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EXISTENCE AND UNIQUENES RESULTS FOR SYSTEMS OF IMPULSIVE FUNCTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION WITH MULTIPLE DELAY

Mohamed Ferhat — Tayeb Blouhi

Abstract. We present some existence and uniqueness results on impulsive functional differential equations with multiple delay with fractional Brownian motion. Our approach is based on the Perov fixed point theorem and a new version of Schaefer's fixed point in generalized metric and Banach spaces.

1. Introduction

Stochastic partial functional differential equations with finite delays driven by fractional Brownian motion (SDEs) are very important as stochastic models of biological, chemical, physical, and economical systems.

The study of impulsive stochastic functional differential equations is a new research area. There are few publications in this theory. The existence of solutions of impulsive differential equations was investigated, for example in [7], [9], [17], [21], [23]–[25], [32], [38], [42], [46] the authors investigated the existence of solutions of nonlinear stochastic differential equations by means of the Banach fixed point theorem. Ouahab [30] obtained existence of solutions of functional

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differential equations by Kakutani's fixed point theorem. It is also worth emphasizing that impulsive differential systems and evolution differential systems are used to describe numerous models of real processes and phenomena appearing in the applied sciences, for instance, in physics, related to chemical technology, population dynamics, biotechnology and economics. Differential equations with impulses were considered for the first time by Milman and Myshkis [27], with impulsive effects and multiple delay, and are of active research which culminated with the monograph by Halanay and Wexler [19]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences may change state abruptly or be subject to short-term perturbations. These perturbations may be seen as impulses. Impulsive problems arise also in various applications in communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine and biology. A comprehensive introduction to the basic theory is well developed in the monographs by Benchohra et al. [2], Graef et al. [17], Laskshmikantham et al. [20], Samoilenko and Perestyuk [39].

Random differential and integral equations play an important role in characterizing many social, physical, biological and engineering problems; see for instance the monograph by Da Prato and Zabczyk [12], Gard [15], Gikhman and Skorokhod [16], Sobczyk [40], Tsokos and Padgett [41], and references therein. Recently, stability of stochastic differential equations with Markovian switching has received a lot of attention [22], [33], [45]. For example, a stochastic model for drug distribution in a biological system was described by Tsokos and Padgett [41] to a closed system with a simplified heat, one organ or capillary bed, and re-circulation of a blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. For the basic theory concerning stochastic differential equations see the monographs by Bharucha–Reid [3], Mao [25], Øksendal [31].

In this article, our main objective is to establish sufficient conditions for the existence of solutions of the following first order stochastic impulsive functional equation with multiple delay:

$$
\begin{cases}
dx(t) = \left(\displaystyle\sum_{i=1}^{n_*} x(t-T_i) + g^1(t,x_t,y_t)\right) dt + f^1(t) d^\circ B^{H_1}(t), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J,\ t \neq t_k, \\
dy(t) = \left(\displaystyle\sum_{i=1}^{n_*} y(t-T_i) + g^2(t,x_t,y_t)\right) dt + f^2(t)\, d^\circ B^{H_1}(t), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J,\ t \neq t_k, \\
\Delta x(t) = I_k(x(t_k)), & t = t_k,\ k = 1,2,\ldots,m, \\
\Delta y(t) = \overline{I}_k(y(t_k)), & t = t_k, k = 1,2,\ldots,m, \\
x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0], \\
y(t) = \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0],
\end{cases}
\tag{1.1}
$$

where $n_* \in \{1, 2, \ldots\}$, $r = \max_{1 \le i \le n_*} T_i$, considered with respect to a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ furnished with a family of right continuous and increasing σ-algebras $\{\mathcal{F}_t : t \in J\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. The impulse times t_k satisfy $0 = t_0 < t_1 < \ldots < t_m < T$. As for y_t we mean the segment solution which is defined in the usual way, that is, if $y(\,\cdot\,,\,\cdot\,)\colon [-r, b] \times \Omega \to \mathbb{R}^n$, then, for any $t \ge 0$, $y_t(\,\cdot\,,\,\cdot\,)\colon [-r, 0] \times \Omega \to \mathbb{R}^n$ is given by

$$y_t(\theta, \omega) = y(t + \theta, \omega), \quad \text{for } \theta \in [-r, 0],\ \omega \in \Omega.$$

Here $y_t(\,\cdot\,)$ represents the history of the state from time $t - r$, up to the present time t. Before describing the properties fulfilled by the operators f^i, g^i and $I_k, \overline{I}_k$, we need to introduce some notation and describe some spaces. Let $\mathcal{D}_{\mathcal{F}_0}$ be the space defined by

$$\mathcal{D}_{\mathcal{F}_0} = \big\{\phi\colon [-r, 0] \times \Omega \to \mathbb{R}^n \text{ is continuous everywhere except for a finite number of points } \phi(t_k^-) \text{ and } \phi(t_k^+) \text{ with } \phi(t_k) = \phi(t_k^-)\big\}$$

endowed with the norm

$$\|\phi(t)\|_{\mathcal{D}_{\mathcal{F}_0}} = \left(\int_{-r}^{0} |\phi(t)|^2\, dt\right)^{1/2}.$$

Now, for a given $b > 0$, we define

$$\mathcal{D}_{\mathcal{F}_0} = \Bigg\{y\colon [-r, b] \times \Omega \to \mathbb{R}^n,\ y_k \in C(J_k, \mathbb{R}^n) \text{ for } k = 1, \ldots m,\ y_0 \in \mathcal{D}_{\mathcal{F}_0}, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+) \text{ with } y(t_k) = y(t_k^-),\ k = 1, \ldots, m, \text{ and } \sup_{t \in [0,b]} \mathbb{E}(|y(t)|^2) < \infty,\ \int_{-r}^{0} |\phi(t)|^2 dt < \infty\Bigg\},$$

endowed with the norm

$$\|y\|_{\mathcal{D}_{\mathcal{F}_b}} = \|\phi\|_{\mathcal{D}_{\mathcal{F}_0}} + \sup_{0 \le s \le b} (\mathbb{E}\, \|y(s)\|^2)^{1/2},$$

where y_k denotes the restriction of y to $J_k = (t_{k-1}, t_k]$, $k = 1, \ldots, m$, and $J_0 = [-r, 0]$.

We will consider an initial data $\phi \in \mathcal{D}_{\mathcal{F}_0}$ where $J := [0, b]$. Let $f^1, f^2\colon J \to \mathbb{R}^n$ be Carathéodory functions, $g^1, g^2\colon J \times \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathbb{R}^n$, and $B^H = (B^{H,j} : j = 1, \ldots, d)$ be a cylindrical fractional Brownian motion with Hurst parameter $H_1, H_2 \in (1/2, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $l = 1, 2$, $I_k, \overline{I}_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, \ldots, m$, and $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$. The notations

$$y(t_k^+) = \lim_{h \to 0^+} y(t_k + h) \qquad \text{and} \qquad y(t_k^-) = \lim_{h \to 0^+} y(t_k - h)$$

stand for the right and the left hand limits of the function y at $t = t_k$, respectively.

It is obvious that the system (1.1) can be seen as a fixed point problem:

$$
\begin{cases}
dz(t) = \bigg(\displaystyle\sum_{i=1}^{n_*} z(t-T_i) + g(t,z_t)\bigg)\,dt + f(t)\,d^\circ B^H(s), & \\
 & t \in [0,b],\ t \neq t_k, \\
\Delta z(t) = I_k^*(z(t_k)), & t = t_k,\ k = 1,2,\ldots,m, \\
z(t) = \widetilde{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0],
\end{cases} \tag{1.2}
$$

where

$$
z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \qquad f(t) = \begin{bmatrix} f^1(t) \\ f^2(t) \end{bmatrix},
$$

$$
g(t,z_t) = \begin{bmatrix} g^1(t,x_t,y_t) \\ g^2(t,x_t,y_t) \end{bmatrix}, \qquad \widetilde{\phi}(t) = \begin{bmatrix} \phi(t) \\ \overline{\phi}(t) \end{bmatrix}.
$$

This paper is motivated by [18], [42] and we generalize the existence and uniqueness of solution results to impulsive stochastic differential equations under non-Lipschitz condition and Lipschitz condition. In Section 2, we introduce all the background material used in this paper such as stochastic calculus and some properties of generalized Banach spaces. In Section 3 we establish a version of Perov's fixed point theorem and prove another result on the existence of solutions to problem (1.1). In Section 4 we prove some existence results based on a nonlinear alternative of Leray–Schauder type theorem in generalized Banach spaces. An example is provided in the last section to illustrate the theory.

2. Preliminaries

In this section, we introduce some notations, and recall some definitions, and preliminary facts which are used throughout this paper. Actually we will borrow them from [6]. Although we could simply refer to this paper whenever we need it, we prefer to include this summary in order to make our paper as much self-contained as possible.

2.1. Some results on stochastic integrals. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (i.e., right continuity and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). For a stochastic process $x(\,\cdot\,,\,\cdot\,)\colon [0,T]\times\Omega \to \mathbb{R}^n$ we will write $x(t)$ (or simply x when no confusion is possible) instead of $x(t,w)$. SDEs with respect to fBm have been interpreted via various stochastic integrals, such as the Wick integral, the Wiener integral, the Skorohod integral, and path-wise integrals [10], [16], [28], [37].

Definition 2.1. The fractional Brownian motion $(B^H(t))$ with Hurst index H is a centered self-similar Gaussian process $B^H = B^H(t)$, $t \in \mathbb{R}^+$, on $(\Omega,\mathcal{F},\mathbb{P})$ with the following properties:

(a) $B^H(0) = 0$;
(b) $\mathbb{E}(B^H(t)) = 0,\ \ t \in \mathbb{R}^+$;
(c) $\mathbb{E}(B^H(t)B^H(s)) = (|t|^{2H} + |s|^{2H} + |t-s|^{2H})/2,\ \ t, s \in \mathbb{R}^+$.

For $H = 1/2$, this is the usual Brownian motion.

We recall some stochastic integration with respect to the fractional Brownian motion [14], [37]. Let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be Borel measurable and $1/2 \le H < 1$. Let $\phi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be given by

$$\phi(t,s) = H(2H-1)|t-s|^{2H-2}, \quad t, s \in \mathbb{R}^+.$$

Then we define

$$L^2_\phi = \left\{ f \,:\, |f|^2_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f(t)f(s)\phi(t,s)\,ds\,dt < \infty \right\}.$$

If we equip L^2_ϕ with the inner product

$$\langle f_1, f_2\rangle_\phi = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} f_1(t)f_2(s)\phi(t,s)\,ds\,dt,$$

then $L^2_\phi(\mathbb{R}^+)$ becomes a separable Hilbert space. Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B^H(\psi_1), \ldots, B^H(\psi_n)),$$

where $n \ge 1$, $f \in C_b^\infty(\mathbb{R}^n)$ (i.e., all partial derivatives of f are bounded), $\psi_i \in \mathcal{H}$, $\mathcal{H}$ is a Hilbert space [1]. The derivative operator D_t^H of a smooth and cylindrical random variable F is defined as the $\mathcal{H}$-valued random variable:

$$D_t^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(B^H(\psi_1), \ldots, B^H(\psi_n)\big)\psi_i.$$

Introduce the Malliavin ϕ-derivative of F:

$$D_t^\phi F = \int_{\mathbb{R}^+} \phi(t,v) D_v^H F\,dv.$$

We refer to [4] for more details.

Definition 2.2. Let $u(t)$ be a stochastic process with integrable trajectories.

(a) The symmetric integral of $u(t)$ with respect to $B^H(t)$ is defined as the limit in probability, as ε tends to zero, of

$$\frac{1}{2\varepsilon}\int_0^T u(s)\big(B^H(s+\varepsilon) - B^H(s-\varepsilon)\big)\,ds,$$

provided it exists; we denote it by $\int_0^T u(s)\,d^\circ B^H(s)$.

(b) The forward integral of $x(t)$ with respect to B^H is defined as the limit in probability, as ε tends to zero, of

$$\frac{1}{\varepsilon}\int_0^T u(s)\frac{B^H(s+\varepsilon) - B^H(s)}{\varepsilon}\,ds,$$

provided it limit exists; we denote it by $\int_0^T u(s)\,d^- B^H(s)$.

(c) The backward integral of $u(t)$ with respect to B^H is defined as the limit in probability, as ε tends to zero, of

$$\frac{1}{\varepsilon}\int_0^T u(s)\frac{B^H(s-\varepsilon)-B^H(s)}{\varepsilon}\,ds,$$

provided it exists; we denote it by $\int_0^T u(s)\,d^+ B^H(s)$.

Remark 2.3 ([4]). Let $\mathcal{L}_\phi(0,T)$ be the family of stochastic processes $u(t)$ on $[0,T]$ such that $u(t)\in\mathcal{L}_\phi(0,T)$ if $\mathbb{E}\,|u(t)|^2_\phi<\infty$. Assume that $u(t)$ is a stochastic process in $L(0,T)$ that satisfies

$$\int_{[0,T]}\int_{[0,T]}|D^H_s u(t)||t-s|^{2H-2}\,ds\,dt<\infty.$$

Then the symmetric integral exists and the following relation holds:

$$\int_0^T u(s)\,d^\circ B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T (D^\phi_s u(t))\,ds, \tag{2.1}$$

where $\diamond$ denotes the Wick product, $1/2<H<1$.

Remark 2.4 ([4]). If $u(t)\in\mathcal{L}_\phi(0,T)$, the definition of the forward and backward integrals with respect to fBm is as follows:

$$\int_0^T u(s)\,d^- B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T (D^\phi_s u(t))\,ds, \tag{2.2}$$

$$\int_0^T u(s)\,d^+ B^H(s)=\int_0^T u(s)\diamond dB^H(s)+\int_0^T (D^\phi_s u(t))\,ds. \tag{2.3}$$

A detailed proof of Lemma 2.5 can be found in the authors' previous work [43].

Lemma 2.5. *Suppose that $Z(s)$ is a stochastic process in $\mathcal{L}_\phi(0,T)$, and $B^H(t)$, $H>1/2$, is a fractional Brownian motion. For any $0<T<\infty$, there exists a constant $C(H,T)$ such that the following inequality holds:*

$$\mathbb{E}\bigg(\int_0^T Z(s)\,d^\circ B^H(s)\bigg)^2\le 2C(H,T)\bigg(\int_0^T \mathbb{E}\,|Z(s)|^2\,ds\bigg)+4CT^2, \tag{2.4}$$

where $C(H,T)=HT^{2H-1}$.

2.2. Some results on fixed point theorems. In this section we define vector metric spaces and generalized Banach spaces and prove some properties of them. If $x,y\in\mathbb{R}^n$, $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, by $x\le y$ we mean $x_i\le y_i$ for all $i=1,\dots,n$. Also $|x|=(|x_1|,\dots,|x_n|)$ and $\max(x,y)=\max(\max(x_1,y_1),\dots,\max(x_n,y_n))$. If $c\in\mathbb{R}$, then $x\le c$ means $x_i\le c$ for each $i=1,\dots,n$. For $x\in\mathbb{R}^n$, $(x)_i=x_i$, $i=1,\dots,n$.

Definition 2.6. Let E be a vector space on $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. By a vector-valued norm on E we mean a map $\|\cdot\|\colon E\to\mathbb{R}^n_+$ with the following properties:

(a) $\|x\| \geq 0$ for all $x \in E$; if $\|x\| = 0$ then $x = 0$;
(b) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in E$ and $\lambda \in \mathbb{K}$;
(c) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in E$.

The pair $(E, \| \cdot \|)$ is called a generalized normed space. If the generalized metric generated by $\| \cdot \|$ (i.e., $d(x,y) = \|x - y\|$) is complete then the space $(E, \| \cdot \|)$ is called a generalized Banach space, where

$$\|x - y\| = \begin{pmatrix} \|x - y\|_1 \\ \vdots \\ \|x - y\|_n \end{pmatrix}.$$

Notice that $\| \cdot \|$ is a generalized Banach space on E if and only if $\| \cdot \|_i$, $i = 1, \ldots, n$, are norms on E.

Remark 2.7. In generalized metric spaces in the sense of Perov, the notations of convergence sequence, Cauchy sequence, completeness, open subset and closed subset are similar to those for usual metric spaces.

Definition 2.8. A square matrix M of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc.

Lemma 2.9 ([36]). *Let M be a square matrix of nonnegative numbers. The following assertions are equivalent:*

(a) *M is convergent to zero,*
(b) *the matrix $I - M$ is non-singular and*
$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots,$$
(c) *$\|\lambda\| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$,*
(d) *$I - M$ is non-singular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.10. We say that a non-singular matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathcal{M}_{n \times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I, \quad \text{where } |A| = (|a_{ij}|)_{1 \leq i,j \leq n} \in \mathcal{M}_{n \times n}(\mathbb{R}_+).$$

The classical Banach contraction principle was extended for contractive maps on spaces endowed with vector-valued metric by Perov in 1964 [34], Perov and Precup [35]. For a version of Schauder fixed point, see Cristescu [11]. The purpose of this section is to present a version of Schaefer's fixed point theorem and nonlinear alternative of Leary–Schauder type in generalized Banach spaces.

Theorem 2.11 ([34]). *Let (X, d) be a complete generalized metric space with $d \colon X \times X \to \mathbb{R}^n$ and let $N \colon X \to X$ be such that $d(N(x), N(y)) \leq M d(x, y)$, for all $x, y \in X$ and some square matrix M of nonnegative numbers. If the matrix*

M is convergent to zero, that is $M^k \to 0$ as $k \to \infty$, then N has a unique fixed point $x_* \in X$

$$d(N^k(x_0), x_*) \leq M^k (I - M)^{-1} d(N(x_0), x_0), \quad \textit{for every } x_0 \in X \textit{ and } k \geq 1.$$

Theorem 2.12 ([11]). *Let E be a generalized Banach space, $C \subset E$ be a nonempty closed convex subset of E and $N\colon C \to C$ be a continuous operator with relatively compact range. Then N has a fixed point in C.*

As a consequence of the Schauder fixed point theorem we present a version of Schaefer's fixed point theorem and a nonlinear alternative Leary–Schauder type theorem in generalized Banach space [6].

Theorem 2.13. *Let $(E, \|\cdot\|)$ be a generalized Banach space and $N\colon E \to E$ be a continuous compact mapping. Moreover, assume that the set*

$$\mathcal{A} = \{x \in E : x = \lambda N(x) \textit{ for some } \lambda \in (0,1)\},$$

is bounded. Then N has a fixed point.

Definition 2.14. The map $f\colon J \times \mathcal{D}_{\mathcal{F}_0} \to \mathbb{R}^n$ is said to be L^2-Carathéodory if

(a) $t \mapsto f(t,v)$ is measurable for each $v \in \mathcal{D}_{\mathcal{F}_0}$;
(b) $v \mapsto f(t,v)$ is continuous for almost all $t \in J$;
(c) for each $q > 0$, there exists $\alpha_q \in L^1(J, \mathbb{R}^+)$ such that

$$\mathbb{E}\,|f(t,v)|^2 \leq \alpha_q(t), \quad \text{for all } \|v\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq q \text{ and for a.e. } t \in J.$$

The following result is known as the Grönwall–Bihari Theorem.

Lemma 2.15 ([5]). *Let $u, g\colon J \to \mathbb{R}$ be positive real continuous functions. Assume there exist $c > 0$ and a continuous nondecreasing function $h\colon \mathbb{R} \to (0, +\infty)$ such that*

$$u(t) \leq c + \int_a^t g(s) h(u(s))\, ds, \quad \textit{for all } t \in J.$$

Then

$$u(t) \leq H^{-1}\bigg(\int_a^t g(s)\, ds\bigg), \quad \textit{for all } t \in J,$$

provided

$$\int_c^{+\infty} \frac{dy}{h(y)} > \int_a^b g(s)\, ds.$$

Here H^{-1} refers to the inverse of the function

$$H(u) = \int_c^u \frac{dy}{h(y)} \quad \textit{for } u \geq c.$$

3. Existence result

Let us start by defining what we mean by a solution of problem (1.1). $AC^i(J,\mathbb{R}^n)$ is the space of functions $y\colon J \to \mathbb{R}^n$, i times differentiable, whose i-th derivative, $y^{(i)}$, is absolutely continuous.

LEMMA 3.1. *Let $g^i\colon \mathcal{D}_{\mathcal{F}_0} \times \mathcal{D}_{\mathcal{F}_0} \to \mathbb{R}^n$ and $f^i\colon J \to \mathbb{R}^n$, $i = 1,2$, be a continuous function. Let $I_k, \overline{I}_k \in C(\mathbb{R}^n,\mathbb{R}^n)$ for each $k = 1,\ldots,m$ and let $x, y \in \mathcal{D}_{\mathcal{F}_b} \cap AC^1$ be a classical solution of the problem*

$$(3.1)\qquad \begin{cases} dx(t) = \left(\displaystyle\sum_{i=1}^{n_*} x(t-T_i) + g^1(x_t,y_t)\right) dt + f^1(t)\, d^\circ B^{H_1}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J,\ t \neq t_k, \\ dy(t) = \left(\displaystyle\sum_{i=1}^{n_*} y(t-T_i) + g^2(x_t,y_t)\right) dt + f^2(t)\, d^\circ B^{H_2}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in J,\ t \neq t_k, \\ \Delta x(t) = I_k(x(t_k)), \qquad\qquad t = t_k,\ k = 1,2,\ldots,m, \\ \Delta y(t) = \overline{I}_k(y(t_k)), \\ x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \qquad\qquad t \in [-r,0], \\ y(t) = \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \qquad\qquad t \in [-r,0], \end{cases}$$

where $r = \max\limits_{1\le i\le n_*} T_i$ if and only if z is a solution of the impulsive integral functional differential equation

$$(3.2)\qquad \begin{cases} x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, \qquad\qquad t \in [-r,0], \\ x(t) = \phi(0) + \displaystyle\sum_{i=1}^{n_*} \int_{T_i}^{0} \phi(s)\, ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)\, ds \\ \qquad + \displaystyle\int_0^t f^1(s)\, d^\circ B^{H_1}(s) + \int_0^t g^1(x_s,y_s)\, ds + \sum_{0\le t_k \le t} I_k(x(t_k)), \\ \qquad\qquad\qquad\qquad \textit{for } t \in J \textit{ and a.e. } w \in \Omega, \end{cases}$$

and

$$(3.3)\qquad \begin{cases} y(t) = \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, \qquad\qquad t \in [-r,0], \\ y(t) = \overline{\phi}(0) + \displaystyle\sum_{i=1}^{n_*} \int_{T_i}^{0} \overline{\phi}(s)\, ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\, ds \\ \qquad + \displaystyle\int_0^t f^2(s)\, d^\circ B^{H_2}(s) + \int_0^t g^2(x_s,y_s)\, ds + \sum_{0\le t_k \le t} \overline{I}_k(y(t_k)), \\ \qquad\qquad\qquad\qquad \textit{for } t \in J \textit{ and a.e. } w \in \Omega. \end{cases}$$

Proof. Let (x,y) be a possible solution of the problem (3.1). Then $z|_{[-r,t_1]} = (x|_{[-r,t_1]}, y|_{[-r,t_1]})$ is a solution to

$$dx(t) = \bigg(\sum_{i=1}^{n_*} x(t-T_i) + g^1(x_t,y_t)\bigg)\,dt + f^1(t)\,d^\circ B^{H_1}(t), \quad t \in J := [0,b].$$

Assume that $t_k < t \leq t_{k+1}$, $k = 1,\ldots,m$. Integration of the above inequality yields

$$\begin{aligned}
x(t_1^-) - x(0) &= \sum_{i=1}^{n_*}\int_0^{t_1} x(s-T_i)\,ds + \int_0^{t_1} f^1(s)\,d^\circ B^{H_1}(s) + \int_0^{t_1} g^1(x_s,y_s)\,ds,\\
x(t_1^-) - x(0) &= \sum_{i=1}^{n_*}\int_{-T_i}^{t_1-T_i} x(s)\,ds + \int_0^{t_1} f^1(s)\,d^\circ B^{H_1}(s) + \int_0^{t_1} g^1(x_s,y_s)\,ds,\\
x(t_2^-) - x(t_1^+) &= \sum_{i=1}^{n_*}\int_{t_1}^{t_2} x(s-T_i)\,ds + \int_{t_1}^{t_2} f^1(s)\,d^\circ B^{H_1}(s) + \int_{t_1}^{t_2} g^1(x_s,y_s)\,ds,\\
x(t_2^-) - x(t_1^-) &= I_1(x(t_1)) + \sum_{i=1}^{n_*}\int_{t_1-T_i}^{t_2} x(s)\,ds\\
&\quad + \int_{t_1}^{t_2} f^1(s)\,d^\circ B^{H_1}(s) + \int_{t_1}^{t_2} g^1(x_s,y_s)\,ds.
\end{aligned}$$

By recurrence,

$$\begin{aligned}
x(t_k^-) - x(t_{k-1}^+) &= \sum_{i=1}^{n_*}\int_{t_{k-1}}^{t_k} x(s-T_i)\,ds\\
&\quad + \int_{t_{k-1}}^{t_k} f^1(s)\,d^\circ B^{H_1}(s) + \int_{t_{k-1}}^{t_k} g^1(x_s,y_s)\,ds,\\
x(t_k^-) - x(t_{k-1}^-) &= I_k(x(t_k)) + \sum_{i=1}^{n_*}\int_{t_{k-1}-T_i}^{t_k-T_i} x(s)\,ds\\
&\quad + \sum_{l=1}^{\infty}\int_{t_{k-1}}^{t_k} f^1(s)\,d^\circ B^{H_1}(s) + \int_{t_{k-1}}^{t_k} g^1(x_s,y_s)\,ds,\\
x(t) - x(t_k^-) &= I_k(x(t_k)) + \sum_{i=1}^{n_*}\int_{t_k-T_i}^{t-T_i} x(s)\,ds\\
&\quad + \int_{t_k}^{t} f^1(s)\,d^\circ B^{H_1}(s) + \int_{t_k}^{t} g^1(x_s,y_s)\,ds.
\end{aligned}$$

Adding these together, we get

$$\begin{aligned}
x(t) &= x(0) + \sum_{0\leq t_k\leq t} I_k(x(t_k)) + \sum_{i=1}^{n_*}\int_{-T_i}^{t-T_i} x(s)\,ds\\
&\quad + \int_0^t f^1(s)\,d^\circ B^{H_1}(s) + \int_0^t g^1(x_s,y_s)\,ds,
\end{aligned}$$

$$x(t) = \phi(0) + \sum_{0 \le t_k \le t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} x(s)\, ds + \sum_{i=1}^{n_*} \int_{0}^{t-T_i} x(s)\, ds + \int_0^t f^1(s)\, d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)\, ds.$$

If x satisfies the integral equation (3.2), then x is a solution of the problem (3.1). Let $t \in [0,b] \setminus \{t_1, \dots, t_m\}$ and

$$x(t) = \phi(0) + \sum_{0 \le t_k \le t} I_k(x(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\, ds + \sum_{i=1}^{n_*} \int_{0}^{t-T_i} x(s)\, ds + \int_0^t f^1(s)\, d^\circ B^{H_1}(s) + \int_0^t g^1(x_s, y_s)\, ds.$$

Similarly,

$$y(t) = \overline{\phi}(0) + \sum_{0 \le t_k \le t} \overline{I}_k(y(t_k)) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \overline{\phi}(s)\, ds + \sum_{i=1}^{n_*} \int_{0}^{t-T_i} y(s)\, ds + \int_0^t f^2(s)\, d^\circ B^{H_2}(s) + \int_0^t g^2(x_s, y_s)\, ds. \qquad \square$$

This lemma leads to the definition of a solution.

Definition 3.2. Given $\phi, \overline{\phi} \in \mathcal{D}_{\mathcal{F}_0}$, an $\mathbb{R}^n$-valued stochastic process $z = (x,y)$ and $\{z(t) : t \in [-r,b]\}$ is said to be a solution of the problem (1.1) if $z(t)$ is measurable and $\mathcal{F}_t$-adapted, for each $t > 0$, $(x(t), y(t)) = (\phi(t), \overline{\phi}(t))$ on $[-r,0]$, $\Delta z|_{t=t_k} = (I_k(x(t_k^-)), \overline{I}_k(x(t_k^-)))$, $k = 1, \dots, m$, the restriction of $z(\,\cdot\,,\,\cdot\,)$ to $[0,b) \setminus \{t_1, \dots, t_m\}$ is continuous, z satisfies the integral equation

$$\begin{cases} x(t) = \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0], \\ x(t) = \phi(0) + \displaystyle\sum_{i=1}^{n_*} \int_{T_i}^{0} \phi(s)\, ds & \\ \qquad + \displaystyle\sum_{i=1}^{n_*} \int_{0}^{t-T_i} x(s)\, ds + \int_0^t f^1(s)\, d^\circ B^{H_1}(s) & \\ \qquad + \displaystyle\int_0^t g^1(s, x_s, y_s)\, ds \quad + \sum_{0 \le t_k \le t} I_k(x(t_k)), & t \in [0,b], \\ y(t) = \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0], \\ y(t) = \overline{\phi}(0) + \displaystyle\sum_{i=1}^{n_*} \int_{T_i}^{0} \overline{\phi}(s)\, ds & \\ \qquad + \displaystyle\sum_{i=1}^{n_*} \int_{0}^{t-T_i} y(s)\, ds + \int_0^t f^2(s)\, d^\circ B^{H_2}(s) & \\ \qquad + \displaystyle\int_0^t g^2(s, x_s, y_s)\, ds + \sum_{0 \le t_k \le t} \overline{I}_k(y(t_k)), & t \in [0,b]. \end{cases}$$

First, we will list the following hypotheses which will be imposed in our main theorem. Throughout this section:

(H_1) The function $f^i\colon J \to \mathbb{R}^n$ satisfies

$$\int_0^b \|f^i(t)\|^2\,dt < \infty \quad \text{for all } t \in J.$$

(H_2) There exist functions $\alpha_i, \beta_i \in L^1([0,b],\mathbb{R}^+)$ and $R > 0$ such that

$$\mathbb{E}\,|g^i(t,x,y) - g^i(t,\overline{x},\overline{y})|^2 \le \alpha_i(t)\|x-\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \beta_i(t)\|y-\overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}},$$

for all $x, y, \overline{x}, \overline{y} \in \mathcal{D}_{\mathcal{F}_0}$ with $\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \|\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \le R$ and almost every t in $[0,b]$.

(H_3) There exist constants $d_k \ge 0$ and $\overline{d}_k \ge 0$, $k = 1,\ldots,m$, such that

$$|I_k(x) - I_k(\overline{x})|^2 \le d_k|x-\overline{x}|^2 \quad \text{and} \quad |\overline{I}_k(y) - \overline{I}_k(\overline{y})|^2 \le \overline{d}_k|y-\overline{y}|^2$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$ and almost every $t \in [0,b]$.

For our main considerations on the problem (1.1), a Preov fixed point is used to investigate the existence and uniqueness of solutions for systems of impulsive stochastic differential equations.

Theorem 3.3. *Assume that* (H_1)–(H_3) *are satisfied and consider the matrix*

$$M = \begin{pmatrix} \sqrt{\dfrac{1}{\tau} + 3m\displaystyle\sum_{k=1}^m d_k} & \dfrac{1}{\sqrt{\tau}} \\ \dfrac{1}{\sqrt{\tau}} & \sqrt{\dfrac{1}{\tau} + 3m\displaystyle\sum_{k=1}^m \overline{d}_k} \end{pmatrix},$$

where τ is sufficiently large. If M converges to zero, then the problem (1.1) *has a unique solution on* $[-r,b]$.

Proof. Consider the operator $N\colon \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} \to \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ defined by

$$N(x,y) = (N_1(x,y), N_2(x,y)), \qquad (x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b},$$

where

$$N_1(x,y)(t) = \begin{cases} \phi(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0], \\ \phi(0) + \displaystyle\sum_{i=1}^{n_*}\int_{T_i}^0 \phi(s)\,ds \\ \qquad + \displaystyle\sum_{i=1}^{n_*}\int_0^{t-T_i} x(s)\,ds + \int_0^t f^1(s)\,d^\circ B^{H_1}(s) \\ \qquad + \displaystyle\int_0^t g^1(s,x_s,y_s)\,ds + \sum_{0<t_k<t} I_k(x(t_k^-)), & t \in [0,b], \end{cases} \tag{3.4}$$

and

$$(3.5)\quad N_2(x,y)(t) = \begin{cases} \overline{\phi}(t) \in \mathcal{D}_{\mathcal{F}_0}, & t \in [-r,0], \\ \overline{\phi}(0) + \displaystyle\sum_{i=1}^{n_*} \int_{T_i}^{0} \overline{\phi}(s)\, ds \\ \quad + \displaystyle\sum_{i=1}^{n_*} \int_0^{t-T_i} y(s)\, ds + \int_0^t f^2(s)\, d^\circ B^{H_2}(s) \\ \quad + \displaystyle\int_0^t g^2(s,x_s,y_s)\, ds + \sum_{0<t_k<t} \overline{I}_k(y(t_k^-)), & t \in [0,b]. \end{cases}$$

The operator in (3.4) and (3.5) is well defined given $(x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$. We shall use Theorem 2.11 to prove that N has a fixed point. Indeed, let $(x,y),(\overline{x},\overline{y}) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$, then for each $t \in [-r,b]$ we have

$$\begin{aligned} &|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 \\ &\quad = \Big| \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)\, ds + \int_0^t g^1(s,x_s,y_s)\, ds + \sum_{0<t_k<t} I_k(x(t_k^-) \\ &\qquad - \sum_{i=1}^{n_*} \int_0^{t-T_i} \overline{x}(s)\, ds - \int_0^t g^1(s,\overline{x}_s,\overline{y}_s)\, ds - \sum_{0<t_k<t} I_k(\overline{x}(t_k^-)) \Big|^2. \end{aligned}$$

By the inequality (Lemma 2.5), we get

$$\begin{aligned} &\mathbb{E}\,|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 \\ &\quad \le 3b \int_0^t \mathbb{E}\,|g^1(s,x_s,y_s) - g^1(s,\overline{x}_s,\overline{y}_s)|^2\, ds \\ &\qquad + 3m \sum_{k=1}^{m} \mathbb{E}\,|I_k(x(t_k)) - I_k(\overline{x}(t_k))|^2 + 3n_* b \int_0^t \mathbb{E}\,|x(s) - \overline{x}(s)|^2\, ds. \end{aligned}$$

Therefore, by conditions (H_1)–(H_3),

$$\begin{aligned} &\mathbb{E}\,|N_1(x(t),y(t)) - N_1(\overline{x}(t),\overline{y}(t))|^2 \\ &\quad \le 3b \int_{-r}^t \big(\alpha_1(s)\,\mathbb{E}\,|x(s) - \overline{x}(s)|^2 + \beta_1(s)\,\mathbb{E}\,|y(s) - \overline{y}(s)|^2\big)\, ds \\ &\qquad + 3m \sum_{k=1}^{m} d_k \mathbb{E}\,|x(t_k) - \overline{x}(t_k)|^2 + 3n_* b \int_0^t \mathbb{E}|x(s) - \overline{x}(s)|^2\, ds \\ &\quad \le \int_0^t (3b\alpha_1(s) + 3n_* b)\,\mathbb{E}\,|x(s) - \overline{x}(s)|^2 ds \\ &\qquad + \int_0^t (3b\beta_1(s))\,\mathbb{E}\,|y(s) - \overline{y}(s)|^2\, ds + 3m \sum_{k=1}^{m} d_k \mathbb{E}\,|x(t_k) - \overline{x}(t_k)|^2, \end{aligned}$$

and therefore, since $(x(s),y(s))=(\overline{x}(s),\overline{y}(s))$ over the interval $[-r,0]$, by taking supremum in the above inequality,

$$\begin{aligned}
\sup_{h\in[0,t]}&\mathbb{E}\,|N_1(x(h),y(h))-N_1(\overline{x}(h),\overline{y}(h))|^2\\
&\leq \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\sup_{\theta\in[0,s]}\mathbb{E}\,|x(\theta)-\overline{x}(\theta)|^2\,ds\\
&\quad+\int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\sup_{\theta\in[0,s]}\mathbb{E}\,|y(\theta)-\overline{y}(\theta)|^2\,ds\\
&\quad+3m\sum_{k=1}^m d_k e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\mathbb{E}\,|x(t_k)-\overline{x}(t_k)|^2\\
&\leq \int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}\,ds\|x-\overline{x}\|_*^2+\int_0^t \alpha(s)e^{\tau\widehat{\alpha}(s)}\,ds\|y-\overline{y}\|_*^2\\
&\quad+3m\sum_{k=1}^m d_k e^{\tau\widehat{\alpha}(s)}e^{-\tau\widehat{\alpha}(s)}\mathbb{E}\,|x(t_k)-\overline{x}(t_k)|^2\\
&\leq \frac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'\,ds\|x-\overline{x}\|_*^2\\
&\quad+\frac{1}{\tau}\int_0^t (e^{\tau\widehat{\alpha}(s)})'\,ds\|y-\overline{y}\|_*^2+3m\sum_{k=1}^m d_k e^{\tau\widehat{\alpha}(s)}\|x-\overline{x}\|_*^2\\
&\leq \bigg(\frac{1}{\tau}+3m\sum_{k=1}^m d_k\bigg)e^{\tau\widehat{\alpha}(t)}\|x-\overline{x}\|_*^2+\frac{1}{\tau}\,e^{\tau\widehat{\alpha}(t)}\|y-\overline{y}\|_*^2.
\end{aligned}$$

We deduce

$$\begin{aligned}
e^{-\tau\widehat{\alpha}(t)}&\sup_{h\in[0,t]}\mathbb{E}\,|N_1(x(h),y(h))-N_1(\overline{x}(h),\overline{y}(h))|^2\\
&\leq \bigg(\frac{1}{\tau}+3m\sum_{k=1}^m d_k\bigg)\|x-\overline{x}\|_*^2+\frac{1}{\tau}\,\|y-\overline{y}\|_*^2,
\end{aligned}$$

where $\|x\|_*^2$ is the Bielecki-type norm on $\mathcal{D}_{\mathcal{F}_b}$ defined by

$$\|x\|_*^2=\sup_{h\in[0,t]}\mathbb{E}\,|x(h,\,\cdot\,)|^2e^{-\tau\widehat{\alpha}(t)},$$

where

$$\alpha(s)=\begin{cases}0 & \text{for } t\in[-r,0],\\ \max\{3b\alpha_1(s)+3n_*b,3b\beta_1(s)\} & \text{for } t\in[0,b]\end{cases}$$

and

$$\widehat{\alpha}(t)=\int_0^t\alpha(s)\,ds,\quad t\in[-r,b].$$

This yields

$$\|N_1(x,y)-N_1(\overline{x},\overline{y})\|_*^2\leq\bigg(\frac{1}{\tau}+3m\sum_{k=1}^m d_k\bigg)\|x-\overline{x}\|_*^2+\frac{1}{\tau}\,\|y-\overline{y}\|_*^2.$$

Using the fact that for all $a, b \geq 0$ we have $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we conclude that

$$\|N_1(x,y) - N_1(\overline{x},\overline{y})\|_* \leq \left(\sqrt{\frac{1}{\tau} + 3m\sum_{k=1}^{m} d_k}\right)\|x-\overline{x}\|_* + \frac{1}{\sqrt{\tau}}\|y-\overline{y}\|_*. \tag{3.6}$$

Similar computations for N_2 yield

$$\|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \leq \frac{1}{\sqrt{\tau}}\|x-\overline{x}\|_* + \left(\sqrt{\frac{1}{\tau} + 3m\sum_{k=1}^{m} \overline{d}_k}\right)\|y-\overline{y}\|_*. \tag{3.7}$$

Now, (3.6), (3.7) can be put together and be rewritten as

$$\begin{aligned}\|N(x,y) - N(\overline{x},\overline{y})\|_* &= \begin{pmatrix} \|N_1((x,y) - N_1(\overline{x},\overline{y})\|_* \\ \|N_2(x,y) - N_2(\overline{x},\overline{y})\|_* \end{pmatrix} \\ &\leq \begin{pmatrix} \sqrt{\frac{1}{\tau} + 3\sum_{k=1}^{m} d_k} & \frac{1}{\sqrt{\tau}} \\ \frac{1}{\sqrt{\tau}} & \sqrt{\frac{1}{\tau} + 3m\sum_{k=1}^{m} \overline{d}_k} \end{pmatrix} \begin{pmatrix} \|x-\overline{x}\|_* \\ \|y-\overline{y}\|_* \end{pmatrix}.\end{aligned}$$

Hence

$$\|N(x,y) - N(\overline{x},\overline{y})\|_* \leq M \begin{pmatrix} \|x-\overline{x}\|_* \\ \|y-\overline{y}\|_* \end{pmatrix},$$

where

$$M = \begin{pmatrix} \sqrt{\frac{1}{\tau} + 3m\sum_{k=1}^{m} d_k} & \frac{1}{\sqrt{\tau}} \\ \frac{1}{\sqrt{\tau}} & \sqrt{\frac{1}{\tau} + 3m\sum_{k=1}^{m} \overline{d}_k} \end{pmatrix}.$$

We choose τ sufficiently large so that the matrix $\|M\| < 1$. Then M is nonnegative, $I - M$ is nonsingular and $(I-M)^{-1} = 1 + M + M^2 + \ldots$ From Lemma 2.9 we obtain that M converges to zero. As a consequence of Perov's fixed point theorem, N has a unique fixed point $(x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ which is a unique solution of the problem (1.1). The result follows from Perov's fixed point theorem. □

For the next result we can prove the a priori estimates of solution for problem (1.1) by similar arguments to those used to prove Theorem 3.3.

Theorem 3.4. *Assume hypotheses in Theorem* 3.3 *hold, with* (H_2) *replaced by:*

($\overline{\mathrm{H}}_2$) *There exist positive constants* $\widetilde{\alpha}_i, \widetilde{\beta}_i$, $i = 1,2$, *such that*

$$\mathbb{E}\,|g^i(t,x,y) - g^i(t,x,y)|^2 \le \widetilde{\alpha}_i \|x - \overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \widetilde{\beta}_i \|y - \overline{y}\|^2_{\mathcal{D}_{\mathcal{F}_0}},$$

for all $x, y, \overline{x}, \overline{y} \in \mathcal{D}_{\mathcal{F}_0}$ *and almost every* $t \in [0,b]$. *Consider the matrix*

$$M_* = \begin{pmatrix} A_1 & A_2, \\ A_3 & A_4 \end{pmatrix},$$

where

$$A_1 = \sqrt{3b\widetilde{\alpha}_1 + 3n_* b + 3m \sum_{k=1}^{m} d_k}\,, \qquad A_2 = \sqrt{3b\widetilde{\beta}_1}\,,$$

$$A_3 = \sqrt{3b\widetilde{\alpha}_2}\,, \qquad A_4 = \sqrt{3b\widetilde{\beta}_2 + 3n_* b + 3m \sum_{k=1}^{m} \overline{d}_k}\,.$$

If M_* *converges to zero, then the problem* (1.1) *has a unique solution.*

4. Existence results

In this section we present the existence result under a nonlinearity g^i, $i = 1,2$, satisfying a Nagumo-type growth conditions:

(H_4) There exist functions $\Lambda_i \in L^1(J, \mathbb{R}^+)$, $i = 1,2$, such that

$$\sup_{t \in J} \mathbb{E}\,|f^i(t)|^2 \le \Lambda_i(t) \quad \text{for all } t \in J.$$

(H_5) There exist a function $p_i \in L^1(J, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi_i \colon [0,\infty) \to (0,\infty)$, $i = 1,2$, such that

$$\begin{cases} \mathbb{E}\,|g^1(t,x,y)|^2 \le p_1(t)\psi_1\big(\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|y\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big), \\ \mathbb{E}\,|g^2(t,x,y)|^2 \le p_2(t)\psi_2\big(\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|y\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big), \end{cases}$$

with

$$\int_{\upsilon(0)}^{\upsilon(t)} \frac{ds}{s + \Phi(s)} \le \int_0^b m(s)\,ds < \int_{\upsilon(0)}^{\infty} \frac{ds}{s + \Phi(s)},$$

where

$$p(t) = \sum_{i=1}^{2} p_i(t) \quad \text{and} \quad \Phi = \sum_{i=1}^{2} \psi_i(t), \quad \Lambda(t) = \sum_{i=1}^{2} \Lambda_i(t),$$

and $\upsilon(0) = bC_1 + (1 - bC_4)\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)$. Let

$$\begin{aligned} C_1 = 6\big(\mathbb{E}\,|\phi(0)|^2 + \mathbb{E}\,|\overline{\phi}(0)|^2\big) + 6n_* r\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big) \\ + 6m\bigg(\sum_{k=1}^{m} c_k + \sum_{k=1}^{m} \overline{c}_k\bigg) + 48CT^2 + 12\,C(H_*,T)\|\Lambda(t)\|_{L^1}, \end{aligned}$$

$C_* = 6b$, $C_4 = 6n_*$, $m(t) = \max\{bC_*p(t), C_4\}$, and

$$C(H_*, T) = \max\{C(H_1, T), C(H_2, T)\},$$

for all $x, y \in \mathcal{D}_{\mathcal{F}_0}$.

(H$_6$) There exist positive constants $c_k, \widetilde{c}_k$, $k = 1, \dots, m$, such that

$$\mathbb{E}\,|I_k(x_k)|^2 \le c_k, \quad \mathbb{E}\,|\overline{I}_k(y_k)|^2 \le \bar{c}_k \quad \text{for all } x, y \in \mathbb{R}^n.$$

Theorem 4.1. *Assume that* (H$_4$)–(H$_6$) *hold. Then* (1.1) *possesses at least one solution on* $[-r, b]$.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator N defined in Theorem 3.3. In order to apply Theorem (2.13), we first show that N is completely continuous. The proof will be given in several steps.

Step 1. $N = (N_1, N_2)$ is continuous. Let (x_n, y_n) be a sequence such that $(x_n, y_n) \to (x, y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ as $n \to \infty$. Then

$$\begin{aligned}
&|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))| \\
&\quad = \bigg| \sum_{i=1}^{n_*} \int_0^{t-T_i} x_n(s)\, ds + \int_0^t g^1(s, (x_n)_s), (y_n)_s)\, ds + \sum_{0<t_k<t} I_k(x_n(t_k^-)) \\
&\qquad - \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)\, ds - \int_0^t g^1(s, x_s, y_s)\, ds - \sum_{0<t_k<t} I_k(x(t_k^-)) \bigg|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}\,|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 \\
&\quad \le 3\mathbb{E} \left| \int_0^t g^1(s, (x_n)_s), (y_n)_s) - g^1(s, x_s, y_s)\, ds \right|^2 \\
&\qquad + 3m \sum_{k=1}^m \mathbb{E}|I_k(x_n(t_k)) - I_k(x(t_k))|^2 + 3 \sum_{i=1}^{n_*} \int_0^{t-T_i} \mathbb{E}\,|x_n(s) - x(s)|^2\, ds.
\end{aligned}$$

From Lemma 2.5 and thanks to (H$_4$)–(H$_6$) we obtain that for any $t \in [0, b]$,

$$\begin{aligned}
&\mathbb{E}\,|N_1(x_n(t), y_n(t)) - N_1(x(t), y(t))|^2 \\
&\quad \le 3b \int_0^t \mathbb{E}\left|g^1(s, (x_n)_s), (y_n)_s)) - g^1(s, x_s, y_s)\right|^2\, ds \\
&\qquad + 3m \sum_{k=1}^m \mathbb{E}|I_k(x_n(t_k)) - I_k(x(t_k))|^2 + 3 \sum_{i=1}^{n_*} \int_0^{t-T_i} \mathbb{E}\,|x_n(s) - x_n(s)|^2\, ds.
\end{aligned}$$

Since g^i is a Carathéodory function, $i=1,2$, $I_k, \overline{I}_k$ are continuous functions, by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\sup_{t\in J}\mathbb{E}\,|N_1(x_n(t),y_n(t))-N_1(x(t),y(t))|^2 &\leq 3b\mathbb{E}\,\|g^1(\,\cdot\,,x_n,y_n)-g^1(\,\cdot\,,x,y)\|^2_{L^2}\\ &+3m\sum_{k=1}^{m}\mathbb{E}\,|I_k(x_n(t_k))-I_k(x(t_k))|^2+3\sum_{i=1}^{n_*}\int_0^{t-T_i}\mathbb{E}\,|x_n(s)-x(s)|^2\,ds\to 0\end{aligned}$$

as $n\to\infty$. Similarly,

$$\begin{aligned}\sup_{t\in J}\mathbb{E}\,|N_2(x_n(t),y_n(t))-N_2(x(t),y(t))|^2 &\leq 3b\mathbb{E}\,\|g^2(\,\cdot\,,x_n,y_n)-g^2(\,\cdot\,,x,y)\|^2_{L^2}\\ &+3m\sum_{k=1}^{m}\mathbb{E}\,|\overline{I}_k(y_n(t_k))-\overline{I}_k(y(t_k))|^2+3\sum_{i=1}^{n_*}\int_0^{t-T_i}\mathbb{E}\,|y_n(s)-y(s)|^2\,ds\to 0\end{aligned}$$

as $n\to\infty$. Thus N is continuous.

Step 2. N maps bounded sets into bounded sets in $\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}$. Indeed, it is enough to show that for any $q>0$ there exists a positive constant l such that for each $(x,y)\in B_q=\big\{(x,y)\in\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}:\|x\|^2_{\mathcal{D}_{\mathcal{F}_b}}\leq q,\ \|y\|^2_{\mathcal{D}_{\mathcal{F}_b}}\leq q\big\}$, we have

$$\|N(x,y)\|^2_{\mathcal{D}_{\mathcal{F}_b}}\leq l=(l_1,l_2).$$

Then, for each $t\in J$,

$$\begin{aligned}\mathbb{E}\,|N_1(x(t),y(t))|^2\leq{}&6\mathbb{E}\,|\phi(0)|^2+6\sum_{i=1}^{n_*}\int_0^{T_i}\mathbb{E}\,|\phi(-s)|^2\,ds\\ &+6\sum_{i=1}^{n_*}\int_0^{t-T_i}\mathbb{E}\,|x(s)|^2\,ds+6\mathbb{E}\,\bigg|\int_0^t f^1(s)\,d^\circ B^{H_1}(s)\bigg|^2\\ &+6b\int_0^t\mathbb{E}\,|g^1(s,x_s,y_s)|^2\,ds+6m\sum_{0<t_k<t}\mathbb{E}\,|I_k(x(t_k^-)|^2.\end{aligned}$$

This implies by (H$_4$)–(H$_6$) and Lemma 2.5 that for each $t\in J$,

$$\begin{aligned}\mathbb{E}\,|N_1(x(t),y(t))|^2\leq{}&6\mathbb{E}\,|\phi(0)|^2+6\sum_{i=1}^{n_*}\int_0^{T_i}\mathbb{E}\,|\phi(-s)|^2\,ds\\ &+6\sum_{i=1}^{n_*}\int_0^{t-T_i}\mathbb{E}\,|x(s)|^2\,ds+12\,C(H_1,T)\|\Lambda_1\|_{L^1}\\ &+6b\|p_1\|_{L^1}\psi_1(2q)+6m\sum_{k=1}^{m}c_k+24CT^2.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\,|N_1(x(t),y(t))|^2\leq{}&6\mathbb{E}\,|\phi(0)|^2+6n_*r\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}+6n_*qb+12\,C(H_1,T)\|\Lambda_1\|_{L^1}\\ &+6b\|p_3\|_{L^1}\psi_1(2q)d+6m\sum_{k=1}^{m}c_k+24CT^2:=l_1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}\,|N_2(x(t),y(t))|^2 \leq & 6\mathbb{E}\,|\overline{\phi}(0)|^2 + 6n_* r\|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + 6n_* qb + 12\,C(H_2,T)\|\Lambda_2\|_{L^1} \\ & + 6b\|\Lambda_2\|_{L^1}\psi_2(2q) + 6m\sum_{k=1}^{m}\overline{c}_k + 24CT^2 := l_2.\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets of $\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}$. Let B_q be a bounded set in $\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}$ as in Step 2. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, and $u\in B_q$. Thus we have

$$\begin{aligned}&\mathbb{E}\,|N_1(x(\tau_2),y(\tau_2)) - N_1(x(\tau_1),y(\tau_1))|^2 \\ &\quad\leq 4\sum_{i=1}^{n_*}\int_{\tau_1-T_i}^{\tau_2-T_i}\mathbb{E}\,|x(s)|^2\,ds + 4(\tau_2-\tau_2)\int_{\tau_1}^{\tau_2}\mathbb{E}\,|g^1(s,x_s,y_s)|^2\,ds \\ &\qquad + 4\mathbb{E}\left|\int_{\tau_1}^{\tau_2} f^1(s)\,d^\circ B^{H_1}(s)\right|^2 + 4m\sum_{\tau_1<t_k<\tau_2}\mathbb{E}\,|I_k(x(t_k))|^2 \\ &\quad\leq 4\sum_{i=1}^{n_*}\int_{\tau_1-T_i}^{\tau_2-T_i}\mathbb{E}\,|x(s)|^2\,ds + 8C(H_1,T)\int_{\tau_1}^{\tau_2}\mathbb{E}\,|f^1(s)|^2\,ds + 16\,C(\tau_2-\tau_1)^2 \\ &\qquad + 4b\int_{\tau_1}^{\tau_2}\mathbb{E}\,|g^1(s,x_s,y_s)|^2\,ds + 4m\sum_{\tau_1<t_k<\tau_2}\mathbb{E}\,|I_k(x(t_k))|^2 \\ &\quad\leq 4n_*(\tau_2-\tau_1)q + 8C(H_1,T)\int_{\tau_1}^{\tau_2}\Lambda_1(s)\,ds + 16\,C(\tau_2-\tau_1)^2 \\ &\qquad + 4b\psi_1(2q)\int_{\tau_1}^{\tau_2}p_1(s)\,ds + 4m\sum_{\tau_1<t_k<\tau_2}c_k.\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\,|N_2(x(\tau_2),y(\tau_2)) - N_2(x(\tau_1),y(\tau_1))|^2 \\ &\qquad\leq 4n_*(\tau_2-\tau_1)q + 8C(H_2,T)\int_{\tau_1}^{\tau_2}\Lambda_2(s)\,ds \\ &\qquad\quad + 4b\psi_2(2q)\int_{\tau_1}^{\tau_2}p_2(s)\,ds + 4m\sum_{\tau_1<t_k<\tau_2}\overline{c}_k + 16\,C(\tau_2-\tau_1)^2.\end{aligned}$$

The right-hand term tends to zero as $|\tau_2-\tau_1|\to 0$. As a consequence of Steps 1–3 together with the Arzelá–Ascoli theorem, we conclude that N maps B_q into a precompact set in $\mathcal{D}_{\mathcal{F}_b}\times\mathcal{D}_{\mathcal{F}_b}$.

Step 4. $(N(B_q)(t)$ is precompact in $\mathbb{R}^n\times\mathbb{R}^n$. As a consequence of Steps 2 and 3, together with the Arzelá–Ascoli theorem, it suffices to show that N maps B_q into a precompact set in $\mathbb{R}^n\times\mathbb{R}^n$. Let $0<t<b$ be fixed and let ε be a real

number satisfying $0 < \varepsilon < t$. For $(x, y) \in B_q$ we define

$$\begin{aligned} N_1^\varepsilon(x,y) = {}&\phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-\varepsilon-T_i} x(s)\,ds \\ &+ \int_0^{t-\varepsilon} f^1(s)\,d^\circ B^{H_1}(s) + \int_0^{t-\varepsilon} g^1(s, x_s, y_s)\,ds + \sum_{0<t_k<t-\varepsilon} I_k(x(t_k^-)), \end{aligned}$$

and

$$\begin{aligned} N_2^\varepsilon(x,y) = {}&\overline{\phi}(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \overline{\phi}(s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-\varepsilon-T_i} y(s)\,ds \\ &+ \int_0^{t-\varepsilon} f^2(s)\,d^\circ B^{H_2}(s) + \int_0^{t-\varepsilon} g^2(s, x_s, y_s)\,ds + \sum_{0<t_k<t-\varepsilon} \overline{I}_k(y(t_k^-)). \end{aligned}$$

The set $H_\varepsilon = \{N^\varepsilon(x,y)(t) = (N_1^\varepsilon(x,y)(t), N_2^\varepsilon(x,y)(t)) : (x,y) \in B_q\}$, is precompact in $\mathbb{R}^n \times \mathbb{R}^n$ for every ε and $0 < \varepsilon < t$. Moreover, for every $(x,y) \in B_q$, we have

$$\begin{aligned} \mathbb{E}\,\|N_1(x,y) - N_1^\varepsilon(x,y)\|^2 \le {}& 4n_*\varepsilon q + 8C(H_1,T) \int_{t-\varepsilon}^{t} \Lambda_1(s)\,ds \\ &+ 4b\psi_1(2q) \int_{t-\varepsilon}^{t} p_1(s)\,ds + 4m \sum_{t-\varepsilon<t_k<t} c_k + 16\,CT^2\varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbb{E}\,\|N_2(x,y) - N_2^\varepsilon(x,y)\|^2 \le {}& 4n_*\varepsilon q + 8C(H_2,T) \int_{t-\varepsilon}^{t} \Lambda_2(s)\,ds \\ &+ 4b\psi_2(2q) \int_{t-\varepsilon}^{t} p_2(s)\,ds + 4m \sum_{t-\varepsilon<t_k<t} \overline{c}_k + 16\,CT^2\varepsilon. \end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set H_ε. Hence, the set $H = \{N(x,y)(t) = (N_1(x,y)(t), N_2(x,y)(t)) : (x,y) \in B_q\}$ is precompact in $\mathbb{R}^n \times \mathbb{R}^n$ and the right-hand term tends to 0 uniformly in t as $\varepsilon \to 0^+$. Hence the relative compactness of $N(B_q)(t)$ for $t \ge 0$ follows. By the Arzela–Ascoli theorem, we conclude that $N\colon \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} \to \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b}$ is completely continuous.

Step 4. A priori bounds. Now it remains to show that the set

$$\mathcal{A} = \{(x,y) \in \mathcal{D}_{\mathcal{F}_b} \times \mathcal{D}_{\mathcal{F}_b} : (x,y) = \lambda N(x,y),\ \lambda \in (0,1)\}$$

is bounded. Let $(x,y) \in \mathcal{A}$. Then $x = \lambda N_1(x,y)$ and $y = \lambda N_2(x,y)$ for some $0 < \lambda < 1$. Thus, for $t \in J$, we have

$$\begin{aligned} x(t) = \lambda\bigg(&\phi(0) + \sum_{i=1}^{n_*} \int_{-T_i}^{0} \phi(s)\,ds + \sum_{i=1}^{n_*} \int_0^{t-T_i} x(s)\,ds \\ &+ \int_0^{t} f^1(s)\,d^\circ B^{H_1}(s) + \int_0^{t} g^1(s, x_s, y_s)\,ds + \sum_{0<t_k<t} I_k(x(t_k^-))\bigg), \end{aligned}$$

and $\phi(t) \in \mathcal{D}_{\mathcal{F}_0}$ for $t \in [-r, 0]$. This implies, by (H$_4$)–(H$_6$) and Lemma 2.5, that, for each $t \in J$,

$$\begin{aligned}\mathbb{E}\,|x(t)|^2 \leq\, & 6\mathbb{E}\,|\phi(0)|^2 + 6n_* r\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ & + 6n_* \int_0^t \mathbb{E}\,|x(s)|^2\,ds + 12\,C(H_1, T)\int_0^t \Lambda_1(s)\,ds \\ & + 24CT^2 + 6b\int_0^t p_1(s)\psi_1\big(\|x_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|y_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)\,ds + 6m\sum_{k=1}^m c_k,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\,|y(t)|^2 \leq\, & 6\mathbb{E}\,|\overline{\phi}(0)|^2 + 6n_* r\|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ & + 6n_* \int_0^t \mathbb{E}\,|y(s)|^2\,ds + 12\,C(H_2, T)\int_0^t \Lambda_2(s)\,ds \\ & + 24CT^2 + 6b\int_0^t p_2(s)\psi_2\big(\|x_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|y_s\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)\,ds + 6m\sum_{k=1}^m \overline{c}_k.\end{aligned}$$

Consider functions $\mu, \overline{\mu}$ defined on J by

$$\begin{aligned}\mu(t) &= \sup\{\mathbb{E}\,|x(s)|^2 : 0 \leq s \leq t\}, \quad t \in J, \\ \overline{\mu}(t) &= \sup\{\mathbb{E}\,|y(s)|^2 : 0 \leq s \leq t\}, \quad t \in J.\end{aligned}$$

Since

$$\|x_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} = \int_{-r}^0 \mathbb{E}\,|x(s+\theta)|^2\,d\theta = \int_{-r+s}^s \mathbb{E}\,|x(\theta)|^2\,d\theta,$$

then

$$\|x_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \int_{-r}^0 \mathbb{E}\,|x(\theta)|^2\,d\theta + \int_0^s \mathbb{E}\,|x(\theta)|^2\,d\theta \leq \int_{-r}^0 \mathbb{E}\,|x(\theta)|^2\,d\theta + b\mu(t),$$

and

$$\|y_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \int_{-r}^0 \mathbb{E}\,|y(\theta)|^2\,d\theta + b\overline{\mu}(t).$$

Hence

$$\|x_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\mu(t) \quad \text{and} \quad \|y_s\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\overline{\mu}(t).$$

Then, for $t \in J$, we have

$$\begin{aligned}\mu(t) \leq\, & 6\mathbb{E}\,|\phi(0)|^2 + 6n_* r\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ & + 6n_* \int_0^t \mu(s)\,ds + 12\,C(H_1, T)\int_0^t \Lambda_1(s)\,ds + 24CT^2 \\ & + 6b\int_0^t p_1(s)\psi_1\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\mu(t) + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\overline{\mu}(t)\big)\,ds + 6m\sum_{k=1}^m c_k.\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\mu}(t) \le{}& 6\mathbb{E}\,|\overline{\phi}(0)|^2 + 6n_* r\|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &+ 6n_*\int_0^t \overline{\mu}(s)\,ds + 12\,C(H_2,T)\int_0^t \Lambda_2(s)\,ds + 24CT^2 \\ &+ 6b\int_0^t p_2(s)\psi_2\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\mu(t) + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\overline{\mu}(t)\big)\,ds + 6m\sum_{k=1}^m \overline{c}_k.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mu(t)+\overline{\mu}(t) \le{}& 6\big(\mathbb{E}\,|\phi(0)|^2 + \mathbb{E}\,|\overline{\phi}(0)|^2\big) + 6n_* r\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big) \\ &+ 6m\bigg(\sum_{k=1}^m c_k + \sum_{k=1}^m \overline{c}_k\bigg) + 48CT^2 + 6n_*\int_0^t (\mu(s)+\overline{\mu}(s))\,ds \\ &+ 6b\sum_{i=1}^2\int_0^t p_i(s)\psi_i\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + b\mu(t) + ||\overline{\phi}||^2_{\mathcal{D}_{\mathcal{F}_0}} + b\overline{\mu}(t)\big)\,ds \\ &+ 12\,C(H_*,T)\sum_{i=1}^2\int_0^t \Lambda_i(s)\,ds,\end{aligned}$$

where $C(H_*,T) = \max\{C(H_1,T), C(H_2,T)\}$. We denote

$$p(t) = \sum_{i=1}^2 p_i(t) \quad \text{and} \quad \Phi = \sum_{i=1}^2 \psi_i(t), \quad \Lambda(t) = \sum_{i=1}^2 \Lambda_i(t).$$

We have

$$\begin{aligned}\mu(t)+\overline{\mu}(t) \le{}& C_1 + C_4\bigg(\int_0^t \mu(s)+\overline{\mu}(s)\bigg)\,ds \\ &+ C_*\int_0^t p(s)\Phi\big(b(\mu(s)+\overline{\mu}(s)) + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)\,ds.\end{aligned}$$

Put

$$\begin{aligned}C_1 ={}& 6(\mathbb{E}\,|\phi(0)|^2 + \mathbb{E}\,|\overline{\phi}(0)|^2) + 6n_* r\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big) \\ &+ 6m\bigg(\sum_{k=1}^m c_k + \sum_{k=1}^m \overline{c}_k\bigg) + 48CT^2 + 12\,C(H_*,T)\|\Lambda(t)\|_{L^1}\end{aligned}$$

and $C_* = 6b$, $C_4 = 6n_*$. Thus, we have

$$\begin{aligned}b(\mu(t)+\overline{\mu}(t)) &+ \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \\ &\le bC_1 + (1-bC_4)\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big) \\ &\quad + C_4\bigg(\int_0^t b(\mu(s)+\overline{\mu}(s)) + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\bigg)\,ds \\ &\quad + bC_*\int_0^t p(s)\Phi\big(b(\mu(s)+\overline{\mu}(s)) + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)\,ds.\end{aligned}$$

Denote the right-hand side of the above inequality as $\upsilon(t)$. Then we have

$$\upsilon(0) = bC_1 + (1 - bC_4)\big(\|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big),$$
$$b(\mu(t) + \overline{\mu}(t)) + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \upsilon(t) \qquad \text{for a.e. } t \in J,$$

and

$$\begin{aligned}\upsilon'(t) = bC_* p(t)\Phi\big(b(\mu(t) + \overline{\mu}(t)) + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)& \\ + C_4\big(b(\mu(t) + \overline{\mu}(t)) + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}}\big)&.\end{aligned}$$

Using the increasing character of Φ we obtain

$$\upsilon'(t) \leq bC_* p(t)\Phi(\upsilon(t)) + C_4\upsilon(t) \leq m(t)(\upsilon(t) + \Phi(\upsilon(t))) \quad \text{for a.e. } t \in J,$$

where $m(t) = \max\{bC_* p(t), C_4\}$. This implies that, for each $t \in [0, b]$,

$$\int_{\upsilon(0)}^{\upsilon(t)} \frac{ds}{s + \Phi(s)} \leq \int_0^b m(s)\, ds < \int_{\upsilon(0)}^{\infty} \frac{ds}{s + \Phi(s)}.$$

Consequently, there exists a constant K such that

$$b(\mu(t) + \overline{\mu}(t)) + \|\phi\|^2_{\mathcal{D}_{\mathcal{F}_0}} + \|\overline{\phi}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq \upsilon(t) < K \quad \text{for each } \ t \in J.$$

Now from the definition of $\mu, \overline{\mu}$ it follows that

$$\mathbb{E}\,|x(t)|^2 + \mathbb{E}\,|y(t)|^2 \leq \mu(t) + \overline{\mu}(t) \leq \frac{K}{b} \quad \text{for each } t \in J.$$

Consequently, $\|x\|^2_{\mathcal{D}_{\mathcal{F}_b}} \leq K/b$ and $\|y\|^2_{\mathcal{D}_{\mathcal{F}_b}} \leq K/b$. This shows that $\mathcal{A}$ is bounded. As a consequence of Theorem 2.13 we deduce that N has a fixed point (x, y) which is a solution to the problem (1.1). $\square$

5. Some examples

In this, section we give examples to illustrate usefulness of our main results.

Example 5.1. Consider the system

$$(5.1)\qquad \begin{cases} dx(t) = \dfrac{(x_t + y_t)^2}{(t+1)(t+2)}\, dt + x(t-2)\, dt + \sigma_1\, d^\circ B^{H_1}(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in J := [0, b] \setminus \{t_1, t_2, \ldots\}, \\ dy(t) = \dfrac{(x_t + y_t + 1)^2}{(t+1)(t+2)}\, dt + y(t-2)\, dt + \sigma_2\, d^\circ B^{H_2}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in J := [0, b] \setminus \{t_1, t_2, \ldots\}, \\ x(t_k^+) - x(t_k^-) = \overline{a}_1 x(t_k^-), \quad k = 1, \ldots, m, \\ y(t_k^+) - y(t_k^-) = \overline{b}_1 y(t_k^-), \quad k = 1, \ldots, m, \\ x(t) = \phi(t), \qquad t \in [-2, 0], \\ y(t) = \overline{\phi}(t), \qquad t \in [-2, 0], \end{cases}$$

where $b > 1/2$ and $\sigma_1, \sigma_2, \overline{a}_1, \overline{b}_1$ are positive constants,

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ t - 1/2 & \text{if } t \in [-2, 0), \end{cases} \qquad \text{and} \qquad \overline{\phi}(t) = \begin{cases} 0 & \text{if } t = 0, \\ t - 1/3 & \text{if } t \in [-2, 0), \end{cases}$$

where

$$\begin{aligned} g^1(t, x, y) &= \frac{(x+y)^2}{(t+1)(t+2)}, \quad f^1(t) = \sigma_1, \quad I_1(x) = \overline{a}_1 x, \\ g^2(t, x, y) &= \frac{(x+y+1)^2}{(t+1)(t+2)}, \quad f^2(t) = \sigma_2, \quad \overline{I}_1(y) = \overline{b}_1 y. \end{aligned}$$

Let $R > 0$ and $x, \overline{x}, y, \overline{y} \in \mathcal{D}_{\mathcal{F}_0}$ be such that

$$\begin{aligned} |g^1(t, x, y) - g^1(t, \overline{x}, \overline{y})| &\leq \frac{4R}{(t+1)(t+2)} (|x - \overline{x}| + |y - \overline{y}|), \\ |g^2(t, x, y) - g^2(t, \overline{x}, \overline{y})| &\leq \frac{4R+2}{(t+1)(t+2)} (|x - \overline{x}| + |y - \overline{y}|). \end{aligned}$$

So

$$\begin{aligned} |g^1(t, x, y) - g^1(t, \overline{x}, \overline{y})|^2 &\leq 2\bigg(\frac{4R}{(t+1)(t+2)}\bigg)^2 (|x - \overline{x}|^2 + |y - \overline{y}|^2), \\ |g^2(t, x, Y) - g^2(t, \overline{x}, \overline{y})|^2 &\leq 2\bigg(\frac{4R+2}{(t+1)(t+2)}\bigg)^2 (|x - \overline{x}|^2 + |y - \overline{y}|^2). \end{aligned}$$

Let

$$\bigg(\frac{8R}{(t+1)(t+2)}\bigg)^2, \bigg(\frac{4R+2}{(t+1)(t+2)}\bigg)^2 \in L^1([0, b], \mathbb{R}^+)$$

with $\|x\|^2_{\mathcal{D}_{\mathcal{F}_0}}, \|\overline{x}\|^2_{\mathcal{D}_{\mathcal{F}_0}} \leq R$. It is clear that

$$|I_1(x) - I_1(\overline{x})|^2 \leq \overline{a}_1^2 |x - \overline{x}|^2, \qquad |\overline{I}_1(y) - \overline{I}_1(\overline{y})|^2 \leq \overline{b}_1^2 |y - \overline{y}|^2.$$

Thanks to these assumptions, it is straightforward to check that (H_1)–(H_3) hold. Let

$$M_b = \begin{pmatrix} \sqrt{\dfrac{1}{\tau} + 3m\overline{a}_1^2} & \dfrac{1}{\sqrt{\tau}} \\ \dfrac{1}{\sqrt{\tau}} & \sqrt{\dfrac{1}{\tau} + 3m\overline{b}_1^2} \end{pmatrix},$$

where τ is sufficiently large. If M_b converges to zero, then the assumptions in Theorem 3.3 are fulfilled and we can conclude that the system (5.1) has a unique solution.

EXAMPLE 5.2. Consider the system

$$
(5.2)\qquad \begin{cases}
dx(t) = \left(1 - qx_t - y_t - \dfrac{ax_t y_t}{x_t + py_t}\right) dt + x(t-2)\,dt + \sigma_1\, d^\circ B^{H_1}(t), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in J := [0,b] \setminus \{1/2\}, \\
dy(t) = \left(\dfrac{R_0 x_t y_t}{x_t + py_t} - y_t\right) dt + y(t-2)\,dt + \sigma_2 d^\circ B^{H_2}(t), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in J := [0,b] \setminus \{1/2\}, \\
x((1/2)^+) - x((1/2)^-) = a_1 x((1/2)^-), \\
x((1/2)^+) - x((1/2)^-) = b_1 x((1/2)^-), \\
x(t) = \phi(t), \qquad\qquad\qquad\qquad t \in [-2,0], \\
y(t) = \overline{\phi}(t), \qquad\qquad\qquad\qquad t \in [-2,0],
\end{cases}
$$

where $b > 1/2$ and $q, a, p, \sigma_1, \sigma_2, R_0, a_1, b_1$ are positive constants,

$$
\phi(t) = \begin{cases} 0 & \text{if } t = 0, \\ t - 1/2 & \text{if } t \in [-2,0), \end{cases} \qquad \text{and} \qquad \overline{\phi}(t) = \begin{cases} 0 & \text{if } t = 0, \\ t - 1/3, & \text{if } t \in [-2,0), \end{cases}
$$

where

$$
g^1(t,x,y) = 1 - qx - y - \frac{axy}{x+py}, \quad f^1(t,x,y) = \sigma_1, \quad I_1(x) = a_1 x,
$$

$$
g^2(t,x,y) = \frac{R_0 xy}{x+py} - y, \qquad\qquad f^2(t,x,y) = \sigma_2, \quad \overline{I}_1(y) = b_1 y.
$$

for $x, \overline{x}, y, \overline{y} \in \mathcal{D}_{\mathcal{F}_0}$. We have

$$
\begin{aligned}
|g^1(t,x,y) - g^1(t,\overline{x},\overline{y})| &\leq q|x-\overline{x}| + |y-\overline{y}| + \left|\frac{axY}{x+pY} - \frac{a\overline{x}\overline{Y}}{\overline{x}+p\overline{y}}\right| \\
&\leq q|x-\overline{x}| + |y-\overline{y}| + a\left|\frac{x\overline{x}(y-\overline{y})}{(x+py)(\overline{x}+p\overline{y})}\right| + ap\left|\frac{y\overline{y}(x-\overline{X})}{(x+py)(\overline{x}+p\overline{y})}\right| \\
&\leq (q+a)|x-\overline{x}| + \frac{a+p}{p}\,|y-\overline{y}|,
\end{aligned}
$$

and

$$
|g^2(t,x,y) - g^2(t,\overline{x},\overline{y})| \leq |y-\overline{y}| + \left|\frac{R_0 xy}{x+py} - \frac{R_0 \overline{x}\overline{y}}{\overline{x}+p\overline{y}}\right| \leq (1+R_0)|y-\overline{y}| + \frac{R_0}{p^2}\,|x-\overline{x}|.
$$

So

$$
|g^1(t,x,Y) - g^1(t,\overline{x},\overline{y})|^2 \leq 2(q+a)^2|x-\overline{x}|^2 + 2\left(\frac{a+p}{p}\right)^2 |y-\overline{y}|^2,
$$

$$
|g^2(t,x,y) - g^2(t,\overline{x},\overline{y})|^2 \leq 2(1+R_0)^2|y-\overline{y}|^2 + 2\,\frac{R_0^2}{p^4}\,|x-\overline{x}|^2.
$$

Thanks to these assumptions, it is straightforward to check that (H_1), $(\overline{\mathrm{H}}_2)$ and (H_3) hold. Set

$$
M_a = 2\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}
$$

where

$$B_1 = \sqrt{6b(q+a)^2 + 3b + 3a_1}, \qquad B_2 = \sqrt{6b}\,\frac{a+p}{p},$$
$$B_3 = \sqrt{12b}\,\frac{R_0}{p^2}, \qquad B_4 = \sqrt{6b(1+R_0)^2 + 3b + 3b_1}.$$

If M_a converges to zero, then the assumptions in Theorem 3.4 are fulfilled, and we can conclude that the system (5.2) has a unique solution.

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Mohamed Ferhat and Tayeb Blouhi
Department of Mathematics
Usto University
Oran, 31000, ALGERIA

E-mail address: ferhat22@hotmail.fr, blouhitayeb@yahoo.com