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RAYLEIGH–BÉNARD PROBLEM FOR THERMOMICROPOLAR FLUIDS

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Abstract. The two-dimensional Rayleigh–Bénard problem for a thermomicropolar fluids model is considered. The existence of suitable weak solutions which may not be unique, and the existence of the unique strong solution are proved. The global attractor for the m-semiflow associated with weak solutions and the global attractor for semiflow associated with strong solutions are shown to be equal.

1. Introduction

The theory of micropolar fluids is a generalization of the Navier–Stokes model in the sense that it takes into account the microstructure of the fluid. The theory is expected to provide a more realistic mathematical model for the non-Newtonian fluid behaviour observed in certain fluids such as polymers suspensions where polymer chains exhibit a complicated evolution. Colloidal fluids, liquid crystals, polymer suspensions in blood, ferro liquid or nanofluids are examples of applications where the micropolar fluid theory is used. It was introduced by Eringen in [14] and its mathematical analysis is presented in [22].

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The theory of thermomicropolar fluids, proposed by Eringen in [15], extends the theory of micropolar fluid by including the heat conduction and heat dissipation effects.

In particular, the system of equations of the Rayleigh–Bénard problem between two horizontal planes for an incompressible, isotropic thermomicropolar fluid, after a reasonable modification of the complete system of the field equations, reads [16], [25] as

$$
\begin{aligned}
u_t + (u \cdot \nabla) u - (\nu + \nu_r)\Delta u + \frac{1}{\rho}\nabla p &= 2\nu_r \operatorname{rot} \omega + g\overline{\alpha} T e_3, \\
\operatorname{div} u &= 0, \\
j(\omega_t + (u \cdot \nabla)\omega) - \alpha\Delta\omega - \beta\nabla \operatorname{div}\omega + 4\nu_r\omega &= 2\nu_r \operatorname{rot} u, \\
T_t + u \cdot \nabla T - \kappa\Delta T &= \delta \operatorname{rot}\omega \cdot \nabla T.
\end{aligned} \tag{1.1}
$$

In the above system $e_3 = (0, 0, 1)$, ν is the usual kinematic Newtonian viscosity, ν_r is the kinematic microrotation viscosity, $\overline{\alpha}$ is the thermal expansion, g is the gravitational acceleration, j is the moment of inertia, α and β are micropolar material viscosities, κ is the thermal conductivity, and δ is the micropolar thermal conduction. We assume that ν, ν_r, $\overline{\alpha}$, j, α, β, κ and δ are positive constants [15]. The unknowns in the above equations are the velocity vector field u, the pressure p, the microrotation vector field ω and the temperature T.

In this paper, we consider the two-dimensional version of the problem (1.1). For the initial data in L^2 spaces, we prove the existence of weak solutions which may not be unique, see Theorem 5.1. If the initial data belongs to H^1, the existence of a strong solution and weak-strong uniqueness are shown in Theorem 7.2.

We are not able to obtain the weak solution due to the term $\operatorname{rot}\omega \cdot \nabla T$ in the heat equation in the system (1.1). This term makes that the time derivative of temperature belongs only to $L^2(0, \tau; D(A^{3/2})^*)$, where A is the $-\Delta$ operator with the boundary conditions defined below, which is insufficient for the uniqueness. We prove, however, that each weak solution of the system becomes smooth instantaneously.

This result is obtained by the bootstrap argument applied to the first three equations of (1.1) whence we deduce that u and ω regularize. Once ω is more smooth, we can deal with the "bad" term $\operatorname{rot}\omega \cdot \nabla T$ in the last equation of (1.1). Thus, we prove that every weak solution becomes strong instantaneously and, due to the weak-strong uniqueness result, no weak solutions originate from the point on the strong trajectory. Thus, the nonuniqueness of weak solutions can possibly occur only at the initial time point if the initial data is in L^2 and not in H^1. Next, in Sections 8 and 9, we deal with the asymptotic behavior of both weak and strong solutions. To this end, we need a subclass of weak solutions obtained by the Galerkin method which we show to be nonempty. Weak solutions in this class still can be nonunique. The reason is that we need the maximum principle type estimate for temperature, up to the initial time point, for which the regularity $T_t \in L^2(0,\tau;D(A^{3/2})^*)$ is insufficient. This estimate, however holds if our weak solution is the weak limit of the Galerkin problems.

Using the theory of multivalued semiflows we show the existence of the global attractor for weak solutions. We are not assuming anything about continuity of the graph, nor closedness of the multivalued map and hence we do not obtain any invariance of the multivalued attractor.

Next, using the classical theory of dissipative semiflows we show the existence of the global attractor for strong solutions. From the bootstrap arguments and the weak-strong uniqueness property it follows that both attractors coincide, and hence the multivalued attractor turns out to be invariant. This regularization effect turns out to be similar to that in the surface quasi-geostrophic (SQG) equation, where, however, one needs to use de Giorgi iterations and the method of nonlinear lower bounds to obtain the bootstrapping effect [4], [6], [7], [11], [12]. Finally, Section 10 is devoted to the relations between the energy dissipation rate and the Nusselt number.

2. Formulation of the problem

Let us consider a two-dimensional problem in the domain $(-\infty,\infty)\times(0,h)$, and assume that the temperature at the bottom part of the boundary is $T_1 + \triangle T$ with $\triangle T > 0$, and at the top is T_1, with T_1 and $\triangle T$ constants. The introduction of two-dimensional problem is done in [23, p. 488] or [29, p. 1216–1217]. Introducing the dimensionless variables

$$\begin{aligned} &x' = \frac{x}{h}, \qquad t' = \frac{\kappa}{h^2}\,t,\\ &u' = \frac{h}{\kappa}\,u, \qquad p' = \frac{h^2}{\rho\kappa^2}\,p, \qquad \omega' = \frac{h^2}{\kappa}\,\omega, \qquad T' = \frac{T-T_1}{\triangle T}, \end{aligned}$$

and dropping the primes we obtain the dimensionless form of the two-dimensional Rayleigh–Bénard problem

$$\begin{gathered} \frac{1}{\mathrm{Pr}}\,(u_t + (u\cdot\nabla)u + \nabla p) = \Delta u + \frac{N}{1-N}\,(2\operatorname{rot}\omega + \Delta u) + \mathrm{Ra}\,Te_2,\\ \operatorname{div} u = 0,\\ \frac{M}{\mathrm{Pr}}\,(\omega_t + u\cdot\nabla\omega) = L\Delta\omega + 2\,\frac{N}{1-N}\,(\operatorname{rot}\, u - 2\omega),\\ T_t + u\cdot\nabla T = \Delta T + D\operatorname{rot}\omega\cdot\nabla T, \end{gathered} \tag{2.1}$$

where $u=(u_1,u_2)$ is the velocity field, p is the pressure, ω is the microrotation, T is the temperature, and e_2 is the unit upward vector $(0,1)\in\mathbb{R}^2$.

In the above system there appear several dimensionless constants.

$N = \dfrac{\nu_r}{\nu + \nu_r}$, coupling parameter — relation between the Newtonian and microrotation viscosities.

$M = \dfrac{j}{h^2}$, relation between the moment of inertia and geometry.

$L = \dfrac{\alpha}{h^2 \nu}$, couple stress parameter — relation between geometry and properties of the fluid.

$D = \dfrac{\delta}{h^2}$, micropolar heat conduction parameter — relation between micropolar thermal conduction and geometry.

$\mathrm{Pr} = \dfrac{\nu}{\kappa}$, Prandtl number — relation between kinematic viscosity and thermal diffusivity.

$\mathrm{Ra} = \dfrac{\overline{\alpha} g \triangle T h^3}{\nu\kappa}$, Rayleigh number — relation between the buoyancy force and damping coefficients.

To avoid confusion we denote

$$\operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad \operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \qquad \operatorname{rot} \omega = \left(\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1} \right).$$

In the new variables the fluid occupies the (nondimensionalized) region

$$\Omega_\infty = (-\infty, \infty) \times (0, 1).$$

The system (2.1) is equipped with the following boundary conditions:

$$u = 0 \restriction_{x_2=0,1}, \quad \omega \restriction_{x_2=0,1} = 0, \quad T \restriction_{x_2=0} = 1 \quad \text{and} \quad T \restriction_{x_2=1} = 0 \tag{2.2}$$

with l-periodicity in the x_1-direction assumed. The initial conditions are

$$u \restriction_{t=0} = u_0, \quad \omega \restriction_{t=0} = \omega_0, \quad T \restriction_{t=0} = T_0 \quad \text{for } x = (x_1, x_2) \in \Omega_\infty.$$

The above dimensionless problem is a model for the Rayleigh–Bénard heat convection in a layer of thermomicropolar fluid bounded by two horizontal one-dimensional parallel plates at distance h from each other with the bottom heated at temperature $T_1 + \triangle T$ and the top cooled at temperature T_1. Therefore, the fluid motion is induced by the difference of temperatures at the bottom and the top parts of the boundary of the flow domain.

Observe that if in (2.1) the coupling parameter N and the micropolar heat conduction parameter D equal to zero then the velocity field u and the temperature T become independent of the microrotation field ω and satisfy the Boussinesq system of equations of classical hydrodynamics.

When $D = 0$, system (2.1) becomes the Boussinesq system of equations for micropolar fluid model. Its mathematical theory involving existence of global in time solutions, existence of global attractors, nonlinear stability, bifurcation problem, etc. was considered in [18], [29], [30]. As in the micropolar fluid model [14], [16], [22] the presence of microrotation, measured by N, $0 \le N < 1$, stabilizes the fluid motion (in comparison to the classical Boussinesq model with $N = 0$), since the upper bounds of the averaged heat transfer in the vertical direction, measured by the Nusselt number Nu, suggest that Nu decreases when N increases. We know that if the micropolar parameters N and L are large enough, the convective part of the upward heat transfer may be completely blocked (then Nu $= 1$) [18]. The physical explanation may be such that with the increase of the kinematic microrotation viscosity coefficient ν_r the friction between the rotating particles increases and impedes the fluid motion (observe that if $\nu_r \to \infty$ then $N \to 1$).

The thermomicropolar fluid model allows to take into account the direct influence of the microrotation field ω on the temperature distribution. The term $D \operatorname{rot} \omega \cdot \nabla T$ in the last equation of (2.1) comes from the presence of an additional term in the heat flux vector field q in this model, given by $q = -\kappa \nabla T + \gamma(T) \operatorname{rot} \omega$. With this q, assuming that $\gamma(T) = -\delta T$, the equation for the temperature has the form (1.1) and, in dimensionless variables, we obtain (2.1).

3. Preliminaries

We define $\Omega \subset \Omega_\infty$ to be a rectangular box of length equal to the period $\Omega = (0, l) \times (0, 1)$, for some $l > 0$. We introduce

$$\begin{aligned}
\widetilde{V}_S &= \big\{u \in C^\infty(\overline{\Omega})^2 \mid \operatorname{div} u = 0,\ u \restriction_{x_2=0,1} = 0, \\
&\qquad u \text{ and all its derivatives are } l\text{-periodic in } x_1\text{-direction}\big\}, \\
\widetilde{V} &= \{\omega \in C^\infty(\overline{\Omega}) \mid \omega \restriction_{x_2=0,1} = 0, \\
&\qquad \omega \text{ and all its derivatives are } l\text{-periodic in } x_1\text{-direction}\}, \\
H_S &= \text{closure of } \widetilde{V}_S \text{ in } L^2(\Omega)^2, \quad H = L^2(\Omega), \\
V_S &= \text{closure of } \widetilde{V}_S \text{ in } H^1(\Omega)^2, \quad \text{and} \quad V = \text{closure of } \widetilde{V} \text{ in } H^1(\Omega).
\end{aligned}$$

Spaces H and H_S are Hilbert spaces with the inner product

$$(u, v) = (u, v)_{L^2} = \int_\Omega u(x) \cdot v(x)\, dx$$

and the corresponding norms $|v| = (v, v)^{1/2}$ for $v \in H, H_S$.

Spaces V and V_S are Hilbert spaces with the norms $\|v\| = (\nabla v, \nabla v)^{1/2}$ for $v \in V, V_S$.

Remark 3.1. We use the uniform notation in the whole paper. The spaces H, V, $D(A)$ are related with the Laplace operator whereas the spaces H_S, V_S, $D(A_S)$ are associated with the Stokes operator.

We have the Poincaré inequality $\lambda_1|v|^2 \le \|v\|^2$, for $v \in V, V_S$, where the optimal (largest) constant is equal to π^2. By $\langle\,\cdot\,,\,\cdot\,\rangle$ we will denote the duality pairings between various spaces and their duals.

We define the standard trilinear forms (see [26] or [31])

$$b_S(u,v,w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \quad \text{and} \quad b(u,v,w) = \sum_{i=1}^{2} \int_\Omega u_i \frac{\partial v}{\partial x_i} w.$$

We introduce the Laplace operator associated with our boundary conditions $A\colon D(A) \to H$, where

$$D(A) = \{v \in V \mid -\Delta v \in H\} \quad \text{and} \quad Av = -\Delta v, \quad v \in D(A).$$

Clearly, the eigenvectors $\{v_k\}_{k\ge 1} \subset D(A)$ of A form the orthonormal basis of H and

$$Av_k = \beta_k v_k \quad \text{and} \quad 0 < \beta_1 \le \ldots \le \beta_k \to \infty. \tag{3.1}$$

Since the domain Ω is a rectangle, these eigenvectors and eigenvalues can be explicitly determined: for $n \in \mathbb{Z}$, $m \ge 1$ there holds

$$\begin{aligned} \beta_{nm} &= \left(\frac{2n\pi}{l}\right)^2 + (2m\pi)^2, \\ v_{nm} &= \sqrt{\frac{2}{l}}\left[\sin\left(\frac{2n\pi}{l}\,x_1\right) + \cos\left(\frac{2n\pi}{l}\,x_1\right)\right]\sin(m\pi x_2). \end{aligned} \tag{3.2}$$

Note that each v_{nm} belongs to $C^\infty(\overline{\Omega})$. We can relabel them so that $\{v_k\}_{k\ge1}$ satisfy (3.1).

Since we know the exact formulas for eigenfunctions we can prove the following result. The proofs is standard so it is omitted.

Lemma 3.2. *Let $f \in H$ and let $u \in D(A)$ satisfy $Au = f$. Then $u \in H^2(\Omega)$ and $\|u\|_{H^2} \le C|f|$ for some constant $C > 0$. As a result, norms $\|u\|_{H^2}$ and $|Au|$ are equivalent on $D(A)$.*

We introduce the fractional power of the Laplacian, see [26, Chapters 3 and 6]. Let

$$D(A^{3/2}) = \left\{ u \in H \;\middle|\; u = \sum_{k\ge1}(u,v_k)v_k \text{ (in } H\text{)}, \sum_{k\ge1}(u,v_k)^2\beta_k^3 < \infty \right\},$$

and define

$$A^{3/2}u = \sum_{k\ge1} \beta_k^{3/2}(u,v_k)v_k.$$

To make $D(A^{3/2})$ a Hilbert space we equip it with the inner product

$$(u, v)_{D(A^{3/2})} = (A^{3/2}u, A^{3/2}v)$$

which gives the corresponding norm $\|u\|_{D(A^{3/2})} = |A^{3/2}u|$.

Lemma 3.3. *We have* $D(A^{3/2}) \subset H^3(\Omega)$ *and norms* $\|\cdot\|_{H^3}$, $\|\cdot\|_{D(A^{3/2})}$ *are equivalent on* $D(A^{3/2})$.

Now, we introduce the Stokes operator A_S, see [26] or [31] for details. Let

$$D(A_S) = \{u \in V_S \mid \exists\, w \in H_S,\ \forall\, \varphi \in V_S\ (w, \varphi) = (\nabla u, \nabla \varphi)\}$$

and define $A_S(u) = w$.

It is known that $A_S = -P\Delta$, where P stands for the Helmholtz–Leray projector from $L^2(\Omega)^2$ onto H_S. The eigenfunctions of the operator A_S form the orthonormal basis of H_S and they are smooth. In fact one can provide explicit formulas for those eigenfunctions, see [27] where those formulas are derived for the three-dimensional case with similar boundary conditions. Moreover, the result analogous to Lemma 3.2 holds also for the Stokes operator.

We present two versions of the Gagliardo–Nirenberg inequality — see [1, Theorem 5.8]:

$$\|v\|_{L^4} \le k_1 |v|^{1/2} \|v\|_{H^1}^{1/2}, \quad \text{for } v \in H^1(\Omega), \tag{3.3}$$

$$\|v\|_{L^\infty} \le k_2 |v|^{1/2} \|v\|_{H^2}^{1/2}, \quad \text{for } v \in H^2(\Omega). \tag{3.4}$$

Combining (3.3) with the Poincaré inequality yields the so-called Ladyzhenskaya inequality

$$\|v\|_{L^4} \le k_3 |v|^{1/2} \|v\|^{1/2}, \quad \text{for } v \in V, V_S, \tag{3.5}$$

and combining (3.3) with Lemma 3.2 gives

$$\|\nabla v\|_{L^4} \le k_4 \|v\|^{1/2} |Av|^{1/2}, \qquad v \in D(A). \tag{3.6}$$

Similarly, combining (3.4) with regularity theorems for the Laplace operator (Lemma 3.2) and for the Stokes operator we get so-called Agmon's inequalities

$$\|v\|_{L^\infty} \le k_5 |v|^{1/2} |Av|^{1/2}, \quad \text{for } v \in D(A), \tag{3.7}$$

$$\|u\|_{L^\infty} \le k_6 |u|^{1/2} |A_S u|^{1/2}, \quad \text{for } u \in D(A_S). \tag{3.8}$$

We remark that in all proofs by C we will denote a generic constant which, in Sections 4, 6, and 7 depends on Ω, constants present in the equations, final time τ, and the initial data, and in Sections 8 and 9 depends only on Ω and the constants present in the equations. By ε we will denote an arbitrarily small constant, and by $C(\varepsilon)$ a constant which depends only on ε and the problem data.

4. Weak solutions: definitions

Following a traditional approach, we change the temperature equation from (2.1) so that the perturbative variable will satisfy the homogeneous boundary conditions

$$\theta(x_1, x_2, t) = T(x_1, x_2, t) - (1 - x_2).$$

We also change the pressure p to $p - \mathrm{Pr}\,\mathrm{Ra}\,(x_2 - x_2^2/2)$ in the velocity equation in (2.1). These transform (2.1) into

$$\frac{1}{\mathrm{Pr}}\,(u_t + (u\cdot\nabla)u + \nabla p) = \Delta u + \frac{N}{1-N}\,(2\operatorname{rot}\omega + \Delta u) + \mathrm{Ra}\,\theta e_2, \tag{4.1}$$

$$\operatorname{div} u = 0, \tag{4.2}$$

$$\frac{M}{\mathrm{Pr}}\,(\omega_t + u\cdot\nabla\omega) = L\Delta\omega + 2\frac{N}{1-N}\,(\operatorname{rot}\, u - 2\omega), \tag{4.3}$$

$$\theta_t + u\cdot\nabla\theta = \Delta\theta + D\operatorname{rot}\omega\cdot\nabla\theta + D\,\frac{\partial\omega}{\partial x_1} + u_2 \tag{4.4}$$

on $\Omega\times(0,\infty)$, equipped with the boundary conditions

$$u\restriction_{x_2=0,1} = 0, \qquad \omega\restriction_{x_2=0,1} = 0, \qquad \theta\restriction_{x_2=0,1} = 0 \tag{4.5}$$

and periodic in the horizontal direction. The initial conditions now read

$$u(0) = u_0, \qquad \omega(0) = \omega_0, \qquad \theta(0) = \theta_0 = T_0 - (1 - x_2). \tag{4.6}$$

We define three classes of weak solutions. In the first one, given by Definition 4.1 we impose only that the equations are satisfied in the weak sense, and solutions at zero are equal to the initial datum.

Definition 4.1. Let $\tau > 0$, $u_0 \in H_S$, $\omega_0 \in H$ and $\theta_0 \in H$. By a weak solution of the problem (4.1)–(4.6) we mean a triple of functions (u, ω, θ),

$$\begin{aligned} &u \in L^2(0,\tau;V_S)\cap C([0,\tau],H_S)\cap W^{1,2}(0,\tau;V_S^*),\\ &\omega \in L^2(0,\tau;V)\cap C([0,\tau],H)\cap W^{1,2}(0,\tau;V^*),\\ &\theta \in L^2(0,\tau;V)\cap C_w([0,\tau];H)\cap W^{1,2}(0,\tau;D(A^{3/2})^*)\end{aligned} \tag{4.7}$$

such that $u(0) = u_0$, $\omega(0) = \omega_0$, $\theta(0) = \theta_0$ satisfying the following identities ([1]):

$$\begin{aligned}\frac{1}{\mathrm{Pr}}\bigg(\frac{d}{dt}(u(t),\varphi) + b_S(u(t),u(t),\varphi)\bigg) + (\nabla u(t),\nabla\varphi)\\ + \frac{N}{1-N}\,\big[(\nabla u(t),\nabla\varphi) - 2(\operatorname{rot}\,\omega(t),\varphi)\big] = \mathrm{Ra}\,(\theta(t)e_2,\varphi)\end{aligned} \tag{4.8}$$

([1]) Recall that the symbol $C_w([0,\tau];H)$ stands for the space of functions $f\colon [0,\tau]\to H$ such that for every $h\in H$ the map $[0,\tau]\ni t\mapsto (f(t),h)\in\mathbb{R}$ is continuous.

for every $\varphi \in V_S$,

$$\frac{M}{\mathrm{Pr}}\left(\frac{d}{dt}(\omega(t),\psi) + b(u(t),\omega(t),\psi)\right) + \frac{N}{1-N}\left[4(\omega(t),\psi) - 2(\operatorname{rot}\, u(t),\psi)\right] + L(\nabla\omega(t),\nabla\psi) = 0 \tag{4.9}$$

for every $\psi \in V$,

$$\frac{d}{dt}(\theta(t),\eta) + b(u(t),\theta(t),\eta) + (\nabla\theta(t),\nabla\eta) = -D(\theta(t),\operatorname{rot}\omega(t)\cdot\nabla\eta) + D\left(\frac{\partial\omega}{\partial x_1}(t),\eta\right) + (u_2(t),\eta) \tag{4.10}$$

for every $\eta \in D(A^{3/2})$, in the sense of scalar distributions on $(0,\tau)$.

Remark 4.2. Clearly, the regularity of the weak solution imposed in the above definition implies that $\theta \in L^\infty(0,\tau;H)$.

In the second class of weak solutions we additionally assume that the temperature θ is continuous at $t=0$ with values in H.

Definition 4.3. Let $\tau > 0$, $u_0 \in H_S$, $\omega_0 \in H$ and $\theta_0 \in H$. By a weak solution of the problem (4.1)–(4.6) with temperature continuous at zero we mean a triple of functions (u,ω,θ) satisfying Definition 4.1 such that, additionally,

$$\limsup_{t\to 0^+} |\theta(t)| \le |\theta_0|. \tag{4.11}$$

Finally, we define a class of weak solutions which are the limits of the Galerkin approximations.

Definition 4.4. Let $\tau > 0$, $u_0 \in H_S$, $\omega_0 \in H$ and $\theta_0 \in H$. By a weak solution of the Galerkin type of the problem (4.1)–(4.6) we mean a triple of functions (u,ω,θ) satisfying Definition 4.1 which are weak limits of the Galerkin approximations (u^m,ω^m,θ^m) in the spaces spanned by the eigenfunctions of the operators A_S and A, i.e.

$$\begin{aligned}
u^m &\to u && \text{weakly in } L^2(0,\tau;V_S) \text{ and weakly-* in } L^\infty(0,\tau;H_S),\\
u^m_t &\to u_t && \text{weakly in } L^2(0,\tau;V_S^*),\\
\omega^m &\to \omega && \text{weakly in } L^2(0,\tau;V) \text{ and weakly-* in } L^\infty(0,\tau;H),\\
\omega^m_t &\to \omega_t && \text{weakly in } L^2(0,\tau;V^*),\\
\theta^m &\to \theta && \text{weakly in } L^2(0,\tau;V) \text{ and weakly-* in } L^\infty(0,\tau;H),\\
\theta^m_t &\to \theta_t && \text{weakly in } L^2(0,\tau;D(A^{3/2})^*).
\end{aligned}$$

Remark 4.5. We will prove in Section 5 the existence of a weak solution of the Galerkin type, given by Definition 4.4. We will also prove that every weak solution of the Galerkin type has temperature continuous at zero, i.e. satisfies condition (4.11). Hence we have the implications

$$(u,\omega,\theta) \text{ is of the Galerkin type } \Rightarrow (u,\omega,\theta) \text{ has temperature continous at zero,}$$
$$(u,\omega,\theta) \text{ has temperature continuous at zero } \Rightarrow (u,\omega,\theta) \text{ is a weak solution.}$$

It is unknown if there exist weak solutions which do not have temperature continuous at zero, or if there are weak solutions with temperature continuous at zero which are not of the Galerkin type, i.e. both inclusions can, in principle, be strict. Since we prove the existence of the Galerkin type weak solution, it follows that for a given initial datum all three classes are nonempty. We also remark that for neither of these three classes we are able to obtain uniqueness of solutions. It is unknown how to approach numerically solutions which are non-unique and are not limits of the approximative finite dimensional problems. Therefore, from the physical point of view, it is reasonable to consider only the solutions which are limits of Galerkin problems. To give our results more mathematical generality, however, we obtain our results for the two wider classes. In particular in Sections 6 and 7 we obtain the bootstrapping results which are valid for all weak solutions which not necessarily have temperature continuous at zero and are not necessarily limits of the Galerkin approximations. In order to obtain the existence of the global attractor for weak solution we will need (4.11), cf. Section 8, so the global attractor will be given for weak solutions with temperature continuous at zero. Since this attractor coincides with the global attractor for the strong solutions, cf. Section 9, it is also the global attractor for the Galerkin type weak solutions.

5. Existence of weak solutions

In this section we prove the existence of the Galerkin type weak solution to the problem (4.1)–(4.6) in the spaces spanned by the eigenfunctions of A_S and A. As the technique is standard, we will show only the a priori estimates. The reader unfamiliar with this technique is encouraged to see [26] or [31].

Theorem 5.1. *Let $u_0 \in H_S$, ω_0, $\theta_0 \in H$, and $\tau > 0$. There exists a weak solution of the Galerkin type to* (4.1)–(4.6)*. Moreover, each solution of the Galerkin type has the temperature continuous at zero.*

Proof. Take the inner product in $L^2(\Omega)^2$ of (4.1) and u, to get

$$\begin{aligned}(5.1)\quad \frac{1}{\mathrm{Pr}}\left(\frac{\partial}{\partial t}u,u\right)+\frac{1}{\mathrm{Pr}}\,b_S(u,u,u)+(A_Su,u)\\ +\frac{N}{1-N}\big[(A_Su,u)-2(\operatorname{rot}\,\omega,u)\big]=\mathrm{Ra}\,(\theta e_2,u).\end{aligned}$$

We have
$$\left(\frac{\partial}{\partial t}u, u\right) = \frac{1}{2}\frac{d}{dt}|u|^2, \qquad (A_S u, u) = \|u\|^2.$$
Since
$$b_S(u, v, w) = -b_S(u, w, v), \quad u, v, w \in V_S,$$
we obtain

$$b_S(u, u, u) = 0. \tag{5.2}$$

We estimate

$$\mathrm{Ra}\,(\theta e_2, u) \le \mathrm{Ra}\,|\theta||u| \le C(\varepsilon)|\theta|^2 + \varepsilon|u|^2 \le C(\varepsilon)|\theta|^2 + \varepsilon\|u\|^2, \tag{5.3}$$

where we used the Poincaré inequality. As a result (5.1) turns into

$$\frac{1}{\mathrm{Pr}}\frac{d}{dt}|u|^2 + \|u\|^2 + 2\frac{N}{1-N}\left[\|u\|^2 - 2(\mathrm{rot}\ \omega, u)\right] \le C|\theta|^2. \tag{5.4}$$

Multiply (4.3) by ω and integrate over Ω

$$\frac{M}{\mathrm{Pr}}\left(\frac{\partial}{\partial t}\omega, \omega\right) + \frac{M}{\mathrm{Pr}}b(u, \omega, \omega) + \frac{N}{1-N}\left[4|\omega|^2 - 2(\mathrm{rot}\ u, \omega)\right] + L(A\omega, \omega) = 0. \tag{5.5}$$

As above, we have
$$\left(\frac{\partial}{\partial t}\omega, \omega\right) = \frac{1}{2}\frac{d}{dt}|\omega|^2, \qquad b(u, \omega, \omega) = 0, \qquad (A\omega, \omega) = \|\omega\|^2.$$
and from (5.5), we obtain

$$\frac{M}{\mathrm{Pr}}\frac{d}{dt}|\omega|^2 + 2L\|\omega\|^2 + 2\frac{N}{1-N}\left[4|\omega|^2 - 2(\mathrm{rot}\ u, \omega)\right] = 0. \tag{5.6}$$

Using the fact that $\|u\|^2 = |\mathrm{rot}\, u|^2$ and $(\mathrm{rot}\, u, \omega) = (\mathrm{rot}\, \omega, u)$, we add (5.4), (5.6) which yields

$$\frac{1}{\mathrm{Pr}}\frac{d}{dt}(|u|^2 + M|\omega|^2) + \|u\|^2 + 2L\|\omega\|^2 + 2\frac{N}{1-N}|2\omega - \mathrm{rot}\, u|^2 \le C|\theta|^2. \tag{5.7}$$

Multiply (4.4) by θ and integrate over Ω

$$\left(\frac{\partial}{\partial t}\theta, \theta\right) + b(u, \theta, \theta) + (A\theta, \theta) = D\int_\Omega (\mathrm{rot}\ \omega \cdot \nabla\theta)\theta + D\left(\frac{\partial \omega}{\partial x_1}, \theta\right) + (u_2, \theta). \tag{5.8}$$

We have
$$\left(\frac{\partial}{\partial t}\theta, \theta\right) = \frac{1}{2}\frac{d}{dt}|\theta|^2, \qquad b(u, \theta, \theta) = 0, \qquad (A\theta, \theta) = \|\theta\|^2,$$
and we estimate the following two terms:
$$\left(\frac{\partial \omega}{\partial x_1}, \theta\right) \le |\theta|\left|\frac{\partial \omega}{\partial x_1}\right| \le |\theta|\|\omega\| \le C(\varepsilon)|\theta|^2 + \varepsilon\|\omega\|^2,$$
and
$$(u_2, \theta) \le |u_2||\theta| \le |u||\theta| \le C(\varepsilon)|u|^2 + \varepsilon\|\theta\|^2.$$

We assume for the moment that θ is smooth enough so that the integral

$$\int_\Omega (\operatorname{rot} \omega \cdot \nabla\theta)\theta$$

is well-defined. In fact, we estimate the Galerkin–Faedo approximations $\theta^m(x,t) = \sum_{j=1}^{m} \theta^{mj}(t)v_j(x)$ of θ which are of class $C^\infty(\overline{\Omega})$ with respect to the variable x, see (3.2). Integrating by parts leads to

$$\begin{aligned}\int_\Omega (\operatorname{rot} \omega \cdot \nabla\theta)\theta &= \frac{1}{2}\int_\Omega \bigg(\frac{\partial}{\partial x_2}\,\omega, -\frac{\partial}{\partial x_1}\,\omega\bigg)\cdot \nabla(\theta^2) \\ &= -\frac{1}{2}\int_\Omega \omega \operatorname{rot}(\nabla(\theta^2)) + \frac{1}{2}\int_{\partial\Omega} \omega\bigg(\frac{\partial\theta}{\partial x_1}\,\theta n_2 - \frac{\partial\theta}{\partial x_2}\,\theta n_1\bigg)\,dS =: I_1 + I_2.\end{aligned}$$

The symmetry of second derivatives implies $I_1 = 0$. Since each $\partial v_k/\partial x_2$ is l-periodic in x_1 (see (3.2)), the boundary conditions yield $I_2 = 0$. Therefore, (5.8) leads to

$$\frac{d}{dt}\,|\theta|^2 + \|\theta\|^2 \le C(\varepsilon)|\theta|^2 + \varepsilon\|\omega\|^2 + C|u|^2. \tag{5.9}$$

Adding (5.7) and (5.9) yields

$$\frac{d}{dt}(|u|^2 + M|\omega|^2 + |\theta|^2) + c_1(\|u\|^2 + \|\omega\|^2 + \|\theta\|^2) \le c_2(|u|^2 + M|\omega|^2 + |\theta|^2), \tag{5.10}$$

where c_1, c_2 are positive constants dependent only on the problem data. Write

$$y(t) = |u(t)|^2 + M|\omega(t)|^2 + |\theta(t)|^2 \quad \text{and} \quad \alpha(t) = \|u(t)\|^2 + \|\omega(t)\|^2 + \|\theta(t)\|^2.$$

Multiply (5.10) by $\exp(-c_2t)$

$$\frac{d}{dt}(y(t)e^{-c_2t}) + c_1e^{-c_2t}\alpha(t) \le 0$$

and integrate from 0 to some $s > 0$

$$y(s) + c_1\int_0^s e^{c_2(s-t)}\alpha(t)\,dt \le e^{c_2s}y(0).$$

Recall that

$$y(0) = (\le)|u_0|^2 + M|\omega_0|^2 + |\theta_0|^2.$$

Fix $\tau > 0$, the above inequality implies that

$$u \in L^\infty(0,\tau;H_S) \cap L^2(0,\tau;V_S), \qquad \omega,\theta \in L^\infty(0,\tau;H) \cap L^2(0,\tau;V). \tag{5.11}$$

The proof of the continuity of the functions

$$[0,\tau] \ni t \mapsto u(t) \in H_S \quad \text{and} \quad [0,\tau] \ni t \mapsto \omega(t) \in H$$

is standard: we show that the time derivatives have the regularity $u_t \in L^2(0,\tau;V_S^*)$ and $\omega_t \in L^2(0,\tau;V^*)$ and use [17, Theorem 3, Chapter 5.9]. We will only show that $u_t \in L^2(0,\tau;V_S^*)$, the proof that $\omega_t \in L^2(0,\tau;V)$ is similar. Taking the duality of (4.1) with $v(t)$ where $v \in L^2(0,\tau;V_S)$ we obtain

$$\langle u_t, v\rangle = -b_S(u,u,v) - \Pr\frac{1}{1-N}(\nabla u,\nabla v) + 2\Pr\frac{N}{1-N}(\operatorname{rot}\omega, v) + \operatorname{Ra}\Pr(\theta e_2, v). \tag{5.12}$$

We use the inequality (3.5) to get

$$|b_S(u,u,v)| = |b_S(u,v,u)| \le \|u\|_{L^4}\|v\|\|u\|_{L^4} \le C|u|\|u\|\|v\|.$$

From (5.12) we get

$$\int_0^\tau \langle u_t, v\rangle \le C\big(\|u\|_{L^\infty(0,\tau;H_S)}\|u\|_{L^2(0,\tau;V_S)} + \|u\|_{L^2(0,\tau;V_S)} + \|\omega\|_{L^2(0,\tau;V)} + \|\theta\|_{L^2(0,\tau;V)}\big)\|v\|_{L^2(0,\tau;V_S)},$$

where we used the Poincaré inequality in the last two terms. Hence and by (5.11), we have

$$\|u_t\|_{L^2(0,\tau;V_S^*)} \le C,$$

where C depends only on Ω, u_0, ω_0, θ_0 and $\tau > 0$.

Recall that we have

$$\theta_t = -u\cdot\nabla\theta + \Delta\theta + D\operatorname{rot}\omega\cdot\nabla\theta + D\frac{\partial\omega}{\partial x_1} + u_2. \tag{5.13}$$

We show that $\theta_t \in L^2(0,\tau;D(A^{3/2})^*)$. Every term from the right-hand side of (5.13) can be bounded in $L^2(0,\tau;V^*)$ as in (5.12) except from

$$D\operatorname{rot}\omega\cdot\nabla\theta.$$

We write $X = D(A^{3/2})$ for short. Take $\eta \in X$, integrate by parts and apply the Hölder inequality to get

$$\begin{aligned}\langle \operatorname{rot}\omega\cdot\nabla\theta,\eta\rangle_{X^*\times X} &= \int_\Omega(\operatorname{rot}\,\omega\cdot\nabla\theta)\eta = -\int_\Omega(\operatorname{rot}\,\omega\cdot\nabla\eta)\theta \\ &\le |\theta|\bigg(\int_\Omega|\operatorname{rot}\,\omega\cdot\nabla\eta|^2\bigg)^{1/2} \le |\theta|\|\omega\|\|\nabla\eta\|_{L^\infty(\Omega)}.\end{aligned}$$

In view of Lemma 3.3, $\nabla\eta \in H^2(\Omega)^2$ so the Sobolev embedding yields

$$\|\nabla\eta\|_{L^\infty(\Omega)^2} \le C\|\nabla\eta\|_{H^2(\Omega)^2} \le C\|\eta\|_{H^3(\Omega)} \le C\|\eta\|_X,$$

where the last inequality follows from Lemma 3.3. Summing up, we have

$$\langle \operatorname{rot}\omega\cdot\nabla\theta,\eta\rangle_{X^*\times X} \le C|\theta|\|\omega\|\|\eta\|_X$$

so

$$\|\operatorname{rot}\,\omega\cdot\nabla\theta\|_{X^*} \le C|\theta|\|\omega\|.$$

Therefore, we have

$$\|\operatorname{rot}\,\omega\cdot\nabla\theta\|_{L^2(0,\tau;X^*)}\leq C\bigg(\int_0^\tau|\theta|^2\|\omega\|^2\bigg)^{1/2}\leq C\|\theta\|_{L^\infty(0,\tau;H)}\|\omega\|_{L^2(0,\tau;V)}.$$

The continuous embedding $V^*\subset X^*$ yields the continuous embedding

$$L^2(0,\tau;V^*)\subset L^2(0,\tau;X^*)$$

and all the bounds on the components of (5.13) made in the space $L^2(0,\tau;V^*)$ are valid in the space $L^2(0,\tau;X^*)$. As a result, we get

$$\|\theta_t\|_{L^2(0,\tau;X^*)}\leq C,$$

where C depends only on Ω, $\tau>0$ and initial conditions (4.6). Finally, the fact that $\theta\in C_w([0,\tau];H)$ follows from the Lions–Magenes lemma, see [21, Chapter 3, Lemma 8.1]. We conclude the proof by showing that the obtained weak solution satisfies (4.11). Integrating (5.10) in time from 0 to t we obtain (we use the index m to denote the m-th Galerkin approximation)

$$\begin{aligned}|u^m(t)|^2+M|\omega^m(t)|^2+|\theta^m(t)|^2+c_1\int_0^t(\|u^m(s)\|^2+\|\omega^m(s)\|^2+\|\theta^m(s)\|^2)\\ \leq|u^m(0)|^2+M|\omega^m(0)|^2+|\theta^m(0)|^2+c_2\int_0^t(|u^m(s)|^2+M|\omega^m(s)|^2+|\theta^m(s)|^2).\end{aligned}$$

Let m tend to infinity. We can pass to the limit with all terms on the right-hand side as we have the strong convergence in the corresponding norms (we use the Aubin–Lions lemma to pass to the limit in the time integrals). As for the terms on the left-hand side we only have weak convergence but we can use the weak lower-semicontinuity of the corresponding norms to deduce that

$$\begin{aligned}|u(t)|^2+M|\omega(t)|^2+|\theta(t)|^2+c_1\int_0^t(\|u(s)\|^2+\|\omega(s)\|^2+\|\theta(s)\|^2)&\\ \leq|u_0|^2+M|\omega_0|^2+|\theta_0|^2+c_2\int_0^t(|u(s)|^2+M|\omega(s)|^2+|\theta(s)|^2).&\end{aligned}$$

The assertion follows by letting t tend to zero. □

6. Bootstrapping

The weak solution obtained in the previous section can be nonunique. In this section we demonstrate that the unknowns u, ω, and θ become more smooth after arbitrarily small time. We cannot, however, bootstrap the regularity of all three functions (u,ω,θ) immediately. Since the "bad" term $\operatorname{rot}\omega\cdot\nabla\theta$ appears in the equation for θ we first observe that if (u,ω,θ) is the weak solution, then we can fix θ and construct the Galerkin approximation for the functions u,ω only. Then u,ω are defined uniquely for a given θ and this observation allows us to bootstrap u and ω in the original triple (u,ω,θ). Extra regularity of u,ω immediately implies the extra regularity of θ.

We start from defining an auxiliary problem, where temperature is prescribed and we look only for u and ω. This auxiliary problem, together with Theorem 6.2 on the uniqueness of its solutions are useful in the proof of the first bootstrapping result, namely Theorem 6.3.

DEFINITION 6.1. Let $\tau > 0$, $u_0 \in H_S$, $\omega_0 \in H$ and $\theta \in L^\infty(0,\tau;H)$. We look for a pair (u,ω),

$$u \in L^2(0,\tau;V_S) \cap C([0,\tau],H_S) \cap W^{1,2}(0,\tau;V_S^*),$$
$$\omega \in L^2(0,\tau;V) \cap C([0,\tau],H) \cap W^{1,2}(0,\tau;V^*)$$

such that $u(0)=u_0$, $\omega(0)=\omega_0$, satisfying the following identities:

$$\begin{aligned}\frac{1}{\mathrm{Pr}}\bigg(\frac{d}{dt}(u(t),\varphi)) + b_S(u(t),u(t),\varphi)\bigg) + (\nabla u(t),\nabla\varphi)\\ + \frac{N}{1-N}\big[(\nabla u(t),\nabla\varphi) - 2(\operatorname{rot}\,\omega(t),\varphi)\big] = \mathrm{Ra}\,(\theta(t)e_2,\varphi)\end{aligned} \tag{6.1}$$

for every $\varphi \in V_S$,

$$\begin{aligned}\frac{M}{\mathrm{Pr}}\bigg(\frac{d}{dt}(\omega(t),\psi)) + b(u(t),\omega(t),\psi)\bigg)\\ + \frac{N}{1-N}\big[4(\omega(t),\psi) - 2(\operatorname{rot}\,u(t),\psi)\big] + L(\nabla\omega(t),\nabla\psi) = 0\end{aligned} \tag{6.2}$$

for every $\psi \in V$ in the sense of scalar distributions on $(0,\tau)$.

THEOREM 6.2. *Solution u,ω to the problem given in Definition* 6.1 *is unique and is a limit of the Galerkin approximation built on the spaces $V^n = \operatorname{span}\{v_1, \dots, v_n\}$, $V_S^n = \operatorname{span}\{v_1^S,\dots,v_n^S\}$ spanned by the eigenfunctions of the operators A and A_S.*

PROOF. The proof, by the Faedo–Galerkin method, follows the lines of the proof of Theorem 5.1 and is standard, so we omit it here. To prove the uniqueness, assume that $(\overline{u},\overline{\omega})$ and $(\widehat{u},\widehat{\omega})$ are two solutions with the same initial datum. Subtracting (6.1) written for $\overline{u}$ and $\widehat{u}$ and taking $\overline{u}-\widehat{u}$ as the test function we obtain, denoting $u=\overline{u}-\widehat{u}$ and $\omega=\overline{\omega}-\widehat{\omega}$

$$\frac{1}{\mathrm{Pr}}\bigg(\frac{1}{2}\frac{d}{dt}\,|u|^2 + b_S(u,\widehat{u},u)\bigg) + \|u\|^2 + \frac{N}{1-N}\big[|\operatorname{rot} u|^2 - 2(\operatorname{rot}\,\omega,u)\big] = 0.$$

Proceeding similarly with (6.2) we obtain

$$\frac{M}{\mathrm{Pr}}\bigg(\frac{1}{2}\frac{d}{dt}\,|\omega|^2 + b(u,\widehat{\omega},\omega)\bigg) + \frac{N}{1-N}\big[4|\omega|^2 - 2(\operatorname{rot}\,u,\omega)\big] + L\|\omega\|^2 = 0.$$

The Ladyzhenskaya inequality (3.5) implies the bounds

$$-b_S(u,\widehat{u},u) \le C|u|\|u\|\|\widehat{u}\| \quad\text{and}\quad -b(u,\widehat{\omega},\omega) \le C|u|^{1/2}\|u\|^{1/2}\|\widehat{\omega}\||\omega|^{1/2}\|\omega\|^{1/2},$$

whence, after adding the two energy equations we get

$$\frac{1}{2\,\mathrm{Pr}}\frac{d}{dt}(|u|^2+M|\omega|^2)+\frac{N}{1-N}\,|2\omega-\operatorname{rot}u|^2+L\|\omega\|^2+\|u\|^2 \le C(|u|\|u\|\|\widehat{u}\|+|u|\|u\|\|\widehat{\omega}\|+|\omega|\|\omega\|\|\widehat{\omega}\|).$$

Simple calculations yield

$$\frac{d}{dt}(|u|^2+M|\omega|^2)\le C(|u|^2\|\widehat{u}\|^2+|u|^2\|\widehat{\omega}\|^2+|\omega|^2\|\widehat{\omega}\|^2),$$

and the assertion follows from the Gronwall lemma as $\widehat{u}\in L^2(0,\tau;V)$ and $\widehat{\omega}\in L^2(0,\tau;V)$. □

The above theorem is useful in the proof of the following result.

Theorem 6.3. *Let (u,ω,θ) solve the problem* (4.1)–(4.6) *in the sense of Definition* 4.1. *Then, for any $0<\rho<\tau$, we have*

$$u\in L^2(\rho,\tau;D(A_S))\cap C([\rho,\tau],V_S)\cap W^{1,2}(\rho,\tau;H_S),$$
$$\omega\in L^2(\rho,\tau;D(A))\cap C([\rho,\tau],V)\cap W^{1,2}(\rho,\tau;H),$$
$$\theta\in L^2(\rho,\tau;V)\cap C([\rho,\tau],H)\cap W^{1,2}(\rho,\tau;V^*).$$

Proof. We “freeze” $\theta\in L^\infty(0,\tau;H)$. Due to Theorem 6.2 the pair (u,ω) which, together with this θ, constitutes the solution of the problem given in Definition 4.1, is unique, and is a limit of the Galerkin approximation. The next estimates will be derived for the Galerkin equations for (6.1) and (6.2), we omit the index m in order to avoid technicalities. Testing the Galerkin equations with A_Su and $A\omega$, respectively, gives

$$\frac{1}{\mathrm{Pr}}\bigg(\frac{1}{2}\frac{d}{dt}\|u\|^2+b_S(u,u,A_Su)\bigg)+|A_Su|^2+\frac{N}{1-N}\,\big[|A_Su|^2-2(\operatorname{rot}\,\omega,A_Su)\big] = \mathrm{Ra}\,(\theta e_2,A_Su),$$

$$\frac{M}{\mathrm{Pr}}\bigg(\frac{1}{2}\frac{d}{dt}\|\omega\|^2+b(u,\omega,A\omega)\bigg)+\frac{N}{1-N}\,\big[4\|\omega\|^2-2(\operatorname{rot}\,u,A\omega)\big]+L|A\omega|^2=0.$$

Using the Ladyzhenskaya and Agmon inequalities (3.5), (3.6), and (3.8), we get the bounds

$$-b_S(u,u,A_Su)\le C|u|^{1/2}\|u\||A_Su|^{3/2}\le\varepsilon|A_Su|^2+C(\varepsilon)|u|^2\|u\|^4,$$
$$-b(u,\omega,A\omega)\le C|u|^{1/2}\|u\|^{1/2}\|\omega\|^{1/2}|A\omega|^{3/2}\le\varepsilon|A\omega|^2+C(\varepsilon)|u|^2\|u\|^2\|\omega\|^2,$$

whence from two above equations we obtain, after adding them, the following bound:

$$\begin{aligned}\frac{1}{2\,\mathrm{Pr}}\frac{d}{dt}(\|u\|^2+M\|\omega\|^2)+|A_Su|^2+L|A\omega|^2 &\le C(\varepsilon)|\theta|^2+\varepsilon|A_Su|^2+\varepsilon|A_Su|^2+C(\varepsilon)|u|^2\|u\|^4+\varepsilon|A_Su|^2\\ &\quad+C(\varepsilon)\|\omega\|^2+\varepsilon|A\omega|^2+C(\varepsilon)|u|^2\|u\|^2\|\omega\|^2+\varepsilon|A\omega|^2+C(\varepsilon)\|u\|^2.\end{aligned}$$

After the appropriate choice of ε, we obtain

$$\frac{1}{\Pr}\frac{d}{dt}(\|u\|^2+M\|\omega\|^2)+|A_Su|^2+L|A\omega|^2 \leq C(|\theta|^2+|u|^2\|u\|^4+\|\omega\|^2+|u|^2\|u\|^2\|\omega\|^2+\|u\|^2). \tag{6.3}$$

Denoting $y(t)=\|u\|^2+M\|\omega\|^2$, and dropping the terms with A_Su, $A\omega$ we get

$$\frac{d}{dt}\,y(t)\leq C|\theta|^2+C(|u|^2\|u\|^2+1)y(t).$$

We want to use the uniform Gronwall Lemma, see e.g. [31, Lemma 1.1, p. 91]. For any $s\in[0,\tau)$ and $t\in(0,\tau-s]$ we have

$$\int_s^{s+t} y(r)\leq \|u\|^2_{L^2(0,\tau;V_S)}+M\|\omega\|^2_{L^2(0,\tau;V)},$$

$$\int_s^{s+t}|\theta(r)|^2\leq \tau\|\theta\|^2_{L^\infty(0,\tau;H)},$$

$$\int_s^{s+t}(|u|^2\|u\|^2+1)\leq \tau+\|u\|^2_{L^\infty(0,\tau;H_S)}\|u\|^2_{L^2(0,\tau;V_S)}.$$

We deduce the bound $y(t)\leq C(1+1/t)$ valid for $t\in(0,\tau]$. Integrating (6.3) from ρ to τ we get

$$\int_\rho^\tau(|A_Su|^2+L|A\omega|^2)\leq C\left(1+\frac{1}{\rho}\right)+C\|\theta\|^2_{L^\infty(0,\tau;H)} + C\|u\|^2_{L^\infty(0,\tau;H_S)}\int_\rho^\tau\left(1+\frac{1}{t}\right)^2+C\int_\rho^\tau\left(1+\frac{1}{t}\right)\leq C\left(1+\frac{1}{\rho}-\ln\rho\right).$$

We have proved that $u\in L^\infty(\rho,\tau;V_S)\cap L^2(\rho,\tau;D(A_S))$ and $\omega\in L^\infty(\rho,\tau;V)\cap L^2(\rho,\tau;D(A))$. These bounds hold for the Galerkin problems, but they are preserved in their weak limit. We will show that $u_t\in L^2(\rho,\tau;H_S)$ (the proof that $\omega_t\in L^2(\rho,\tau;H)$ is analogous). In the sense of distributions we have the equality

$$u_t=-\Pr A_Su-\frac{\Pr N}{1-N}A_Su+\frac{2\Pr N}{1-N}\operatorname{rot}\omega+\Pr\mathrm{Ra}\,\theta e_2+b_S(u,u,\,\cdot\,).$$

It is sufficient to show that $b_S(u,u,\,\cdot\,)$ defines a linear and continuous functional on $L^2(\rho,\tau;H_S)$. Let $\phi\in L^2(\rho,\tau;H_S)$. We have, by (3.8),

$$\int_\rho^\tau b_S(u(t),u(t),\phi(t))\leq C\int_\rho^\tau|A_Su|^{1/2}|u|^{1/2}\|u\||\phi| \leq C\|\phi\|_{L^2(\rho,\tau;H_S)}\|u\|^{1/2}_{L^\infty(\rho,\tau;H_S)}\|u\|_{L^\infty(\rho,\tau;V_S)}\|u\|_{L^2(\rho,\tau;D(A_S))},$$

and the assertion holds.

We pass to the proof that the extra regularity of (u,ω) implies the extra regularity of θ. It is enough to prove that $\theta_t\in L^2(\rho,\tau;V^*)$. In the sense of distributions we have the equality

$$\langle \theta_t, \eta \rangle = -D(\theta, \operatorname{rot} \omega \cdot \nabla \eta) + D\bigg(\frac{\partial \omega}{\partial x_1}, \eta\bigg) + (u_2, \eta) - b(u, \theta, \eta) - (\nabla \theta, \nabla \eta).$$

All terms on the right-hand side of the last equation constitute linear and continuous functionals of the variable η on the space $L^2(\rho, \tau; V)$ and hence $\theta_t \in L^2(\rho, \tau; V^*)$. We provide the proof only for the term $(\theta, \operatorname{rot} \omega \cdot \nabla \eta)$. We have, by (3.5) and (3.6),

$$\begin{aligned}\int_\rho^\tau (\theta(t), \operatorname{rot} \omega(t) \cdot \nabla \eta(t)) &\leq C \int_\rho^\tau |\theta|^{1/2} \|\theta\|^{1/2} \|\omega\|^{1/2} |A\omega|^{1/2} \|\eta\| \\ &\leq C \|\theta\|^{1/2}_{L^\infty(\rho,\tau;H)} \|\omega\|^{1/2}_{L^\infty(\rho,\tau;V)} \|\theta\|^{1/2}_{L^2(\rho,\tau;V)} \|\omega\|^{1/2}_{L^2(\rho,\tau;D(A))} \|\eta\|_{L^2(\rho,\tau;V)},\end{aligned}$$

and the proof is complete. □

If the initial data has the regularity $u_0 \in V_S$ and $\omega_0 \in V$ then we do not need to use the uniform Gronwall lemma in the proof of the last theorem, and the regularity holds on the whole interval $(0, \tau)$. Hence, the following result holds.

Theorem 6.4. *Let (u, ω, θ) solve the problem given in Definition* 4.1 *with the initial data $u_0 \in V_S$, $\omega_0 \in V$, $\theta_0 \in H$. Then each weak solution has the regularity*

$$\begin{aligned} u &\in L^2(0, \tau; D(A_S)) \cap C([0, \tau], V_S) \cap W^{1,2}(0, \tau; H_S), \\ \omega &\in L^2(0, \tau; D(A)) \cap C([0, \tau], V) \cap W^{1,2}(0, \tau; H), \\ \theta &\in L^2(0, \tau; V) \cap C([0, \tau], H) \cap W^{1,2}(0, \tau; V^*). \end{aligned}$$

Moreover, the solution is unique.

Proof. The proof that each weak solution has the desired regularity follows the lines of the proof of Theorem 6.3. We only prove the uniqueness. To this end let $(\overline{u}, \overline{\omega}, \overline{\theta})$ and $(\widehat{u}, \widehat{\omega}, \widehat{\theta})$ be two solutions with the same initial data and denote $(u, \omega, \theta) = (\overline{u} - \widehat{u}, \overline{\omega} - \widehat{\omega}, \overline{\theta} - \widehat{\theta})$. We subtract the equation (4.8) written for $\overline{u}$ and $\widehat{u}$ and test the difference with $A_S u$ which gives

$$\begin{aligned}\frac{1}{2\operatorname{Pr}} \frac{d}{dt} \|u\|^2 + \frac{1}{\operatorname{Pr}} (b_S(\overline{u}, \overline{u}, A_S u) - b_S(\widehat{u}, \widehat{u}, A_S u)) + |A_S u|^2 \\ + \frac{N}{1-N} \big[|A_S u|^2 - 2(\operatorname{rot} \omega, A_S u)\big] = \operatorname{Ra} (\theta e_2, A_S u).\end{aligned}$$

Similarly testing the subtracted two equations (4.9) with $A\omega$ we get

$$\begin{aligned}\frac{M}{2\operatorname{Pr}} \frac{d}{dt} \|\omega\|^2 + \frac{1}{\operatorname{Pr}} (b(\overline{u}, \overline{\omega}, A\omega) - b(\widehat{u}, \widehat{\omega}, A\omega)) \\ + \frac{N}{1-N} \big[4\|\omega\|^2 - 2(\operatorname{rot} u, A\omega)\big] + L|A\omega|^2 = 0.\end{aligned}$$

Subtracting (4.10) written for $\overline{\theta}$ and $\widehat{\theta}$ and testing with θ we obtain

$$\frac{1}{2}\frac{d}{dt}|\theta|^2 + \|\theta\|^2 + b(\overline{u},\overline{\theta},\theta) - b(\widehat{u},\widehat{\theta},\theta)$$
$$= -D\big[(\overline{\theta}, \operatorname{rot}\overline{\omega}\cdot\nabla\theta) - (\widehat{\theta}, \operatorname{rot}\widehat{\omega}\cdot\nabla\theta)\big] + D\left(\frac{\partial\omega}{\partial x_1},\theta\right) + (u_2,\theta).$$

We must estimate four differences of the trilinear terms. We have

$$\begin{aligned}
-(b_S(\overline{u},\overline{u},A_Su) - b_S(\widehat{u},\widehat{u},A_Su)) &= -\,(b_S(\overline{u},u,A_Su) + b_S(u,\widehat{u},A_Su))\\
&\le C(\|u\|^{1/2}\|\overline{u}\||A_Su|^{3/2} + |u|^{1/2}\|\widehat{u}\||A_Su|^{3/2})\\
&\le \varepsilon|A_Su|^2 + C(\varepsilon)\|u\|^2(\|\overline{u}\|^4 + \|\widehat{u}\|^4),\\
-(b(\overline{u},\overline{\omega},A\omega) - b(\widehat{u},\widehat{\omega},A\omega)) &= -\,(b(\overline{u},\omega,A\omega) + b(u,\widehat{\omega},A\omega))\\
&\le C(\|\overline{u}\|\|\omega\|^{1/2}|A\omega|^{3/2} + \|u\||A\widehat{\omega}||A\omega|)\\
&\le \varepsilon|A\omega|^2 + C(\varepsilon)(\|\omega\|^2\|\overline{u}\|^4 + \|u\|^2|A\widehat{\omega}|^2),\\
-(b(\overline{u},\overline{\theta},\theta) - b(\widehat{u},\widehat{\theta},\theta)) &= -\,b(u,\widehat{\theta},\theta) = b(u,\theta,\widehat{\theta})\\
&\le C\|u\|\|\theta\|\|\widehat{\theta}\| \le \varepsilon\|\theta\|^2 + C(\varepsilon)\|u\|^2\|\widehat{\theta}\|^2,\\
-\big[(\overline{\theta}, \operatorname{rot}\overline{\omega}\cdot\nabla\theta) - (\widehat{\theta}, \operatorname{rot}\widehat{\omega}\cdot\nabla\theta)\big] &= -\,(\overline{\theta}, \operatorname{rot}\omega\cdot\nabla\theta)\\
&\le C|\overline{\theta}|^{1/2}\|\overline{\theta}\|^{1/2}\|\omega\|^{1/2}|A\omega|^{1/2}\|\theta\|\\
&\le \varepsilon\|\theta\|^2 + \varepsilon|A\omega|^2 + C(\varepsilon)|\overline{\theta}|^2\|\overline{\theta}\|^2\|\omega\|^2.
\end{aligned}$$

The bilinear terms can be estimated in the following way

$$\begin{aligned}
\frac{2N}{1-N}(\operatorname{rot}\,\omega, A_Su) &\le \varepsilon|A_Su|^2 + C(\varepsilon)\|\omega\|^2,\\
\mathrm{Ra}\,(\theta e_2, A_Su) &\le \varepsilon|A_Su|^2 + C(\varepsilon)|\theta|^2,\\
\frac{2N}{1-N}(\operatorname{rot}\,u, A\omega) &\le \varepsilon|A\omega|^2 + C(\varepsilon)\|u\|^2,\\
D\left(\frac{\partial\omega}{\partial x_1},\theta\right) &\le \varepsilon\|\theta\|^2 + C(\varepsilon)\|\omega\|^2,\\
(u_2,\theta) &\le \varepsilon\|\theta\|^2 + C(\varepsilon)\|u\|^2.
\end{aligned}$$

Combining all above estimates and choosing appropriately small ε yields

$$\frac{d}{dt}\left(\frac{1}{\mathrm{Pr}}\|u\|^2 + \frac{M}{\mathrm{Pr}}\|\omega\|^2 + |\theta|^2\right) + |A_Su|^2 + L|A\omega|^2 + \|\theta\|^2$$
$$\le C|\theta|^2 + C\|\omega\|^2(1 + \|\overline{u}\|^4 + |\overline{\theta}|^2\|\overline{\theta}\|^2) + C\|u\|^2(1 + \|\widehat{u}\|^4 + \|\overline{u}\|^4 + |A\widehat{\omega}|^2 + \|\widehat{\theta}\|^2)$$
$$= a(t)\left(\frac{1}{\mathrm{Pr}}\|u\|^2 + \frac{M}{\mathrm{Pr}}\|\omega\|^2 + |\theta|^2\right),$$

where $a(t) \in L^1(0,\tau)$ is nonnegative. The assertion follows from the Gronwall lemma. □

We continue the bootstrapping by showing the additional regularity result on the temperature. We prove the following theorem.

Theorem 6.5. *Let* (u,ω,θ) *solve the problem given in Definition* 4.1 *with the initial data* $u_0 \in V_S$, $\omega_0 \in V$, $\theta_0 \in H$. *Then, for each* $\rho > 0$, *there holds*

$$\theta \in L^2(\rho,\tau;D(A)) \cap C([\rho,\tau],V) \cap W^{1,2}(\rho,\tau;H).$$

If, moreover, $\theta_0 \in V$, *then*

$$\theta \in L^2(0,\tau;D(A)) \cap C([0,\tau],V) \cap W^{1,2}(0,\tau;H).$$

Proof. The estimates are done on the level of Galerkin projections and they are preserved in weak limits. We test (4.10) with $A\theta$ which yields, after integrating by parts the term $(\theta, \operatorname{rot}\omega\cdot\nabla A\theta)$

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + b(u,\theta,A\theta) + |A\theta|^2 = -D(\operatorname{rot}\,\omega\cdot\nabla\theta, A\theta) + D\bigg(\frac{\partial\omega}{\partial x_1}, A\theta\bigg) + (u_2, A\theta).$$

We estimate

$$\begin{aligned} |b(u,\theta,A\theta)| &\le C|u|^{1/2}\|u\|^{1/2}\|\theta\|^{1/2}|A\theta|^{3/2} \le \varepsilon|A\theta|^2 + C(\varepsilon)|u|^2\|u\|^2\|\theta\|^2,\\ D\bigg(\frac{\partial\omega}{\partial x_1}, A\theta\bigg) &\le D\|\omega\||A\theta| \le \varepsilon|A\theta|^2 + C(\varepsilon)\|\omega\|^2,\\ (u_2,A\theta) &\le |u||A\theta| \le \varepsilon|A\theta|^2 + C(\varepsilon)|u|^2. \end{aligned}$$

By the Hölder inequality and (3.6) we obtain

$$\begin{aligned} D\int_\Omega(\operatorname{rot}\,\omega\cdot\nabla\theta)A\theta &\le C\|\omega\|^{1/2}|A\omega|^{1/2}\|\theta\|^{1/2}|A\theta|^{3/2}\\ &\le \varepsilon|A\theta|^2 + C(\varepsilon)\|\omega\|^2|A\omega|^2\|\theta\|^2. \end{aligned}$$

With these bounds the following estimate holds:

$$\frac{d}{dt}\|\theta\|^2 + |A\theta|^2 \le C(|u|^2\|u\|^2 + \|\omega\|^2|A\omega|^2)\|\theta\|^2 + C \le \|\omega\|^2 + |u|^2). \tag{6.4}$$

Clearly, we have the sufficient bounds on $|u|^2\|u\|^2 + \|\omega\|^2|A\omega|^2$ and $\|\omega\|^2+|u|^2$ to apply the Gronwall lemma (if $\theta_0\in V$) and uniform Gronwall lemma (if $\theta_0 \in H$) to get the required regularity of θ. To get the regularity of θ_t note that as

$$\theta_t = -u\cdot\nabla\theta - A\theta + D\operatorname{rot}\omega\cdot\nabla\theta + D\,\frac{\partial\omega}{\partial x_1} + u_2,$$

we need to show that two nonlinear terms belong to L^2 with respect to time with values in H. Indeed, we use the embedding $H^1(\Omega)\subset L^4(\Omega)$ and Lemma 3.2 to get

$$\begin{aligned} \int_\rho^\tau\int_\Omega |u\cdot\nabla\theta|^2 &\le \int_\rho^\tau \|u\|_{L^4}^2\|\nabla\theta\|_{L^4}^2\\ &\le C\int_\rho^\tau \|u\|^2|A\theta|^2 \le C\|u\|^2_{L^\infty(\rho,\tau;V_S)}\|\theta\|^2_{L^2(\rho,\tau;D(A))} \end{aligned}$$

and, similarly, by (3.6),

$$\int_\rho^\tau \int_\Omega |\operatorname{rot}\,\omega\cdot\nabla\theta| \le \int_\rho^\tau \|\nabla\omega\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 \le C\int_\rho^\tau \|\omega\||A\omega|\|\theta\||A\theta|$$
$$\le C\|\omega\|_{L^\infty(\rho,\tau;V)}\|\theta\|_{L^\infty(\rho,\tau;V)}\|\omega\|_{L^2(\varepsilon,\tau;D(A))}\|\theta\|_{L^2(\rho,\tau;D(A))},$$

and the regularity from Theorem 6.4 yields the assertion. □

We summarize all results of this section in the following theorem which will be useful later.

THEOREM 6.6. *Let (u,ω,θ) solve the problem given in Definition* 4.1 *with the initial data $(u_0,\omega_0,\theta_0)\in H_S\times H\times H$. Then, for any $\rho>0$, we have*

$$u\in L^2(\rho,\tau;D(A_S))\cap C([\rho,\tau],V_S)\cap W^{1,2}(\rho,\tau;H_S),$$
$$\omega,\theta\in L^2(\rho,\tau;D(A))\cap C([\rho,\tau],V)\cap W^{1,2}(\rho,\tau;H).$$

If $(u_0,\omega_0,\theta_0)\in V_S\times V\times V$ then the above regularities hold with $\rho=0$.

7. Strong solutions and further bootstrapping

This section is devoted to the strong solutions for the considered problem. We formulate the definition of the strong solution and prove the theorem on its existence, uniqueness, and continuous dependence on the data. We also establish the relation between strong and weak solutions. Moreover we prove that the regularity can be further increased, indeed, for the the strong solutions we instantaneously get $(u_t,\omega_t,\theta_t)\in C([\rho,\tau];H_S\times H\times H)$ and $(u,\omega,\theta)\in C_w([\rho,\tau];D(A_S)\times D(A)\times D(A))$. Although we stop at this point, it is possible to continue the bootstrapping procedure and establish the H^s regularity of solutions for arbitrarily large s. We also remark that the estimates needed to obtain the regularity $(u,\omega,\theta)\in C_w([\rho,\tau];D(A_S)\times D(A)\times D(A))$ will be later useful to obtain the compactness in $V_S\times V\times V$ of the absorbing sets.

DEFINITION 7.1. Let $\tau>0$, $u_0\in V_S$, $\omega_0\in V$ and $\theta_0\in V$. By a strong solution of problems (4.1)–(4.6) we mean a triple of functions (u,ω,θ),

$$u\in L^2(0,\tau;D(A_S))\cap C([0,\tau],V_S)\cap W^{1,2}(0,\tau;H_S),$$
$$\omega,\theta\in L^2(0,\tau;D(A))\cap C([0,\tau],V)\cap W^{1,2}(0,\tau;H),$$

such that $u(0)=u_0$, $\omega(0)=\omega_0$, $\theta(0)=\theta_0$ and the following identities hold

$$\frac{1}{\Pr}\left(\frac{d}{dt}(u(t),\varphi)+b_S(u(t),u(t),\varphi)\right)+(-\Delta u(t),\varphi)$$
$$+\frac{N}{1-N}[(-\Delta u(t),\varphi)-2(\operatorname{rot}\,\omega(t),\varphi)]=\operatorname{Ra}(\theta(t)e_2,\varphi) \tag{7.1}$$

for every $\varphi \in H_S$,

$$(7.2)\quad \frac{M}{\Pr}\bigg(\frac{d}{dt}(\omega(t),\psi) + b(u(t),\omega(t),\psi)\bigg) + \frac{N}{1-N}\big[4(\omega(t),\psi) - 2(\operatorname{rot} u(t),\psi)\big] + L(-\Delta\omega(t),\psi) = 0,$$

for every $\psi \in H$,

$$(7.3)\quad \frac{d}{dt}(\theta(t),\eta) + b(u(t),\theta(t),\eta) + (-\Delta\theta(t),\eta) = D(\operatorname{rot}\,\omega(t)\cdot\nabla\theta(t),\eta) + D\bigg(\frac{\partial\omega}{\partial x_1}(t),\eta\bigg) + (u_2(t),\eta)$$

for every $\eta \in H$, in the sense of scalar distributions on $(0,\tau)$.

THEOREM 7.2. *Let $u_0 \in V_S$, $\omega_0 \in V$, $\theta_0 \in V$ and $\tau > 0$. There is a unique strong solution (u,ω,θ) in the sense of Definition* 7.1. *Moreover, the strong solution depends continuously on the initial data, namely, for each $t \geq 0$, the map*

$$V_S \times V \times V \ni (u_0,\omega_0,\theta_0) \mapsto (u(t),\omega(t),\theta(t)) \in V_S \times V \times V$$

is continuous. The uniqueness of the strong solution holds also in the class of weak solutions given by Definition 4.1. *Moreover, if $(u_0,\omega_0,\theta_0) \in H_S \times H \times H$ and (u,ω,θ) is the weak solution with this initial data, then for any $\rho > 0$ this solution restricted to $[\rho,\tau]$ is strong.*

PROOF. Theorem 5.1 implies the existence of the weak solution for the given initial data. In view of Theorem 6.6 for $\rho = 0$ this weak solution has the desired regularity. We can integrate by parts all terms in the definition of the weak solution and the density of embeddings $V_S \subset H_S$ and $V \subset H$ implies that this solution is in fact strong. The uniqueness follows immediately from Theorem 6.4. The weak-strong uniqueness follows from the fact that for $(u_0,\omega_0,\theta_0) \in V_S \times V \times V$ each weak solution is strong and strong solutions are unique. The fact that weak solution restricted to $[\rho,\tau]$ is strong follows immediately from Theorem 6.6 for $\rho > 0$. It remains to show the continuous dependence on the data and by the estimates used in the proof of Theorem 6.4 it suffices to show the continuous dependence of θ. Let $(\overline{u}_0,\overline{\omega}_0,\overline{\theta}_0)$ and $(\widehat{u}_0,\widehat{\omega}_0,\widehat{\theta}_0)$ be two initial states, and $(\overline{u},\overline{\omega},\overline{\theta})$, $(\widehat{u},\widehat{\omega},\widehat{\theta})$ be two corresponding strong solutions. If we set $(u,\omega,\theta) = (\overline{u}-\widehat{u},\overline{\omega}-\widehat{\omega},\overline{\theta}-\widehat{\theta})$, then θ satisfies

$$\theta_t - \Delta\theta = D\operatorname{rot}\overline{\omega}\cdot\nabla\overline{\theta} - D\operatorname{rot}\widehat{\omega}\cdot\nabla\widehat{\theta} + D\,\frac{\partial\omega}{\partial x_1} + u_2 - \overline{u}\cdot\nabla\overline{\theta} + \widehat{u}\cdot\nabla\widehat{\theta}.$$

We test this equation with $A\theta$ which yields

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + |A\theta|^2 = D\left(\frac{\partial\omega}{\partial x_1}, A\theta\right) + (u_2, A\theta)$$
$$+ D\int_\Omega (\operatorname{rot}\,\omega\cdot\nabla\overline{\theta})A\theta + D\int_\Omega (\operatorname{rot}\,\widehat{\omega}\cdot\nabla\theta)A\theta - b(u,\overline{\theta},A\theta) - b(\widehat{u},\theta,A\theta).$$

We estimate as above

$$\begin{aligned}\frac{d}{dt}\|\theta\|^2 + 2|A\theta|^2 &\le \varepsilon|A\theta|^2 + C(\varepsilon)\|\omega\|^2 + \varepsilon|A\theta|^2 + C(\varepsilon)|u|^2\\ &\quad + C\|\omega\|^{1/2}|A\omega|^{1/2}\|\overline{\theta}\|^{1/2}|A\overline{\theta}|^{1/2}|A\theta|\\ &\quad + C\|\widehat{\omega}\|^{1/2}|A\widehat{\omega}|^{1/2}\|\theta\|^{1/2}|A\theta|^{3/2} + C\|u\||A\overline{\theta}||A\theta| + C\|\widehat{u}\|\|\theta\|^{1/2}|A\theta|^{3/2}\\ &\le \varepsilon|A\theta|^2 + C(\varepsilon)\|\omega\|^2 + C(\varepsilon)\|u\|^2 + \varepsilon|A\theta|^2 + C(\varepsilon)\|\omega\||A\omega|\|\overline{\theta}\||A\overline{\theta}|\\ &\quad + \varepsilon|A\theta|^2 + C(\varepsilon)\|\widehat{\omega}\|^2|A\widehat{\omega}|^2\|\theta\|^2 + \varepsilon|A\theta|^2\\ &\quad + C(\varepsilon)\|u\|^2|A\overline{\theta}|^2 + \varepsilon|A\theta|^2 + C(\varepsilon)\|\widehat{u}\|^4\|\theta\|^2\\ &\le \varepsilon|A\theta|^2 + \varepsilon|A\omega|^2 + C\|\omega\|^2 + C\|u\|^2 + C\|\omega\|^2\|\overline{\theta}\|^2|A\overline{\theta}|^2\\ &\quad + C\|\widehat{\omega}\|^2|A\widehat{\omega}|^2\|\theta\|^2 + C\|u\|^2|A\overline{\theta}|^2 + C\|\widehat{u}\|^4\|\theta\|^2.\end{aligned}$$

The assertion follows by combining this bound with the estimates from the proof of Theorem 6.4. □

In the next theorem we establish additional instantaneous regularity of strong solutions. Although it can be considered as the regularity result of independent interest, its estimates will be later useful to get the set which is absorbing for the strong solutions and compact in $V_S \times V \times V$. Note that some of the estimates of the next theorem are similar to the ones from [34].

THEOREM 7.3. *Let $(u_0,\omega_0,\theta_0) \in V_S \times V \times V$ and let (u,ω,θ) be the strong solution with the initial data (u_0,ω_0,θ_0). For each $\rho > 0$ we have $(u_t,\omega_t,\theta_t) \in C([\rho,\tau]; H_S \times H \times H)$ and $(u,\omega,\theta) \in C_w([\rho,\tau]; D(A_S)\times D(A)\times D(A))$.*

PROOF. Differentiating the system (4.1)–(4.4) with respect to time we get

$$\frac{1}{\Pr}(u_{tt} + (u_t\cdot\nabla)u + (u\cdot\nabla)u_t + \nabla p_t) = \Delta u_t + \frac{N}{1-N}(2\operatorname{rot}\omega_t + \Delta u_t) + e_2 \mathrm{Ra}\,\theta_t,$$
$$\operatorname{div} u_t = 0,$$
$$\frac{M}{\Pr}(\omega_{tt} + u_t\cdot\nabla\omega + u\cdot\nabla\omega_t) = L\Delta\omega_t + 2\frac{N}{1-N}(\operatorname{rot}\,u_t - 2\omega_t),$$
$$\theta_{tt} + u_t\cdot\nabla\theta + u\cdot\nabla\theta_t = \Delta\theta_t + D\operatorname{rot}\omega_t\cdot\nabla\theta + D\operatorname{rot}\omega\cdot\nabla\theta_t + D\frac{\partial\omega_t}{\partial x_1} + u_{2t}.$$

We test the first of the above three equations with u_t, the third one with ω_t, and the last one with θ_t and add the three resultant equations which gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\bigg(\frac{1}{\mathrm{Pr}}|u_t|^2+\frac{M}{\mathrm{Pr}}|\omega_t|^2+|\theta_t|^2\bigg)\\&+b_S(u_t,u,u_t)+b(u_t,\omega,\omega_t)+b(u_t,\theta,\theta_t)+\|u_t\|^2+L\|\omega_t\|^2+\|\theta_t\|^2\\&\quad+\frac{N}{1-N}(\|u_t\|^2-2(\operatorname{rot}\,\omega_t,u_t)-2(\operatorname{rot}\,u_t,\omega_t)+4|\omega_t|^2)\\&\qquad=D(\operatorname{rot}\,\omega_t\cdot\nabla\theta,\omega_t)+D\bigg(\frac{\partial\omega_t}{\partial x_1},\theta_t\bigg)+(1+\mathrm{Ra})(\theta_t e_2,u_t).\end{aligned}$$

Clearly $\|u_t\|^2-2(\operatorname{rot}\,\omega_t,u_t)-2(\operatorname{rot}\,u_t,\omega_t)+4|\omega_t|^2=|\omega_t-2\operatorname{rot}u_t|^2$. We estimate the remaining terms

$$\begin{aligned}|b_S(u_t,u,u_t)|&\le C|u_t|\|u_t\|\|u\|\le\varepsilon\|u_t\|^2+C(\varepsilon)\|u\|^2|u_t|^2,\\|b(u_t,\omega,\omega_t)|&\le C|u_t|^{1/2}\|u_t\|^{1/2}\|\omega\||\omega_t|^{1/2}\|\omega_t\|^{1/2}\\&\le C|u_t|\|u_t\|\|\omega\|+C|\omega_t|\|\omega_t\|\|\omega\|\\&\le\varepsilon\|u_t\|^2+\varepsilon\|\omega_t\|^2+C(\varepsilon)\|\omega\|^2(|u_t|^2+|\omega_t|^2),\\|b(u_t,\theta,\theta_t)|&\le C|u_t|^{1/2}\|u_t\|^{1/2}\|\theta\||\theta_t|^{1/2}\|\theta_t\|^{1/2}\\&\le C|u_t|\|u_t\|\|\theta\|+C|\theta_t|\|\theta_t\|\|\theta\|,\\&\le\varepsilon\|u_t\|^2+\varepsilon\|\theta_t\|^2+C(\varepsilon)\|\theta\|^2(|u_t|^2+|\theta_t|^2),\\D(\operatorname{rot}\,\omega_t\cdot\nabla\theta,\omega_t)&\le C\|\omega_t\|^{3/2}|A\theta|^{1/2}\|\theta\|^{1/2}|\omega_t|^{1/2}\\&\le\varepsilon\|\omega_t\|^2+C(\varepsilon)|A\theta|^2\|\theta\|^2|\omega_t|^2,\\D\bigg(\frac{\partial\omega_t}{\partial x_1},\theta_t\bigg)&\le\varepsilon\|\omega_t\|^2+C(\varepsilon)|\theta_t|^2,\\(1+\mathrm{Ra})(\theta_t e_2,u_t)&\le\varepsilon\|\theta_t\|^2+C(\varepsilon)|u_t|^2.\end{aligned}$$

Putting together all the bounds we obtain

$$\begin{aligned}\frac{d}{dt}&\bigg(\frac{1}{\mathrm{Pr}}|u_t|^2+\frac{M}{\mathrm{Pr}}|\omega_t|^2+|\theta_t|^2\bigg)+\|u_t\|^2+L\|\omega_t\|^2+\|\theta_t\|^2\\&\le C(1+\|u\|^2+\|\omega\|^2+\|\theta\|^2)|u_t|^2\\&\quad+C(\|\omega\|^2+|A\theta|^2\|\theta\|^2)|\omega_t|^2+C(\|\theta\|^2+1)|\theta_t|^2\\&\le C(1+\|u\|^2+\|\omega\|^2+\|\theta\|^2+|A\theta|^2\|\theta\|^2)\\&\quad\cdot\bigg(\frac{1}{\mathrm{Pr}}|u_t|^2+\frac{M}{\mathrm{Pr}}|\omega_t|^2+|\theta_t|^2\bigg).\end{aligned}\tag{7.4}$$

Since we have $(|u_t|^2/\mathrm{Pr}+M|\omega_t|^2/\mathrm{Pr}+|\theta_t|^2)\in L^1(0,\tau)$ and $1+\|u\|^2+\|\omega\|^2+\|\theta\|^2+|A\theta|^2\|\theta\|^2\in L^1(0,\tau)$, we apply the uniform Gronwall lemma which implies that $(u_t,\omega_t,\theta_t)\in L^\infty(\rho,\tau;H_S\times H\times H)$. Integrating from ρ to τ (7.4), it follows that $(u_t,\omega_t,\theta_t)\in L^2(\rho,\tau;V_S\times V\times V)$.

For u_{tt} and ω_{tt} there hold the bounds in $L^2(\rho,\tau;V_S^*)$ and $L^2(\rho,\tau;V^*)$, respectively, which imply that $(u_t,\omega_t)\in C([\rho,\tau];H_S\times H)$. Estimates are similar to those in the proof of Theorem 5.1. We provide the estimate only for u_{tt} (the estimate of ω_{tt} is similar to that of u_{tt}). Assuming that $v\in L^2(\rho,\tau;V_S)$ we obtain

$$\begin{aligned}\langle u_{tt},v\rangle = -b_S(u_t,u,v)-b_S(u,u_t,v)&\\ -\frac{\Pr}{1-N}(\nabla u_t,\nabla v)+\frac{2\Pr N}{1-N}(\operatorname{rot}\omega_t,v)+\mathrm{Ra}\ \Pr(\theta_t e_2,v).&\end{aligned}$$

All terms on the right-hand side constitute linear and continuous functionals of the variable v on the space $L^2(\rho,\tau;V_S)$ and their norms in $L^2(\rho,\tau;V_S^*)$ are bounded. Indeed, for linear terms the assertion is obvious, while for terms with b_S we have the bounds

$$\begin{aligned}\int_\rho^\tau |b_S(u_t,u,v)| &\le C\int_\rho^\tau \|u\|\|u_t\|\|v\|\\ &\le C\|u\|_{L^\infty(\rho,\tau;V_S)}\|u_t\|_{L^2(\rho,\tau;V_S)}\|v\|_{L^2(\rho,\tau;V_S)},\\ \int_\rho^\tau |b_S(u,u_t,v)| &\le C\int_\rho^\tau \|u\|\|u_t\|\|v\|\\ &\le C\|u\|_{L^\infty(\rho,\tau;V_S)}\|u_t\|_{L^2(\rho,\tau;V_S)}\|v\|_{L^2(\rho,\tau;V_S)}.\end{aligned}$$

To deal with θ_{tt} we will show the bound in $L^2(\rho,\tau;V^*)$ which will imply that $\theta_t\in C([\rho,\tau];H)$. Taking the test function $\phi\in L^2(\rho,\tau;V)$ we obtain

$$\begin{aligned}\langle\theta_{tt},\phi\rangle = -b(u_t,\theta,\phi)-b(u,\theta_t,\phi)-(\nabla\theta_t,\nabla\phi)&\\ +D\langle\operatorname{rot}\omega_t\cdot\nabla\theta,\phi\rangle+D\langle\operatorname{rot}\omega\cdot\nabla\theta_t,\phi\rangle+D\left(\frac{\partial\omega}{\partial x_1},\phi\right)+(u_2,\phi).&\end{aligned}$$

We only show the bounds for the nonlinear terms

$$\begin{aligned}\int_\rho^\tau |b(u_t,\theta,\phi)| &\le C\int_\rho^\tau \|\theta\|\|u_t\|\|\phi\| \le C\|\theta\|_{L^\infty(\rho,\tau;V)}\|u_t\|_{L^2(\rho,\tau;V_S)}\|\phi\|_{L^2(\rho,\tau;V)},\\ \int_\rho^\tau |b(u,\theta_t,\phi)| &\le C\int_\rho^\tau \|u\|\|\theta_t\|\|\phi\| \le C\|u\|_{L^\infty(\rho,\tau;V_S)}\|\theta_t\|_{L^2(\rho,\tau;V)}\|\phi\|_{L^2(\rho,\tau;V)}.\end{aligned}$$

To derive the estimates in last two nonlinear terms we first integrate by parts

$$\begin{aligned}\int_\Omega \operatorname{rot}\omega_t\cdot\nabla\theta\phi &= -\int_\Omega \omega_t\nabla\theta\cdot\operatorname{rot}\phi \le C|\omega_t|^{1/2}\|\omega_t\|^{1/2}|A\theta|^{1/2}\|\theta\|^{1/2}\|\phi\|,\\ \int_\Omega \operatorname{rot}\omega\cdot\nabla\theta_t\phi &= -\int_\Omega \operatorname{rot}\omega\cdot\nabla\phi\,\theta_t \le C|\theta_t|^{1/2}\|\theta_t\|^{1/2}|A\omega|^{1/2}\|\omega\|^{1/2}\|\phi\|.\end{aligned}$$

Next we estimate the time integrals

$$\int_\rho^\tau \int_\Omega \operatorname{rot}\omega_t \cdot \nabla\theta\phi \\ \leq C\|\omega_t\|^{1/2}_{L^\infty(\rho,\tau;H)}\|\theta\|^{1/2}_{L^\infty(\rho,\tau;V)}\|\omega_t\|^{1/2}_{L^2(\rho,\tau;V)}|\theta|^{1/2}_{L^2(\rho,\tau;D(A))}\|\phi\|_{L^2(\rho,\tau;V)},$$

$$\int_\rho^\tau \int_\Omega \operatorname{rot}\omega \cdot \nabla\theta_t\phi \\ \leq C\|\theta_t\|^{1/2}_{L^\infty(\rho,\tau;H)}\|\omega\|^{1/2}_{L^\infty(\rho,\tau;V)}\|\theta_t\|^{1/2}_{L^2(\rho,\tau;V)}|\omega|^{1/2}_{L^2(\rho,\tau;D(A))}\|\phi\|_{L^2(\rho,\tau;V)}.$$

We have proved that it makes sense to consider pointwise values of $(u_t, \omega_t, \theta_t)$ on $[\rho, \tau]$ as elements in $H_S \times H \times H$. We come back to (6.3). It implies that

$$|A_S u|^2 + L|A\omega|^2 \\ \leq C(|\theta|^2 + |u|^2\|u\|^4 + \|\omega\|^2 + |u|^2\|u\|^2\|\omega\|^2 + \|u\|^2 + |u_t|^2 + |\omega_t|^2),$$

and since the right-hand side belongs to $L^\infty(\rho,\tau)$ it follows that $|A_S u|^2 + L|A\omega|^2 \in L^\infty(\rho,\tau)$. To obtain the bound for θ, note that (6.4) yields

$$|A\theta|^2 \leq C(|u|^2\|u\|^2 + \|\omega\|^2|A\omega|^2)\|\theta\|^2 + C(\|\omega\|^2 + |u|^2 + |\theta_t|^2),$$

and the proof is complete as from previous estimates the right-hand side belongs to $L^\infty(\rho,\tau)$. The assertion follows from the Lions–Magenes lemma in [21]. □

Note that although we stop on $D(A_S) \times D(A) \times D(A)$, it is possible to continuous the bootstrapping and obtain the regularity results in higher order Sobolev norms. These regularity results would later lead to higher regularity results on the attractor. Since this results are technical and required estimates are analogous to the ones that we derive in the present paper, we decide to stop at H^2 regularity.

8. Global attractor for weak solutions

Denote by $\mathcal{H}$ the product Hilbert space $H_S \times H \times H$ and by $\mathcal{V}$ the space $V_S \times V \times V$. We define the family of multivalued maps $\{S_\mathcal{H}(t)\}_{t\geq 0}$ with $S_\mathcal{H}(t)\colon \mathcal{H} \to 2^\mathcal{H}$ by

$$S_\mathcal{H}(t)(u_0, \omega_0, \theta_0) := \{(u(t), \omega(t), \theta(t))\}$$

where (u, ω, θ) is (possibly nonunique) weak solution *with temperature continuous at zero* (i.e. given by Definition 4.3) to the problem (4.1)–(4.6) with the initial data $(u_0, \omega_0, \theta_0) \in \mathcal{H}$. This class is nonempty by Theorem 5.1. Moreover, all bootstrapping results of previous sections are valid for a wider class of weak solutions given by Definition 4.1, so they also hold for weak solutions with temperature continuous at zero. We refer the reader to the appendix for the necessary abstract theorems and definitions.

We will prove the following theorem.

Theorem 8.1. *The family of maps* $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$ *is an m-semiflow which has a global attractor* $\mathcal{A}_{\mathcal{H}}$ (*in the sense of Definition* A.2). *This attractor is a bounded set in* $D(A_S)\times D(A)\times D(A)$.

The above theorem is a direct consequence of Theorem A.3 and Lemmas 8.2 and 8.3 below. Note that Theorem A.3 provides the existence of the attractor which is not necessarily the invariant set. By definition it is only compact and it is the smallest closed set in $\mathcal{H}$ which attracts (also in $\mathcal{H}$) all sets from $\mathcal{B}(\mathcal{H})$ ([2]). The reason for choosing such approach is the difficulty in proving the continuity or closedness of the graph for the m-semiflow $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$. However, in Section 9 we obtain the global attractor $\mathcal{A}_{\mathcal{V}}$ for strong solutions which is invariant and shown to be equal to $\mathcal{A}_{\mathcal{H}}$. This, eventually, yields the invariance of $\mathcal{A}_{\mathcal{H}}$ as well.

Lemma 8.2. *The family* $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$ *is an m-semiflow* (*in the sense of Definition* A.1).

Proof. Non-emptiness of $S_{\mathcal{H}}(t)(u_0,\omega_0,\theta_0)$ follows from Theorem 5.1. The same theorem implies that (a) of Definition A.1 holds. To prove (b) let $s>0$ and let $(u(t+s),\omega(t+s),\theta(t+s))\in S_{\mathcal{H}}(t+s)(u_0,\omega_0,\theta_0)$, with (u,ω,θ) being the weak solution. Theorem 7.2 implies that the function $(u(\,\cdot+s),\omega(\,\cdot+s),\theta(\,\cdot+s))$ is a strong solution on the interval $[0,t-s]$ with the initial data $(u(s),\omega(s),\theta(s))$ and hence it is also a weak solution (note that for strong solutions we have $\theta(\,\cdot\,+s)\in C([0,t-s];H)$ and this immediately implies that (4.11) holds at the initial point which implies (b)). □

Lemma 8.3. *The m-semiflow* $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$ *has an absorbing set* B_0 *which is closed and bounded in* $D(A_S)\times D(A)\times D(A)$.

Proof. We note that throughout the proof C and κ will denote generic positive constants which vary from line to line (or even in the same line) independent of time and the initial data. We will prove that there exists a closed ball B_0 in $D(A_S)\times D(A)\times D(A)$ such that $S_{\mathcal{H}}(t)B\subset B_0$ for any $B\in\mathcal{B}(\mathcal{H})$ and for all $t\geq t_0(B)>0$. To this end take $B\in\mathcal{B}(\mathcal{H})$ and the weak solution (u,ω,θ) with the initial data $(u_0,\omega_0,\theta_0)\in B$.

Step 1. Maximum principle in L^2 *for temperature.* Let $\rho\in(0,\tau)$. We know that $T(x_1,x_2,t)=\theta(x_1,x_2,t)+(1-x_2)\in L^2(\rho,\tau;H^2(\Omega))\cap C([\rho,\tau],H^1(\Omega))\cap W^{1,2}(\rho,\tau;H)$ and T satisfies the boundary conditions (2.2) so

$$(T-1)_+\in C([\rho,\tau],V)\cap W^{1,2}(\rho,\tau;H).$$

We have

$$T_t+u\cdot\nabla T-\Delta T=D\,\mathrm{rot}\,\omega\cdot\nabla T$$

([2]) The symbol $\mathcal{B}(\mathcal{H})$ denotes the family of all bounded subsets of $\mathcal{H}$.

as an equality in $L^2(\rho,\tau;H)$. We multiply it by $(T-1)_+$ and integrate over Ω:

$$\frac{1}{2}\frac{d}{dt}\,|(T-1)_+|^2+\|(T-1)_+\|^2=0, \tag{8.1}$$

because $\int_\Omega (u\cdot\nabla T)(T-1)_+=0$ and $\int_\Omega(\operatorname{rot}\,\omega\cdot\nabla T)(T-1)_+=0$. Indeed, the value of the first integral is well-known. As for the second one, we have

$$\int_\Omega |(\operatorname{rot}\,\omega\cdot\nabla T)(T-1)_+|\le C\|\operatorname{rot}\,\omega\|_{L^4}\|T\|\|(T-1)_+\|_{L^4}\le C|A\omega|\|T\|^2,$$

so it is well-defined and (due to boundary conditions and symmetry of second derivatives)

$$\begin{aligned}\int_\Omega(\operatorname{rot}\,\omega\cdot\nabla T)(T-1)_+&=\int_{T>1}(\operatorname{rot}\,\omega\cdot\nabla(T-1))(T-1)\\&=\frac{1}{2}\int_\Omega \operatorname{rot}\omega\cdot\nabla(T-1)_+^2\\&=-\frac{1}{2}\int_\Omega(\operatorname{div}\,\operatorname{rot}\omega)(T-1)_+^2+\frac{1}{2}\int_{\partial\Omega}(T-1)_+^2(\operatorname{rot}\,\omega\cdot\overrightarrow{n})\,dS=0.\end{aligned}$$

We use the Poincaré inequality in (8.1) and the Gronwall inequality to get

$$|(T-1)_+(t)|^2\le|(T(\rho)-1)_+|^2e^{-C(t-\rho)}\le C|T(\rho)|^2e^{-Ct},\quad t\ge\rho.$$

In a similar way we may show that

$$|T_-(t)|^2\le|(T(\rho))_-|^2e^{-C(t-\rho)}\le C|T(\rho)|^2e^{-Ct},\quad t\ge\rho.$$

Therefore, for $t\ge\rho$,

$$\begin{aligned}|T(t)|^2&=\int_{\{T<0\}}T^2+\int_{\{0\le T\le 1\}}T^2+\int_{\{T>1\}}T^2\\&\le|T_-|^2+|\Omega|+\int_{\{T>1\}}(T-1)^2+\int_{\{T>1\}}2T-\int_{\{T>1\}}1\\&\le|T_-|^2+|\Omega|+|(T-1)_+|^2+\int_{\{T>1\}}2\bigg(\frac{1}{\sqrt{2}}T\bigg)\sqrt{2}\end{aligned}$$

(by the Young inequality)

$$\le|T_-|^2+|\Omega|+|(T-1)_+|^2+\frac{1}{2}\int_\Omega T^2+2|\Omega|$$

and so

$$|T(t)|^2\le 2(|T_-(t)|^2+|(T-1)_+(t)|^2+3|\Omega|)\le C|T(\rho)|^2e^{-Ct}+C.$$

As $|T(\rho)|\le|\theta(\rho)|+C$ and $|\theta(t)|\le|T(t)|+C$, we deduce that

$$|\theta(t)|^2\le C|\theta(\rho)|^2e^{-Ct}+C\quad\text{for all }\rho>0,\ t\ge\rho.$$

By (4.11) we take $\limsup_{\rho\to0^+}$, which gives

$$|\theta(t)|^2\le C|\theta_0|^2e^{-Ct}+C\quad\text{for every }t>0.$$

Step 2. Bounds for $|u(t)|$ and $|\omega(t)|$ and absorbing set in $\mathcal{H}$. By (5.7) we get

$$\frac{1}{\Pr}\frac{d}{dt}(|u|^2 + M|\omega|^2) + \|u\|^2 + 2L\|\omega\|^2 \le C|\theta|^2 \le C + C|\theta_0|^2 e^{-Ct}$$

so we can apply the Poincaré inequality and get

$$y'(s) + \kappa y(s) \le C + C|\theta_0|^2 e^{-Cs},$$

where $y(s) := |u(s)|^2 + M|\omega(s)|^2$ and κ is a constant. By the Gronwall lemma and some simple calculations we get

$$y(t) \le C + y(0)e^{-Ct} + C|\theta_0|^2 e^{-Ct}.$$

It follows that

$$|\theta(t)|^2 + |u(t)|^2 + |\omega(t)|^2 \le C + C(|u_0|^2 + |\omega_0|^2 + |\theta_0|^2)e^{-Ct}.$$

Hence, there exists B_1, a closed and bounded set in $\mathcal{H}$ (actually a ball) such that for every $B \in \mathcal{B}(\mathcal{H})$ we can find $t_1(B)$ such that for every $t \ge t_1$ we have $(u(t), \omega(t), \theta(t)) \in B_1$ for all solutions starting from B.

Step 3. Bounds for $\|u(t)\|$, $\|\omega(t)\|$, and $\|\theta(t)\|$ and absorbing set in $\mathcal{V}$. From Step 2, for $t \ge t_1(B)$,

$$|u(t)|^2 + |\omega(t)|^2 + |\theta(t)|^2 \le C,$$

for all solutions starting from B. Assume that $t \ge t_1$. From (5.10), noting that the constants in (5.10) are independent of time and initial data, we get

$$\frac{d}{dt}(|u|^2 + M|\omega|^2 + |\theta|^2) + C(\|u\|^2 + \|\omega\|^2 + \|\theta\|^2) \le C$$

We can integrate this inequality between t and $t+1$

$$|u(t+1)|^2 + M|\omega(t+1)|^2 + |\theta(t+1)|^2 + C\int_t^{t+1} \|u(s)\|^2 + \|\omega(s)\|^2 + \|\theta(s)\|^2$$
$$\le |u(t)|^2 + M|\omega(t)|^2 + |\theta(t)|^2 + C \le C$$

so, for $t \ge t_1$,

$$\int_t^{t+1} \|u(s)\|^2 + \|\omega(s)\|^2 + \|\theta(s)\|^2 \le C.$$

Now, from (6.3) we get, for $t \ge t_1$,

$$\frac{d}{dt}(\|u\|^2 + M\|\omega\|^2) + \Pr|A_S u|^2 + \Pr L|A\omega|^2 \le C(1 + \|u\|^4 + \|\omega\|^2 + \|u\|^2\|\omega\|^2 + \|u\|^2), \tag{8.2}$$

and

$$\frac{d}{dt}(\|u\|^2 + M\|\omega\|^2) \le C + C(\|u\|^2 + M\|\omega\|^2)(\|u\|^2 + 1).$$

If we denote

$$y(t) = \|u(t)\|^2 + M\|\omega(t)\|^2, \qquad g(t) = C(\|u(t)\|^2 + 1), \qquad h(t) = C,$$

then y, g and h are locally integrable and there are constants $a_1, a_2, a_3 \geq 0$ such that

$$\int_t^{t+1} y(s)\,ds \leq a_1, \qquad \int_t^{t+1} g(s)\,ds \leq a_2, \qquad \int_t^{t+1} h(s)\,ds \leq a_3, \quad t \geq t_1.$$

We may use the uniform Gronwall lemma to get

$$\|u(t)\|^2 + M\|\omega(t)\|^2 \leq (a_2 + a_3)e^{a_1} = C \quad \text{for every } t \geq t_1 + 1.$$

From now on all bounds are derived for $t \geq t_1 + 1$. From (8.2), we have

$$\frac{d}{dt}(\|u\|^2 + M\|\omega\|^2) + \operatorname{Pr}|A_S u|^2 + \operatorname{Pr} L|A\omega|^2 \leq C.$$

We integrate this estimate between t and $t+1$ to get the bound

$$\|u(t+1)\|^2 + M\|\omega(t+1)\|^2 + \operatorname{Pr}\int_t^{t+1} |A_S u(s)|^2 + L|A\omega(s)|^2\,ds \leq C,$$

hence

$$\int_t^{t+1} |A_S u(s)|^2 + |A\omega(s)|^2\,ds \leq C.$$

Estimate (6.4) implies that

$$\frac{d}{dt}\|\theta\|^2 + |A\theta|^2 \leq C(1 + |A\omega|^2)\|\theta\|^2 + C. \tag{8.3}$$

Denoting $y(t) = \|\theta(t)\|^2$, $g(t) = C(1 + |A\omega(t)|^2)$, $h(t) = C$, we get $y'(t) \leq g(t)y(t) + h(t)$. There are positive constants a_1', a_2', a_3' such that

$$\int_t^{t+1} y(s)\,ds \leq a_1', \qquad \int_t^{t+1} g(s)\,ds \leq a_2', \qquad \int_t^{t+1} h(s)\,ds \leq a_3',$$

hence, by the uniform Gronwall lemma, $\|\theta(t)\|^2 \leq C$ for $t \geq t_1 + 2$. We integrate the estimate (8.3) between t and $t+1$, $t \geq t_1 + 2$ to get the bound

$$\int_t^{t+1} |A\theta(s)|^2\,ds \leq C.$$

Summing up, there exists a time $t_1 = t_1(B) > 0$ such that for $t \geq t_1 + 2$

$$\|u(t)\|^2 + \|\omega(t)\|^2 + \|\theta(t)\|^2 \leq C, \tag{8.4}$$

for all solutions starting from B. This proves that there exists B_2, a closed and bounded absorbing set in $\mathcal{V}$.

Step 4. Bounds for $|A_S u|$, $|A\omega|$, $|A\theta|$ and absorbing set in $D(A_S) \times D(A) \times D(A)$. We know that, for $t \geq t_1 + 2$, (8.4) holds. The next estimates will be derived for $t \geq t_1 + 2$. We test (7.1) with u_t, (7.2) with ω_t, and (7.3) with θ_t which yields

$$
\begin{aligned}
&\frac{1}{\mathrm{Pr}}(|u_t|^2 + b_S(u,u,u_t)) + \frac{1}{2}\frac{d}{dt}\,\|u\|^2 + \frac{N}{1-N}\left(\frac{1}{2}\frac{d}{dt}\,\|u\|^2 - 2(\,\mathrm{rot}\;\omega, u_t)\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \mathrm{Ra}\,(\theta e_2, u_t),\\
&\frac{M}{\mathrm{Pr}}(|\omega_t|^2 + b(u,\omega,\omega_t)) + \frac{2N}{1-N}\left(\frac{d}{dt}\,|\omega|^2 - (\,\mathrm{rot}\;u,\omega_t)\right) + L\,\frac{1}{2}\frac{d}{dt}\,\|\omega\|^2 = 0,\\
&|\theta_t|^2 + b(u,\theta,\theta_t) + \frac{1}{2}\frac{d}{dt}\,\|\theta\|^2 = D(\,\mathrm{rot}\;\omega\cdot\nabla\theta,\theta_t) + (u_2,\theta_t) + D\left(\frac{\partial\omega}{\partial x_1},\theta_t\right).
\end{aligned}
$$

We use the following estimates

$$
\begin{aligned}
(\,\mathrm{rot}\;\omega, u_t) &\leq \varepsilon|u_t|^2 + C(\varepsilon)\|\omega\|^2, \qquad (\theta e_2, u_t) \leq \varepsilon|u_t|^2 + C(\varepsilon)|\theta|^2,\\
(\,\mathrm{rot}\;u, \omega_t) &\leq \varepsilon|\omega_t|^2 + C(\varepsilon)\|u\|^2, \qquad (u_2, \theta_t) \leq \varepsilon|\theta_t|^2 + C(\varepsilon)|u|^2,\\
\left(\frac{\partial\omega}{\partial x_1},\theta_t\right) &\leq \varepsilon|\theta_t|^2 + C(\varepsilon)\|\omega\|^2,\\
|b_S(u,u,u_t)| &\leq C|u_t||u|^{1/2}\|u\||A_S u|^{1/2} \leq \varepsilon|u_t|^2 + C(\varepsilon)|u|\|u\|^2|A_S u|\\
&\leq \varepsilon|u_t|^2 + C(\varepsilon)|u|^2\|u\|^4 + C(\varepsilon)|A_S u|^2,\\
|b(u,\omega,\omega_t)| &\leq C|u|^{1/2}\|u\|^{1/2}\|\omega\|^{1/2}|A\omega|^{1/2}|\omega_t|\\
&\leq \varepsilon|\omega_t|^2 + C(\varepsilon)|u|\|u\|\|\omega\||A\omega|\\
&\leq \varepsilon|\omega_t|^2 + C(\varepsilon)|u|^2\|u\|^2\|\omega\|^2 + C(\varepsilon)|A\omega|^2\\
|b(u,\theta,\theta_t)| &\leq C|u|^{1/2}\|u\|^{1/2}\|\theta\|^{1/2}|A\theta|^{1/2}|\theta_t|\\
&\leq \varepsilon|\theta_t|^2 + C(\varepsilon)|u|\|u\|\|\theta\||A\theta|\\
&\leq \varepsilon|\theta_t|^2 + C(\varepsilon)|u|^2\|u\|^2\|\theta\|^2 + C(\varepsilon)|A\theta|^2\\
|(\,\mathrm{rot}\;\omega\cdot\nabla\theta,\theta_t)| &\leq C\|\omega\|^{1/2}|A\omega|^{1/2}\|\theta\|^{1/2}|A\theta|^{1/2}|\theta_t|\\
&\leq \varepsilon|\theta_t|^2 + C(\varepsilon)\|\omega\||A\omega|\|\theta\||A\theta|\\
&\leq \varepsilon|\theta_t|^2 + C(\varepsilon)\|\omega\|^2|A\omega|^2 + C(\varepsilon)\|\theta\|^2|A\theta|^2,
\end{aligned}
$$

whence the above system implies

$$
\begin{aligned}
&\frac{1}{\mathrm{Pr}}\,|u_t|^2 + \frac{1}{1-N}\frac{d}{dt}\,\|u\|^2 \leq C(|\theta|^2 + \|\omega\|^2 + |u|^2\|u\|^4 + |A_S u|^2),\\
&\frac{M}{\mathrm{Pr}}\,|\omega_t|^2 + \frac{4N}{1-N}\frac{d}{dt}\,|\omega|^2 + L\,\frac{d}{dt}\,\|\omega\|^2\\
&\qquad\qquad\qquad \leq C(\|u\|^2 + |u|^2\|u\|^2\|\omega\|^2 + |A\omega|^2 0,\\
&|\theta_t|^2 + \frac{d}{dt}\,\|\theta\|^2 \leq C\big(\|\omega\|^2|A\omega|^2 + \|\theta\|^2|A\theta|^2 + |u|^2\\
&\qquad\qquad\qquad + \|\omega\|^2 + |u|^2\|u\|^2\|\theta\|^2 + |A\theta|^2\big).
\end{aligned}
\tag{8.5}
$$

The bounds (8.5) imply that

$$\begin{aligned}\frac{1}{\mathrm{Pr}}\,|u_t|^2+\frac{1}{1-N}\frac{d}{dt}\,\|u\|^2 &\le C(1+|A_Su|^2),\\ \frac{M}{\mathrm{Pr}}\,|\omega_t|^2+\frac{4N}{1-N}\frac{d}{dt}\,|\omega|^2+L\,\frac{d}{dt}\,\|\omega\|^2 &\le C(1+|A\omega|^2),\\ |\theta_t|^2+\frac{d}{dt}\,\|\theta\|^2 &\le C(1+|A\omega|^2+|A\theta|^2).\end{aligned}$$

Integrating these bounds over the interval $(t,t+1)$ we deduce that

$$\int_t^{t+1}|u_t(s)|^2+|\omega_t(s)|^2+|\theta_t(s)|^2\;ds\le C.$$

From (7.4) we have the estimate

$$\frac{d}{dt}\bigg(\frac{1}{\mathrm{Pr}}\,|u_t|^2+\frac{M}{\mathrm{Pr}}\,|\omega_t|^2+|\theta_t|^2\bigg)\le C(1+|A\theta|^2)\bigg(\frac{1}{\mathrm{Pr}}\,|u_t|^2+\frac{M}{\mathrm{Pr}}\,|\omega_t|^2+|\theta_t|^2\bigg).$$

Denoting

$$y(t)=\frac{1}{\mathrm{Pr}}\,|u_t(t)|^2+\frac{M}{\mathrm{Pr}}\,|\omega_t(t)|^2+|\theta_t(t)|^2\quad\text{and}\quad g(t)=C(1+|A\theta(t)|^2),$$

we can use the uniform Gronwall lemma to deduce that $|u_t|^2+|\omega_t|^2+|\theta_t|^2\le C$, for all $t\ge t_1+3$. From (6.3) it follows that

$$\begin{aligned}|A_Su|^2+L|A\omega|^2\le C(|\theta|^2+|u|^2\|u\|^4+\|\omega\|^2+|u|^2\|u\|^2\|\omega\|^2+\|u\|^2)\\ -\frac{2}{\mathrm{Pr}}(A_Su,u_t)-\frac{2M}{\mathrm{Pr}}(A\omega,\omega_t),\end{aligned}$$

whence $|A_Su|^2+|A\omega|^2\le C$, whenever $t\ge t_1+3$. Finally (6.4) implies that

$$|A\theta|^2\le C(|u|^2\|u\|^2+\|\omega\|^2|A\omega|^2)\|\theta\|^2+C\le\|\omega\|^2+|u|^2)-2(A\theta,\theta_t),$$

whence we deduce that $|A\theta|^2\le C$, for $t\ge t_1+3$, and the proof is complete. □

9. Global attractor for strong solutions

Define $S_{\mathcal{V}}(t)=S_{\mathcal{H}}(t)\restriction_{\mathcal{V}}$, the restriction to $\mathcal{V}$ of the m-semiflow considered in Section 8. Theorem 7.2 implies that $S_{\mathcal{V}}$ is single-valued and

$$S_{\mathcal{V}}(u_0,\omega_0,\theta_0)=(u(t),\omega(t),\theta(t))\in\mathcal{V},$$

where (u,ω,θ) is the unique strong solution with the initial data (u_0,ω_0,θ_0). Moreover, by Theorem 7.2, for every $t\ge 0$, the map $S_{\mathcal{V}}(t)$ is continuous in $\mathcal{V}$. We have also the following lemma.

Lemma 9.1. *The family of maps* $\{S_{\mathcal{V}}(t)\}_{t\ge0}$ *is a semiflow* (*in the sense of Definition* A.4).

To prove the existence of a global attractor for strong solutions (cf. Definition A.5 and Theorem A.6) it is enough to show the existence of a compact absorbing set for $S_{\mathcal{V}}$. It follows from Lemma 8.3. Indeed, the same set which is absorbing for $S_{\mathcal{H}}$ is also absorbing for $S_{\mathcal{V}}$. Thus, we have

LEMMA 9.2. *The semiflow $\{S_{\mathcal{V}}(t)\}_{t\geq 0}$ has an absorbing set B_0 which is closed and bounded in $D(A_S)\times D(A)\times D(A)$ and therefore compact in $\mathcal{V}$.*

Now, we can immediately use Theorem A.6 to deduce the following result.

THEOREM 9.3. *The semiflow $\{S_{\mathcal{V}}(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}_V$, which is a bounded set in $D(A_S)\times D(A)\times D(A)$.*

As to the relation between $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{A}_{\mathcal{V}}$ we have the following

THEOREM 9.4. *The sets $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{A}_{\mathcal{V}}$ coincide.*

PROOF. We know that $\operatorname{dist}_{\mathcal{H}}(S_{\mathcal{H}}(t)\mathcal{A}_{\mathcal{V}},\mathcal{A}_{\mathcal{H}})\to 0$ as $t\to\infty$. But on $\mathcal{A}_{\mathcal{V}}$ the mappings $S_{\mathcal{H}}(t)$ and $S_{\mathcal{V}}(t)$ coincide, and $\mathcal{A}_{\mathcal{V}}$ is invariant under $S_{\mathcal{V}}(t)$, so $\operatorname{dist}_{\mathcal{H}}(\mathcal{A}_{\mathcal{V}},\mathcal{A}_{\mathcal{H}})=0$. This implies that $\mathcal{A}_{\mathcal{V}}\subset\mathcal{A}_{\mathcal{H}}$ as both sets are closed in $\mathcal{H}$. For the opposite assertion we will show that $\mathcal{A}_{\mathcal{V}}$, which is compact in $\mathcal{H}$, is attracting in $\mathcal{H}$ and hence $\mathcal{A}_{\mathcal{H}}$, the smallest closed attracting set in $\mathcal{H}$, must be its subset. To this end let $B\subset\mathcal{B}(\mathcal{H})$ and $t\geq t_1(B)+2$, cf. Step 3 of Lemma 8.3. We have

$$\operatorname{dist}_{\mathcal{H}}(S_{\mathcal{H}}(t)B,\mathcal{A}_{\mathcal{V}})\leq\operatorname{dist}_{\mathcal{H}}(S_{\mathcal{H}}(t-t_1(B)-2)S_{\mathcal{H}}(t_1(B)+2)B,\mathcal{A}_{\mathcal{V}}).$$

But $S_{\mathcal{H}}(t_1(B)+2)B\subset B_2$, where B_2 is bounded and absorbing in $\mathcal{V}$. On B_2, $S_{\mathcal{H}}(t)=S_{\mathcal{V}}(t)$, so

$$\begin{aligned}\operatorname{dist}_{\mathcal{H}}(S_{\mathcal{H}}(t)B,\mathcal{A}_{\mathcal{V}})&\leq\operatorname{dist}_{\mathcal{H}}(S_{\mathcal{V}}(t-t_1(B)-2)B_2,\mathcal{A}_{\mathcal{V}})\\&\leq C\operatorname{dist}_{\mathcal{V}}(S_{\mathcal{V}}(t-t_1(B)-2)B_2,\mathcal{A}_{\mathcal{V}}).\end{aligned}$$

Passing with t to infinity we obtain the assertion of the theorem. □

We can now summarize all the obtained results in the following theorem.

THEOREM 9.5. *Weak solutions with temperature continuous at zero given by Definition 4.3 generate an m-semiflow $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$ and strong solutions given by Definition 7.1 generate a semiflow $\{S_{\mathcal{V}}(t)\}_{t\geq 0}$. We have $S_{\mathcal{V}}(t)=S_{\mathcal{H}}(t)|_{\mathcal{V}}$ for all $t\geq 0$. There exists the invariant set $\mathcal{A}$ compact in $\mathcal{V}$ and bounded in $D(A_S)\times D(A)\times D(A)$ which is the global attractor both for $\{S_{\mathcal{H}}(t)\}_{t\geq 0}$ and $\{S_{\mathcal{V}}(t)\}_{t\geq 0}$.*

10. Relations between heat transport and dissipation rate

For a function $g\colon [0,\infty) \to \mathbb{R}$ define

$$\langle g \rangle = \limsup_{t\to\infty} \frac{1}{t}\int_0^t g(\tau)\,d\tau.$$

Let

$$\mathrm{Nu} = 1 + \left\langle \frac{1}{l}\int_\Omega u_2(x,t)T(x,t)\,dx \right\rangle$$

be the Nusselt number (note that l = measure of Ω), and $\varepsilon = \mathrm{Pr}\langle \|u\|^2\rangle / l$ be the time averaged dissipation rate, both expressed in terms of nondimensional variables.

Our aim is to establish relations between Nu and ε for the thermomicropolar model (1.1). It is known that for the classical Boussinesq model the relation is given by the equation (see [8])

$$\varepsilon = \mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1). \tag{10.1}$$

We denote

$$J(t) = \int_\Omega u_2(x,t)T(x,t)\,dx.$$

From the first three equations of (2.1) we have

$$\frac{d}{dt}\frac{1}{2}\,|u|^2 + \mathrm{Pr}\,\|u\|^2 + \frac{\mathrm{Pr}\,N}{1-N}(\|u\|^2 - 2(\operatorname{rot}\,\omega, u)) = \mathrm{Pr}\,\mathrm{Ra}\,J, \tag{10.2}$$

and

$$\frac{d}{dt}\frac{M}{2}\,|\omega|^2 + L\,\mathrm{Pr}\,\|\omega\|^2 + \frac{\mathrm{Pr}\,N}{1-N}(4|\omega|^2 - 2(\operatorname{rot}\,u,\omega)) = 0. \tag{10.3}$$

Using $(\operatorname{rot}\,\omega,u) = (\operatorname{rot}\,u,\omega)$, eliminating these terms from the above equations, and dividing by l we obtain

$$\frac{d}{dt}\frac{1}{2l}\,|u|^2 + \frac{\mathrm{Pr}}{l(1-N)}\,\|u\|^2 = \frac{d}{dt}\frac{M}{2l}\,|\omega|^2 + \frac{L\,\mathrm{Pr}}{l}\,\|\omega\|^2 + \frac{4\,\mathrm{Pr}\,N}{l(1-N)}\,|\omega|^2 + \frac{\mathrm{Pr}\,\mathrm{Ra}}{l}\,J.$$

Taking the average $\langle\,\cdot\,\rangle$ and using the boundedness of $|u(t)|$ and $|\omega(t)|$ we obtain

$$\varepsilon = (1-N)\left\langle \frac{L\,\mathrm{Pr}}{l}\,\|\omega\|^2 + \frac{4\,\mathrm{Pr}\,N}{l(1-N)}\,|\omega|^2 + \frac{\mathrm{Pr}\,\mathrm{Ra}}{l}J \right\rangle. \tag{10.4}$$

Observe that if we set $N=0$ and $L=0$, the above equation gives the classical relation known for the Boussinesq model,

$$\varepsilon = \frac{\mathrm{Pr}\,\mathrm{Ra}}{l}\,\langle J\rangle = \mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1).$$

From (10.4) we obtain

$$\varepsilon \ge (1-N)\,\frac{\mathrm{Pr}\,\mathrm{Ra}}{l}\,\langle J\rangle = (1-N)\,\mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1) \tag{10.5}$$

or

$$\mathrm{Nu} \le \frac{\varepsilon}{(1-N)\,\mathrm{Pr}\,\mathrm{Ra}} + 1.$$

Now, we estimate ε from below. Adding (10.2) and (10.3) and using $(\operatorname{rot} \omega, u) = (\operatorname{rot} u, \omega)$ and $|\operatorname{rot} u|^2 = \|u\|^2$ we obtain after a few calculations

$$\frac{1}{2\,\mathrm{Pr}}\frac{d}{dt}(|u|^2 + M|\omega|^2) + \|u\|^2 + L\|\omega\|^2 + \frac{N}{1-N}|\operatorname{rot} u - 2\omega|^2 = \mathrm{Ra}\,J.$$

Taking the average over time variable and multiplying by Pr/l we have

$$\left\langle \frac{\mathrm{Pr}}{l}\|u\|^2 + \frac{\mathrm{Pr}\,L}{l}\|\omega\|^2 + \frac{\mathrm{Pr}\,N}{l(1-N)}|\operatorname{rot} u - 2\omega|^2 \right\rangle = \frac{\mathrm{Pr}\,\mathrm{Ra}}{l}\langle J\rangle = \mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1).$$

Therefore,

$$\varepsilon \le \mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1). \tag{10.6}$$

Finally, from (10.5) and (10.6) we have the following relations between the Nusselt number and the dissipation rate for the thermomicropolar fluid,

$$(1-N)\,\mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1) \le \varepsilon \le \mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1) \tag{10.7}$$

or

$$\frac{\varepsilon}{\mathrm{Pr}\,\mathrm{Ra}} + 1 \le \mathrm{Nu} \le \frac{\varepsilon}{(1-N)\,\mathrm{Pr}\,\mathrm{Ra}} + 1.$$

The same relations hold for the reduced model, namely, model (2.1) with $D = 0$ and clearly, for $N = 0$, we get the usual relation (10.1) for the classical Boussinesq model.

The numbers ε and Nu can depend on the parameters of the model, in particular on the coupling number N and micropolar damping L. In our physical interpretation the coupling number is responsible for the friction between the particles. If $N \to 1$ and $L \to \infty$ (this happens when the kinematic microrotation viscosity $\nu_r \to \infty$ and micropolar damping $\alpha \to \infty$) the increasing friction between the particles make the motion slower and more quiet, and possibly may stop it completely. This would be the situation when $\varepsilon \to 0$ and $\mathrm{Nu} \to 1$. Looking at the second inequality in (10.7) we may ask if it is true that

$$\mathrm{Pr}\,\mathrm{Ra}\,(\mathrm{Nu}-1) \to 0, \quad \text{as } N \to 1.$$

Then we would have that $\mathrm{Nu}(L,N) - 1 \sim o(1)$ as $N \to 1$ and $L \to \infty$ (or, more accurately, as Nu can depend on other parameters than N and L only, including the initial data, Nu has the upper bound which is $o(1)+1$). The positive answer holds for the case $D = 0$. In fact, in [18] it is proved that assuming $\mathrm{Ra} > 128$, for sufficiently large N and L we have $\mathrm{Nu} = 1$, the type of result which have no counterpart in the theory of homogeneous Navier–Stokes fluids.

Appendix A. Attractors for single- and multivalued semiflows

We recall some useful definitions and results from the theory of *m-semiflows*, cf. [9], [10], [19], [24]. In the following, X is a Banach space. By $\mathcal{B}(X)$ we denote the family of nonempty and bounded subsets of X.

Definition A.1. A family $\{S(t)\}_{t\geq 0}$ of multivalued maps $S(t)\colon X\to 2^X\setminus\{\emptyset\}$ is an m-semiflow if

(a) For any $x\in X$ we have $S(0)x=\{x\}$.
(b) For any s, $t\geq 0$ and $x\in X$ we have $S(t+s)x\subset S(t)S(s)x$ ([3]).

A set $B_0\in\mathcal{B}(X)$ is *absorbing* if for every bounded set $B\subset X$ there exists $t_B\geq 0$ such that $\bigcup_{t\geq t_B} S(t)B\subset B_0$. As customary, the main object of study is the so-called global attractor, whose attraction property is defined in terms of the Hausdorff semidistance in X, namely $\operatorname{dist}_X(A,B)=\sup_{x\in A}\inf_{y\in B}\|x-y\|_X$.

Definition A.2. The set $\mathcal{A}\subset X$ is a global attractor for an m-semiflow $\{S(t)\}_{t\geq 0}$ if

(a) $\mathcal{A}$ is a compact set in X.
(b) $\mathcal{A}$ uniformly attracts all bounded sets in X, i.e. $\lim_{t\to\infty}\operatorname{dist}_X(S(t)B,\mathcal{A})=0$ for every $B\in\mathcal{B}(X)$.
(c) $\mathcal{A}$ is the smallest (in the sense of inclusion) closed set which has the property (b).

Note that we do not impose *any continuity or closed graph type condition* on $S(t)$. This way, the attractor will be the minimal closed attracting set, but it does not have to be invariant (neither positively nor negatively semi-invariant), see [5], [10]. In our case, the following sufficient condition for the existence of the global attractor is enough. The theorem was proved in [10, Proposition 4.2], where more general formalism of pullback attractors is considered.

Theorem A.3. *If the m-semiflow $\{S(t)\}_{t\geq 0}$ possesses a compact absorbing set B_0 then it has a global attractor $\mathcal{A}$.*

Clearly we have $\mathcal{A}\subset B_0$ as B_0 is the compact absorbing (and hence also attracting) set.

We also briefly recall some notions concerning singe-valued semiflows and their attractors. Note that, in contrast to the multivalued case, we include the continuity in the definition of the semiflow.

Definition A.4. A family $\{S(t)\}_{t\geq 0}$ of maps $S(t)\colon X\to X$ is a semiflow if

(a) For any $x\in X$ we have $S(0)x=x$.
(b) For any s, $t\geq 0$ and $x\in X$ we have $S(t+s)x=S(t)S(s)x$.
(c) For any $t\geq 0$ the mapping $S(t)$ is continuous.

Definition A.5. The set $\mathcal{A}\subset X$ is a global attractor for a semiflow $\{S(t)\}_{t\geq 0}$ if

([3]) Here, by a mild abuse of notation, we write $S(t)S(s)(x)$ instead of a formally correct $S(t)(S(s)(x))$.

(a) $\mathcal{A}$ is a compact set in X.
(b) $\mathcal{A}$ uniformly attracts all bounded sets in X, i.e. $\lim_{t\to\infty} \operatorname{dist}_X(S(t)B, \mathcal{A}) = 0$ for every $B \in \mathcal{B}(X)$.
(c) $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

The following result on the existence of global attractor is classical, cf. [20], [26], [31].

THEOREM A.6. *If the semiflow $\{S(t)\}_{t\geq 0}$ possesses a compact absorbing set B_0, then it has a global attractor $\mathcal{A}$.*

It is well known that $\mathcal{A}$ is the smallest closed attracting set, so we must have $\mathcal{A} \subset B_0$, as B_0 is closed and attracting.

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