Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 2, 2018, 729–738
DOI: 10.12775/TMNA.2018.010

HETEROCLINIC SOLUTIONS OF ALLEN–CAHN TYPE EQUATIONS WITH A GENERAL ELLIPTIC OPERATOR

Karol Wroński

Abstract. We consider a generalization of the Allen–Cahn type equation in divergence form $-\mathrm{div}(\nabla G(\nabla u(x,y))) + F_u(x,y,u(x,y)) = 0$. This is more general than the usual Laplace operator. We prove the existence and regularity of heteroclinic solutions under standard ellipticity and m-growth conditions.

1. Introduction

The Allen–Cahn equation is a well-known elliptic partial differential equation considered by many authors in the form:

$$-\Delta u(x,y) + F_u(x,y,u) = 0$$

where F is a double-well potential of u and has some other standard properties like periodicity in x and y (see the next section for details). Here we are not interested in the Dirichlet problem but in the existence of heteroclinic solutions in the whole of $\mathbb{R}^2$. This problem was widely studied and there are many articles that contain the existence theorems about such solutions. As an example we can take [11] where the authors show the existence and multiplicity of heteroclinic and some other special types of solutions. Earlier in [1] and [2] the problem was solved in a more simple form where $F(x,u) = f(x)F(u)$.

2010 *Mathematics Subject Classification.* 35J20, 35B65, 35B40.
Key words and phrases. Heteroclinic solutions; Allen–Cahn equation.

In this article it is shown that the Laplace operator in the Allen–Cahn equation can be replaced with a much more general elliptic operator in a divergence form which needs only to have properties which are usually called "standard growth conditions". This was already done by the authors of [6] but they assumed quadratic growth of the operator and worked in $W^{1,2}_{\mathrm{loc}}$. All results on Allen–Cahn equation and many similar problems refer to the famous article by Moser [9] later generalized by Bangert [3] where he considered minima of a very general functional in the form $\int F(x,u(x),u_x(x))\,dx$ periodic in first $n+1$ variables and having quadratic growth in last n variables. Some interesting results for double-well type potentials were also obtained by Valdinoci [12] and Bessi [4].

Here it will be proved that the Allen–Cahn equation with a generalized elliptic operator with higher growth also has heteroclinic solutions. Our result is different from all cited above as they consider Laplace operator or operators with quadratic growth. We mainly generalize some results of [11] and the methods of proofs are in many situations the same. However it is not a straightforward generalization as $-\mathrm{div}(\nabla G(\nabla u(x,y)))$ is not a linear operator — this difficulty is seen especially in Lemma 5.2. Also regularity results need some additional assumptions, namely (G1) and (G2).

2. Preliminaries

Consider a quasilinear elliptic equation in the divergence form:

$$-\mathrm{div}(\nabla G(\nabla u(x,y))) + F_u(x,y,u(x,y)) = 0 \tag{AC}$$

where $u\colon \mathbb{R}^2 \to \mathbb{R}$ and $F \in C^2(\mathbb{R}^3,\mathbb{R})$ satisfies the following conditions customary for an Allen–Cahn problem:

- (F_1) F is 1-periodic in x and y,
- (F_2) $F(x,y,0) = F(x,y,1) = 0$ for all $(x,y) \in \mathbb{R}^2$,
- (F_3) $F(x,y,s) > 0$ for all $(x,y) \in \mathbb{R}^2$ and $s \in (0,1)$,
- (F_4) $F(x,y,s) \geqslant 0$ for all $(x,y,s) \in \mathbb{R}^3$.

As an example of such F one can take $F(x,y,s) = s^2(1-s)^2$ or $\sin^2(\pi s)$.

We also assume that $G \in C^{2,\alpha}(\mathbb{R}^2,\mathbb{R})$ satisfies the following standard growth and ellipticity conditions:

- (G_1) $\nu_1|p|^m \le G(p) \le \nu_2(1+|p|)^m$ for some positive constants ν_1,ν_2 and for every $p \in \mathbb{R}^2$,
- (G_2) $\displaystyle \nu_1(1+|p|)^{m-2}\sum_{i=1,2}\xi_i^2 \le \sum_{i,j\in\{1,2\}}\frac{\partial^2 G}{\partial p_i \partial p_j}(p)\xi_i\xi_j \le \nu_2(1+|p|)^{m-2}\sum_{i=1,2}\xi_i^2,$

  for some positive constants ν_1,ν_2 and every p, $\xi \in \mathbb{R}^2$,
- (G_3) $\displaystyle \left|\frac{\partial G}{\partial p_i}(p)\right| \le \mu(1+|p|)^{m-1}$ for some positive μ and every $p \in \mathbb{R}^2$.

An easy example of a function satisfying such conditions is $G(p) = |p|^2 + |p|^m$. Unfortunately we cannot simply take $G(p) = |p|^m$ (for which $\operatorname{div}(\nabla G(\nabla u))$ is equal to the m-Laplacian operator) because then (G_2) would not be satisfied. When $m = 2$ and $G = \|\cdot\|^2$ it is easy to see that $\operatorname{div}(\nabla G(\nabla u(x,y))) = \Delta u(x,y)$ and the equation (AC) becomes the standard Allen–Cahn problem considered in [11]. In that paper the variational problem is solved using the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times[0,1])$ and its subspace $E_1(\mathbb{R}\times[0,1])$ consisting of functions u such that

$$\lim_{x\to-\infty} u(x,y) = 0, \qquad \lim_{x\to\infty} u(x,y) = 1$$

and periodic in y.

In this work we are concerned with a generalization of Allen–Cahn equation where the function G in the elliptic operator can be much more complicated then $\|\cdot\|^2$. For that reason instead of $W^{1,2}$ we use Sobolev spaces $W^{1,m}$ with m as in growth conditions on G. Notice that we do not assume $m > 2$ but only $m > 1$, so elements of $W^{1,m}$ may even be not continuous but in fact we obtain $C^{2,\alpha}$ solutions using some regularity theorems.

It is easy to see that the problem (AC) has two trivial constant solutions equal to 0 and 1. We will search for solutions of equation (AC) which are heteroclinic in x (i.e. convergent to 0 as $x\to-\infty$ and to 1 as $x\to\infty$) and periodic in y. As the domain of solutions is $\mathbb{R}^2$ and the heteroclinic solutions are not integrable on their domain there is a problem in the variational formulation of (AC). To solve this we introduce a space E_1 which contains functions $u\in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^2)$ that are 1-periodic in y and $\|\nabla u\|_{L^m(\mathbb{R}\times[0,1])} < \infty$. The space E_1 is equipped with the norm

$$\|u\|_{E_1} = \|u\|_{L^m([0,1]^2)} + \|\nabla u\|_{L^m(\mathbb{R}\times[0,1])}. \tag{2.1}$$

Convergence in the norm $\|\cdot\|_{E_1}$ obviously implies convergence in metric of $W^{1,m}_{\mathrm{loc}}(\mathbb{R}^2)$. Consequently, E_1 is a closed normed subspace of the complete and reflexive space $W^{1,m}_{\mathrm{loc}}(\mathbb{R}^2)$ so it is also a reflexive Banach space. On such a space we can formulate variational problem. To do this we introduce some notation. For every function $u\colon\mathbb{R}^2\to\mathbb{R}$ and every integer k we define a function $\tau_k u$ by

$$\tau_k u(x,y) = u(x-k,y). \tag{2.2}$$

The operator L is given by $L(u) = G(\nabla u) + F(x,y,u)$ and the functional I is defined by

$$I(u) = \int_0^1 dy \int_{\mathbb{R}} L(u)\,dx. \tag{2.3}$$

Sometimes we will write I in an alternative way: $a_k(u) = \int_0^1 dy\int_k^{k+1} L(u)\,dx$ and then $I(u) = \sum_{k\in\mathbb{Z}} a_k(u)$.

We will search for a minimum of I on a set Γ of heteroclinic functions:

$$\Gamma(0,1) = \Big\{ u \in E_1 : \tau_{-1}u \geqslant u \text{ a.e.} \\ \wedge \lim_{k\to-\infty} \tau_{-k}u = 0 \wedge \lim_{k\to+\infty} \tau_{-k}u = 1 \text{ in } L^m_{\mathrm{loc}} \Big\}. \tag{2.4}$$

We also need to define the minimum of I as $c(0,1) = \min_{u\in\Gamma(0,1)} I(u)$. Note that $u \in \Gamma(0,1)$ if and only if $\tau_k u \in \Gamma(0,1)$ and $I(u) = I(\tau_k u)$.

Now we can state the main theorem of this article:

Theorem 2.1. *There exists a function $v \in \Gamma(0,1)$ which is a classical solution of* (AC) *such that $I(v) = c(0,1)$ and $0 < v < 1$.*

3. Solving a variational problem

It is obvious that $I(v) \geq 0$ for all $v \in \Gamma(0,1)$ (because G and F are non-negative) so there exists the infimum $c(0,1) \geq 0$. By $\{u_j\}$ we will denote the minimizing sequence of I in $\Gamma(0,1)$. The sequence $I(u_j)$ is convergent so there exists M such that $I(u_j) < M$ for all j. This implies that $\{u_j\}$ is a bounded sequence in E_1 because

$$\int_0^1 dy \int_{\mathbb{R}} \|\nabla u_j\|^m \, dx \leq \int_0^1 dy \int_{\mathbb{R}} G(\nabla u_j) + F(x,y,u_j) \, dx = I(u_j) < M$$

and $\iint_{[0,1]^2} |u_j| \, dx \, dy$ is bounded by 1 as $u_j \in \Gamma(0,1)$.

The minimizing sequence $\{u_j\}$ can be chosen in many ways because applying τ_k to u_j does not change values of $I(u_j)$. Therefore we can assume that u_j were chosen to satisfy inequalities:

$$\iint\limits_{[0,1]^2} u_j \, dx \, dy > \frac{1}{2} \quad \text{and} \quad \int_0^1 dy \int_{k-1}^k u_j \, dx \leq \frac{1}{2} \quad \text{for all } k \leq 0. \tag{3.1}$$

This is possible because every u_j can be replaced by $\tau_{-k_j} u_j$ where k_j is the smallest k such that

$$\int_0^1 dy \int_{k-1}^k u_j \, dx > \frac{1}{2}.$$

Such k exists because

$$\int_0^1 dy \int_{k-1}^k u_j \, dx \to 0 \quad \text{as } k_j \to -\infty$$

and

$$\int_0^1 dy \int_{k-1}^k u_j \, dx \to 1 \quad \text{as } k_j \to +\infty.$$

This normalization was done in order to use it in the proof of Lemma 3.1.

As E_1 is reflexive there exist $v \in E_1$ and a subsequence of $\{u_j\}$ (still denoted by $\{u_j\}$) which converges weakly to v in E_1. From this subsequence we can obviously choose another subsequence convergent strongly in L^m_{loc} and pointwise almost everywhere.

Unfortunately I is an improper integral so it does not need to have all good properties typical for functionals written by an integral over a bounded set. This is the reason why we need to prove many facts that would be obvious for minima of some better functionals.

Below we show that the limit of the minimizing sequence is in fact a minimum of I. We first prove that $I(v) \leq M$ (so $I(v)$ is finite). For every j, n we have

$$\int_0^1 dy \int_{-n}^{n} L(u_j)\, dx < M.$$

By the weak lower semicontinuity of this functional for bounded domains (see for example Theorem 8.11 of [5]) we get

$$\int_0^1 dy \int_{-n}^{n} L(v)\, dx \leq M \quad \text{for every } n.$$

Letting $n \to \infty$ we conclude that

$$\int_0^1 dy \int_{\mathbb{R}} L(v)\, dx \leq M.$$

LEMMA 3.1. *$v \in \Gamma(0,1)$.*

PROOF. For every j, $\tau_{-1}u_j \geq u_j$ and $u_j \to v$ almost everywhere so $\tau_{-1}v \geq v$ almost everywhere. We shall prove that $\tau_{-k}v \to 1$ in L^m_{loc} as $k \to +\infty$. The sequence $\tau_{-k}v$ is bounded in $W^{1,m}_{\mathrm{loc}}(\mathbb{R}\times[0,1])$ because $I(\tau_{-k}v) = I(v) \leq M$. There exists v^*_∞ such that $\tau_{-k}v \to v^*_\infty$ weakly in $W^{1,m}_{\mathrm{loc}}(\mathbb{R}\times[0,1])$, strongly in L^m_{loc} and $v^*_\infty \in W^{1,m}([0,1]^2)$. Also, we have that $a_k(v) = a_0(\tau_{-k}v) \to a_0(v^*_\infty)$ because $I(v)$ is finite and therefore $a_k(v) \to 0$. By the definition of a_0: $v^*_\infty = \mathrm{const}$ and

$$\iint_{[0,1]^2} F(x,y,v^*_\infty)\, dx\, dy = 0.$$

This means that $v^*_\infty = 0$ or $v^*_\infty = 1$. From (3.1) we obtain $v^*_\infty = 1$. $\square$

Now we show that $I(v) = c(0,1)$. It is obvious that $I(v) \geq c(0,1)$. For a fixed $\varepsilon > 0$ and sufficiently large j we have

$$\sum_{-n}^{n} a_k(u_j) \leq I(u_j) \leq c(0,1) + \varepsilon \quad \text{for every } n.$$

Taking $j \to \infty$ we obtain

$$\sum_{-n}^{n} a_k(v) \leq c(0,1) + \varepsilon \quad \text{for every } n.$$

When $n \to \infty$ we get $I(v) \leq c(0,1) + \varepsilon$ for every $\varepsilon > 0$ so $I(v) = c(0,1)$.

Definition 3.2. For every $r \leq 1/2$ and $z \in \mathbb{R} \times [0,1]$ we define a set

$$Z(B_r(z), v) = \{u \in E_1 : u = v \text{ on } B_{1/2} - B_r(z)\}$$

and on this set we define

$$\Phi_{B_r(z),v}(u) = \iint_{B_r(z)} L(u)\,dx\,dy.$$

It is easy to check that $Z(B_r(z), v)$ is a closed subset of E_1. The minimum of $\Phi_{B_r(z),v}$ on $Z(B_r(z), v)$ will be called $c(B_r(z), v)$ and by $M(B_r(z), v)$ we will call the set of $w \in Z(B_r(z), v)$ for which $\Phi_{B_r(z),v}(w) = c(B_r(z), v)$.

Proposition 3.3. *For every $z \in \mathbb{R} \times [0,1]$ and $\varepsilon \in (0,1/2)$ v is the unique minimum of $\Phi_{B_\varepsilon(z),v}$ on $Z(B_\varepsilon(z), v)$.*

The proof will be done in Section 5.

We now apply Proposition 3.3 to obtain that v is a weak solution of (AC) on $B_\epsilon(z)$. For every $\varphi \in C^1(B_\epsilon(z))$ for which $\operatorname{supp}\varphi \subset B_\epsilon(z)$ we have $v + t\varphi \in Z(B_\varepsilon, v)$ so $\Phi_{B_\varepsilon(z),v}(v + t\varphi) \geq \Phi_{B_\varepsilon(z),v}(v)$. As a consequence we get

$$\frac{d}{dt}\,\Phi_{B_\epsilon(z),v}(v + t\varphi)\bigg|_{t=0} = \iint_{B_\epsilon(z)} \nabla G(\nabla v)\nabla\varphi + F_u(x,y,u)\varphi\,dx\,dy = 0$$

and therefore v is a weak solution of (AC) on every $B_{1/2}(z)$.

4. Regularity of weak solutions

The first problem in proving the regularity results comes from the fact that equation (AC) is formulated on $\mathbb{R}^2$. To solve it we introduced the functional I defined using one integral over $\mathbb{R} \times [0,1]$ and so we cannot state whether its minimum is a weak solution.

At this moment we can apply Theorem 3.1 of [7] to show that the found minimum v is Hölder continuous.

To obtain a weak solution we used a functional Φ defined by integration over a bounded set. In the first step we search for its minimum on a set $Z(B_r(z), v)$. Notice that by Definition 3.2 on $B_{1/2}(z) \setminus B_r(z)$ one has $u = v$. Therefore we consider the Dirichlet problem with Hölder continuous boundary data.

Well-known facts from regularity theory (see for example [8, Chapter 5, Theorem 6.1]) say that the solution of this local problem is in fact a classical $C^{2,\alpha}$ solution on $B_r(z)$. For that purpose we need growth conditions (G_1)–(G_3). Note that we do not assume anything on the growth of F whereas in "natural growth conditions" in [8] there are some assumptions on it. This is due to the fact that when searching for heteroclinic solutions we know that, for $s \in (0,1)$, $F(x,y,s)$ is positive, bounded and has bounded derivatives.

Regularity of a minimum of $\Phi_{B_\varepsilon(z),v}$ is important in proofs of some lemmas in Section 5, especially we need a $C^{2,\alpha}$ solution in Lemma 5.2 where the maximum principle is used. Notice that before we prove Proposition 3.3 we do not know if v is a minimum of $\Phi_{B_\varepsilon(z),v}$ so now it does not have to be $C^{2,\alpha}$. Proposition 3.3 states that unique minimum of $\Phi_{B_\varepsilon(z),v}$ is in fact equal to v and therefore the proposition has to be proved to state that $v \in C^{2,\alpha}$.

5. Properties of weak solutions (proof of Proposition 3.3)

In this section some properties of local weak solutions of (AC) will be derived. In conclusion we will get the proof of Proposition 3.3. The proofs of Lemmas 5.1, 5.3–5.5 are almost the same as in [11] but we still write them in order to show that the more general setting of the problem does not change the methods.

We already know that $\tau_{-k}v \to 0$ (respectively to 1) in L^2 as $k \to -\infty$ (respectively to $+\infty$). Due to the regularity of v this convergence can be replaced by pointwise limits: $\lim_{x\to-\infty} v(x,y) = 0$ and $\lim_{x\to+\infty} v(x,y) = 1$. To finish the proof of Theorem 2.1 we only need to show the sharp inequalities: $0 < v < 1$ and $v < \tau_{-1}v$. This will be done after Lemma 5.3.

LEMMA 5.1. *For every radius $r < 1/2$ there exists $w \in Z(B_r(z), v)$ such that*

$$\Phi_{B_r(z),v}(w) = c(B_r(z), v).$$

PROOF. The procedure is almost the same as in the case of finding a weak solution v in E_1. We choose a minimizing sequence $\{u_j\} \subset Z(B_r(z), v)$. This sequence is bounded in the reflexive space $W^{1,m}(B_{1/2}(z))$, so there exists a subsequence weakly convergent to some $w \in W^{1,m}(B_{\frac{1}{2}}(z))$. As $Z(B_r(z), v)$ is convex and closed in $W^{1,m}(B_{1/2}(z))$ we get that $w \in Z(B_r(z), v)$. $\Phi_{B_r(z),v}$ is defined by the integral on a bounded domain so $\Phi_{B_r(z),v}(w) = c(B_r(z), v)$. The same arguments that we used before show that w is a classical C^2 solution. $\square$

The next result is based on Lemma 4.2 of [9].

LEMMA 5.2. *$M(B_r(z), v)$ is an ordered set, i.e. if $\varphi, \psi \in M(B_r(z), v)$ and $\varphi \not\equiv \psi$ then $\varphi < \psi$ or $\varphi > \psi$ in $B_r(z)$.*

PROOF. Let us define $\chi = \max\{\varphi, \psi\}$ and $\xi = \min\{\varphi, \psi\}$. Notice that

$$\Phi_{B_r(z),v}(\chi) + \Phi_{B_r(z),v}(\xi) = \Phi_{B_r(z),v}(\varphi) + \Phi_{B_r(z),v}(\psi) = 2c(B_r(z), v)$$

hence χ and ξ belong to $M(B_r(z), v)$. It is also obvious that $\chi \geq \xi$ and if $\varphi \not\equiv \psi$ then $\chi \not\equiv \xi$. By Theorem 2.5.3 of [10] either $\chi > \xi$ or $\chi \equiv \xi$ in $B_r(z)$. If $\chi > \xi$ then there is no point $z_0 \in B_r(z)$ where $\varphi(z_0) = \psi(z_0)$ so by continuity we have $\varphi < \psi$ or $\varphi > \psi$ in $B_r(z)$. In the case when $\chi \equiv \xi$ we easily get $\varphi \equiv \psi$. $\square$

LEMMA 5.3. *$M(B_r(z), v)$ contains the smallest element* (*in the sense of the order defined in Lemma* 5.2).

Proof. For every $\xi \in B_r(z)$ take $w_z(\xi) = \inf_{w \in M(B_r(z),v)} w(\xi)$. We will prove that this infimum is in fact a minimum and thus $w_z \in M(B_r(z), v)$.

Let us assume that there exist $\xi_0 \in B_r(z)$ and $w \in M(B_r(z), v)$ such that $w_z(\xi_0) = w(\xi_0)$. By Lemma 5.2 if it were true for ξ_0 then $w_z \equiv w$ and the proof is completed.

If such ξ_0 does not exist we take any $\xi_0 \in B_r(z)$ and a sequence of $w_n(\xi_0)$ convergent to $w_z(\xi_0)$. Order in $M(B_r(z), v)$ makes the sequence $\{w_n\}$ nonincreasing. It is also bounded in $W^{1,m}(B_{1/2}(z))$, hence weakly convergent to w_z which is in $M(B_r(z), v)$ (because it is convex and closed in $W^{1,m}(B_r(z))$). □

For the next lemma we will need some new notation. Let us define points $z_n = z + (n, 0)$ for $n \in \mathbb{Z}$ and a new function $\widetilde{v}$ such that for every $j \in \mathbb{Z}$ we have $\widetilde{v} = w_{z_j}$ in $B_{1/2}(z_j)$ and $\widetilde{v} = v$ in $\mathbb{R} \times [0,1] - \bigcup_{n \in \mathbb{Z}} B_\varepsilon(z_n)$.

Lemma 5.4. *The function $\widetilde{v}$ defined above is an element of $\Gamma(0,1)$.*

Proof. We prove the inequality $\widetilde{v} \le \tau_{-1}\widetilde{v}$. If it were not true then for some j there would exist a point $(x_0, y_0) \in B_\epsilon(z_j)$ such that

$$w_{z_j}(x_0, y_0) > \tau_{-1} w_{z_j}(x_0, y_0) = w_{z_{j+1}}(x_0 + 1, y_0).$$

For every $(x, y) \in B_{1/2}(z_j)$ we can define

$$\psi(x, y) = w_{z_{j+1}}(x + 1, y), \qquad \phi(x, y) = w_{z_j}(x, y)$$

and $\chi = \max\{\psi, \phi\}$, $\xi = \min\{\psi, \phi\}$. Inequalities $\xi = \phi = v \le \tau_{-1} v = \psi = \chi$ hold on $B_{1/2}(z_j) - B_\varepsilon(z_j)$. Moreover, $\xi \in Z(B_\varepsilon(z_j), v)$ and $\xi \in Z(B_\varepsilon(z_{j+1}), v)$ so

$$\Phi_{B_\varepsilon(z_j)}(\xi) + \Phi_{B_\varepsilon(z_j)}(\chi) = \Phi_{B_\varepsilon(z_j)}(\phi) + \Phi_{B_\epsilon(z_j)}(\psi) = 2c(B_\epsilon(z_j), v)$$

and

$$\Phi_{B_\varepsilon(z_j)}(\chi) = \Phi_{B_\varepsilon(z_{j+1})}(\tau_1 \chi).$$

Consequently, we have $\chi \in M(B_\varepsilon(z_j), v)$ and $\tau_1 \chi \in M(B_\varepsilon(z_{j+1}), v)$. The definition of $\widetilde{v}$ and Lemma 5.2 give $\chi \ge w_{z_j} = \phi$ and therefore

$$w_{z_j}(x_0, y_0) \le \chi(x_0, y_0) \le \psi(x_0, y_0) = w_{z_{j+1}}(x_0 + 1, y_0)$$

which contradicts the definition of (x_0, y_0). □

As a consequence of the above lemma $I(v) \le I(\widetilde{v})$ and hence for all $j \in \mathbb{Z}$ we have:

$$\iint_{B_\epsilon(z_j)} L(v)\,dx\,dy \le \iint_{B_\epsilon(z_j)} L(\widetilde{v})\,dx\,dy = \iint_{B_\epsilon(z_j)} L(w_{z_j})\,dx\,dy = c(b_\varepsilon(z_j)).$$

Taking $j = 0$ we get

$$\iint_{B_\varepsilon(z)} L(v)\,dx\,dy = c(b_\varepsilon(z))$$

and taking $\varepsilon = r$ we have $\Psi_{B_r(z)}(v) = c(B_r(z))$.

The last lemma in this section finishes the proof of Proposition 3.3.

Lemma 5.5. *Function v is the only element in $Z(B_r(z), v)$ which minimizes $\Phi_{B_r(z)}$.*

Proof. For every domain $D \subset B_r(z)$ we can find a minimizer ψ of Ψ_D such that $\psi = v$ in $B_r(z) \setminus D$. We need to show that ψ is unique and $\psi = v$. To do it we start from the obvious inequality $\Psi_D(\psi) \le \Psi_D(v)$. This also implies $\Psi_{B_r(z)}(\psi) \le \Psi_{B_r(z)}(v)$. Since $\psi \in Z(B_r(z), v)$ we get $\Psi_{B_r(z)}(\psi) \ge \Psi_{B_r(z)}(v)$ and thus $\psi \in M(B_r(z), v)$. As $\psi = v$ in $B_r(z) \setminus D$ by Lemma 5.2 we have $\psi = v$ in $B_r(z)$. □

6. Finishing the proof of Theorem 2.1

We already know that $v \in E_1$ is also an element of $\Gamma[0,1]$ and therefore v is a classical solution of (AC). It remains to show sharp inequalities: $0 < v < 1$ and $\tau_{-1}v < v$. Analogously to the proof of Lemma 5.2 we find that both inequalities are consequences of Theorem 2.5.3 of [10].

Acknowledgements. The author would like to thank professor Marek Izydorek for suggesting the problem and many fruitful conversations.

References

[1] F. Alessio, L. Jeanjean and P. Montecchiari, *Stationary layered solutions in $\mathbb{R}^2$ for a class of non autonomous Allen–Cahn equations*, Calc. Var. Partial Differential Equations **11** (2000), 177–202.

[2] F. Alessio, L. Jeanjean and P. Montecchiari, *Existence of infinitely many stationary layered solutions in $\mathbb{R}^2$ for a class of periodic Allen–Cahn equations*, Com. Partial Differential Equations **27** (2002), 1537–1574.

[3] V. Bangert, *On minimal laminations of the torus*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), 95–138.

[4] U. Bessi, *Slope-changing solutions of elliptic problems on $\mathbb{R}^n$*, Nonlinear Anal. **68** (2008), 3923–3947.

[5] B. Dacorogna, *Direct Methods in the Calculus of Variations*, Springer, New York, 2007.

[6] R. de la Llave and E. Valdinoci, *Multiplicity results for interfaces of Ginzburg–Landau–Allen–Cahn equations in periodic media*, Adv. Math. **215** (2007), 379–426.

[7] M. Giaquinta and E. Giusti, *On the regularity of the minima of variatonal integrals*, Acta Math. **148** (1982), 31–46.

[8] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[9] J. Moser, *Minimal solutions of variational problems on a torus*, Ann. Inst. H. Poincaré Anal. Non Linéaire **3** (1986), 229–272.

[10] P. Pucci and J.B. Serrin, *The Maximum Principle*, Birkhäuser, Basel, 2007.

[11] P.H. Rabinowitz and E. Stredulinsky, *Mixed states for an Allen–Cahn type equation*, Comm. Pure Appl. Math. **56** (2003), 1078–1134.

[12] E. Valdinoci, *Plane-like minimizers in periodic media: jet flows and Ginzburg–Landau-type functionals*, J. Reine Angew. Math. **574** (2001), 147–185.

Manuscript received October 12, 2017

accepted November 30, 2017

Karol Wroński
Department of Technical Physics
and Applied Mathematics
Gdańsk University of Technology
Narutowicza 11/12
80-233 Gdańsk, POLAND

E-mail address: karwrons@pg.gda.pl