

BLOWUP VERSUS GLOBAL IN TIME EXISTENCE OF SOLUTIONS FOR NONLINEAR HEAT EQUATIONS

PIOTR BILER

In memory of Marek Burnat

ABSTRACT. This note is devoted to a simple proof of blowup of solutions for a nonlinear heat equation. The criterion for a blowup is expressed in terms of a Morrey space norm and is in a sense complementary to conditions guaranteeing the global in time existence of solutions. The method goes back to H. Fujita and extends to other nonlinear parabolic equations.

1. Introduction

In this paper we consider the Cauchy problem for the simplest example of a semilinear parabolic equation in \mathbb{R}^d , $d \geq 1$, $p > 1$,

$$(1.1) \quad u_t = \Delta u + |u|^{p-1}u, \quad x \in \mathbb{R}^d, \quad t > 0,$$

$$(1.2) \quad u(x, 0) = u_0(x).$$

This problem has been thoroughly studied beginning with [18], [20], [21], and many fine properties of its solutions are known. For the reference, see the extensive monograph [32] and a recent paper [33].

2010 *Mathematics Subject Classification.* 35B44, 35K55.

Key words and phrases. Nonlinear heat equation; blowup of solutions; global existence of solutions.

The author, partially supported by the NCN grants 2013/09/B/ST1/04412, 2016/23/B/ST1/00434, thanks Ignacio Guerra for interesting conversations leading to revisiting Fujita's method, Philippe Souplet, Mikołaj Sierżęga and the referees for many pertinent remarks.

The purpose of this note is to give a short proof of nonexistence of global in time (and sometimes also local in time) positive solutions to problem (1.1)–(1.2) together with criteria for blowup expressed in terms of the norms of critical Morrey spaces. Those criteria are, in a sense, complementary to assumptions guaranteeing the global in time existence of solutions. The idea of the proof goes back to the seminal paper [18] but the straightforward connection to the Morrey spaces norms seems to be missed out up to now. An idea in [13, proof of Theorem 3] in the context of radial solutions of the Keller–Segel model of chemotaxis is also reminiscent of that. Recently, this approach has been systematically developed for (radially symmetric solutions of) the chemotaxis models with different diffusion operators in [12] as an alternative to other proofs based on considerations of (new) moments in [5], [6], [8]–[10]. Here, for the nonlinear heat equation generally we do not use geometric assumptions on solutions such as radial symmetry. This method classically introduced by [18] is flexible enough, and extends also to other nonlinear parabolic problems, see remarks at the end of this paper.

Whenever problem (1.1)–(1.2) admits a singular stationary solution, this plays an important role in determining when solutions with u_0 featuring singularities either lead to global in time solutions or they blow up in a finite time. The form of this solution is well known, we recall this below.

THEOREM 1.1 (Singular stationary solutions). *For $d \geq 3$ and $p > d/(d-2)$ the function*

$$u_C(x) = c|x|^{-2/(p-1)}$$

with the constant

$$c = \left(\frac{2}{p-1} \left(d - \frac{2p}{p-1} \right) \right)^{1/(p-1)}$$

is a stationary positive (weak and pointwise) solution of equation (1.1).

PROOF. The exponent $\gamma = 2/(p-1)$ is uniquely determined by the requirement $\gamma + 2 = p\gamma$, and the constant c is determined by the relation

$$(1.3) \quad c^{p-1} = \gamma(d - p\gamma) = \gamma(d - 2 - \gamma).$$

Since $p\gamma < d$, this is a distributional solution of equation (1.1). \square

2. Global in time solutions

A typical example of a global in time existence result is the following.

THEOREM 2.1. *Suppose that $d \geq 3$, $p > d/(d-2)$, and u_0 satisfies the estimate*

$$0 \leq u_0(x) < \delta u_C(x) = \delta c|x|^{-2/(p-1)}$$

for some $\delta < 1$. Then any solution $u \in C^2(\mathbb{R}^d \times [0, T])$ of problem (1.1)–(1.2) with the property $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)}u(x, t) = 0$ satisfied uniformly on $[0, T]$, exists globally in time and is bounded by u_C

$$0 \leq u(x, t) < \delta u_C(x) \quad \text{for each } t > 0 \text{ and } x \in \mathbb{R}^d.$$

PROOF. This result can be considered as a kind of a comparison principle for equation (1.1) when a subcritical solution u , $0 \leq u(x, t) < \delta u_C(x)$ (continuous off the origin with a proper decay at infinity) is compared with δu_C , and then this can be continued onto some interval $[T, T + h]$, and further, step by step, onto the whole half-line $[0, \infty)$. For analogous results in chemotaxis theory with either Brownian or fractional diffusion, see [10], [7].

We sketch the proof skipping some technical details related to the local existence of solutions with the assumed regularity and decay. Suppose *a contrario* that $u(x_0, t_0) = \delta u_C(x_0)$ for some $x_0 \in \mathbb{R}^d$ with minimal $|x_0|$ and the least $t_0 > 0$. Consider the auxiliary function

$$(2.1) \quad z(x, t) = |x|^{2/(p-1)}u(x, t).$$

Under the *a contrario* assumption $z(x_0, t_0) = \delta c$, and if t_0 is the first moment when $z(x, t)$ hits the level $\delta c \in \mathbb{R}$, we have

$$(2.2) \quad \nabla z(x_0, t_0) = 0, \quad \Delta z(x_0, t_0) \leq 0,$$

since x_0 is the point where the maximum of z (equal to δc) is attained. Let us compute

$$(2.3) \quad \begin{aligned} \left. \frac{\partial}{\partial t} z(x_0, t) \right|_{t=t_0} &= |x_0|^{2/(p-1)}(\Delta u + u^p) \\ &= |x_0|^{2/(p-1)}(\Delta(|x|^{-2/(p-1)}z(x, t_0))\Big|_{x=x_0} + |x_0|^{-2/(p-1)}z(x_0, t_0)^p) \\ &= (\Delta z(x_0, t_0) - c^{p-1}z(x_0, t_0) + z(x_0, t_0)^p) \\ &\leq z(x_0, t_0)(z(x_0, t_0)^{p-1} - c^{p-1}) < 0 \end{aligned}$$

since

$$\begin{aligned} \nabla(|x|^{-\gamma}z) &= |x|^{-\gamma}\nabla z - \gamma|x|^{-\gamma-2}x z, \\ \Delta(|x|^{-\gamma}z) &= |x|^{-\gamma}\Delta z - 2\gamma|x|^{-\gamma-2}x \cdot \nabla z + \gamma(\gamma + 2 - d)z, \end{aligned}$$

relations (2.2) hold, and recall (1.3) to see that $\gamma(\gamma + 2 - d) = -c^{p-1}$. The inequality

$$\left. \frac{\partial}{\partial t} z(x_0, t) \right|_{t=t_0} < 0$$

contradicts the assumption that z hits for the first time the *constant* level δc at $t = t_0$. □

REMARK 2.2. This kind of result is not, of course, new (cf. [23, Theorem A]) but the proof seems to be somewhat novel. Similar pointwise arguments are powerful tools and, as such, they have been used in different contexts as e.g. fluid dynamics and chemotaxis theory: [14], [16], [17], [22], and free boundary problems: [15]. If u_0 is radially symmetric and $u_0(x) < \delta u_C(x)$ for some $\delta < 1$, then the solution of (1.1)–(1.2) exists globally in time, see [29, Theorem 1.1] and also [33, Remark 3.1 (iv)]. Related results are in [27, Lemma 2.2], and stability of the singular solution is studied in [30]. Results for not necessarily radial solutions starting either below or slightly above the singular solution u_C are in [19, Theorem 10.4] (reported in [32, Theorem 20.5]), and in [34, Theorem 1.1]. Note that solutions of the Cauchy problem (1.1)–(1.2) in the latter case are nonunique.

3. Solutions exploding in a finite time

Our goal here is to show a finite time blowup (Theorem 3.1), and that the critical size of some functional norm (Remarks 3.4 and 3.5 on Morrey spaces) of initial data leading to a blowup is asymptotically close to the optimal size of this norm guaranteeing the existence of a global in time regular positive solution (Theorem 3.9). The first argument is essentially that of [18]. The considerations in [20, 21] employed some (quite complicated) moments and energies of solutions with Gaussian weights but not exactly the quantity (3.5). Finally, Remarks 3.6, 3.10 and 3.11 comment on further properties of solutions of problem (1.1)–(1.2) related to Morrey spaces.

THEOREM 3.1. *Suppose that $u_0 \geq 0$ satisfies the condition*

$$(3.1) \quad \sup_{T>0} T^{1/(p-1)} \|e^{T\Delta} u_0\|_{\infty} > \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

Then, any local in time weak solution $u = u(x, t)$ of the Cauchy problem (1.1)–(1.2) with $p > 1$ cannot be continued beyond $t = T$.

PROOF. Note that here we assume merely $p > 1$. For a fixed $T > 0$ consider the weight function $G = G(x, t)$ which solves the backward heat equation with the unit Dirac measure as the final time condition at $t = T$

$$(3.2) \quad G_t + \Delta G = 0, \quad G(\cdot, T) = \delta_0.$$

Clearly, we have a solution

$$(3.3) \quad G(x, t) = (4\pi(T-t))^{-d/2} \exp\left(-\frac{|x|^2}{4(T-t)}\right),$$

the unique nonnegative one, satisfying moreover

$$(3.4) \quad \int_{\mathbb{R}^d} G(x, t) dx = 1 \quad \text{for each } t \in [0, T].$$

We consider $t \in [0, T)$, and define for a solution u of (1.1)–(1.2) which is supposed to exist on $[0, T)$ the moment

$$(3.5) \quad W(t) = \int_{\mathbb{R}^d} G(x, t)u(x, t) dx = e^{(T-t)\Delta}u(t)(0),$$

where $e^{t\Delta}$, $t > 0$, denotes the heat semigroup on \mathbb{R}^d defined with the Gauss–Weierstrass kernel. Evidently, we have

$$(3.6) \quad W(0) \in [0, \infty) \quad \text{for } u_0 \geq 0 \quad \text{and} \quad W(0) \in (0, \infty) \quad \text{if } 0 \leq u_0 \not\equiv 0$$

and, moreover,

$$(3.7) \quad W(0) = e^{T\Delta}u_0(0).$$

Since G decays exponentially in x as $|x| \rightarrow \infty$, the moment W is well defined (at least) for (weak, pointwise, distributional) solutions $u = u(x, t)$ which are polynomially bounded in x as $|x| \rightarrow \infty$. The evolution of the moment W is governed by the identity

$$(3.8) \quad \begin{aligned} \frac{dW(t)}{dt} &= \int_{\mathbb{R}^d} (Gu_t + G_t u) dx \\ &= \int_{\mathbb{R}^d} G(\Delta u + u^p) dx - \int_{\mathbb{R}^d} \Delta G u dx \\ &= \int_{\mathbb{R}^d} \Delta G u dx + \int_{\mathbb{R}^d} Gu^p dx - \int_{\mathbb{R}^d} \Delta G u dx \\ &= \int_{\mathbb{R}^d} Gu^p dx \geq \left(\int_{\mathbb{R}^d} Gu dx \right)^p, \end{aligned}$$

where the third line means that u is a weak solution, and the last line follows by the Hölder inequality and property (3.4). Now, the differential inequality

$$(3.9) \quad \frac{dW(t)}{dt} \geq (W(t))^p$$

(with the strict inequality for nonconstant functions $u(t)$) shows that $W(t)$ increases, and after integrating immediately leads to

$$W(0)^{1-p} - W(t)^{1-p} \geq (p-1)t$$

for all $t \in [0, T)$, so that

$$(3.10) \quad W(t) \geq (W(0)^{1-p} - (p-1)t)^{-1/(p-1)}$$

again for all $t \in [0, T)$. Now, passing to the limit $t \nearrow T$, it is clear that if

$$(3.11) \quad W(0) = e^{T\Delta}u_0(0) > ((p-1)T)^{-1/(p-1)},$$

then we arrive at a contradiction with property (3.6), so that the solution blows up not later than $t = T$. In other words, if the condition

$$(3.12) \quad T^{1/(p-1)} e^{T\Delta}u_0(0) > \left(\frac{1}{p-1} \right)^{1/(p-1)}$$

is satisfied, then the solution u cannot exist for $t = T$. By the translational invariance of equation (1.1), for positive u_0 condition (3.12) is equivalent to inequality (3.1). Note that a different approach to that is in [38, Theorem 1]. \square

REMARK 3.2. Observe that for any positive initial condition $u_0 \not\equiv 0$ there is $N > 0$ such that condition (3.11) is satisfied for Nu_0 , cf. [38, Corollary 1.1]. Condition (3.11) is also valid for each constant $u_0 > 0$ and suitably large T .

REMARK 3.3. It is clear that if $d < 2/(p-1)$, then each positive $u_0 \not\equiv 0$ leads to a blowing up solution, as it has been proved in [18, Theorem 1]. Indeed,

$$\begin{aligned} \sup_{T>0} T^{1/(p-1)} \|e^{T\Delta} u_0\|_\infty &= \lim_{T \rightarrow \infty} T^{1/(p-1)} \|e^{T\Delta} u_0\|_\infty \\ &= \lim_{T \rightarrow \infty} T^{-d/2+1/(p-1)} (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-|x|^2/(4T)} u_0(x) dx \\ &= \lim_{T \rightarrow \infty} T^{-d/2+1/(p-1)} (4\pi)^{-d/2} \|u_0\|_1 = \infty, \end{aligned}$$

so a sufficient condition (3.1) for blowup holds. Here, we offer a proof of the analogous result if $d = 2/(p-1)$, an alternative to the one given in [37, Theorem 1]. Since condition (3.1) for $d/2 = 1/(p-1)$ reads

$$(3.13) \quad (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-|y|^2/(4T)} u_0(y) dy > \left(\frac{1}{p-1} \right)^{1/(p-1)},$$

and $\lim_{T \rightarrow \infty} W(0) = (4\pi)^{-d/2} \|u_0\|_1$, it suffices to show that $\|u(t)\|_1$ becomes large enough for some $t \geq 0$, and replace the initial condition u_0 by $u(t)$.

Problem (1.1)–(1.2) for positive solutions can be rewritten in the mild form as

$$(3.14) \quad u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} u(\tau)^p d\tau.$$

Let us write the Duhamel formula (3.14) for $t = T$ in a more detailed way as

$$\begin{aligned} (3.15) \quad u(x, T) - e^{T\Delta} u_0(x) &= \int_0^T \int_{\mathbb{R}^d} (4\pi(T-\tau))^{-d/2} e^{-|x-y|^2/4(T-\tau)} u(y, \tau)^p dy d\tau \\ &\geq \int_0^T \left(\int_{\mathbb{R}^d} (4\pi(T-\tau))^{-d/2} \right. \\ &\quad \cdot \left. \frac{1}{2} (e^{-|x-y|^2/4(T-\tau)} + e^{-|x+y|^2/4(T-\tau)}) u(y, \tau) dy \right)^p d\tau \\ &\geq \int_0^T \left(\int_{\mathbb{R}^d} (4\pi(T-\tau))^{-d/2} e^{-|x|^2/4(T-\tau)} e^{-|y|^2/4(T-\tau)} u(y, \tau) dy \right)^p d\tau \end{aligned}$$

by the Hölder inequality and by the Cauchy inequality

$$\frac{1}{2} (e^{x \cdot y/2(T-\tau)} + e^{-x \cdot y/2(T-\tau)}) \geq 1.$$

Now, integrating over $x \in \mathbb{R}^d$, we obtain as a consequence of (3.15)

$$\begin{aligned} \|u(T)\|_1 &\geq \|u_0\|_1 + \int_0^T \int_{\mathbb{R}^d} e^{-p|x|^2/4(T-\tau)} W(\tau)^p dx d\tau \\ &\geq \|u_0\|_1 + W(0)^p \int_0^T \left(\frac{4\pi}{p}\right)^{d/2} (T-\tau)^{d/2} d\tau \\ &\geq \|u_0\|_1 + W(0)^p T^{d/2+1} \frac{2}{d+2} \left(\frac{4\pi}{p}\right)^{d/2} \\ &= \|u_0\|_1 + \tilde{c}(T^{1/(p-1)}W(0))^p = \|u_0\|_1 + \tilde{c} \left(\int_{\mathbb{R}^d} e^{-|y|^2/(4T)} u_0(y) dy \right)^p \end{aligned}$$

for some \tilde{c} independent of u_0 . Observe that the norm $\|u(t)\|_1$ increases in time. Therefore, by a shift of time, we have

$$\|u(t+T)\|_1 \geq \|u(t)\|_1 (1 + \tilde{c}\|u_0\|^{p-1})$$

and it is clear that for some t the norm $\|u(t)\|_1$ becomes large enough in order to condition (3.1) holds with time shifted from 0 to t , see also (3.13). Therefore, $u(t)$ blows up in a finite time.

REMARK 3.4. Here we discuss some questions related to applicability of Morrey spaces in the analysis of optimal conditions for local in time existence of solutions as well as for the finite time blowup of solutions. Recall that (homogeneous) Morrey spaces over \mathbb{R}^d modeled on $L^q(\mathbb{R}^d)$, $q \geq 1$, are defined by their norms

$$(3.16) \quad \|u\|_{M_q^s} \equiv \left(\sup_{R>0, x \in \mathbb{R}^d} R^{d(q/s-1)} \int_{\{|y-x|<R\}} |u(y)|^q dy \right)^{1/q} < \infty.$$

Caution. The notation for Morrey spaces used elsewhere might be different, e.g. M_q^s is denoted by $M^{q,\lambda}$ with $\lambda = dq/s$ in [33].

The most frequent situation is when $q = 1$ and we consider $M_1^s \equiv M^s$. The spaces $M^{d(p-1)/2}(\mathbb{R}^d)$ and $M_q^{d(p-1)/2}(\mathbb{R}^d)$, $q > 1$, are critical (i.e. invariant by scalings that conserve equation (1.1)) in the study of equation (1.1), see [33], and we refer to [3], [9], [7], [25] for analogous examples in chemotaxis theory.

For instance, if the norm $\|u_0\|_{M_q^{d(p-1)/2}}$ (for a number $q > 1$) is small enough, then a solution of problem (1.1)–(1.2) is global in time, see [33, Proposition 6.1] (and for the chemotaxis system cf. [3, Theorem 1]). The former result can be proved directly (while the proof in [33] was by contradiction) using the Picard iterations of the mapping

$$\mathcal{N}(u)(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} (|u|^{p-1}u)(\tau) d\tau,$$

$u_{n+1} = \mathcal{N}(u_n)$, $n = 1, 2, \dots$, with $u_0 \in M_q^{d(p-1)/2}(\mathbb{R}^d)$ ($q > 1$, $p > 1 + 2q/d$) small enough. They are convergent in the norm

$$\|u(t)\| = \sup_{t>0} t^{(1-q/r)/(p-1)} |u(t)|_{M_r^{d(p-1)/(2q)}}$$

for $\max\{p, q\} < r < pq$, since $\|e^{t\Delta}u_0\| < \infty$. For useful estimates of the heat semigroup in Morrey spaces we refer the reader to [35, Theorem 3.8].

Clearly, assumption (2.1) in Theorem 2.1 implies

$$|u_0|_{M^{d(p-1)/2}} < \frac{c\sigma_d}{d - 2/(p-1)},$$

and the conclusion (2.1) reads

$$|u(t)|_{M^{d(p-1)/2}} < \frac{c\sigma_d}{d - 2/(p-1)}.$$

REMARK 3.5. Note that the quantity $\sup_{t>0} t^{1/(p-1)} \|e^{t\Delta}u_0\|_\infty$ in (3.1) is equivalent to the norm of the Besov space $B_{\infty,\infty}^{-2/(p-1)}(\mathbb{R}^d)$, e.g. [33, Remark 4.2]. Moreover, for positive functions and $p > 1 + 2/d$, the quantity (3.1) is equivalent to the Morrey space norm $M^{d(p-1)/2}(\mathbb{R}^d)$, see [25, Proposition 2 B)]. Therefore, a sufficient condition for blowup of solutions of problem (1.1)–(1.2) becomes

$$|u_0|_{M^{d(p-1)/2}} \gg 1.$$

REMARK 3.6. Related results on the local and global existence of solutions to (1.1)–(1.2), as well as the existence of initial traces (i.e. u_0) for nonnegative solutions ($u = u(t)$ defined on $(0, T)$) are in [1], [2]. In particular, the estimates [1, (1.4), Proposition 4.3, p. 380] show that a necessary condition for the existence of a local in time solution reads: $u_0 \in M^{d(p-1)/2}(\mathbb{R}^d)$, and each positive solution satisfies $u(t) \in M^{d(p-1)/2}(\mathbb{R}^d)$ uniformly on $(0, T)$. Since these papers have dealt with general parabolic problems, neither the optimal constants have been derived in a particular case of (1.1), nor L^∞ -blowup has been shown for sufficiently large norm $u_0 \in M^{d(p-1)/2}(\mathbb{R}^d)$.

REMARK 3.7. Note that at the blowup time T we have $\lim_{t \nearrow T} \|u(t)\|_\infty = \infty$ but some other norms — in particular $|u(T)|_{M^{d(p-1)/2}}$ — can remain bounded, cf. also [33, Remark 6.1 (iv)].

Taking into account the above remarks on the critical Morrey space, we formulate the following partial dichotomy result

COROLLARY 3.8. *There exist two positive constants c_d and C_d such that, if $p > 1 + 2/d$ and for some $q > 1$, then*

- (a) $|u_0|_{M_q^{d(p-1)/2}} < c_d$ implies that problem (1.1)–(1.2) has a global in time solution;

(b) $|u_0|_{M^{d(p-1)/2}} > C_d$ implies that each solution of problem (1.1)–(1.2) blows up in a finite time.

Below, we determine two threshold numbers measuring how big must be either u_0 compared to u_C or the Morrey norm $M^{d(p-1)/2}(\mathbb{R}^d)$ of a radial u_0 in order to a solution of (1.1)–(1.2) blows up in a finite time.

THEOREM 3.9. (a) Let $d \geq 3$, $p > d/(d-2)$ (so that $d-\gamma > 0$), and define the threshold number

$$\mathcal{N} = \inf\{N : \text{solution with the initial datum satisfying } u_0(x) \geq Nu_C(x) \text{ blows up in a finite time}\}.$$

Then, the relation $\mathcal{N} \rightarrow 1$ as $d \rightarrow \infty$ holds. More precisely,

$$(3.17) \quad 1 < \mathcal{N} \leq 2^{1/(p-1)} \left(d - \frac{2p}{p-1}\right)^{-1/(p-1)} \frac{\Gamma(d/2)}{\Gamma(d/2 - 1/(p-1))}$$

$$(3.18) \quad \approx \left(1 - \frac{2p}{d(p-1)}\right)^{-1/(p-1)}$$

as $d \rightarrow \infty$.

(b) Moreover, there exists $\mathcal{M} > 0$ such that if $u_0 \geq 0$ is radially symmetric and such that $|u_0|_{M^{d(p-1)/2}} > \mathcal{M}$, then the solution with u_0 as the initial data blows up in a finite time. A rough estimate from above for the threshold value of \mathcal{M} when $p > (d+2)/d$ (i.e. exactly when $d-\gamma > 0$) is the following

$$(3.19) \quad \mathcal{M} \leq \left(\frac{1}{p-1}\right)^{1/(p-1)} (4\pi)^{d/2} e^{(d-\gamma)/2} (2(d-\gamma))^{\gamma-d/2} \\ \approx \left(\frac{1}{p-1}\right)^{1/(p-1)} \sigma_d \sqrt{\frac{\pi}{2}} \left(\frac{d}{2} - \frac{1}{p-1}\right)^{1/(p-1)+1/2}$$

as $d \rightarrow \infty$.

PROOF. (a) We compute

$$(3.20) \quad t^{1/(p-1)} (4\pi t)^{-d/2} \int |x|^{-\gamma} \exp\left(-\frac{|x|^2}{4t}\right) dx \\ = 2^{-\gamma-1} \pi^{-d/2} \sigma_d \int_0^\infty e^{-z} z^{(d-\gamma)/2-1} dz = 2^{-\gamma} \frac{\Gamma((d-\gamma)/2)}{\Gamma(d/2)},$$

where we used the fact that the area σ_d of the unit sphere in \mathbb{R}^d is equal to

$$(3.21) \quad \sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)},$$

and we obtain (3.17). The asymptotic formula in (3.18) is a consequence of the Stirling formula for the Gamma function

$$(3.22) \quad \Gamma(z+1) \approx \sqrt{2\pi z} z^z e^{-z} \quad \text{and} \quad \frac{\Gamma(z+a)}{\Gamma(z)} \approx z^a \quad \text{as } z \rightarrow \infty,$$

see e.g. [36]. Indeed,

$$(3.23) \quad \mathcal{N} \approx 2^{1/(p-1)} \left(d - \frac{2p}{p-1} \right)^{-1/(p-1)} \frac{\Gamma(d/2)}{\Gamma(d/2 - 1/(p-1))} \\ \approx 2^{1/(p-1)} \left(d - \frac{2p}{p-1} \right)^{-1/(p-1)} \left(\frac{d}{2} \right)^{1/(p-1)}$$

holds, i.e. the discrepancy between the multiples of $|x|^{-2/(p-1)}$ in the initial data corresponding to global and blowing up solutions converges to 1 as $d \rightarrow \infty$.

(b) For a radially symmetric function $u_0 \geq 0$ define its radial distribution function

$$M(r) = \sigma_d \int_0^r u_0(\varrho) \varrho^{d-1} d\varrho,$$

so that $\sigma_d u_0(r) r^{d-1} = M_r(r)$. It is not hard to see that for radial $u_0 \geq 0$ we have

$$\|u_0\|_{M^{d(p-1)/2}} = \sup_{R>0} R^{\gamma-d} \int_{\{|x|<R\}} u_0(x) dx.$$

If $u_0 \in M^{d(p-1)/2}(\mathbb{R}^d)$ with $\|u_0\|_{M^{d(p-1)/2}} = \mathcal{M}$, then for each $\varepsilon > 0$ there exists $R > 0$ such that $M(R) \geq (\mathcal{M} - \varepsilon) R^{d-\gamma}$. We need to estimate $T^{1/(p-1)} W(0) = T^{1/(p-1)} e^{T\Delta} u_0(0)$ from below, and we do it in a rough way as

$$(3.24) \quad T^{1/(p-1)} W(0) = T^{1/(p-1)} \int_0^\infty (4\pi T)^{-d/2} e^{-r^2/4T} M_r(r) dr \\ \geq T^{1/(p-1)-d/2} (4\pi)^{-d/2} \int_0^R e^{-r^2/4T} M_r(r) dr \\ = T^{1/(p-1)-d/2} (4\pi)^{-d/2} \left(e^{-R^2/4T} M(R) + \int_0^R e^{-r^2/4T} \frac{r}{2T} M(r) dr \right) \\ \geq T^{1/(p-1)-d/2} (4\pi)^{-d/2} (\mathcal{M} - \varepsilon) R^{d-\gamma} e^{-R^2/4T}.$$

Since $\max_{T>0} T^{1/(p-1)-d/2} e^{-R^2/4T}$ is attained when $R^2/(4T) = (d-\gamma)/2$ and this quantity is equal to $(R^2 e / (2(d-\gamma)))^{(\gamma-d)/2}$, we need to determine when

$$(4\pi)^{-d/2} (\mathcal{M} - \varepsilon) \left(\frac{2(d-\gamma)}{e} \right)^{(d-\gamma)/2} > \left(\frac{1}{p-1} \right)^{1/(p-1)}.$$

This leads, by (3.21) and (3.22), to

$$\mathcal{M} \gtrsim 2^\gamma \left(\frac{1}{p-1} \right)^{1/(p-1)} \frac{\sigma_d}{2} \frac{\Gamma(d/2)}{\Gamma((d-\gamma)/2 + 1)} \sqrt{2\pi} \sqrt{\frac{d-\gamma}{2}} \\ \approx 2^\gamma \left(\frac{1}{p-1} \right)^{1/(p-1)} \sigma_d \left(\frac{d-\gamma}{2} \right)^{(\gamma-1)/2} \sqrt{\frac{\pi}{2}}.$$

Note that, compared to $|u_C|_{M^{d(p-1)/2}} = (c\sigma_d)/(d - \gamma)$, the number \mathcal{M} is much bigger:

$$\frac{\mathcal{M}}{|u_C|_{M^{d(p-1)/2}}} \approx \frac{((d - \gamma)/2)^{(\gamma+1)/2}}{((d - p\gamma)/2)^{\gamma/2}} \sqrt{2\pi} \approx \sqrt{\pi d},$$

and, of course, $\mathcal{M} \leq C_d$ by the definition of C_d in Corollary 3.8 (b). □

REMARK 3.10. It is well known, cf. [24] and [32, Theorem 17.12], that (under some supplementary assumptions) if

$$\lim_{|x| \rightarrow \infty} u_0(x) |x|^{2/(p-1)} = \infty,$$

then solutions of (1.1)–(1.2) cannot be global in time. On the other hand, the authors of [31] showed that for some $u_0(x) \sim |x|^{-2/(p-1)}$ global in time solutions of (1.1)–(1.2) exist and are unbounded as $t \rightarrow \infty$.

A very short proof of the former result follows from the observation that such an initial condition u_0 satisfies $|u_0|_{M^{d(p-1)/2}} = \infty$, hence condition (3.1) holds. Indeed, inequality

$$\int_{\{|x| \leq R\}} u_0(x) dx \geq C \int_{R_0}^R r^{-\gamma+d-1} dr \approx CR^{d-\gamma}$$

is satisfied with arbitrarily large constants C and suitably big fixed R_0 , so that

$$\lim_{R \rightarrow \infty} R^{\gamma-d} \int_{|x| \leq R} u_0(x) dx \geq C.$$

REMARK 3.11. If $u_0 \geq 0$ is such that

$$\limsup_{t \searrow 0} t^{1/(p-1)} \|e^{t\Delta} u_0\|_\infty > \left(\frac{1}{p-1}\right)^{1/(p-1)},$$

then the statement: *there exists $T > 0$ and a solution of problem (1.1)–(1.2) on $(0, T)$* is not true. One is tempted to say that there is an instantaneous blowup of solution with such an initial data, but the correct statement is rather: *there is no continuity property of solutions with respect to (large) initial data in the space $M^{d(p-1)/2}(\mathbb{R}^d)$ when, e.g. the initial data with large $M^{d(p-1)/2}(\mathbb{R}^d)$ norms are truncated on levels growing to infinity. In other words: for large initial data the problem (1.1)–(1.2) is ill-posed in $\mathcal{C}([0, T], M^{d(p-1)/2}(\mathbb{R}^d))$ for any $T > 0$.*

Extensions and generalizations. The method of proving blowup with the use of the Gaussian moment (3.5) extends to other nonlinear equations with the heat operator in the principal part. For instance, the equation

$$u_t = \Delta u + Q(x) |u|^{p-1} u$$

considered in [26] leads to the following sufficient for blowup of its positive solutions

$$W(0) \geq (4\pi)^{-d/2} \left(\int_0^T (p-1)(T-t)^{d(p-1)/2} \cdot \left(\int_{\mathbb{R}^d} Q(x)^{-1/(p-1)} e^{-|x|^2/4(T-t)} dx \right)^{1-p} dt \right)^{-1/(p-1)}.$$

These integrals can be calculated explicitly, e.g. for $Q(x) = |x|^\beta$, $\beta \in \mathbb{R}$.

Another example for which sufficient criteria for blowup can be easily established with this method is the quasilinear equation

$$u_t = \Delta u + \nabla \cdot (x|u|^{p-1}u),$$

and, of course, radially symmetric solutions of the Keller–Segel system in chemotaxis theory as it was mentioned before.

Added in proofs. Related results for a nonlinear heat equations with non-local diffusion operators are in recent papers [4] and [11].

REFERENCES

- [1] D. ANDREUCCI AND E. DiBENEDETTO, *On the Cauchy problem and initial traces for a class of evolution equations with strongly nonlinear sources*, Ann. Sci. Norm. Super. Pisa Cl. Sci. (4) **18** (1991), 363–441.
- [2] P. BARAS AND M. PIERRE, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré Anal. non Linéaire **2** (1985), 185–212.
- [3] P. BILER, *The Cauchy problem and self-similar solutions for a nonlinear parabolic equation*, Studia Math. **114** (1995), 181–205.
- [4] P. BILER, *Blowup of solutions for nonlinear nonlocal heat equations*, arXiv:1807.03569, 1–12. (submitted)
- [5] P. BILER, T. CIEŚLAK, G. KARCH AND J. ZIENKIEWICZ, *Local criteria for blowup of solutions in two-dimensional chemotaxis models*, Discrete Contin. Dyn. Syst. A **37** (2017), 1841–1856.
- [6] P. BILER AND G. KARCH, *Blowup of solutions to generalized Keller–Segel model*, J. Evol. Equ. **10** (2010), 247–262.
- [7] P. BILER, G. KARCH AND D. PILARCZYK, *Global radial solutions in classical Keller–Segel chemotaxis model*, arXiv:1807.02628. 1–20. (submitted)
- [8] P. BILER, G. KARCH AND J. ZIENKIEWICZ, *Optimal criteria for blowup of radial and N -symmetric solutions of chemotaxis systems*, Nonlinearity **28** (2015), 4369–4387.
- [9] P. BILER, G. KARCH AND J. ZIENKIEWICZ, *Morrey spaces norms and criteria for blowup in chemotaxis models*, Networks and Non Homogeneous Media **11** (2016), 239–250.
- [10] P. BILER, G. KARCH AND J. ZIENKIEWICZ, *Large global-in-time solutions to a nonlocal model of chemotaxis*, Adv. Math. **330** (2018), 834–875.
- [11] P. BILER AND D. PILARCZYK, *Around a singular solution to a nonlocal nonlinear heat equation*, arXiv:1807.03567, 1–20. (submitted)

- [12] P. BILER AND J. ZIENKIEWICZ, *Blowing up radial solutions in the minimal Keller–Segel chemotaxis model*, arXiv:1807.02633, 1–20. (submitted).
- [13] M.P. BRENNER, P. CONSTANTIN, L.P. KADANOFF, A. SCHENKEL AND S.C. VENKATARAMANI, *Diffusion, attraction and collapse*, *Nonlinearity* **12** (1999), 1071–1098.
- [14] J. BURCZAK AND R. GRANERO-BELINCHÓN, *Global solutions for a supercritical drift-diffusion equation*, *Adv. Math.* **295** (2016), 334–367.
- [15] P. CONSTANTIN, F. GANCEDO, R. SHVYDKOY AND V. VICOL, *Global regularity for 2D Muskat equations with finite slope*, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), 1041–1074.
- [16] A. CÓRDOBA AND D. CÓRDOBA, *A maximum principle applied to quasi-geostrophic equations*, *Commun. Math. Phys.* **249** (2004), 511–528.
- [17] P. CONSTANTIN AND V. VICOL, *Nonlinear maximum principles for dissipative linear non-local operators and applications*, *Geom. Funct. Anal.* **22** (2012), 1289–1321.
- [18] H. FUJITA, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$* , *J. Fac. Sci. Univ. Tokyo Sect. I* **13** (1966), 109–124.
- [19] V.A. GALAKTIONOV AND J.L. VÁZQUEZ, *Continuation of blowup solutions of nonlinear heat equations in several space dimensions*, *Comm. Pure Appl. Math.* **50** (1997), 1–67.
- [20] Y. GIGA AND R.V. KOHN, *Asymptotically self-similar blow-up of semilinear heat equations*, *Comm. Pure Appl. Math.* **38** (1985), 297–319.
- [21] Y. GIGA AND R.V. KOHN, *Nondegeneracy of blowup for semilinear heat equations*, *Comm. Pure Appl. Math.* **42** (1989), 845–884.
- [22] R. GRANERO-BELINCHÓN AND R. ORIVE-ILLERA, *An aggregation equation with a nonlocal flux*, *Nonlinear Anal.* **108** (2014), 260–274.
- [23] C. GUI, W.-M. NI AND X. WANG, *On the stability and instability of positive steady states of a semilinear heat equation in \mathbb{R}^n* , *Comm. Pure Appl. Math.* **45** (1992), 1153–1181.
- [24] T.-Y. LEE AND W.-M. NI, *Global existence, large time behavior and life span of solutions of a semilinear parabolic Cauchy problem*, *Trans. Amer. Math. Soc.* **333** (1992), 365–378.
- [25] P.-G. LEMARIÉ-RIEUSSET, *Small data in an optimal Banach space for the parabolic-parabolic and parabolic-elliptic Keller–Segel equations in the whole space*, *Adv. Differential Equations* **18** (2013), 1189–1208.
- [26] X. LI AND Z. XIANG, *Existence and nonexistence of local/global solutions for a nonhomogeneous heat equation*, *Comm. Pure Appl. Analysis* **13** (2014), 1465–1480.
- [27] N. MIZOGUCHI, *On the behavior of solutions for a semilinear parabolic equation with supercritical nonlinearity*, *Math. Z.* **239** (2002), 215–229.
- [28] N. MIZOGUCHI, *Blowup behavior of solutions for a semilinear heat equation with supercritical nonlinearity*, *J. Differential Equations* **205** (2004), 298–328.
- [29] N. MIZOGUCHI, *Boundedness of global solutions for a supercritical semilinear heat equation and its application*, *Indiana Univ. Math. J.* **54** (2005), 1047–1059.
- [30] P. POLÁČIK AND E. YANAGIDA, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, *Math. Ann.* **327** (2003), 745–771.
- [31] P. POLÁČIK AND E. YANAGIDA, *Global unbounded solutions of the Fujita equation in the intermediate range*, *Math. Ann.* **360** (2014), 255–266.
- [32] P. QUITTNER AND PH. SOUPLLET, *Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States*, Birkhäuser Advanced Texts, 2007.
- [33] PH. SOUPLLET, *Morrey spaces and classification of global solutions for a supercritical semilinear heat equation in \mathbb{R}^n* , *J. Funct. Anal.* **272** (2017), 2005–2037.

- [34] PH. SOUPLLET AND F.B. WEISSLER, *Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 213–235.
- [35] M.E. TAYLOR, *Analysis on Morrey spaces and applications to Navier–Stokes and other evolution equations*, Comm. Partial Differential Equations **17** (1992), 1407–1456.
- [36] F.G. TRICOMI AND A. ERDÉLYI, *The asymptotic expansion of a ratio of Gamma functions*, Pacific J. Math. **1** (1951), 133–142.
- [37] F.B. WEISSLER, *Existence and non-existence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.
- [38] F.B. WEISSLER, *L^p -energy and blow-up for a semilinear heat equation*, Proc. Symp. Pure Math. **45** (1986), no. 2, 545–551.

Manuscript received May 16, 2017

accepted August 8, 2017

PIOTR BILER
Instytut Matematyczny
Uniwersytet Wrocławski
pl. Grunwaldzki 2/4
50-384 Wrocław, POLAND

E-mail address: Piotr.Biler@math.uni.wroc.pl