \text{for any } u \in H_0^1(0,1). \tag{2.2}$$

In view of (2.1), we know that

$$\|u\| := \bigg(\int_0^1 |u'(t)|^2\,dt\bigg)^{1/2}$$

is equivalent to the norm $\|u\|_{H_0^1}$ in $H_0^1(0,T)$. Let $\|u\|_\infty := \max\limits_{t\in[0,1]} |u(t)|$. Then, for $u \in H_0^1(0,T)$, we have

$$\|u\|_\infty \leq \|u\|. \tag{2.3}$$

In fact, for any $t \in [0,1]$, using Höder's inequality,

$$|u(t)| = \bigg|u(0) + \int_0^t u'(s)\,ds\bigg| \leq \int_0^1 |u'(t)|\,dt \leq \bigg(\int_0^1 |u'(t)|^2\,dt\bigg)^{1/2}.$$

Lemma 2.1 ([15, Theorem 1.1]). *If φ is sequentially weakly lower semi-continuous on a reflexive Banach space X and has a bounded minimizing sequence, then φ has a minimum on X.*

Definition 2.2. Let $\varphi\colon X \to \mathbb{R}$ differentiable and $c \in \mathbb{R}$. We say that φ satisfies the $(\mathrm{PS})_c$-condition if the existence of a sequence $\{u_k\}$ in X such that

$$\varphi(u_k) \to c, \qquad \varphi'(u_k) \to 0$$

as $k \to \infty$, implies that c is a critical value of φ.

Lemma 2.3 ([15, Theorem 4.4]). *Let X be a Banach space, $\varphi\colon X \to \mathbb{R}$ a function bounded from below and differentiable on X. Assume that φ satisfies the $(\mathrm{PS})_c$-condition with $c = \inf\limits_X \varphi$, then φ has a minimum on X.*

Lemma 2.4 ([25, Theorem 38.A]). *For the functional $F\colon M \subseteq X \to [-\infty, +\infty]$ with $M \neq \emptyset$, $\min\limits_{u\in M} F(u) = \alpha$ has a solution in case the following hold*:

(a) *X is a real reflexive Banach space*;
(b) *M is bounded and weak sequentially closed*;
(c) *F is sequentially weakly lower semi-continuous on M.*

Lemma 2.5 ([15, Theorem 4.10]). *Let E be a Banach space and $\varphi \in C^1(E,\mathbb{R})$. Assume that there exist $u_0 \in E$, $u_1 \in E$ and a bounded open neighbourhood Ω of u_0 such that $u_1 \in E \setminus \Omega$ and $\inf_{\partial\Omega} \varphi > \max\{\varphi(u_0), \varphi(u_1)\}$. Let*

$$\Gamma = \{g \in C([0,1],E) : g(0) = u_0,\ g(1) = u_1\} \quad \textit{and} \quad c = \inf_{g\in\Gamma} \max_{s\in[0,1]} \varphi(g(s)).$$

If φ satisfies the $(\mathrm{PS})_c$-condition, then c is a critical value of φ and

$$c > \max\{\varphi(u_0), \varphi(u_1)\}.$$

Definition 2.6. φ satisfies the Cerami condition, denoted by (C), if any sequence $\{x_n\}$ in X such that $\{\varphi(x_n)\}$ is bounded and $\|\varphi'(x_n)\|(1+\|x_n\|) \to 0$ has a convergent subsequence.

The Cerami condition [8] is weaker than the Palais–Smale condition and it was used by Bartolo, Benci and Fortunato to prove a deformation lemma (Theorem 1.3 in [7]) which allows rather general minimax results.

Consider

$$\begin{cases} -u''(t) = f_{\varepsilon,\lambda}(t,u(t)), & t \in \Omega, \quad (2.4a)\\ \Delta(u'(t_i)) = I_{i,\varepsilon}(u(t_i)), & i = 1,\ldots,p, \quad (2.4b)\\ u(0) = u(1) = 0, & \quad (2.4c) \end{cases}$$

following the ideas of the variational approach to impulsive differential equations of [16], [22], multiply (2.4a) by $v \in H_0^1(0,T)$ and integrate between 0 and 1, we find that

$$\int_0^1 u''(t)v(t)\,dt = -\int_0^1 f_{\varepsilon,\lambda}(t,u(t))v(t)\,dt. \tag{2.5}$$

In view of (2.4c), we have

$$\begin{aligned} \int_0^1 u''(t)v(t)\,dt &= \int_0^{t_1} u''(t)v(t)\,dt \\ &\quad + \sum_{i=1}^{p-1} \int_{t_i}^{t_{i+1}} u''(t)v(t)\,dt + \int_{t_p}^1 u''(t)v(t)\,dt \\ &= -\int_0^1 u'(t)v'(t)\,dt + \sum_{i=1}^{p} \big[u'(t_i^-) - u'(t_i^+)\big]v(t_i), \end{aligned} \tag{2.6}$$

which combined with (2.4b) and (2.5) yields to

$$\int_0^1 u'(t)v'(t)\,dt + \sum_{i=1}^{p} I_{i,\varepsilon}(u(t_i))v(t_i) = \int_0^1 f_{\varepsilon,\lambda}(t,u(t))v(t)\,dt. \tag{2.7}$$

Considering the aforementioned equality, we introduce the following concept of a weak solution for (2.4).

Definition 2.7. A function $u \in H_0^1(0,1)$ is a weak solution of (2.4) if (2.7) holds for any $v \in H_0^1(0,1)$.

Definition 2.8. A function $u \in C[0,1]$ satisfying the boundary condition (2.4c) is said to be a classical solution of (2.4) if, for each $i = 0, \dots, p$, $u|_{(t_i,t_{i+1})} \in C^2(t_i, t_{i+1})$ satisfies the equation (2.4a) on (t_i, t_{i+1}), the limits $u'(t_i^-)$ and $u'(t_i^+)$ exist for each $i = 1, \dots, p$ and satisfy the impulsive condition (2.4b).

Lemma 2.9. *If $u \in H_0^1(0,1)$ is a weak solution of* (2.4)*, then u is a classical solution of* (2.4)*.*

Proof. It follows from $u \in H_0^1(0,1)$ that $u \in C[0,1]$ satisfies (2.4c). For each $i = 0, \dots, p$, let

$$K_i := \{v \in H_0^1(0,1) \mid v(t) = 0 \text{ for every } t \in [0,t_i] \cup [t_{i+1}, 1]\},$$

by (2.7), we find that

$$\int_{t_i}^{t_{i+1}} u'(t)v'(t)\,dt = \int_{t_i}^{t_{i+1}} f_{\varepsilon,\lambda}(t, u(t))v(t)\,dt, \quad \text{for any } v \in K_i. \tag{2.8}$$

Since $f_{\varepsilon,\lambda} \in C((0,1) \times \mathbb{R})$, we find that $(u')'|_{(t_i,t_{i+1})} \in C(t_i, t_{i+1})$ and thus $u|_{(t_i,t_{i+1})} \in C^2(t_i, t_{i+1})$. Integrating (2.8) by parts we obtain

$$\int_{t_i}^{t_{i+1}} \big(u''(t) + f_{\varepsilon,\lambda}(t, u(t))\big)v(t)\,dt = 0, \quad \text{for any } v \in K_i.$$

and hence

$$-u''(t) = f_{\varepsilon,\lambda}(t, u(t)), \quad \text{for a.e. } t \in (t_i, t_{i+1}),$$

which combined with $f_{\varepsilon,\lambda} \in C((0,1) \times \mathbb{R})$ and $u|_{(t_i,t_{i+1})} \in C^2(t_i, t_{i+1})$ yields to

$$-u''(t) = f_{\varepsilon,\lambda}(t, u(t)), \quad \text{for any } t \in (t_i, t_{i+1}). \tag{2.9}$$

For any $x_1, x_2 \in (t_i, t_{i+1})$, we find that

$$|u'(x_2) - u'(x_1)| = |u''(\xi)||x_2 - x_1| \le C|x_2 - x_1|,$$

for some $\xi \in (x_1, x_2)$. Thus for any $\varepsilon > 0$, there exists $\delta = \varepsilon/C$ such that

$$|u'(x_2) - u'(x_1)| < \varepsilon, \quad \text{for any } x_1, x_2 \in (t_i, t_i + \delta).$$

Thus $u'(t_i^+)$ exists. Similarly $u'(t_{i+1}^-)$ exists. For any $v \in H_0^1(0,T)$, multiply (2.9) by v and integrate between 0 and 1, by (2.6), we obtain

$$\int_0^1 f_{\varepsilon,\lambda}(t, u(t))v(t)\,dt = -\int_0^1 u''(t)v(t)dt = \int_0^1 u'(t)v'(t)dt + \sum_{i=1}^{p} \Delta(u'(t_i))v(t_i),$$

which combined with (2.7) yields to

$$\sum_{i=1}^{p} \Delta(u'(t_i))v(t_i) = \sum_{i=1}^{p} I_{i,\varepsilon}(u(t_i))v(t_i).$$

Since v is arbitrary, we deduce that $\Delta(u'(t_i)) = I_{i,\varepsilon}(u(t_i))$ for each $i = 1, \dots, p$. □

Lemma 2.10. *If $u \in H_0^1(0,1)$ is a classical solution of* (2.4)*, then $u(t) \ge \varphi_\varepsilon(t)$ for any $t \in [0,1]$, and hence u is a positive classical solution of* (1.1)*.*

Proof. Suppose that there exists a $t_* \in (0,1)$ such that

$$u(t_*) < \varepsilon\, t_*(1-t_*). \tag{2.10}$$

By $f_{\varepsilon,\lambda} > 0$ and $I_{i,\varepsilon} \le 0$, we have $u'(t)$ is decreasing on Ω and $u'(t_i^+) \le u'(t_i^-)$, so $u(t)$ is concave on $[0,1]$. To prove the claim, suppose that $0 \le x_1 < x_2 < x_3 \le 1$. Then there exists $y_1 \in (x_1,x_2)$ such that

$$\frac{u(x_2)-u(x_1)}{x_2-x_1} \ge u'(y_1). \tag{2.11}$$

In fact, when there is no impulse in (x_1,x_2), we have (2.11) holds clearly; when $x_1 < s_1 < \ldots < s_m < x_2$, where $\{s_1,\ldots,s_m\} \subseteq \{t_1,\ldots,t_p\}$, then there exist $k_1 \in (x_1,s_1),\ldots,k_m \in (s_{m-1},s_m)$ and $k_{m+1} \in (s_m,x_2)$ such that

$$\begin{aligned}\frac{u(x_2)-u(x_1)}{x_2-x_1} &= \frac{x_2-s_m}{x_2-x_1}\,u'(k_{m+1}) + \sum_{j=2}^{m} \frac{s_j-s_{j-1}}{x_2-x_1}\,u'(k_j) + \frac{s_1-x_1}{x_2-x_1}u'(k_1)\\ &\ge \frac{x_2-s_m}{x_2-x_1}\,u'(k_{m+1}) + \sum_{j=2}^{m} \frac{s_j-s_{j-1}}{x_2-x_1}\,u'(k_{m+1}) + \frac{s_1-x_1}{x_2-x_1}\,u'(k_{m+1}) = u'(k_{m+1}).\end{aligned}$$

Similarly there exists $y_2 \in (x_2,x_3)$ such that

$$\frac{u(x_3)-u(x_2)}{x_3-x_2} \le u'(y_2).$$

Since $u'(t)$ is decreasing on Ω and $u'(t_i^+) \le u'(t_i^-)$, we have $u'(y_1) \ge u'(y_2)$. So

$$\frac{u(x_3)-u(x_2)}{x_3-x_2} \le \frac{u(x_2)-u(x_1)}{x_2-x_1}.$$

Since $u(t)$ is concave on $[0,1]$ and $u(0) = u(1) = 0$, we have $u(t) \ge 0$ for $t \in [0,1]$. Furthermore, there exists a $t^* \in [0,1]$ such that $u(t^*) = \|u\|_\infty$. When $t^* \in (0,1)$, by concavity of $u(t)$ on $[0,1]$ and $u(0) = u(1) = 0$, we have

$$\begin{aligned}u(t) &= u\bigg(\bigg(1-\frac{t}{t^*}\bigg)0 + \frac{t}{t^*}\,t^*\bigg)\\ &\ge \bigg(1-\frac{t}{t^*}\bigg)u(0) + \frac{t}{t^*}\,u(t^*) = \frac{t}{t^*}u(t^*) \ge t(1-t)\,u(t^*),\end{aligned}$$

for any $t \in [0,t^*]$, and

$$\begin{aligned}u(t) &= u\bigg(\frac{1-t}{1-t^*}\,t^* + \frac{t-t^*}{1-t^*}\,1\bigg)\\ &\ge \frac{1-t}{1-t^*}\,u(t^*) + \frac{t-t^*}{1-t^*}\,u(1) \ = \frac{1-t}{1-t^*}\,u(t^*) \ge t(1-t)\,u(t^*),\end{aligned}$$

for any $t \in [t^*,1]$. When $t^* = 0$ or $t^* = 1$, we find that $\|u\|_\infty = 0$, so $u(t) \equiv 0$. Thus

$$u(t) \ge \|u\|_\infty t(1-t), \quad \text{for any } t \in [0,1]. \tag{2.12}$$

In view of (2.10) and (2.12), we have

$$\varepsilon t_*(1-t_*) > u(t_*) \geq \|u\|_\infty t_*(1-t_*).$$

Thus $\|u\|_\infty < \varepsilon$. So it follows from (1.4) that

$$u''(t) = -f_{\varepsilon,\lambda}(t,u(t)) \leq -2\varepsilon = \varphi_\varepsilon''(t), \quad \text{for any } t \in \Omega.$$

What is more, $\Delta\big[u'(t_i) - \varphi_\varepsilon'(t_i)\big] = I_{i,\varepsilon}(u(t_i)) \leq 0$. Thus $u'(t) - \varphi_\varepsilon'(t)$ is nonincreasing on Ω and $u'(t_i^+) - \varphi_\varepsilon'(t_i^+) \leq u'(t_i^-) - \varphi_\varepsilon'(t_i^-)$, so $u(t) - \varphi_\varepsilon(t)$ is concave on $[0,1]$, which combined with $[u(0) - \varphi_\varepsilon(0)] = [u(1) - \varphi_\varepsilon(1)] = 0$ yields to $u(t) \geq \varphi_\varepsilon(t)$ on [0,1], which contradicts to (2.10). $\square$

Consider the functional $\Phi \in C^1(H_0^1;\mathbb{R})$ defined by

$$\Phi(u) := \frac{1}{2}\int_0^1 |u'(t)|^2\,dt - \int_0^1 F_{\varepsilon,\lambda}(t,u(t))\,dt + \phi(u)$$

where

$$F_{\varepsilon,\lambda}(t,x) := \int_\varepsilon^x f_{\varepsilon,\lambda}(t,y)\,dy \quad \text{and} \quad \phi(u) := \sum_{i=1}^p \int_\varepsilon^{u(t_i)} I_{i,\varepsilon}(x)\,dx.$$

And

$$\langle \Phi'(u), v\rangle = \int_0^1 u'(t)v'(t)\,dt - \int_0^1 f_{\varepsilon,\lambda}(t,u(t))v(t)\,dt + \sum_{i=1}^p I_{i,\varepsilon}(u(t_i))v(t_i).$$

Thus critical points of Φ correspond to weak solutions of the problem (2.4). And it follows from the continuity of $f_{\varepsilon,\lambda}$ and $I_{i,\varepsilon}$ that $\Phi(u)$ is sequentially weakly lower semi-continuous on $H_0^1(0,T)$ as the sum of a convex continuous function and of two weakly continuous functions.

Notice that $f_{\varepsilon,\lambda} > 0$ and $I_{i,\varepsilon} \leq 0$, we have

$$F_{\varepsilon,\lambda}(t,x) = -\int_x^\varepsilon f_{\varepsilon,\lambda}(t,y)\,dy \leq 0, \quad (t,x) \in (0,1)\times(-\infty,\varepsilon) \tag{2.13}$$

and

$$\int_\varepsilon^x I_{i,\varepsilon}(y)\,dy \geq 0, \quad x \in (-\infty,\varepsilon). \tag{2.14}$$

For $(t,x) \in (0,1)\times[\varepsilon,+\infty)$, we have $f_{\varepsilon,\lambda}(t,x) = \lambda f(t,x) + x^{-\alpha}$, so

$$f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x) = G_{\varepsilon,\lambda}(t,x), \quad (t,x) \in (0,1)\times[\varepsilon,+\infty) \tag{2.15}$$

and

$$\frac{\partial}{\partial x}\bigg[\frac{F_{\varepsilon,\lambda}(t,x)}{x^2}\bigg] = \frac{1}{x^3}G_{\varepsilon,\lambda}(t,x), \quad (t,x) \in (0,1)\times[\varepsilon,+\infty), \tag{2.16}$$

where

$$G_{\varepsilon,\lambda}(t,x) := \lambda\bigg(f(t,x)x - 2\int_\varepsilon^x f(t,y)\,dy\bigg) - \frac{1+\alpha}{1-\alpha}x^{1-\alpha} + \frac{2\varepsilon^{1-\alpha}}{1-\alpha}.$$

If we assume (H2), then

$$\text{(2.17)} \qquad \lim_{x\to+\infty}\frac{F_{\varepsilon,\lambda}(t,x)}{x^2}=\lim_{x\to+\infty}\bigg(\frac{1}{2}\,\lambda\,\frac{f(t,x)}{x}+\frac{1}{2}\frac{1}{x^{1+\alpha}}\bigg)=\frac{1}{2}\,\lambda\pi^2,$$

which combined with (2.16) yields to

$$\text{(2.18)} \quad \frac{1}{2}\,\lambda\pi^2-\frac{F_{\varepsilon,\lambda}(t,x)}{x^2}=\int_x^{+\infty}\frac{\partial}{\partial y}\bigg[\frac{F_{\varepsilon,\lambda}(t,y)}{y^2}\bigg]\,dy=\int_x^{+\infty}\frac{1}{y^3}\,G_{\varepsilon,\lambda}(t,y)\,dy,$$

for $(t,x)\in(0,1)\times[\varepsilon,+\infty)$.

If we assume (H8), then

$$\int_\varepsilon^x I_{i,\varepsilon}(y)\,dy=\int_\varepsilon^x I_i(y)\,dy\geq-\frac{a_i}{\gamma_i+1}\,x^{\gamma_i+1}-b_ix-C,\quad x\in[\varepsilon,+\infty),$$

which combined with (2.14) yields to

$$\text{(2.19)} \qquad \begin{aligned}\phi(u)&=\sum_{u(t_i)\geq\varepsilon}\int_\varepsilon^{u(t_i)}I_{i,\varepsilon}(x)\,dx+\sum_{u(t_i)<\varepsilon}\int_\varepsilon^{u(t_i)}I_{i,\varepsilon}(x)\,dx\\&\geq-\sum_{i=1}^p\frac{a_i}{\gamma_i+1}\,\|u\|_\infty^{\gamma_i+1}-C\|u\|_\infty-C.\end{aligned}$$

Lemma 2.11. *Assume* (H2) *and* (H8) *hold, then* Φ *satisfies* (C) *provided* (H6) *or* (H7) *holds.*

Proof. Suppose that $\{u_n\}$ is a sequence in $H_0^1(0,1)$ such that

$$\text{(2.20)} \qquad \{\Phi(u_n)\}\text{ is bounded and }\|\Phi'(u_n)\|(1+\|u_n\|)\to0.$$

By a standard argument it suffices to show that $\{u_n\}$ is bounded in $H_0^1(0,1)$ when verifying (C). In view of $I_{i,\varepsilon}\leq0$ and $f_{\varepsilon,\lambda}>0$, we have

$$\begin{aligned}\|u_n^-\|^2=-\langle\Phi'(u_n),u_n^-\rangle-\int_{u_n(t)<0}f_{\varepsilon,\lambda}(t,u_n(t))u_n^-(t)\,dt+\sum_{i=1}^pI_{i,\varepsilon}(u_n(t_i))u_n^-(t_i)\\\leq\|\Phi'(u_n)\|\|u_n^-\|.\end{aligned}$$

So $\|u_n^-\|$ is bounded, and hence there exists a constant $C_0>0$ such that

$$\text{(2.21)} \qquad u_n(t)\geq-u_n^-(t)\geq-C_0,\quad\text{for any }t\in[0,1].$$

To prove $\{u_n\}$ is bounded in $H_0^1(0,1)$ we argue by contradiction. Suppose that $\|u_n\|\to+\infty$. Let

$$\text{(2.22)} \qquad v_n(t):=\frac{u_n(t)}{\|u_n\|}=\frac{u_n^+(t)}{\|u_n\|}-\frac{u_n^-(t)}{\|u_n\|}.$$

Since $H_0^1(0,1)$ is a reflexive Banach space, $\|v_n\|=1$ and $\|u_n^-\|$ is bounded, passing to a subsequence if necessary (denoted again by $\{v_n\}$), we have $\{v_n\}$

converges to some $v \geq 0$ weakly in $H_0^1(0,1)$, strongly in $L^2(0,1)$, and $\{v_n\}$ converges uniformly to the v on $[0,1]$ (see Proposition 1.2 in [15]). Then

$$1 - \int_0^1 v_n'(t)v'(t)\,dt = \frac{1}{\|u_n\|}\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)]\,dt + \frac{1}{\|u_n\|}\langle \Phi'(u_n), v_n - v\rangle - \frac{1}{\|u_n\|}\sum_{i=1}^{p} I_{i,\varepsilon}(u_n(t_i))[v_n(t_i)-v(t_i)]. \tag{2.23}$$

For $u_n(t) < \varepsilon$, it follows from (1.4) that

$$\left|\frac{1}{\|u_n\|} f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)]\right| \leq \frac{1}{\|u_n\|}\big(\varphi_\varepsilon^{-\alpha}(t) + \lambda C_\varepsilon\big)|v_n(t)-v(t)|. \tag{2.24}$$

In view of (H2) and $f \in C([0,1]\times[0,+\infty))$, we find that

$$f(t,x) \leq \frac{3\pi^2}{2}x + C, \quad (t,x)\in(0,1)\times[\varepsilon,+\infty), \tag{2.25}$$

then, for $u_n(t) \geq \varepsilon$, we have

$$\begin{aligned}\left|\frac{1}{\|u_n\|} f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)]\right| &= \frac{1}{\|u_n\|}\big(\lambda f(t,u_n(t)) + u_n^{-\alpha}(t)\big)|v_n(t)-v(t)| \\ &\leq \left(\lambda\frac{3\pi^2}{2} + (\lambda C + \varepsilon^{-\alpha})\frac{1}{\|u_n\|}\right)|v_n(t)-v(t)|,\end{aligned}$$

which combined with (2.24) yields to

$$\lim_{n\to+\infty}\frac{1}{\|u_n\|} f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)] = 0, \quad \text{for a.e. } t\in(0,1)$$

and

$$\left|\frac{1}{\|u_n\|} f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)]\right| \leq C\big(\varphi_\varepsilon^{-\alpha}(t)+1\big) \in L^1(0,1),$$

for large n. So

$$\frac{1}{\|u_n\|}\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))[v_n(t)-v(t)]\,dt \to 0 \quad \text{as } n\to\infty. \tag{2.26}$$

It follows from $I_{i,\varepsilon}\in C(\mathbb{R};(-\infty,0])$ and (H8) that

$$|I_{i,\varepsilon}(x)| = \begin{cases} |I_i(x)| \leq a_i x^{\gamma_i} + b_i, & x \geq \varepsilon, \\ |I_i(x)| \leq \max\limits_{[0,\varepsilon]}|I_i(x)|, & \varphi_\varepsilon(t_i) \leq x < \varepsilon, \\ |I_i(\varphi_\varepsilon(t_i))|, & x < \varphi_\varepsilon(t_i), \end{cases}$$

and hence $|I_{i,\varepsilon}(x)| \leq a_i|x|^{\gamma_i} + C$ for $x\in\mathbb{R}$, then

$$\left|\frac{1}{\|u_n\|}\sum_{i=1}^{p} I_{i,\varepsilon}(u_n(t_i))[v_n(t_i)-v(t_i)]\right| \leq \sum_{i\in\mathcal{A}} a_i|v_n(t_i)-v(t_i)| + \sum_{i\in\mathcal{B}} a_i\frac{|v_n(t_i)-v(t_i)|}{\|u_n\|^{1-\gamma_i}} + C\sum_{i=1}^{p}\frac{|v_n(t_i)-v(t_i)|}{\|u_n\|},$$

thus

$$\frac{1}{\|u_n\|}\sum_{i=1}^{p} I_{i,\varepsilon}(u_n(t_i))[v_n(t_i)-v(t_i)] \to 0 \quad \text{as } n\to\infty.$$

In view of the above display and (2.26), passing to the limit in (2.23) gives $\|v\|=1$, so there exists a $\Omega_1 \subseteq (0,1)$ such that $\operatorname{meas}(\Omega_1)>0$ and $v(t)>0$ for $t\in\Omega_1$, and hence $u_n(t)=v_n(t)\|u_n\|\to+\infty$ for $t\in\Omega_1$.

In the following, two cases are considered, respectively.

Case 1. (H6) holds. By (1.8), for each $i=1,\dots,p$ there exists $x_1>\varepsilon$ such that

$$I_i(x)x - 2\int_{\varepsilon}^{x} I_i(y)dy \le C, \quad \text{for any } x\ge x_1,$$

which combined with the continuity of $I_{i,\varepsilon}$ and (2.21) yields to

$$I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - 2\int_{\varepsilon}^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx \le C, \tag{2.27}$$

It follows from (1.7) that

$$\lim_{x\to+\infty}\left[\lambda\frac{1}{x^{1-\alpha}}\left(f(t,x)x-2\int_{\varepsilon}^{x} f(t,y)\,dy\right)-\frac{1+\alpha}{1-\alpha}\right]x^{1-\alpha}=+\infty$$

uniformly for $t\in(0,1)$, which combined with (2.15) yields to

$$\lim_{x\to+\infty}\big(f_{\varepsilon,\lambda}(t,x)x-2F_{\varepsilon,\lambda}(t,x)\big)=+\infty \quad \text{uniformly for } t\in(0,1). \tag{2.28}$$

In view of (2.15), (2.28) and $f\in C([0,1]\times[0,+\infty))$, we have

$$f_{\varepsilon,\lambda}(t,x)x-2F_{\varepsilon,\lambda}(t,x)\ge -C, \quad (t,x)\in(0,1)\times[\varepsilon,+\infty). \tag{2.29}$$

So it follows from $f_{\varepsilon,\lambda}>0$, (1.4) and (2.29) that

$$\begin{aligned} f_{\varepsilon,\lambda}(t,x)x-2F_{\varepsilon,\lambda}(t,x) &\\ &\ge \begin{cases} -(\varphi_\varepsilon^{-\alpha}(t)+\lambda C_\varepsilon)|x|, & (t,x)\in(0,1)\times[-\infty,0),\\ 0, & (t,x)\in(0,1)\times[0,\varepsilon),\\ -C, & (t,x)\in(0,1)\times[\varepsilon,+\infty),\end{cases}\end{aligned} \tag{2.30}$$

then we have

$$\begin{aligned}&\int_{v=0} f_{\varepsilon,\lambda}(t,u_n(t))u_n(t)-2F_{\varepsilon,\lambda}(t,u_n(t))\,dt\\ &\ge \int_{v=0,\,u_n<0} -(\varphi_\varepsilon^{-\alpha}(t)+\lambda C_\varepsilon)|u_n^-(t)|\,dt+\int_{v=0,\,0\le u_n<\varepsilon} 0\,dt+\int_{v=0,\,u_n\ge\varepsilon} -C\,dt\\ &\ge -\int_0^1 \big(\varphi_\varepsilon^{-\alpha}(t)+\lambda C_\varepsilon\big)\,dt\|u_n^-\|-C,\end{aligned}$$

which combined with the boundedness of $\{\|u_n^-\|\}$ yields that

$$\int_{v=0} f_{\varepsilon,\lambda}(t,u_n(t))u_n(t)-2F_{\varepsilon,\lambda}(t,u_n(t))\,dt\ge -C. \tag{2.31}$$

In view of (2.28), (2.30) and Fatou's lemma, we have

$$\int_{v>0} f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt \to +\infty \quad \text{as } n\to\infty.$$

So, by the above display and (2.31), we have

$$\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt = \int_{v>0} + \int_{v=0} \to +\infty$$

as $n\to\infty$, which combined with (2.27) yields to

$$\begin{aligned}\frac{1}{2}\langle\Phi'(u_n),u_n\rangle - \Phi(u_n) = -\frac{1}{2}\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt \\ + \frac{1}{2}\sum_{i=1}^p\bigg(I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - 2\int_\varepsilon^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx\bigg) \to -\infty\end{aligned}$$

as $n\to\infty$, which contradicts (2.20).

Case 2. (H7) holds. By (1.10), for each $i=1,\ldots,p$ there exists $x_1>\varepsilon$ such that

$$I_i(x)x - 2\int_\varepsilon^x I_i(y)\,dy \ge -C, \quad \text{for any } x\ge x_1,$$

which combined with the continuity of $I_{i,\varepsilon}$ and (2.21) yields to

$$I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - 2\int_\varepsilon^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx \ge -C, \tag{2.32}$$

It follows from (1.9) that

$$\lim_{x\to+\infty}\bigg[\lambda\frac{1}{x^{1-\alpha}}\bigg(f(t,x)x - 2\int_\varepsilon^x f(t,y)\,dy\bigg) - \frac{1+\alpha}{1-\alpha}\bigg]x^{1-\alpha} = -\infty$$

uniformly for $t\in(0,1)$, which combined with (2.15) yields to

$$\lim_{x\to+\infty} f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x) = -\infty \quad \text{uniformly for } t\in(0,1). \tag{2.33}$$

In view of (2.15), (2.33) and $f\in C([0,1]\times[0,+\infty))$, we have

$$f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x) \le C, \quad (t,x)\in(0,1)\times[\varepsilon,+\infty). \tag{2.34}$$

So it follows from $f_{\varepsilon,\lambda}>0$, (1.4) and (2.34) that

$$\begin{aligned}&f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x) \\ &\le \begin{cases} 2(\varphi_\varepsilon^{-\alpha}(t)+\lambda C_\varepsilon)(\varepsilon+|x|), & (t,x)\in(0,1)\times[-\infty,0), \\ C(\varphi_\varepsilon^{-\alpha}(t)+1), & (t,x)\in(0,1)\times[0,\varepsilon), \\ C, & (t,x)\in(0,1)\times[\varepsilon,+\infty),\end{cases}\end{aligned} \tag{2.35}$$

which combined with $\|u_n^-\|$ is bounded yields to

$$\int_{v=0} f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt \le C. \tag{2.36}$$

In view of (2.33), (2.35) and Fatou's lemma, we have

$$\int_{v>0} f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt \to -\infty \quad \text{as } n\to\infty.$$

In view of the above display and (2.36), we have,

$$\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt = \int_{v>0} + \int_{v=0} \to -\infty$$

as $n\to\infty$, which combined with (2.32) yields to

$$\begin{aligned}\frac{1}{2}\langle \Phi'(u_n),u_n\rangle - \Phi(u_n) &= -\frac{1}{2}\int_0^1 f_{\varepsilon,\lambda}(t,u_n(t))u_n(t) - 2F_{\varepsilon,\lambda}(t,u_n(t))\,dt \\ &\quad + \frac{1}{2}\sum_{i=1}^{p}\bigg(I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - 2\int_\varepsilon^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx\bigg) \to +\infty\end{aligned}$$

as $n\to\infty$, which contradicts (2.20). □

3. Main results

In this section, the existence and the multiplicity of weak solutions for the problem (2.4) will be discussed, respectively.

3.1. Existence of one solution.

Theorem 3.1. *Suppose that* (H1) *and* (H8) *hold, then the problem* (2.4) *has a weak solution* $w\in H_0^1(0,1)$ *such that* $\Phi(w) = \inf_{H_0^1(0,T)} \Phi(u)$ *provided*

$$0<\lambda<\frac{\pi^2}{a}\bigg(1-\sum_{i\in\mathcal{A}} a_i\bigg). \tag{3.1}$$

Proof. It follows from assumption (H1) that

$$F_{\varepsilon,\lambda}(t,x) = \int_\varepsilon^x \lambda f(t,y) + y^{-\alpha}\,dy \le \frac{1}{2}a\lambda x^2 + Cx + Cx^{1-\alpha} + C,$$

for $(t,x)\in(0,1)\times[\varepsilon,+\infty)$, which combined with (2.1), (2.3) and (2.13) yields to

$$\begin{aligned}\int_0^1 F_{\varepsilon,\lambda}(t,u(t))\,dt &= \int_{u(t)<\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt + \int_{u(t)\ge\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt \\ &\le \frac{a\lambda}{2}\int_0^1 u^2(t)\,dt + C\|u\|_\infty + C\|u\|_\infty^{1-\alpha} + C \\ &\le \frac{a\lambda}{2\pi^2}\|u\|^2 + C\|u\| + C\|u\|^{1-\alpha} + C.\end{aligned}$$

In view of the above inequality and (2.19), we have

$$\Phi(u) \ge \frac{1}{2}\bigg(1-\frac{a\lambda}{\pi^2}-\sum_{i\in\mathcal{A}} a_i\bigg)\|u\|^2 - \sum_{i\in\mathcal{B}}\frac{a_i}{\gamma_i+1}\|u\|^{\gamma_i+1} - C\|u\| - C\|u\|^{1-\alpha} - C,$$

so $\Phi(u)\to+\infty$ as $\|u\|\to\infty$ by (3.1). Thus Lemma 2.1 implies the result. □

Theorem 3.2. *Suppose that* (H2), (H5) *and* (H8) *hold, then the problem* (2.4) *has a weak solution* $w \in H_0^1(0,1)$ *such that* $\Phi(w) = \inf_{H_0^1(0,T)} \Phi(u)$ *provided*

$$0 < \lambda < 1 - \sum_{i\in\mathcal{A}} a_i. \tag{3.2}$$

Proof. In view of (H5) and $f \in C([0,1] \times [0,+\infty))$, we find that

$$f(t,x)x - 2\int_\varepsilon^x f(t,y)\,dy \geq -Cx^\beta - C, \quad (t,x) \in (0,1) \times [\varepsilon,+\infty),$$

which combined with (2.18) yields to

$$\frac{1}{2}\lambda\pi^2 - \frac{F_{\varepsilon,\lambda}(t,x)}{x^2} \geq -C\bigg(\frac{1}{x^{2-\beta}} + \frac{1}{x^2} + \frac{1}{x^{1+\alpha}}\bigg), \quad (t,x) \in (0,1) \times [\varepsilon,+\infty).$$

Thus it follows from the above inequality and (2.13) that

$$\begin{aligned} -\int_0^1 F_{\varepsilon,\lambda}(t,u(t))\,dt &= -\int_{u(t)\geq\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt - \int_{u(t)<\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt \\ &\geq -\frac{1}{2}\lambda\pi^2\int_0^1 u^2(t)\,dt - C\|u\|_\infty^\beta - C\|u\|_\infty^{1-\alpha} - C, \end{aligned}$$

which combined with (2.1), (2.3) and (2.19) yields to

$$\Phi(u) \geq \frac{1}{2}\bigg(1-\lambda-\sum_{i\in\mathcal{A}} a_i\bigg)\|u\|^2 - \sum_{i\in\mathcal{B}} \frac{a_i}{\gamma_i+1}\,\|u\|^{\gamma_i+1} - C\|u\|^\beta - C\|u\|^{1-\alpha} - C\|u\| - C.$$

So $\Phi(u) \to +\infty$ as $\|u\| \to \infty$ by (3.2). Thus Lemma 2.1 implies the result. □

Theorem 3.3. *Suppose that* (H2), (H6) *and* (H8) *hold with* $b_i \equiv 0$ *and* $\mathcal{A} = \{1,\ldots,p\}$, *then the problem* (2.4) *has a weak solution* $w \in H_0^1(0,1)$ *such that* $\Phi(w) = \inf_{H_0^1(0,T)} \Phi(u)$ *provided* $0 < \lambda = 1 - \sum_{i\in\mathcal{A}} a_i$.

Proof. In view of (2.14) and (H8) holds with $b_i \equiv 0$ and $\mathcal{A} = \{1,\ldots,p\}$,

$$\phi(u) = \bigg(\sum_{u(t_i)<\varepsilon} + \sum_{u(t_i)\geq\varepsilon}\bigg)\int_\varepsilon^{u(t_i)} I_{i,\varepsilon}(x)\,dx \geq -\frac{1}{2}\sum_{i=1}^p a_i\|u\|_\infty^2 - C. \tag{3.3}$$

For $(t,x) \in (0,1) \times [\varepsilon,+\infty)$, it follows from (2.29) that

$$\frac{\partial}{\partial x}\bigg[\frac{F_{\varepsilon,\lambda}(t,x)}{x^2}\bigg] = \frac{1}{x^3}\big(f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x)\big) \geq -Cx^{-3},$$

which combined with (2.17) yields to

$$\frac{1}{2}\lambda\pi^2 - \frac{F_{\varepsilon,\lambda}(t,x)}{x^2} = \int_x^{+\infty} \frac{\partial}{\partial y}\bigg[\frac{F_{\varepsilon,\lambda}(t,y)}{y^2}\bigg]\,dy \geq -C\frac{1}{x^2}.$$

Then it follows from the above inequality and (2.13) that

$$-\int_0^1 F_{\varepsilon,\lambda}(t,u(t))\,dt = -\int_{u(t)\geq\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt - \int_{u(t)<\varepsilon} F_{\varepsilon,\lambda}(t,u(t))\,dt \geq -\frac{1}{2}\lambda\pi^2\int_0^1 u^2(t)\,dt - C,$$

which combined with (2.1) and (3.3) yields to

$$\Phi(u) \geq \frac{1}{2}\bigg(1-\lambda-\sum_{i\in\mathcal{A}} a_i\bigg)\|u\|^2 - C = -C.$$

What is more, Lemma 2.11 implies that Φ satisfies (C). So it follows from Lemma 2.3 that the result holds. □

3.2. Existence of two solutions. Let B_r be the open ball in $H_0^1(0,T)$ with radius $r>0$ and centered at 0 and let ∂B_r and $\overline{B_r}$ denote the boundary and closure of B_r, respectively.

Lemma 3.4. *If* $0<\lambda<\sup_{r>0} h(r)$*, then there exist* $r_0\in(0,+\infty)$ *and* $w_1\in B_{r_0}$ *such that* $\Phi(w_1)=\min_{\overline{B_{r_0}}}\Phi(u)$ *and*

$$\Phi(w_1) < \inf_{\partial B_{r_0}} \Phi(u). \tag{3.4}$$

Proof. By $\lambda<\sup_{r>0} h(r)$, there exists a $r_0\in(0,+\infty)$ such that

$$\lambda \max_{[0,1]\times[0,r_0]} f(t,x)x < r_0^2 - r_0^{1-\alpha} + \sum_{i=1}^p \min_{[0,r_0]} I_i(x)x. \tag{3.5}$$

Since $\overline{B_{r_0}}$ is a closed convex set, $\overline{B_{r_0}}$ is weak sequentially closed. Thus it follows from Lemma 2.4 that there exists a $w_1\in\overline{B_{r_0}}$ such that $\Phi(w_1)=\min_{\overline{B_{r_0}}}\Phi(u)$, and hence w_1 is a weak solution of (2.4). So, by Lemmas 2.9 and 2.10, w_1 is a positive classical solution of (1.1). Suppose $w_1\in\partial B_{r_0}$, then

$$-\int_0^1 w_1''(t)w_1(t)\,dt - \int_0^1 w_1^{1-\alpha}(t)\,dt = \lambda\int_0^1 f(t,w_1(t))w_1(t)\,dt,$$

which combined with (2.3) and (2.6) yields to

$$\begin{aligned} r_0^2 = \|w_1\|^2 &= \int_0^1 w_1'(t)w_1'(t)\,dt \\ &= \lambda\int_0^1 f(t,w_1(t))w_1(t)\,dt + \int_0^1 w_1^{1-\alpha}(t)\,dt - \sum_{i=1}^p I_i(w_1(t_i))w_1(t_i) \\ &\leq \lambda\max_{[0,1]\times[0,r_0]} f(t,x)x + r_0^{1-\alpha} - \sum_{i=1}^p \min_{[0,r_0]} I_i(x)x, \end{aligned}$$

which contradicts (3.5). So $w_1 \in B_{r_0}$ and $\Phi(w_1) < \Phi(u)$ for any $u \in \partial B_{r_0}$, and hence (3.4) holds. □

In view of Lemma 3.4, we will only need to check that Φ satisfies (C) and $\Phi(r\sin(\pi t)) \to -\infty$ as $r \to +\infty$, then Lemma 2.5 will give a second critical point $w_2 \in H_0^1(0,1)$ such that

$$\Phi(w_2) = \inf_{g\in\Gamma} \max_{s\in[0,1]} \Phi(g(s)),$$

where $\Gamma = \{g \in C([0,1], H_0^1(0,1)) : g(0) = w_1,\ g(1) = r\sin(\pi t)\}$.

Since $I_{i,\varepsilon} \in C(\mathbb{R}; (-\infty, 0])$, we find that, for any $r > 0$,

$$\begin{aligned} \phi(r\sin(\pi t)) &= \sum_{r\sin(\pi t_i)\le\varepsilon} + \sum_{r\sin(\pi t_i)>\varepsilon} \int_\varepsilon^{r\sin(\pi t_i)} I_{i,\varepsilon}(x)\,dx \\ &\le \sum_{r\sin(\pi t_i)\le\varepsilon} \int_\varepsilon^{r\sin(\pi t_i)} I_{i,\varepsilon}(x)\,dx \\ &\le \sum_{i=1}^{p} \max_{0\le x\le\varepsilon} \{-I_{i,\varepsilon}(x)\}\varepsilon \le C. \end{aligned} \tag{3.6}$$

Theorem 3.5. *Suppose that* (H2), (H7) *and* (H8) *hold, then the problem* (2.4) *has two weak solutions in* $H_0^1(0,T)$ *provided* $1 \le \lambda < \sup_{r>0} h(r)$.

Proof. It follows from (1.9) that

$$\lim_{x\to+\infty} \left[\lambda \frac{1}{x^{1-\alpha}} \left(f(t,x)x - 2\int_\varepsilon^x f(t,y)\,dy \right) - \frac{1+\alpha}{1-\alpha}\right] x^{1-\alpha} = -\infty$$

uniformly for $t \in (0,1)$, then $G_{\varepsilon,\lambda}(t,x) \to -\infty$ as $x \to +\infty$ uniformly for $t \in (0,1)$. So

$$\lim_{x\to+\infty} x^2 \int_x^{+\infty} \frac{1}{y^3}\, G_{\varepsilon,\lambda}(t,y)\,dy = -\infty \quad \text{uniformly for } t \in (0,1). \tag{3.7}$$

For $(t,x) \in (0,1) \times [\varepsilon, +\infty)$, it follows from (2.18) that

$$\frac{1}{2}\pi^2 x^2 - F_{\varepsilon,\lambda}(t,x) = \frac{1}{2}\,\pi^2(1-\lambda)x^2 + x^2 \int_x^{+\infty} \frac{1}{y^3}\, G_{\varepsilon,\lambda}(t,y)\,dy.$$

So $\lambda \ge 1$ and (3.7) imply that

$$\lim_{x\to+\infty} \left(\frac{1}{2}\pi^2 x^2 - F_{\varepsilon,\lambda}(t,x) \right) = -\infty \quad \text{uniformly for } t \in (0,1). \tag{3.8}$$

Furthermore, for $(t,x) \in (0,1) \times [\varepsilon, +\infty)$, it follows from (2.34) that

$$\frac{\partial}{\partial x} \left[\frac{F_{\varepsilon,\lambda}(t,x)}{x^2} \right] = \frac{1}{x^3} \big(f_{\varepsilon,\lambda}(t,x)x - 2F_{\varepsilon,\lambda}(t,x) \big) \le C\, \frac{1}{x^3},$$

which combined with (2.17) yields

$$\frac{1}{2}\,\lambda\pi^2 - \frac{F_{\varepsilon,\lambda}(t,x)}{x^2} = \int_x^{+\infty} \frac{\partial}{\partial y} \left[\frac{F_{\varepsilon,\lambda}(t,y)}{y^2} \right] dy \le C\, \frac{1}{x^2}.$$

So $\lambda \geq 1$ implies that

$$\frac{1}{2}\,\pi^2 x^2 - F_{\varepsilon,\lambda}(t,x) \leq C, \quad (t,x) \in (0,1) \times [\varepsilon, +\infty), \tag{3.9}$$

For $(t,x) \in (0,1) \times [0,\varepsilon)$, by (1.4) we have

$$\frac{1}{2}\,\pi^2 x^2 - F_{\varepsilon,\lambda}(t,x) \leq \frac{1}{2}\,\pi^2 \varepsilon^2 + \varphi_\varepsilon^{-\alpha}(t)\varepsilon + \lambda C_\varepsilon \varepsilon \in L^1(0,1),$$

which combined with (3.8) and (3.9) yields to

$$\int_0^1 \frac{1}{2}\,\pi^2 (r\sin(\pi t))^2 - F_{\varepsilon,\lambda}(t, r\sin(\pi t))\,dt \to -\infty \quad \text{as } r \to +\infty. \tag{3.10}$$

In view of (2.2), we have that

$$\Phi(r\sin(\pi t)) = \int_0^1 \frac{1}{2}\pi^2 (r\sin(\pi t))^2 - F_{\varepsilon,\lambda}(t, r\sin(\pi t))dt + \phi(r\sin(\pi t)).$$

So it follows from (3.6) and (3.10) that $\Phi(r\sin(\pi t)) \to -\infty$ as $r \to +\infty$. By Lemma 2.11, Φ satisfies (C). $\square$

THEOREM 3.6. *Suppose that* (H3) *and* (H8) *hold, then the problem* (2.4) *has two weak solutions in* $H_0^1(0,T)$ *provided* $\pi^2/b < \lambda < \sup_{r>0} h(r)$.

PROOF. By (1.4) and (H3), we find that

$$F_{\varepsilon,\lambda}(t,x) \geq \begin{cases} -\varepsilon\varphi_\varepsilon^{-\alpha}(t) - C, & (t,x) \in (0,1) \times [0,\varepsilon), \\ \dfrac{1}{2}\,\lambda b x^2 - \lambda C x + \dfrac{1}{1-\alpha}\,x^{1-\alpha} - C, & (t,x) \in (0,1) \times [\varepsilon, +\infty). \end{cases}$$

It follows from the above inequality, (2.2), (3.6) and $\lambda > \pi^2/b$ that

$$\begin{aligned}
\Phi(r\sin(\pi t)) &\leq \frac{1}{2}\pi^2 \int_{r\sin(\pi t)<\varepsilon} (r\sin(\pi t))^2\,dt \\
&\quad + \frac{1}{2}\pi^2 \int_{\varepsilon \leq r\sin(\pi t)} (r\sin(\pi t))^2\,dt + \varepsilon \int_{r\sin(\pi t)<\varepsilon} \varphi_\varepsilon^{-\alpha}(t)\,dt \\
&\quad - \int_{\varepsilon \leq r\sin(\pi t)} \frac{1}{2}\,\lambda b (r\sin(\pi t))^2\,dt + \int_{\varepsilon \leq r\sin(\pi t)} \lambda C r\sin(\pi t)\,dt \\
&\quad - \int_{\varepsilon \leq r\sin(\pi t)} \frac{1}{1-\alpha}(r\sin(\pi t))^{1-\alpha}\,dt + C \\
&\leq \frac{1}{2}(\pi^2 - \lambda b) \int_{\varepsilon \leq r\sin(\pi t)} (\sin(\pi t))^2 dt r^2 + Cr + Cr^{1-\alpha} + C,
\end{aligned}$$

so $\Phi(r\sin(\pi t)) \to -\infty$ as $r \to +\infty$.

Notice that the role of (2.25) could be replaced by (H3); then we proceed similarly as in the proof of Lemma 2.11. Suppose that $\{u_n\}$ is a sequence in $H_0^1(0,1)$ such that (2.20) holds, then $\|u_n^-\|$ is bounded and (2.21) holds. If $\|u_n\| \to +\infty$, define v_n by (2.22), passing to a subsequence if necessary (denoted again by $\{v_n\}$), we have $\{v_n\}$ converges to some $v \geq 0$ weakly in $H_0^1(0,1)$, strongly in $L^2(0,1)$, and uniformly on $[0,1]$. And there exists a $\Omega_1 \subseteq (0,1)$ such that $\operatorname{meas}(\Omega_1) > 0$ and $v(t) > 0$ for $t \in \Omega_1$.

It follows from $f_{\varepsilon,\lambda} > 0$, (H3) and $x^{-\alpha}$ is bounded on $[\varepsilon, +\infty)$ that

$$\begin{aligned}\lambda bx - f_{\varepsilon,\lambda}(t,x) \\ &\leq \begin{cases} \lambda bx, & (t,x) \in (0,1)\times(-\infty,\varepsilon), \\ \lambda bx - \lambda f(t,x) - x^{-\alpha} \leq \lambda C + C, & (t,x) \in (0,1)\times[\varepsilon,+\infty), \end{cases}\end{aligned}$$

which combined with (2.21) yields to

$$\begin{aligned}\int_0^1 \big(\lambda b u_n(t) - f_{\varepsilon,\lambda}(t,u_n(t))\big)\sin(\pi t)\,dt \\ &\leq \int_{u_n(t)<\varepsilon} \lambda b u_n(t)\sin(\pi t)\,dt + \int_{\varepsilon\leq u_n(t)} (\lambda C + C)\sin(\pi t)\,dt \leq C.\end{aligned}$$

In view of (2.2), (2.21), $I_{i,\varepsilon} \leq 0$ and the above inequality, we have

$$\begin{aligned}(\lambda b - \pi^2)\int_0^1 v_n(t)\sin(\pi t)\,dt &= \frac{1}{\|u_n\|}\int_0^1 \lambda b u_n(t)\sin(\pi t) - u_n'(t)\sin'(\pi t)\,dt \\ &= \frac{1}{\|u_n\|}\int_0^1 \big(\lambda b u_n(t) - f_{\varepsilon,\lambda}(t,u_n(t))\big)\sin(\pi t)\,dt \\ &\quad + \frac{1}{\|u_n\|}\sum_{i=1}^p I_{i,\varepsilon}(u_n(t_i))\sin(\pi t_i) - \frac{1}{\|u_n\|}\langle \Phi'(u_n), \sin(\pi t)\rangle \\ &\leq \frac{1}{\|u_n\|} C\big(1 + \|\Phi'(u_n)\|\big).\end{aligned}$$

Passing to the limit in the above inequality gives

$$(\lambda b - \pi^2)\int_0^1 v(t)\sin(\pi t)\,dt \leq 0.$$

which is impossible since $\lambda > \pi^2/b$ and $v(t) > 0$ for $t \in \Omega_1$ with $\operatorname{meas}(\Omega_1) > 0$. Thus Φ satisfies (C). □

Theorem 3.7. *Suppose that* (H4) *holds, then the problem* (2.4) *has two weak solutions in* $H_0^1(0,T)$ *provided*

$$\frac{\sigma - 1 + \alpha}{\tau(1-\alpha)} < \lambda < \sup_{r>0} h(r). \tag{3.11}$$

Proof. By (1.5) and (3.11), there exists $x_0 > \varepsilon$ such that, for $(t,x) \in (0,1)\times[x_0,+\infty)$

$$\frac{\lambda}{x^{1-\alpha}}\left(\sigma\int_\varepsilon^x f(t,y)\,dy - f(t,x)x\right) \leq -\frac{\sigma - 1 + \alpha}{1-\alpha},$$

and hence

$$\lambda\frac{1}{x^{1-\alpha}}\left(\sigma\int_\varepsilon^x f(t,y)\,dy - f(t,x)x\right) + \frac{\sigma - 1 + \alpha}{1-\alpha} \leq 0 < \frac{1}{x^{1-\alpha}}\frac{\sigma\varepsilon^{1-\alpha}}{1-\alpha},$$

So, for $(t,x) \in (0,1) \times [x_0,+\infty)$, we have

$$\begin{aligned} (3.12)\quad & \sigma F_{\varepsilon,\lambda}(t,x) - f_{\varepsilon,\lambda}(t,x)x \\ &= \lambda\bigg(\sigma \int_{\varepsilon}^{x} f(t,y)\,dy - f(t,x)x\bigg) + \frac{\sigma - 1 + \alpha}{1-\alpha}\, x^{1-\alpha} - \frac{\sigma\varepsilon^{1-\alpha}}{1-\alpha} < 0. \end{aligned}$$

Then

$$F_{\varepsilon,\lambda}(t,x) \geq \frac{F_{\varepsilon,\lambda}(t,x_0)}{x_0^{\sigma}}\, x^{\sigma}, \quad (t,x) \in (0,1) \times [x_0,+\infty). \tag{3.13}$$

Since $f_{\varepsilon,\lambda} > 0$ and $x_0 > \varepsilon$, we find that $F_{\varepsilon,\lambda}(t,x_0) > 0$ and

$$-F_{\varepsilon,\lambda}(t,x) \leq \begin{cases} \varepsilon(\varphi_{\varepsilon}^{-\alpha}(t) + \lambda C_{\varepsilon}), & (t,x) \in (0,1) \times [0,\varepsilon), \\ 0, & (t,x) \in (0,1) \times [\varepsilon,x_0), \end{cases}$$

by (1.4), then

$$-\int_{r\sin(\pi t) < x_0} F_{\varepsilon,\lambda}(t, r\sin(\pi t))\,dt \leq C.$$

So, it follows from the above inequality, (2.2), (3.6) and (3.13) that

$$\begin{aligned} \Phi(r\sin(\pi t)) &\leq \frac{1}{2}\pi^2 \int_0^1 (\sin(\pi t))^2\,dt r^2 \\ &\quad - \int_{r\sin(\pi t) < x_0} F_{\varepsilon,\lambda}(t, r\sin(\pi t))\,dt \\ &\quad - \int_{x_0 \leq r\sin(\pi t)} F_{\varepsilon,\lambda}(t, r\sin(\pi t))\,dt + C \\ &\leq Cr^2 - \int_{x_0 \leq r\sin(\pi t)} \frac{F_{\varepsilon,\lambda}(t,x_0)}{x_0^{\sigma}} (\sin(\pi t))^{\sigma}\,dt r^{\sigma} + C, \end{aligned}$$

and hence $\Phi(r\sin(\pi t)) \to -\infty$ as $r \to +\infty$.

We proceed similarly as in the proof of Lemma 2.11. Suppose that $\{u_n\}$ is a sequence in $H_0^1(0,1)$ such that (2.20) holds, then $\|u_n^-\|$ is bounded and (2.21) holds. By (1.6), for each $i = 1,\dots,p$ there exists $x_1 > \varepsilon$ such that

$$I_i(x)x - \sigma \int_{\varepsilon}^{x} I_i(y)\,dy \leq C, \quad \text{for any } x \geq x_1,$$

which combined with the continuity of $I_{i,\varepsilon}$ and (2.21) yields to

$$I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - \sigma \int_{\varepsilon}^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx \leq C, \quad \text{for any } n. \tag{3.14}$$

In view of $f_{\varepsilon,\lambda} > 0$, $f \in C([0,1] \times [0,+\infty))$, (1.4) and (3.12), we have

$$\sigma F_{\varepsilon,\lambda}(t,x) - f_{\varepsilon,\lambda}(t,x)x \\ \leq \begin{cases} (\varphi_\varepsilon^{-\alpha}(t) + \lambda C_\varepsilon)|x|, & (t,x) \in (0,1) \times (-\infty,0], \\ 0, & (t,x) \in (0,1) \times (0,\varepsilon), \\ \sigma \displaystyle\int_\varepsilon^x \lambda f(t,y) + y^{-\alpha}\,dy \leq C, & (t,x) \in (0,1) \times [\varepsilon,x_0], \\ 0, & (t,x) \in (0,1) \times [x_0,+\infty), \end{cases}$$

which combined with $\|u_n^-\|$ is bounded yields to

$$\int_0^1 \sigma F_{\varepsilon,\lambda}(t,u_n(t)) - f_{\varepsilon,\lambda}(t,u_n(t))u_n(t)\,dt \leq C, \quad \text{for any } n.$$

In view of the above inequality, (2.20) and (3.14) yields to

$$\begin{aligned}\left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 &= \sigma\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle \\ &\quad + \int_0^1 \sigma F_{\varepsilon,\lambda}(t,u_n(t)) - f_{\varepsilon,\lambda}(t,u_n(t))u_n(t)\,dt \\ &\quad + \sum_{i=1}^p \left(I_{i,\varepsilon}(u_n(t_i))u_n(t_i) - \sigma \int_\varepsilon^{u_n(t_i)} I_{i,\varepsilon}(x)\,dx\right) \leq C.\end{aligned}$$

So $\{u_n\}$ is bounded in $H_0^1(0,1)$ and hence Φ satisfies (C). □

Proof of Theorem 1.1. In view of Lemma 2.9 and Lemma 2.10, the results follow from Theorem 3.1–3.7. □

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