Topological **M**ethods in **N**onlinear **A**nalysis
Volume 52, No. 1, 2018, 111–146
DOI: 10.12775/TMNA.2018.006

REGULARITY PROBLEM FOR $2m$-ORDER QUASILINEAR PARABOLIC SYSTEMS WITH NON SMOOTH IN TIME PRINCIPAL MATRIX. $(A(t), m)$-CALORIC APPROXIMATION METHOD

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Dedicated to the memory of Professor Marek Burnat

Abstract. Partial regularity of solutions to a class of $2m$-order quasilinear parabolic systems and full interior regularity for $2m$-order linear parabolic systems with non smooth in time principal matrices is proved in the paper. The coefficients are assumed to be bounded and measurable in the time variable and VMO-smooth in the space variables uniformly with respect to time. To prove the result, we apply the $(A(t), m)$-caloric approximation method, $m \geq 1$. It is both an extension of the $A(t)$-caloric approximation applied by the authors earlier to study regularity problem for systems of the second order with non-smooth coefficients and an extension of the A-polycaloric lemma proved by V. Bögelein in [6] to systems of $2m$-order.

1. Introduction

In this paper we continue to study partial regularity of weak solutions to quasilinear parabolic systems. We consider a class of $2m$-order systems in the form

$$(1.1)\quad u_t(z) + (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha (A^{\alpha\beta}(z, D^{m-1} u(z)) D^\beta u(z)) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha F_\alpha(z),$$

2010 *Mathematics Subject Classification.* 35K55.
Key words and phrases. High order parabolic systems; regularity problem.
First author was supported by RFFI, grant n. 17-01-00678.

for $z \in Q$, where $m \geq 1$, $z = (x,t) \in Q = \Omega \times (-T,0)$, Ω is a bounded domain in $\mathbb{R}^n, n \geq 2, T > 0$ is an arbitrary fixed number. By u_t we denote the time derivative of a function $u\colon Q \to \mathbb{R}^N, N > 1$ and for multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_0$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$ we denote by

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

the space derivatives. Moreover, $D^j u = \{D^\alpha u\}_{|\alpha|=j}$. Through the paper we use the convention of summation over repeated indices. The functions F_α for $|\alpha| \leq m$ belong to appropriate Campanato spaces.

We assume that the $N \times N$ matrices $A^{\alpha\beta}(x,t,p^{m-1})$, $|\alpha| = |\beta| = m$, $p^{m-1} \in \mathbb{P}_{m-1} = \{p^\alpha \in \mathbb{R}^N : \alpha = (\alpha_1, \dots, \alpha_n),\ |\alpha| = m-1\}$ satisfy the uniform ellipticity conditions with positive numbers μ and ν, $\nu \leq \mu$:

$$\sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}(z,p^{m-1})\xi^\alpha \cdot \xi^\beta \geq \nu\,|\xi|^2, \qquad |\xi|^2 = \sum_{|\alpha|=m} |\xi^\alpha|^2, \tag{1.2}$$

for any systems $\{\xi^\alpha\}_{|\alpha|=m}$ of vectors of $\mathbb{R}^N$, all arguments $p^{m-1} \in \mathbb{P}_{m-1}$ and almost all $z \in Q$.

Moreover, for all $p^{m-1} \in \mathbb{P}_{m-1}$,

$$\|A(\,\cdot\,,p^{m-1})\|_{\infty,Q} = \bigg[\operatorname*{ess\,sup}_{z\in Q} \sum_{k,l=1,\dots,N} \sum_{|\alpha|=|\beta|=m} (A^{\alpha\beta}_{kl}(z,p^{m-1}))^2\bigg]^{1/2} \leq \mu. \tag{1.3}$$

For simplicity we write $A(z,p^{m-1}) \in \{\nu,\mu\}$ provided that $A(z,p^{m-1}) = \{A^{\alpha\beta}(z,p^{m-1})\}_{|\alpha|=|\beta|=m}$ satisfy the assumptions (1.2), (1.3) for any $p^{m-1} \in \mathbb{P}_{m-1}$ and almost all $z \in Q$.

We consider weak solutions u of system (1.1) defined as follows:

Definition 1.1. A function $u \in V(Q) := L^2((-T,0); W_2^m(\Omega))$ is a weak solution to system (1.1) if it satisfies the identity

$$\begin{aligned}\int_Q [-u(z)\cdot\phi_t(z) + \sum_{|\alpha|=|\beta|=m} A^{\alpha\beta}(z, D^{m-1}u(z))D^\beta u(z)\cdot D^\alpha\phi(z)]\,dz \\ = \sum_{|\alpha|\leq m} \int_Q F_\alpha(z)\,\cdot D^\alpha\phi(z)\,dz\end{aligned} \tag{1.4}$$

for all $\phi \in C_0^\infty(Q)$.

In this paper we relax the known conditions on the matrices $A^{\alpha\beta}(z,p^{m-1})$ which guarantee partial regularity of weak solutions u to system (1.1) (see [6]). We estimate the Hausdorff measure of the singular set Σ of u. In particular, we prove that for the linear systems their singular sets Σ are empty. Let us remark that results of this paper in the case $m = 1$ coincide with the results of our earlier paper for quasilinear systems of the second order in [5].

Under similar smoothness conditions in x and t for the principal matrix, the regularity question for a wide class of nonlinear *scalar* equations and $2m$-order parabolic *linear* systems ($m \geq 1$) was studied in a series of the works by N.V. Krylov, H. Dong, D. Kim (see [18], [19], [11]–[13] and references therein). In these works the principal coefficients of the studied systems were also assumed bounded and measurable in t and VMO-smooth in the space variables.

Using a different approach to study *quasilinear systems*, we proved partial regularity of the second order ($m = 1$) parabolic systems in divergence and non divergence forms in [3]–[5] under conditions on the principal matrices analogous to (1.2), (1.3) by so called $A(t)$-caloric approximation method (for application of the method see also [1]). This method is a modification of the A-caloric approximation method by F. Duzaar and G. Mingione [15] (see also [8], [6]) and it is an analogue of elliptic A-harmonic approximation having its origin in E. De Giorgi ideas (see [10], [14]).

To study $2m$-order systems, we prove here a new variant of the $A(t)$-caloric approximation lemma and name it $(A(t), m)$-caloric approximation lemma. We hope that this approach will be helpful to study regularity problem for more general classes of $2m$-order quasilinear and nonlinear parabolic systems.

The paper is organized as follows: in Section 2 we list notation and the main results, Section 3 contains auxiliary results, Section 4 is dedicated to properties of $(A(t), m)$-caloric functions. We prove $(A(t), m)$-caloric lemma in Section 5. Finally, in the last Section 6 we prove Theorems 2.1, 2.2 and 2.4.

2. Notation and main results

In the paper we assume that Ω is an open bounded domain in $\mathbb{R}^n$ and T is a positive number. We will use the following notation:

- $z = (x, t)$,
- $z^0 = (x^0, t^0) \in \Omega \times (-T, 0) = Q \subset \mathbb{R}^{n+1}$,
- $\Gamma = \partial\Omega \times (-T, 0)$,
- $\partial_p Q = \Gamma \cup (\overline{\Omega} \times \{-T\})$,
- $\Lambda_r(t^0) = (t^0 - r^{2m}, t^0)$,
- $B_r(x^0) = \{x \in \mathbb{R}^n; |x - x^0| < r\}$,
- $Q_r(z^0) = B_r(x^0) \times \Lambda_r(t^0)$,
- $\Gamma_r(z^0) = \partial B_r(x^0) \times \Lambda_r(t^0)$,
- $\partial_p Q_r(z^0) = \Gamma_r(z^0) \cup (\overline{B_r(x^0)} \times \{t^0 - r^{2m}\})$.

Throughout the paper we will use the standard notation for the Lebesgue and Sobolev spaces. Note that the Hölder spaces $C^{0,\alpha}(Q)$, Morrey spaces $L^{2,\lambda}(Q)$, and Campanato spaces $\mathcal{L}^{2,\lambda}(Q)$ are considered with respect to the *m-parabolic metric*

$$\delta_m(z^1, z^2) = \max\left\{|x^1 - x^2|, |t^1 - t^2|^{1/2m}\right\}.$$

Thus, for example, $C^{0,\alpha}(Q) = C^{\alpha,\alpha/2m}_{x,t}(Q)$ in the euclidian metric in $\mathbb{R}^{n+1}$.

For a function u in Sobolev spaces $W^m_2(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ we assume that all the derivatives of u up to the order m are square integrable on Ω. Sobolev spaces $W^{m,k}_2(Q)$ consist of all functions u such that all derivatives of u with respect to the space variables up to the order m as well as all derivatives of u with respect to time up to the order k are square integrable on Q. Finally, $W^m_{2,0}(\Omega) = [\overline{C^\infty_0(\Omega)}]_{W^m_2(\Omega)}$.

Further we denote the spaces

$$\begin{aligned} V(Q) &= L^2((-T,0); W^m_2(\Omega)), \\ V(Q_r(z^0)) &= L^2(\Lambda_r(t^0); W^m_2(B_r(x^0)), \\ \overset{0}{V}(Q_r(z^0)) &= L^2(\Lambda_r(t^0); W^m_{2,0}(B_r(x^0)), \end{aligned}$$

for z^0, r such that $Q_r(z^0) \subset\subset Q$.

The space averages and the space-time averages of $u \in L^1(Q_r(z^0))$ are defined by

$$\begin{aligned} (u)_{r,x^0}(t) &= \frac{1}{|B_r(x^0)|} \int_{B_r(x^0)} u(y,t)\,dy, \\ (u)_{r,z^0} &= \frac{1}{|Q_r(z^0)|} \int_{Q_r(z^0)} u(z)\,dz = \fint_{Q_r(z^0)} u(z)\,dz. \end{aligned}$$

Space averages of coefficients $A^{\alpha\beta}$ are defined by

$$(A^{\alpha\beta})_{r,x^0}(t,p^{m-1}) = \frac{1}{|B_r(x^0)|} \int_{B_r(x^0)} A^{\alpha\beta}(y,t,p^{m-1})\,dy.$$

Here $|B_r|$ and $|Q_r|$ stand for the Lebesgue measure of B_r and Q_r in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively.

In what follows we will use the notation Q_r, V_r, $(u)_r$ and $(A^{\alpha\beta})_r$ without denoting center of the ball or cylinder if it does not cause misunderstandings.

Next we formulate the main assumptions on the data. Let for α, β with $|\alpha| = |\beta| = m$ the coefficients $A^{\alpha\beta} = \big(A^{\alpha\beta}_{il}(z,p^{m-1})\big)_{i,l=1,\dots,N}$, with $p^{m-1} \in \mathbb{R}^{m-1}$ are Carathéodory functions on $Q \times \mathbb{P}_{m-1}$ and satisfy the following conditions:

(H1) There are positive constants μ, ν, $\nu \le \mu$ such that $[N \times N]$ matrices $A^{\alpha\beta}$ with $|\alpha| = |\beta| = m$ satisfy the ellipticity condition (1.2) and the boundedness condition (1.3).

(H2) For almost all $z \in Q$, and $p^{m-1}, q^{m-1} \in \mathbb{P}_{m-1}$ it holds

$$|A^{\alpha\beta}(z,p^{m-1}) - A^{\alpha\beta}(z,q^{m-1})| \le \omega(|p^{m-1} - q^{m-1}|^2), \tag{2.1}$$

where $\omega(s)$ is a non decreasing, bounded and concave function on $[0,\infty)$ with $\lim\limits_{s\to 0+} \omega(s) = 0$.

(H3) For all $i, l \leq N$, $|\alpha| = |\beta| = m$, almost all $t \in (-T, 0)$ and all $p^{m-1} \in \mathbb{P}_{m-1}$ the coefficients $A_{il}^{\alpha\beta}(\,\cdot\,, t, p^{m-1})$ belong to $\mathrm{VMO}(\Omega)$ and

$$\sup_{\substack{\rho \leq r \\ Q_r(z^0) \subset Q}} \sup_{p^{m-1} \in \mathbb{P}_{m-1}} \fint_{\Lambda_\rho(t^0)} \left(\fint_{B_\rho(x^0)} |A(y, t, p^{m-1}) - A_{\rho, x^0}(t, p^{m-1})|^2 \, dy \right) dt \\ =: q^2(r) \to 0 \tag{2.2}$$

for $r \to 0+$. Here

$$(A^{\alpha\beta})_{\rho, x^0}(t, p^{m-1}) = \fint_{B_\rho(x^0)} A^{\alpha\beta}(x, t, p^{m-1}) \, dx.$$

(H4) The functions $F_\alpha \in \mathcal{L}^{2, n+2|\alpha|-2+2\gamma}(Q; \delta_m)$, $|\alpha| \leq m$, $\gamma \in (0, 1)$.

Note that the assumption (H2) means continuity of $A^{\alpha\beta}(x, t, p^{m-1})$ in the arguments p^{m-1} which is uniform with respect to $z = (x, t)$ and the assumption (H3) is VMO continuity of $A^{\alpha\beta}$ in x uniform with respect to t and p^{m-1}. We do not assume any additional smoothness in t of the coefficients $A^{\alpha\beta}$.

We can require that conditions (H1)–(H4) are satisfied only locally in Q because in this paper we study the interior partial regularity of weak solutions to system (1.1).

In order to concentrate our attention on the properties of the principal matrices $A^{\alpha\beta}$ we omit additional nonlinear terms of the lower order.

Next we formulate the main results of the paper. To shorten the formulas we will denote for $k = 0, \dots, m$ by

$$|D^k u| = \left(\sum_{\alpha,\, |\alpha| = k} |D^\alpha u|^2 \right)^{1/2}.$$

Theorem 2.1. *Let the assumptions* (H1)–(H4) *hold and $u \in V(Q)$ be a weak solution to* (1.1). *Then there exist numbers $\tau, \theta_0 \in (0, 1)$ and $r_0 > 0$ such that, if $Q_r(z^0) \subset Q$ and*

$$r^{-(n+2(m-1))} \int_{Q_r(z^0)} |D^m u(z)|^2 \, dz < \theta_0 \tag{2.3}$$

with some $r < r_0$, then u and all derivatives $D^k u$ with $k \leq m - 1$, belong to $C^\gamma(\overline{Q}_{\tau r}(z^0))$ with the exponent $\gamma \in (0, 1)$ given in the assumption (H4). *The norms $\|D^k u\|_{C^\gamma(\overline{Q}_{\tau r}(z^0))}$ can be estimated by the data of the problem, $\|u\|_{V(Q)}$ and r^{-1}.*

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *hold and $u \in V(Q)$ be a weak solution to system* (1.1). *Then u and its derivatives $D^k u$, $k \leq m - 1$ are Hölder continuous functions* (*in the parabolic δ_m metric*) *on an open set $Q_0 \subset Q$, $Q_0 = Q \setminus \Sigma$ where Σ is the closed singular set of u and $\mathcal{H}_{n+2(m-1)}(\Sigma; \delta_m) = 0$.*

Remark 2.3. Let all conditions of Theorem 2.1 hold. We put

$$\mathbf{A}(x,t) = A(x,t,D^{m-1}u(x,t)).$$

The matrix $\mathbf{A}$ is bounded in t and VMO-smooth in x on the regular set Q_0 (defined in Theorem 2.2). We can consider u as a weak solution of the linear system

$$u_t(z) + (-1)^m D^m(\mathbf{A}(z)D^m u(z)) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} F_\alpha(z), \quad z \in Q_0, \tag{2.4}$$

and apply results stated in [12] to obtain further smoothness of u (certainly, under appropriate assumptions of the functions F_α).

As a particular case of (1.1) we can consider the linear system

$$u_t + (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(A^{\alpha\beta}(z)D^\beta u) = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha F_\alpha(z), \quad z \in Q, \tag{2.5}$$

Then Theorem 2.1 implies the following result on the regularity of solutions to system (2.5).

Theorem 2.4. *Let the assumptions* (H1), (H3) *hold for the matrix $A(z)$, $z \in Q$, and the functions F_α satisfy conditions* (H4). *Let $u \in V(Q)$ be a weak solution to system* (2.5). *Then u and its derivatives $D^k u$, $k \leq m-1$, are the Hölder continuous functions in Q with the exponent $\gamma \in (0,1)$ where γ is fixed in the assumption* (H4).

Remark 2.5. As we mentioned, regularity results for *linear* parabolic systems of the higher order with non smooth in time coefficients follow from the paper [12] where solvability results for such systems in the Sobolev spaces were stated but here we suggest another approach and consider the right hand sides F_α not in L^q but in Campanato spaces.

3. Auxiliary results

In this section we recall several results needed further.

Lemma 3.1 (Interpolation lemma, see e.g. [6], [7, Lemma B1]). *Let a function $u \in W_2^m(B_r(x^0))$. Then*:

(a) *For any $k \leq m-1$ and any $\varepsilon \in (0,1]$ there is a constant c which depends on n, m such that the following inequality holds*

$$\int_{B_r(x^0)} |D^k u(x)|^2\,dx \leq \varepsilon\, r^{2(m-k)} \int_{B_r(x^0)} |D^m u(x)|^2\,dx + c(n,m)\varepsilon^{-k/(m-k)}\, r^{-2k} \int_{B_r(x^0)} |u(x)|^2\,dx. \tag{3.1}$$

(b) *If we choose r_1, r_2 such that the condition $r/2 \le r_1 < r_2 \le r$ is satisfied, then*

$$\tag{3.2} \begin{aligned} \int_{B_{r_2}(x^0)\setminus B_{r_1}(x^0)} &|D^k u(x)|^2\,dx \\ &\le \varepsilon\,(r_2-r_1)^{2(m-k)} \int_{B_{r_2}(x^0)\setminus B_{r_1}(x^0)} |D^m u(x)|^2\,dx \\ &\quad + c(n,m)\varepsilon^{-k/(m-k)}\,(r_2-r_1)^{-2k} \int_{B_{r_2}(x^0)\setminus B_{r_1}(x^0)} |u(x)|^2\,dx. \end{aligned}$$

LEMMA 3.2 (Caccioppoli inequality). *Let $a_0(z) = \{a_0^{\alpha\beta}(z)\}$ for $|\alpha| = |\beta| = m$ be $N \times N$-matrices, $a_0 \in L^\infty(Q)$, $a_0(z) \in \{\nu,\mu\}$ for almost all $z \in Q$ with some positive constants $\nu \le \mu$. Let $F_\gamma \in L^2(Q)$ for all γ with $|\gamma| \le m$ and $u \in V(Q)$ be a weak solution to the system*

$$\tag{3.3} u_t(z) + (-1)^m D^\alpha(a_0^{\alpha\beta}(z)\,D^\beta u(z)) = \sum_{|\gamma|\le m} (-1)^{|\gamma|} D^\gamma\,F_\gamma(z).$$

Then, for any polynomial $P_{m-1}\colon \mathbb{R}^n \to \mathbb{R}^N$ of the degree less or equal to $m-1$ depending on x only, the following inequality holds

$$\tag{3.4} \begin{aligned} \fint_{Q_{r/2}(z^0)} |D^m u(z)|^2\,dz \le c_{cacc.} r^{-2m} &\fint_{Q_r(z^0)} |u(z) - P_{m-1}(x)|^2\,dz \\ &+ c\sum_{|\alpha|=0}^{m} r^{2(m-|\alpha|)} \fint_{Q_r(z^0)} |F_\alpha(z)|^2\,dz \end{aligned}$$

in any cylinder $Q_r(z^0) \subset\subset Q$. The constants in (3.4) *depend only on ν, μ, m and n.*

We could not find in literature the Caccioppoli and the Poincaré inequalities for systems (3.3) in an appropriate form and give below the proofs of these lemmas.

In the draft of a proof of Caccioppoli inequality we will use Steklov formulation (see ([6], [20]).

PROOF OF LEMMA 3.2. First, we recall that any weak solution $u \in V(Q)$ of system (3.3) is a function from the class $C((-T,0); L^2(\Omega))$ (see, for example [20, Chapter 2]). Moreover, for a fixed polynomial $P_{m-1}(x)$ of the degree less than m, the function $u(z) - P_{m-1}(x)$ is also a weak solution to system (3.3) and the following identity is valid

$$\tag{3.5} \begin{aligned} \int_{Q_r(z^0)} &[-(u-P_{m-1})\cdot\phi_t + a_0 D^m u \cdot D^m\phi]\,dz \\ &+ \int_{B_r(x^0)} (u-P_{m-1})\cdot\phi\,dx\big|^{t=t^0} = \sum_{|\alpha|\le m} \int_{Q_r(z^0)} F_\alpha \cdot D^\alpha\phi\,dz, \end{aligned}$$

for all $\phi \in W_2^1(Q_r(z^0))$, $\phi|_{\partial_p Q_r(z^0)} = 0$, $Q_r(z^0) \subset\subset Q$.

We fix numbers $r_1 < r_2$ such that $r/2 \le r_1 < r_2 \le r$ and an arbitrary polynomial $P_{m-1}(x)$ and consider the function

$$\phi(z) = (u(z) - P_{m-1}(x))\eta^{2m}(x)\,\theta^2(t)$$

where $\eta \in C_0^\infty(B_{r_2}(x^0))$, $\eta(x) = 1$ in $B_{r_1}(x^0)$, $|D^k\eta(x)| \le c/(r_2 - r_1)^k$; $\theta \in C^\infty(\mathbb{R}^1)$, $\theta(t) = 1$ in $\Lambda_{r/2}(t^0)$, $\theta(t) = 0$ for $t \le t^0 - r^{2m}$, $|\theta'(t)| \le c/r^{2m}$. Unfortunately, ϕ is not differentiable in t and thus it cannot be used as a test function directly. The well known Steklov average procedure should be applied beforehand (see, for example, [20, Chapter 3]). To spare the place we omit this procedure and illustrate the idea of the proof by putting the function ϕ in (3.5). Then we obtain the relation

$$\begin{aligned}
&\int_{Q_r(z^0)} -|u(z) - P_{m-1}(x)|^2\eta^{2m}(x)\theta'(t)\,\theta(t)\,dz \\
&\quad + \int_{B_r(x^0)} \frac{|u(z) - P_{m-1}(x)|^2\eta^{2m}(x)}{2}\,dx\,\bigg|^{t=t^0} \\
&\quad + \int_{Q_r(z^0)} a_0 D^m u \cdot (D^m u\,\eta^{2m} + \mathcal{K}(u,\eta))\,\theta^2\,dz \\
&= \int_{Q_r(z^0)} F_m \cdot (D^m u\,\eta^{2m} + \mathcal{K}(u,\eta))\,\theta^2\,dz \\
&\quad + \sum_{k=0}^{m-1} \int_{Q_r(z^0)} F_k \cdot D^k((u - P_{m-1})\,\eta^{2m})\,\theta^2\,dz.
\end{aligned} \tag{3.6}$$

Here we denoted by $F_k = \{F_\alpha\}_{|\alpha|=k}$ and by $\mathcal{K}(u,\eta)$ the terms with the lower order derivatives of u:

$$\mathcal{K}(u,\eta) = \sum_{j=0}^{m-1} D^j(u - P_{m-1})D^{m-j}\eta^{2m}.$$

Using the ellipticity condition and the Cauchy inequality we derive from (3.6) that

$$\begin{aligned}
\int_{Q_r(z^0)} |D^m u|^2\eta^{2m}\theta^2\,dz &\le \frac{c}{r^{2m}} \int_{Q_r(z^0)} |u - P_{m-1}|^2\,dz \\
&\quad + c\int_{Q_r(z^0)} |\mathcal{K}(u,\eta)|^2\theta^2\,dz + c\int_{Q_r(z^0)} |F_m|^2\,dz \\
&\quad + c\sum_{k=0}^{m-1} \int_{Q_r(z^0)} |F_k|\,|D^k[(u - P_{m-1})\eta^{2m}]|\,dz.
\end{aligned} \tag{3.7}$$

Further we write $B_\rho = B_\rho(x^0)$, $\Lambda_\rho = \Lambda_\rho(t^0)$, $Q_\rho = Q_\rho(z^0)$ and put

$$T_r = (B_{r_2} \setminus B_{r_1}) \times \Lambda_r.$$

We recall that $D^j\eta = 0$ on B_{r_1} for all $j = 1, \ldots, m-1$, and estimate the integral

$$
\begin{aligned}
(3.8) \qquad I &:= \int_{Q_r} |\mathcal{K}(u,\eta)|^2\theta^2\, dz \\
&\le c\int_{T_r} \sum_{j=1}^{m-1} \frac{|D^j(u-P_{m-1})|^2\theta^2\, dz}{(r_2-r_1)^{2(m-j)}}\, dz + c\int_{T_r} \frac{|u-P_{m-1}|^2\theta^2}{(r_2-r_1)^{2m}}\, dz
\end{aligned}
$$

Now we integrate inequality (3.2) (with an $\varepsilon > 0$ to be defined later) in $t \in \Lambda_r$ and obtain the relation

$$
\sum_{i=1}^{m-1} \int_{T_r} \frac{|D^i(u-P_{m-1})|^2\theta^2}{(r_2-r_1)^{2(m-i)}}\, dz \le \varepsilon \int_{T_r} |D^m u|^2\theta^2\, dz + c_\varepsilon \int_{T_r} \frac{|u-P_{m-1}|^2}{(r_2-r_1)^{2m}}\, dz.
$$

Then

$$
(3.9) \qquad I \le c_1\varepsilon \int_{T_r} |D^m u|^2\,\theta^2\, dz + c_\varepsilon \int_{T_r} \frac{|u-P_{m-1}|^2}{(r_2-r_1)^{2m}}\, dz.
$$

Further we estimate the integrals with the functions F_k in (3.7) by the Cauchy inequality with a parameter $q \in (0,1)$, we will choose q later. Thus

$$
\begin{aligned}
J_k := \int_{Q_r} |F_k|\,|D^k[(u-P_{m-1})\,\eta^{2m}]|\, dz &\le q \int_{Q_r} \frac{|D^k[(u-P_{m-1})\eta^{2m}]|^2}{(r_2-r_1)^{2(m-k)}}\, dz \\
&\quad + c_q \int_{Q_r} |F_k|^2\, dz (r_2-r_1)^{2(m-k)} =: q\, l_1 + l_2.
\end{aligned}
$$

Here

$$
\begin{aligned}
l_1 &\le c\int_{T_r} (r_2-r_1)^{-2(m-k)} \sum_{j=0}^{k} \frac{|D^j(u-P_{m-1})|^2\theta^2}{(r_2-r_1)^{2(k-j)}}\, dz \\
&\le c\int_{T_r} \sum_{j=1}^{k} \frac{|D^j(u-P_{m-1})|^2\theta^2}{(r_2-r_1)^{2(m-j)}}\, dz + c\int_{Q_r} \frac{|u-P_{m-1}|^2}{(r_2-r_1)^{2m}}\, dz.
\end{aligned}
$$

We apply once more the interpolation inequality (3.2) with $\varepsilon = 1$ to estimate the integrals with $|D^j(u-P_{m-1})|^2$ in the last inequality and obtain that

$$
l_1 \le c\int_{T_r} |D^m u|^2\theta^2\, dz + c\int_{Q_r} \frac{|u-P_{m-1}|^2}{(r_2-r_1)^{2m}}\, dz.
$$

Thus, for $q \in (0,1)$,

$$
\begin{aligned}
(3.10) \qquad J_k &\le c_2\, q \int_{T_r} |D^m u|^2\theta^2\, dz \\
&\quad + c\int_{Q_r} \frac{|u-P_{m-1}|^2}{(r_2-r_1)^{2m}}\, dz + c_q \sum_{k=0}^{m} r^{2(m-k)} \int_{Q_r} |F_k|^2\, dz.
\end{aligned}
$$

Estimates (3.9) and (3.10) help us to deduce from (3.7) the inequality

$$\int_{Q_r} |D^m u|^2 \eta^{2m} \theta^2 \, dz \leq (c_1 \varepsilon + c_2 \, q(m-1)) \int_{T_r} |D^m u|^2 \theta^2 \, dz + c_\varepsilon \int_{Q_r} \frac{|u - P_{m-1}|^2}{(r_2 - r_1)^{2m}} \, dz + c_q \sum_{k=0}^{m} r^{2(m-k)} \int_{Q_r} |F_k|^2 \, dz. \tag{3.11}$$

Now we choose the parameters $\varepsilon, q \in (0,1)$ from the condition

$$c_1 \varepsilon + c_2 \, q(m-1) \leq 1/2.$$

For the function

$$g(\rho) = \int_{\Lambda_r(t^0)} \int_{B_\rho(x^0)} |D^m u|^2 \theta^2 \, dx \, dt, \quad \rho < r,$$

we obtain from (3.11) the inequality

$$g(r_1) \leq 1/2 \, g(r_2) + \frac{M(r)}{(r_2 - r_1)^{2m}} + S(r) \tag{3.12}$$

where

$$M(r) = c \int_{Q_r} |u - P_{m-1}|^2 \, dz, \qquad S(r) = c \sum_{k=0}^{m} r^{2(m-k)} \int_{Q_r} |F_k|^2 \, dz,$$

for $r/2 \leq r_1 < r_2 \leq r$. By the well known lemma (see, for example, Lemma 8.18 in [17]) this inequality implies that

$$g\left(\frac{r}{2}\right) \leq c \, \frac{M(r)}{r^{2m}} + c \, S(r), \tag{3.13}$$

where the constants $c = c(\nu, \mu, m, n)$. Inequality (3.4) follows from (3.13). $\square$

Lemma 3.3 (Poincaré inequality). *Let the assumptions of Lemma* 3.2 *hold. Then*

$$\fint_{Q_r(z^0)} |u(z) - P^*_{m-1,r}(x)|^2 \, dz \leq c_{\text{poinc}} \, r^{2m} \fint_{Q_{2r}(z^0)} |D^m u(z)|^2 \, dz + c \sum_{|\alpha| \leq m} r^{4m - 2|\alpha|} \fint_{Q_{2r}(z^0)} |F_\alpha(z)|^2 \, dz \tag{3.14}$$

for any r such that $Q_{2r}(z^0) \subset\subset Q$. Here $P^*_{m-1,r}(x)$ minimize the integral $\int_{Q_r(z^0)} |u(z) - P_{m-1}(x)|^2 \, dz$ among all polynomials of degree at most $m-1$ and the constants in (3.14) *depend only on ν, μ, m and n.*

Proof. Let u be a weak solution of system (3.3). We fix a cylinder $Q_r(z^0) \subset\subset Q$, and numbers $r_1 < r_2$ where $r/2 \leq r_1 < r_2 \leq r$. We also fix numbers $s \in \Lambda_r(t^0) \setminus \Lambda_{r/2}(t^0)$ and $\tau \in (s, t^0)$. We denote by $\eta = \eta(x)$ a cut-off smooth function for $B_{r_2}(x^0)$, $\eta(x) = 1$ in $B_{r_1}(x^0)$ and $|D^k \eta| \leq c/(r_2 - r_1)^k$, $k \in \mathbb{N}$.

We address to identity (3.5) with the test function

$$\phi(z) = (u(z) - P^*(x;s))\eta^{2m}(x)\chi_\varepsilon(t)$$

where the polynomial $P^*(x;s)$ minimize (for the fixed s) the integral

$$\int_{B_r(x^0)} |u(x,s) - P(x)|^2\,dx$$

among all polynomials of the degree not more than $m-1$. The piece-wise continuous function $\chi_\varepsilon(t) = 1$ for $t \in (s,\tau)$ and $\chi_\varepsilon(t) = 0$ for $t \in \mathbb{R}^1 \setminus (s-\varepsilon, \tau+\varepsilon)$ and $(s-\varepsilon, \tau+\varepsilon) \subset \Lambda_r(t^0)$. (We omit the Steklov average procedure.) After some trivial calculations we tend $\varepsilon \to 0$ and obtain the equality

$$\begin{aligned}\int_{B_{r_2}(x^0)} \frac{1}{2}\,\eta^{2m}(x)|u(x,t) - P^*(x,s)|^2\,dx\Big|_{t=s}^{t=\tau} \\ + \int_s^\tau \int_{B_{r_2}(x^0)} a_0 D^m u \cdot (D^m u\,\eta^{2m} + \mathcal{L}(u,\eta))\,dx\,dt \\ = \int_s^\tau \int_{B_{r_2}(x^0)} F_m \cdot (D^m u\,\eta^{2m} + \mathcal{L}(u,\eta))\,dx\,dt \\ + \sum_{k=0}^{m-1} \int_s^\tau \int_{B_{r_2}(x^0)} F_k \cdot D^k[(u - P^*(x,s))\eta^{2m}]\,dx\,dt.\end{aligned} \tag{3.15}$$

We denoted in (3.15) by $\mathcal{L}(u,\eta)$ the expression

$$\mathcal{L}(u,\eta) := \sum_{k=0}^{m-1} D^k[(u(x,t) - P^*(x;s))D^{m-k}\eta^{2m}(x)].$$

To short the place we will write

$$\begin{gathered} B_\rho = B_\rho(x^0), \qquad Q_\rho = Q_\rho(z^0), \qquad \Lambda_\rho = \Lambda_\rho(t^0), \\ \widehat{u}(z) = u(z) - P^*(x;s), \qquad \Upsilon_r = (B_{r_2} \setminus B_{r_1}) \times (s,\tau). \end{gathered}$$

Now we estimate the terms with $\mathcal{L} = \mathcal{L}(u,\eta)$ in (3.15) as follows:

$$\begin{aligned} M_1 &:= \left| \int_s^\tau \int_{B_{r_2}} a_0 D^m u \cdot \mathcal{L}\,dx\,dt \right| \\ &\le c \int_{\Upsilon_r} \frac{r^m |D^m u|}{(r_2 - r_1)^m}\, \frac{\sum\limits_{k=0}^{m-1} |D^k \widehat{u}|(r_2 - r_1)^k}{r^m}\,dz \\ &\le q \int_{\Upsilon_r} \sum_{k=0}^{m-1} \frac{|D^k\widehat{u}|^2 (r_2 - r_1)^{2k}}{r^{2m}}\,dz + c_q \int_{\Upsilon_r} \frac{r^{2m}|D^m u|^2}{(r_2 - r_1)^{2m}}\,dz. \end{aligned} \tag{3.16}$$

Here we applied the Cauchy inequality with the parameter $q \in (0,1)$ to be defined later.

After integration inequality (3.2) (with $\varepsilon = 1$) in $t \in (s, \tau)$ we obtain that

$$\int_{\Upsilon_r} \sum_{k=1}^{m-1} |D^k \widehat{u}|^2 \, dz (r_2 - r_1)^{2k} \leq c\,(r_2 - r_1)^{2m} \int_{\Upsilon_r} |D^m u|^2 \, dz + c \int_{\Upsilon_r} |\widehat{u}|^2 \, dz. \tag{3.17}$$

Then

$$M_1 \leq c \int_{Q_r} |D^m u|^2 \, dz + c_q \int_{Q_r} \frac{r^{2m} |D^m u|^2}{(r_2 - r_1)^{2m}} \, dz + c_1 \, q \int_{\Upsilon_r} \frac{|\widehat{u}|^2}{r^{2m}} \, dz. \tag{3.18}$$

Further,

$$\begin{aligned} M_2 =: &\left| \int_s^\tau \int_{B_{r_2}} F_m \cdot \mathcal{L} \, dz \right| \\ &\leq c \int_{\Upsilon_r} \frac{|F_m| \, |\widehat{u}|}{(r_2 - r_1)^m} \, dz + c \sum_{k=1}^{m-1} \int_{\Upsilon_r} \frac{r^m |F_m|}{(r_2 - r_1)^m} \, \frac{|D^k \widehat{u}| (r_2 - r_1)^k}{r^m} \, dz \\ &\leq q \int_{\Upsilon_r} \frac{|\widehat{u}|^2}{r^{2m}} \, dz + c_q \frac{r^{2m}}{(r_2 - r_1)^{2m}} \int_{Q_r} |F_m|^2 \, dz \\ &\quad + q \sum_{k=1}^{m-1} \int_{\Upsilon_r} \frac{|D^k \widehat{u}|^2 (r_2 - r_1)^{2k}}{r^{2m}} \, dz \\ &\leq c_q \frac{r^{2m}}{(r_2 - r_1)^{2m}} \int_{Q_r} |F_m|^2 \, dz \\ &\quad + c\, q \frac{(r_2 - r_1)^{2m}}{r^{2m}} \int_{\Upsilon_r} |D^m u|^2 \, dz + c_2 \, q \int_{\Upsilon_r} \frac{|\widehat{u}|^2}{r^{2m}} \, dz. \end{aligned} \tag{3.19}$$

On the last step of relations (3.19) we applied inequality (3.17).

With the help of (3.18) and (3.19) we derive from (3.15) the inequality

$$\begin{aligned} \int_{B_{r_2}} |\widehat{u}(x, \tau)|^2 \eta^{2m} \, dx &\leq \int_{B_{r_2}} |\widehat{u}(x, s)|^2 \, dx \\ &\quad + c_q \frac{r^{2m}}{(r_2 - r_1)^{2m}} \int_{Q_r} |D^m u|^2 \, dz + c_q \frac{r^{2m}}{(r_2 - r_1)^{2m}} \int_{Q_r} |F_m|^2 \, dz \\ &\quad + \sum_{k=0}^{m-1} \int_s^\tau \int_{B_{r_2}} |F_k \cdot D^k (\widehat{u} \eta^{2m})| \, dx dt + \frac{(c_1 + c_2)\, q}{r^{2m}} \int_{\Upsilon_r} |\widehat{u}|^2 \, dz. \end{aligned} \tag{3.20}$$

By the definition of $\widehat{u}$ and due to the minimality property of $P^*(x; s)$ in $B_r(x^0)$ we can write that

$$\int_{B_r} |\widehat{u}(x, s)|^2 \leq \int_{B_r} |u(x, s) - P^0(x; s)|^2 \, dx, \tag{3.21}$$

where $P^0(x; s)$ is the mean value polynomial of the degree $m - 1$ in B_r, i.e.

$$\int_{B_r} D^\alpha (u(x, s) - P^0(x, s)) \, dx = 0, \quad \text{for all } |\alpha| \leq m - 1.$$

For almost all $s \in \Lambda_r \setminus \Lambda_{r/2}$ the function $u(\,\cdot\,,s) \in W_2^m(B_r)$ and further we consider such s.

We apply inequality (3.21) and then few times the Poincaré inequality to the functions $D^\alpha(u(x,s) - P^0(x,s))$, $|\alpha| \le m-1$. The following chain of the inequalities holds:

$$\begin{aligned}(3.22)\qquad &\int_{B_r} |\widehat{u}(x,s)|^2\,dx \le c(n)r^2 \int_{B_r} |D(u(x,s) - P^0(x;s))|^2\,dx\\ &= c(n)r^2 \int_{B_r} |Du(x,s) - (Du)_r(s)|^2\,dz \le \ldots \le c(n,m)r^{2m} \int_{B_r} |D^m u(x,s)|^2\,dx.\end{aligned}$$

Further we estimate the sum M_3 with integrals F_k in (3.20) as follows:

$$\begin{aligned}(3.23)\qquad M_3 &\le c \sum_{k=0}^{m-1} \int_s^\tau \int_{B_{r_2}} |F_k|\, \frac{\Sigma_{j=0}^k |D^j \widehat{u}|}{(r_2-r_1)^{k-j}}\,dx\,dt\\ &\le c_q \sum_{k=0}^{m-1} \int_{Q_r} \frac{|F_k|^2\,dz\,r^{2m}}{(r_2-r_1)^{2k}} + q \sum_{j=0}^{k} \int_{\Upsilon_r} \frac{|D^j\widehat{u}|^2\,dz(r_2-r_1)^{2j}}{r^{2m}}.\end{aligned}$$

Now we estimate the integrals with $|D^j\widehat{u}|^2$ in (3.23) by (3.17). Then

$$\begin{aligned}(3.24)\qquad M_3 \le c_q &\sum_{k=0}^{m-1} \int_{Q_r} \frac{|F_k|^2\,dz\,r^{2m}}{(r_2-r_1)^{2k}}\\ &+ c\,q \frac{(r_2-r_1)^{2m}}{r^{2m}} \int_{\Upsilon_r} |D^m u|^2\,dz + \frac{c_3\,q}{r^{2m}} \int_{\Upsilon_r} |\widehat{u}|^2\,dz.\end{aligned}$$

Taking into account inequalities (3.21) and (3.24) we obtain from (3.20) that

$$\begin{aligned}(3.25)\qquad &\int_{B_{r_1}} |u(x,\tau) - P^*(x;s)|^2\,dx\\ &\le c\,r^{2m} \int_{B_r} |D^m u(x,s)|^2\,dx + \frac{c_4\,q}{r^{2m}} \int_s^{t^0} \int_{B_{r_2}\setminus B_{r_1}} |\widehat{u}|^2\,dx\,dt\\ &+ c_q \frac{r^{2m}}{(r_2-r_1)^{2m}} \int_{Q_r} |D^m u|^2\,dz + c_q \sum_{k=0}^{m} \frac{r^{2m}}{(r_2-r_1)^{2k}} \int_{Q_r} |F_k|^2\,dz,\end{aligned}$$

where $c_4 = c_1 + c_2 + c_3$. As the right-hand side of (3.25) does not depend on τ we derive from the last inequality that

$$\begin{aligned}(3.26)\qquad &\sup_{\tau\in(s,t^0)} \int_{B_{r_1}} |u(x,\tau) - P^*(x;s)|^2\,dx\\ &\le c_4\,q \sup_{t\in(s,t^0)} \int_{B_{r_2}\setminus B_{r_1}} |u(x,t) - P^*(x;s)|^2\,dx + c\,r^{2m} \int_{B_r} |D^m u(x,s)|^2\,dx\\ &+ c_q \frac{r^{2m}}{(r_2-r_1)^{2m}} \int_{Q_r} |D^m u|^2\,dz + c_q \sum_{k=0}^{m} \left(\frac{r^{2m}}{(r_2-r_1)^{2k}} \int_{Q_r} |F_k|^2\,dz \right).\end{aligned}$$

We put

$$g(\rho) = \sup_{t\in(s,t^0)} \int_{B_\rho(x^0)} |u(x,t) - P^*(x,s)|^2 \, dx$$

and choose $q < 1$ from the condition $c_4\, q \le 1/2$. Then inequality (3.26) implies that

$$g(r_1) \le \frac{1}{2}\, g(r_2) + A_0(r;s) + \frac{A_1(r)}{(r_2 - r_1)^{2m}} + \sum_{k=0}^{m} \frac{\mathbb{Z}_k(r)}{(r_2-r_1)^{2k}}, \tag{3.27}$$

for $r/2 \le r_1 < r_2 \le r$, where

$$\begin{aligned} A_0(r;s) &= c r^{2m} \int_{B_r} |D^m u(x,s)|^2 \, dx, \\ A_1(r) &= c_q r^{2m} \int_{Q_r} |D^m u|^2 \, dz, \\ \mathbb{Z}_k(r) &= c_q r^{2m} \int_{Q_r} |F_k|^2 \, dz. \end{aligned}$$

Further we exploit the same assertion as in the proof of Lemma 3.2 (see, for example, [17, Lemma 8.18]) and obtain that

$$g(r_1) \le c\, A_0(r,s) + c \frac{A_1(r)}{(r_2 - r_1)^{2m}} + c \sum_{k=0}^{m} \frac{\mathbb{Z}_k(r)}{(r_2-r_1)^{2k}}$$

for all r_1, r_2 such that $r/2 \le r_1 < r_2 \le r$. Here the constants c may depend on m, n, k. Thus, for $r_1 = r/2$ and $r_2 = r$, we obtain the inequality

$$\begin{aligned} \sup_{t\in(s,t^0)} \int_{B_{r/2}(x^0)} |u(x,t) - P^*(x;s)|^2 \, dx &\le c\, r^{2m} \int_{B_r(x^0)} |D^m u(x,s)|^2 \, dx \\ &\quad + c \int_{Q_r(z^0)} |D^m u|^2 \, dz + c \sum_{k=0}^{m} r^{2(m-k)} \int_{Q_r(z^0)} |F_k|^2 \, dz. \end{aligned} \tag{3.28}$$

Let $P^*_{m-1}(x)$ be the polynomial minimizing the integral

$$\int_{Q_{r/2}(z^0)} |u(z) - P_{m-1}(x)|^2 \, dz$$

among all polynomials of the degree not more than $m-1$. Then (for the fixed earlier s) the following inequality holds:

$$\begin{aligned} \int_{Q_{r/2}(z^0)} |u(z) - P^*_{m-1}(x)|^2 \, dz &\le \int_{Q_{r/2}(z^0)} |u(z) - P^*(x;s)|^2 \, dz \\ &\le \left(\frac{r}{2}\right)^{2m} \sup_{t\in(s,t^0)} \int_{B_{r/2}(x^0)} |u(x,t) - P^*(x;s)|^2 \, dx \end{aligned} \tag{3.29}$$

$$\leq cr^{4m} \int_{B_r(x^0)} |D^m u(x,s)|^2 \, dx \\ + cr^{2m} \int_{Q_r(z^0)} |D^m u|^2 \, dz + c \sum_{k=0}^{m} r^{2(2m-k)} \int_{Q_r(z^0)} |F_k|^2 \, dz.$$

Now we integrate the inequality (3.29) in $s \in \Lambda_r(t^0) \setminus \Lambda_{r/2}(t^0)$ and divide by its measure. Then inequality (3.14) follows. □

REMARK 3.4. Let the assumption (H1) hold and $F_\alpha \in L^2(Q)$ for all $|\alpha| \leq m$. We can consider a weak solution $u \in V(Q)$ to system (1.1) as a solution to system (3.3) with the matrix $a_0^{\alpha\beta}(z) = A^{\alpha\beta}(z, D^{m-1}u(z))$. It follows from Lemmas 3.2 and 3.3 that the function u satisfies Caccioppoli and Poincaré inequalities (3.4) and (3.14).

4. Properties of the $(A(t), m)$-caloric functions

We consider a cylinder $Q_R \subset Q$ and positive definite $N \times N$ matrices $A^{\alpha\beta}(t) = (A_{ik}^{\alpha\beta}(t))_{i,k \leq N}$ with $A_{ik}^{\alpha\beta}(t) \in L^\infty(\Lambda_R)$ for $|\alpha| = |\beta| = m$, satisfying the condition (H1) for almost all $t \in \Lambda_R$.

DEFINITION 4.1. We say that h is an $(A(t), m)$-caloric function in $Q_R(z^0)$ if it is a weak solution to the system

$$h_t - A(t)D^{2m}h = 0, \qquad z \in Q_R(z^0). \tag{4.1}$$

According to the Definition 1.1, a weak solution $h \in V(Q_R(z^0))$ to system (4.1) satisfies the identity

$$\int_{Q_R(z^0)} [-h(z) \cdot \phi_t(z) + A(t)D^m h(z) \cdot D^m \phi(z)] \, dz = 0 \tag{4.2}$$

for all $\phi \in C_0^\infty(Q_R(z^0))$. Obviously, any $(A(t), m)$-caloric function satisfies Caccioppoli and Poincaré inequalities (3.4) and (3.14). Moreover, weak solutions h of the linear parabolic systems (4.1) have an additional smoothness in any Q_r for $r < R$. We summarize smoothness results about $(A(t), m)$-caloric functions in the following propositions.

LEMMA 4.2. *Let $h \in V(Q_R)$ be an $(A(t), m)$-caloric function in Q_R. Then h belongs to the space $W_2^{2m,1}(Q_r)$ and solves system* (4.1) *for almost all $z \in Q_R$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ and $r < r_0 < R$ the functions $w(z) = D^\alpha h$ are $(A(t), m)$-caloric functions in Q_R and satisfy the inequality*

$$\sup_{t \in \Lambda_r} \int_{B_r} |w(x,t)|^2 \, dx + \int_{Q_r} |D^m w(z)|^2 \, dz \leq \frac{C}{(r_0 - r)^2} \int_{Q_{r_0}} |w(z)|^2 dz \tag{4.3}$$

for $r < r_0$. Moreover, functions $w = D^\alpha h$ have derivatives $D^\beta w$ with respect to the space variables of arbitrary order $|\beta|$ and the first derivative with respect to *time in $L^2(Q_r)$ which satisfy the estimate*

$$
\sup_{\Lambda_r} \int_{B_r} |D^\beta w(x,t)|^2\,dx + \int_{Q_r} (|D^\beta w(z)|^2 + |(D^\beta w)_t(z)|^2)\,dz \\
\leq c(\beta, (R-r)^{-1}) \int_{Q_R} |h(z)|^2\,dz. \tag{4.4}
$$

The functions w are continuous on $\overline{Q_r}$ and $(w)_t \in L^\infty(Q_r)$ for any multiindex α.

Proof. The proof of this lemma is similar to the proof of Lemma 4 in [5].

In the standard way we derive estimate (4.3) for $w = h$, i.e. for $|\alpha| = 0$ (see Lemma 4 in [5] or proof of Lemma 1 in [12]). Then we choose $Q_r(z^0) \subset\subset Q_{r_0}(z^0) \subset\subset Q_R(z^0)$ and index $i \in \{1, \dots, n\}$ and for σ, $|\sigma| \leq R - r_0$, we consider the difference

$$
w_i(\sigma; x, t) = \frac{h(x + \sigma \mathbf{e}_i, t) - h(x,t)}{\sigma}, \qquad z \in Q_{r_0}(z^0), \tag{4.5}
$$

where $\mathbf{e}_i$, $i = 1, \dots, n$, form the canonical basis in $\mathbb{R}^n$. As the coefficients $A(t)$ do not depend on x, the function $w_i(\sigma; x, t) \in V(Q_{r_0}(z^0))$ is $(A(t), m)$-caloric and it satisfies inequality (4.3), i.e.

$$
\int_{Q_r(z^0)} |D^m w_i(\sigma; x, t)(z)|^2\,dz \leq \frac{c}{(r_0 - r)^2} \int_{Q_{r_0}(z^0)} |w_i(\sigma; x, t)(z)|^2\,dz \\
\leq \frac{c}{(r_0 - r)^2} \int_{Q_R(z^0)} \left|\frac{\partial h}{\partial x_i}\right|^2 dz. \tag{4.6}
$$

The right-hand side of the last inequality is independent on σ, and we can pass $\sigma \to 0$ and deduce that there exist the derivatives $D^m(h_{x_i}) \in L^2(Q_r(z^0))$, for all $i = 1, \dots, n$, satisfying the estimate

$$
\sup_{\Lambda_r(t^0)} \int_{B_r(x^0)} |\nabla h(x,t)|^2\,dx + \int_{Q_r(z^0)} |D^{m+1} h|^2\,dz \\
\leq \frac{c}{(r_0 - r)^2} \int_{Q_{r_0}(z^0)} |\nabla h|^2\,dz \leq c_0 \int_{Q_R(z^0)} |h|^2\,dz, \tag{4.7}
$$

the constant c_0 depends on $(R - r_0)^{-1}$ and $(r_0 - r)^{-1}$. As the number $r < R$ was fixed arbitrarily, the derivative $D^{m+1}h$ belongs to $L^2_{\mathrm{loc}}(Q_R(z^0))$. Each function $h_{x_i} \in V(Q_r(z^0))$ is $(A(t), m)$-caloric function and we can repeat our considerations to justify that there exist the space derivatives of $h \in L^2_{\mathrm{loc}}(Q_R(z^0))$ of any order and these derivatives satisfy estimate (4.3).

We can rewrite the identity (4.2) into the form

$$
\int_{Q_R(z^0)} [-h \cdot \phi_t + (-1)^m A(t) D^{2m} h \cdot \phi]\,dz = 0 \quad \text{for all } \phi \in C_0^\infty(Q_R(z^0)).
$$

The existence of $h_t \in L^2_{\mathrm{loc}}(Q_R(z^0))$ follows directly from the definition of weak derivative. We can conclude that

$$h_t + (-1)^m A(t) D^{2m} h = 0 \quad \text{a.e. in } Q_R(z^0), \tag{4.8}$$

and

$$\|h_t\|_{L^2(Q_r(z^0))} \le c\,\|D^{2m}h\|_{L^2(Q_r(z^0))} \le c((R-r)^{-1})\,\|h\|_{L^2(Q_R(z^0))}. \tag{4.9}$$

As all functions $w(z) = D^\alpha h(z)$, $|\alpha| < \infty$, are $(A(t), m)$-caloric functions in $Q_r(z^0)$ then

$$\|(D^\alpha h)_t\|_{L^2(Q_r(z^0))} \le c\|D^{2m+|\alpha|}h\|_{L^2(Q_r(z^0))} \le c(\alpha, (R-r)^{-1})\|h\|_{L^2(Q_R(z^0))}.$$

It means that estimate (4.4) holds.

Estimate (4.3) for $w = D^\alpha h$ guarantees that

$$\sup_{\Lambda_r(t^0)} \|D^\alpha h(\,\cdot\,, t)\|_{L^2(B_r(x^0))} \le c\,\|h\|_{L^2(Q_R(z^0))}.$$

Thus $h(\,\cdot\,,t) \in W^k_2(B_r(x^0))$ uniformly with respect to $t \in \Lambda_r(t^0)$. If $2k > n$ then by the embedding theorem the function h is a continuous function of x for any fixed $t \in \Lambda_r(t^0)$ and

$$\sup_{t\in\Lambda_r(t^0)} \|h(\,\cdot\,,t)\|_{C(\overline{B_r(x^0)})} \le c\,\|h\|_{L^2(Q_R(z^0))}.$$

Moreover,

$$\begin{aligned}
|D^\alpha h(x,t) - D^\alpha h(x,\tau)|^2 &= \left| \int_\tau^t (D^\alpha h)_s(x,s)\,ds \right|^2 \\
&\le |t-\tau| \int_{\Lambda_r(t^0)} |(D^\alpha h)_s(x,s)|^2\,ds \\
&\le |t-\tau| \int_{\Lambda_r} \sup_{x\in B_r(x^0)} |(D^\alpha h)_s(x,s)|^2\,ds \\
&\le |t-\tau|\, c(|\alpha|, k) \int_{\Lambda_r} \|(D^\alpha h)_s\|^2_{W^k_2(B_r(x^0))}\,ds \\
&\le |t-\tau|\, c(|\alpha|, k, (R-r)^{-1}) \|h\|^2_{L_2(Q_R(z^0))} \to 0
\end{aligned}$$

as $\tau \to t$. We used estimate (4.4) in the last step. It means that $D^\alpha h \in C(\overline{Q_r(z^0)})$ for all $|\alpha| < \infty$, and (4.1) implies that h_t and $(D^\alpha h)_t \in L^\infty(Q_r(z^0))$ for all α with $|\alpha| < \infty$. $\square$

Next we deduce Campanato type estimates for $(A(t), m)$-caloric functions. (For the case $m = 1$ see [9].)

Recall that for a fixed $u \in V(Q_r)$ we denoted by $P^*_{m-1,r}(x)$ the polynomial minimizing the integral $\int_{Q_r} |u(z) - P_{m-1}(x)|^2\,dz$ among all polynomials of degree less or equal to $m-1$.

Lemma 4.3 (Campanato type estimates). *Let h be an $(A(t), m)$-caloric function on Q_R and $0 < \rho < r \le R$. Then, for $\rho < r \le R$, the following inequalities hold*

$$\fint_{Q_\rho} |D^m h(z)|^2 \, dz \le c \fint_{Q_r} |D^m h(z)|^2 \, dz, \tag{4.10}$$

$$\fint_{Q_\rho} |h(z) - P^*_{m-1,\rho}(x)|^2 \, dz \le c \left(\frac{\rho}{r}\right)^{2m} \fint_{Q_r} |h(z) - P^*_{m-1,r}(x)|^2 \, dz. \tag{4.11}$$

Proof. First we prove inequality (4.10). For $\rho \le r/2$ and $2k > n$ we have the inequalities

$$\begin{aligned}\fint_{Q_\rho} |D^m h(z)|^2 \, dz &\le \sup_{t \in \Lambda_{r/2}} \|D^m h(\,\cdot\,, t)\|^2_{L^\infty(B_{r/2})} \\ &\le c(r^{-1}) \sup_{t \in \Lambda_{r/2}} \|D^m h(\,\cdot\,, t)\|^2_{W^k_2(B_{r/2})} \le c(r^{-1}) \sum_{i=0}^{k} \|D^{i+m} h(\,\cdot\,, t)\|^2_{L^2(B_{r/2})}.\end{aligned}$$

By (4.4) the last expression is estimated by $c(r^{-1})\|D^m h\|^2_{L^2(Q_r)}$ and similarity transformation implies that $c(r^{-1}) = c_0(m, \nu, \mu)$. For $\rho \ge r/2$ inequality (4.10) is evident. Thus, inequality (4.10) is proved for all $\rho \le r$.

Let now $\rho \le r/4$. Using the Poincaré inequality (3.14) with $F_\alpha = 0$, estimate (4.10), and then the Caccioppoli inequality, we obtain that

$$\begin{aligned}\fint_{Q_\rho} |h(z) - P^*_{m-1,\rho}(x)|^2 \, dz &\le c\rho^{2m} \fint_{Q_{2\rho}} |D^m h(z)|^2 \, dz \\ &\le c\rho^{2m} \fint_{Q_{r/2}} |D^m h(z)|^2 \, dz \le c\left(\frac{\rho}{r}\right)^{2m} \fint_{Q_r} |h(z) - P^*_{m-1,r}(x)|^2 \, dz\end{aligned}$$

which proves inequality (4.11) for $\rho \le r/4$.

For $\rho \ge r/4$ inequality (4.11) is evident. □

5. $(A(t), m)$-caloric lemma

In this section we will consider a fixed cylinder $Q_r(z^0) \subset Q$ and for simplicity we will leave out the notation of the center z^0 of the cylinder as well as t^0 and x^0 in the notation of time intervals and balls. Further we denote for $t \in \Lambda_r$ by $A(t)$ the matrices $\{A^{\alpha\beta}(t)\}$ for α, β such that $|\alpha| = |\beta| = m$, where $A^{\alpha\beta}(t) = \{A^{\alpha\beta}_{kl}\}$ for $k, l = 1, \dots, N$.

Lemma 5.1. *Let μ, ν be positive numbers, $\nu \le \mu$, m, n, N belong to $\mathbb{N}$ and $m \ge 1$, $n \ge 2$, $N \ge 1$. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever matrices $A^{\alpha\beta}(t)$ (with entries $A^{\alpha\beta}_{kl} \in L^\infty(\Lambda_r; \mathbb{R})$ for $|\alpha| = |\beta| = m$; $k, l \le N$) satisfy the condition $A(t) \in \{\nu, \mu\}$ for almost all $t \in \Lambda_r$, then for any*

$u \in V(Q_r)$ such that

$$\fint_{Q_r} \sum_{|\alpha|\le m} r^{2(|\alpha|-m)} |D^\alpha u|^2\, dz \le 1, \tag{5.1}$$

$$\left| \fint_{Q_r} [-u(z)\cdot \phi_t(z) + (A^{\alpha\beta}(t) D^\beta u(z)\cdot D^\alpha \phi(z))]\, dz \right| \le \delta \sup_{z\in Q_r} |D^m \phi(z)|, \tag{5.2}$$

for all $\phi \in C_0^\infty(Q_r)$, there exists an $(A(t),m)$-caloric function $h \in V(Q_{r/2})$ so that

$$\fint_{Q_{r/2}} \sum_{|\alpha|\le m} r^{2(|\alpha|-m)} |D^\alpha h(z)|^2\, dz \le 2^{n+2m+2}, \tag{5.3}$$

$$\fint_{Q_{r/2}} \sum_{|\beta|\le m-1} |D^\beta u(z) - D^\beta h(z)|^2\, r^{2(|\beta|-m)}\, dz \le \varepsilon. \tag{5.4}$$

Proof. Without loss of generality we can prove the lemma for $r = 1$; otherwise we rescale u to cylinder $Q_1(0)$ via

$$U(y,\tau) = \frac{1}{r^m} u(x^0 + ry, t^0 + r^{2m} t)$$

which satisfies the assumptions of the lemma on $Q_1(0)$, set the matrix $\widetilde{A}(\tau) = A(t^0 + r^{2m}\tau)$ and find an $(\widetilde{A}(\tau), m)$-caloric function $H(y,\tau)$ on $Q_{1/2}(0)$ and rescale it back to $Q_{r/2}(z^0)$ so that $h(x,t) = r^m H((x-x^0)/r, (t-t^0)/r^{2m})$. Then h is $(A(t),m)$-caloric on $Q_{r/2}(z^0)$, $A(t) = \widetilde{A}((t-t^0)/r^{2m})$.

Assume that, by contradiction, the assertion of the lemma were false. Then we could find an $\varepsilon > 0$ and a sequence of matrices $A_{(k)}(t)$ satisfying conditions (1.2), (1.3) with uniform ellipticity constant ν and uniform upper bound μ and a sequence of functions $u_k \in V(Q_1)$ such that the estimates

$$\fint_{Q_1} \sum_{|\alpha|\le m} |D^\alpha u_k(z)|^2\, dz \le 1, \tag{5.5}$$

$$\left| \fint_{Q_1} (u_k(z)\cdot \varphi_t(z) - (A^{\alpha\beta}_{(k)}(t) D^\beta u_k(z)\cdot D^\alpha \varphi(z))\, dz \right| \le \frac{1}{k} \sup_{Q_1} |D^m \varphi(z)|, \tag{5.6}$$

for $k \in \mathbb{N}$ and for all $\phi \in C_0^\infty(Q_1)$ hold, but at the same time

$$\fint_{Q_{1/2}} \sum_{|\beta|\le m-1} |D^\beta (u_k(z) - h(z))|^2\, dz > \varepsilon \tag{5.7}$$

for all $h \in H_k$. Here

$$H_k = \Bigg\{ v \in V(Q_{1/2}) : v \text{ is } (A_{(k)}(t), m)\text{-caloric in } Q_{1/2} \\ \text{and} \quad \fint_{Q_{1/2}} \sum_{|\beta|\le m} |D^\beta h(z)|^2\, dz \le 2^{(n+2m+2)} \Bigg\}.$$

Note that from (5.5) it follows that (passing to a not relabelled subsequence) $u_k \rightharpoonup u$, $D^\beta u_k \rightharpoonup D^\beta u$ in $L^2(Q_1)$ for all $|\beta| \le m$ and

$$\fint_{Q_1} \sum_{|\alpha|\le m} |D^\alpha u(z)|^2\,dz \le 1. \tag{5.8}$$

Step 1. Strong convergence of a subsequence. We prove strong convergence of the (again not relabelled) subsequence (u_k) in $L^2(\Lambda_1; W_2^{m-1}(B_1))$, i.e.

$$\|u_k - u\|_{L^2(\Lambda_1; W_2^{m-1}(B_1))}, \quad k \to \infty. \tag{5.9}$$

The proof can be achieved almost in the same way as in [15] and we give an explanation for reader's convenience.

First of all, it follows from (5.6) that

$$\begin{aligned}\bigg|\int_{Q_1} u_k\,\phi_t\,dz\bigg| &\le \mu \int_{Q_1} |D^m u_k|\,|D^m\phi|\,dz + \frac{\omega_n}{k}\sup_{Q_1}|D^m\phi| \\ &\le \mu\,\|D^m u_k\|_{2,Q_1}\,\|D^m\phi\|_{2,Q_1} + \frac{\omega_n}{k}\sup_{Q_1}|D^m\phi|.\end{aligned}$$

(Note that only equiboundedness of $A_{(k)}(t)$ was used). Here and below ω_n stands for the measure of unit ball in R^n.

From the last inequality and (5.5) we get that

$$\bigg|\int_{Q_1} u_k\,\phi_t\,dz\bigg| \le \sqrt{\omega_n}\mu\,\|D^m\phi\|_{2,Q_1} + \frac{\omega_n}{k}\sup_{Q_1}|D^m\phi|. \tag{5.10}$$

For $t, t+h \in (-1,0)$, $h > 0$, and for sufficiently small $\sigma > 0$ we denote by $\chi_\sigma(\tau)$ a continuous piecewise linear function such that $\chi_\sigma(\tau) = 1$ for $\tau \in [t, t+h]$, $\chi_\sigma(\tau) = 0$ for $\tau \in [0, t-\sigma]$ and for $\tau \in [t+h+\sigma, 1]$. We remark that $\chi_\sigma'(\tau) = 1/\sigma$ on the interval $(t-\sigma, t)$ and $\chi_\sigma'(\tau) = -1/\sigma$ on $(t+h, t+h+\sigma)$.

We choose an arbitrary function $\psi \in C_0^\infty(B_1; \mathbb{R}^N)$ and put $\phi(z) = \chi_\sigma(t)\,\psi(x)$ as a test function in (5.10). Then

$$\begin{aligned}&\bigg|\int_{B_1}\bigg(\fint_{t-\sigma}^{t} u_k(x,\tau)\,d\tau - \fint_{t+h}^{t+h+\sigma} u_k(x,\tau)\,d\tau\bigg)\psi(x)\,dx\bigg| \\ &\quad\le \sqrt{\omega_n}\,\mu\bigg(\int_{t-\sigma}^{t+h+\sigma}\chi_\sigma^2(\tau)\,d\tau\bigg)^{1/2}\|D^m\psi\|_{2,B_1} + \frac{\omega_n}{k}\sup_{B_1}|D^m\psi| \\ &\quad\le \sqrt{\omega_n}\mu(h+2\sigma)^{1/2}\|D^m\psi\|_{2,B_1} + \frac{\omega_n}{k}\sup_{B_1}|D^m\psi|.\end{aligned} \tag{5.11}$$

By the embedding theorem it holds

$$\|D^m\psi\|_{L_\infty,B_1} \le c(n,m,l)\,\|\psi\|_{\overset{\circ}{W}{}_2^{l}(B_1)}$$

whenever $l > n/2 + m$. Now, we pass with $\sigma \to 0$ in (5.11) and get that, for almost all $t \in (-T,0), t+h \in (-T,0)$, the limits exist. Thus we get that, for all

$l > n/2 + m$, it holds:

$$|L(\psi)| := \left| \int_{B_1} \psi(x) \cdot (u_k(x,t+h) - u_k(x,t))\, dx \right| \le c_1 [h^{1/2} + 1/k] \, \|\psi\|_{\overset{o}{W}{}^l_2(B_1)} \tag{5.12}$$

for almost all $t, t+h \in \Lambda_1$. Here c_1 depends on n, m, l, μ.

Estimate (5.12) means that $L(\psi)$ is a linear bounded functional in $\overset{o}{W}{}^l_2(B_1)$ and its norm can be estimated by

$$\|u_k(x,t+h) - u_k(x,t)\|_{W_2^{-l}(B_1)} \le c_1 \, [\sqrt{h} + 1/k]. \tag{5.13}$$

Further we denote $X = W_2^m(B_1)$, $\mathbb{B} = W_2^{m-1}(B_1)$ and $Y = W_2^{-l}(B_1)$ and apply the results of J. Simon [21]. First, because of the compact embedding of X in $\mathbb{B}$ and continuous embedding of $\mathbb{B}$ in Y, for any $\eta > 0$ there exists number $M(\eta) > 0$ such that

$$\|v\|^2_{\mathbb{B}} \le \eta \, \|v\|^2_X + M(\eta)\, \|v\|^2_Y \quad \text{for all } v \in X. \tag{5.14}$$

We choose $v = u_k(\,\cdot\,, t+h) - u_k(\,\cdot\,, t)$ in (5.14) and integrate this inequality over $t \in (-1, -h)$:

$$\begin{aligned} &\int_{-1}^{-h} \|u_k(\,\cdot\,,t+h) - u_k(\,\cdot\,,t)\|^2_{W_2^{m-1}(B_1)} \, dt \\ &\quad \le \eta \int_{-1}^{-h} \|u_k(\,\cdot\,,t+h) - u_k(\,\cdot\,,t)\|^2_{W_2^{m}(B_1)} \, dt \\ &\qquad + M(\eta) \int_{-1}^{-h} \|u_k(\,\cdot\,,t+h) - u_k(\,\cdot\,,t)\|^2_{W_2^{-l}(B_1)} \, dt \\ &\quad \le 4\omega_n \, \eta + c_2(\eta)[h + 1/k], \end{aligned} \tag{5.15}$$

where $c_2(\eta) = 2M(\eta)\, c_1^2$. We claim that

$$\int_{-1}^{-h} \|u_k(\,\cdot\,,t+h) - u_k(\,\cdot\,,t)\|^2_{W_2^{m-1}(B_1)} \, dt \to 0, \quad \text{as } h \to 0 \tag{5.16}$$

uniformly with respect to $k \in \mathbb{N}$. Indeed, for a fixed $\varepsilon > 0$, we choose $\eta > 0$ to satisfy the inequality

$$4\,\omega_n \, \eta < \varepsilon/3. \tag{5.17}$$

Then we fix $k_0 \in \mathbb{N}$ so that $c_2(\eta)/k_0^2 < \varepsilon/3$. Thus also $c_2(\eta)/k^2 < \varepsilon/3$ for all $k \ge k_0$.

Let $h_1 > 0$ be so small that for all $k = 1, \dots, k_0 - 1$ and all positive $h \le h_1$ it holds

$$\int_{-1}^{-h} \|u_k(\,\cdot\,,t+h) - u_k(\,\cdot\,,t)\|^2_{W_2^{m-1}(B_1)} \, dt < \varepsilon.$$

At last we choose positive $h_2 \leq h_1$ so small that $c_2(\eta)\, h_2 < \varepsilon/3$. This choice provides the estimate

$$\int_{-1}^{-h} \|u_k(\,\cdot\,, t+h) - u_k(\,\cdot\,, t)\|^2_{W_2^{m-1}(B_1)}\, dt < \varepsilon \tag{5.18}$$

for all $h \leq h_2$ and all $k \in \mathbb{N}$. Thus (5.16) holds, and we can apply Theorem 5 in [21] with the spaces $X \overset{c}{\hookrightarrow} \mathbb{B} \hookrightarrow Y$ introduced earlier. The set $F = \{u_k(z)\}_{k\in\mathbb{N}}$ satisfies the conditions of Theorem 5 in [21] thus it is relatively compact in $\mathbb{B}$ and there exists a subsequence of u_k (we do not rename it) such that

$$\|u_k - u_s\|_{L^2(\Lambda_1; W_2^{m-1}(B_1))} \to 0, \quad \text{as } k, s \to \infty. \tag{5.19}$$

The relation (5.9) follows from the completeness of $L^2(\Lambda_1; W_2^{m-1}(B_1))$.

Step 2. Limit equation. As the sequence $A_{(k)}(t)$ is uniformly bounded in $L^\infty(\Lambda_1)$ there exist matrices $A(t) = \{A^{\alpha\beta}(t)\} \in L^\infty(\Lambda_1)$, $|\alpha| = |\beta| = m$ and a (not relabelled) subsequence $A_{(k)}(t)$ so that $A_{(k)}(t)$ weakly* converge to $A(t)$ in $L^\infty(\Lambda_1)$ and $||A(t)||_{L^\infty(\Lambda_1)} \leq \liminf_{k\to\infty} ||A_{(k)}(t)||_{L^\infty(\Lambda_1)} \leq \mu$. For a fixed positive h and for almost all $t \in (-1, -h)$ we have

$$\frac{1}{h}\int_t^{t+h} \sum_{|\alpha|=|\beta|=m} (A^{\alpha\beta}_{(k)}(\tau)\xi^\beta \cdot \xi^\alpha)\, d\tau \geq \nu\, |\xi|^2, \qquad |\xi|^2 = \sum_{|\alpha|=m} |\xi^\alpha|^2,$$

for all ξ^α, $\xi^\beta \in \mathbb{R}^N$. Passing with $k \to \infty$ we get

$$\frac{1}{h}\int_t^{t+h} (A^{\alpha\beta}(\tau)\xi^\beta \cdot \xi^\alpha)\, d\tau \geq \nu|\xi|^2.$$

If now $h \to 0+$ then we obtain that $(A^{\alpha\beta}(t)\xi^\beta \cdot \xi^\alpha) \geq \nu|\xi|^2$ for almost all $t \in \Lambda_1$. As φ has a compact support in Q_1 we can rewrite (5.6) as

$$\left| \fint_{Q_r} [u_k(z) \cdot \phi_t(z) + A^{\alpha\beta}_{(k)}(t) u_k(z) \cdot D^{\alpha+\beta}\phi(z)]\, dz \right| \leq \frac{1}{k} \sup_{z\in Q_r} |D^m\phi(z)|,$$

for all $\phi \in C_0^\infty(Q_r)$.

According to Step 1 it holds $u_k \to u$ in $L^2(\Lambda_1; W_2^{m-1}(B_1))$, $A_{(k)} \rightharpoonup^* A$ in $L^\infty(\Lambda_1)$ and thus u is a very weak solution to the problem

$$\int_{Q_1} (u(z) \cdot \varphi_t(z) - A^{\alpha\beta}(t) u(z) \cdot D^{\alpha+\beta}\varphi(z))\, dz = 0,$$

for all $\varphi \in C_0^\infty(Q_1)$. The derivatives $D^\alpha u$, $|\alpha| \leq m$ belongs to $L^2(Q_1)$, and the matrices $A^{\alpha\beta}$ do not depend on the space variables. It means that we can rewrite the equation back to get that $u \in V(Q_1)$ satisfies the identity

$$\int_{Q_1} (u \cdot \varphi_t - A^{\alpha\beta}(t) D^\beta u \cdot D^\alpha \varphi)\, dz = 0 \tag{5.20}$$

for all $\varphi \in C_0^\infty(Q_1)$. Thus, u is the $(A(t), m)$-caloric function.

Step 3. Smoothness of u. As u is $(A(t),m)$-caloric function in Q_1, the assertion of Lemma 4.2 is valid. In particular, $D^{2m}u$, $u_t \in L^2(Q_{1/2})$, and the formula (4.4) yields the inequality

$$\int_{Q_{1/2}} |(|D^{2m}u(z)|^2 + |u_t(z)|^2)\, dz \le c \int_{Q_1} |u(z)|^2\, dz \le c_3. \tag{5.21}$$

Here the last inequality follows from (5.8).

Step 4. Auxiliary problems. For $k \in \mathbb{N}$ we denote by v_k a weak solution to the problem

$$\begin{aligned} &v_t(z) + (-1)^m D^\alpha (A^{\alpha\beta}_{(k)}(t) D^\beta v(z)) \\ &\qquad = (-1)^m D^\alpha ((A^{\alpha\beta}_{(k)}(t) - A^{\alpha\beta}(t)) D^\beta u(z)) =: \Phi_k(z), \quad \text{for } z \in Q_{1/2}, \\ &\qquad |D^s v| = 0 \quad \text{on } \Gamma_{1/2},\ s = 0, \dots, m-1,\ v|_{t=-(1/2)^{2m}} = 0. \end{aligned} \tag{5.22}$$

The right-hand side Φ_k belongs to $L^2(Q_{1/2})$ and the existence of weak solutions $v_k \in W^{2m,1}_2(Q_{1/2})$ follows (for example as a special case from Theorem 6 in [13]). Moreover, the estimate

$$\|(v_k)_t\|^2_{2,Q_{1/2}} + \|D^{2m}v_k\|^2_{2,Q_{1/2}} \le c\, \|D^{2m}u\|^2_{2,Q_{1/2}} \le c\, c_3 =: c_4. \tag{5.23}$$

holds. This estimate guarantees existence of a subsequence of $\{v_k\}$ weakly convergent in $W^{2m,1}_2(Q_{1/2})$ to a function $v \in W^{2m,1}_2(Q_{1/2})$. In particular, it implies the convergence of v_k to v in $L^2(Q_{1/2})$ (once more for not relabelled subsequence).

Now we use weak formulation of problem (5.22) for the solution v_k with the test function v_k and get the estimate

$$\begin{aligned} &\sup_{\Lambda_{1/2}} \int_{B_{1/2}} |v_k(x,t)|^2\, dx + \nu \int_{Q_{1/2}} |D^m v_k(z)|^2\, dz \\ &\qquad \le \left| \int_{Q_{1/2}} D^\alpha (A^{\alpha\beta}_{(k)}(t) - A^{\alpha\beta}(t)) D^\beta u(z) \cdot v_k(z)\, dz \right|. \end{aligned} \tag{5.24}$$

Note that the right-hand side of (5.24) (we denote it by J_k) tends to zero when $k \to \infty$. Indeed,

$$\begin{aligned} J_k = \bigg| &\int_{Q_{1/2}} (A^{\alpha\beta}_{(k)}(t) - A^{\alpha\beta}(t)) D^{\alpha+\beta} u \cdot v\, dz \\ &+ \int_{Q_{1/2}} (A^{\alpha\beta}_{(k)}(t) - A^{\alpha\beta}(t)) D^{\alpha+\beta} u \cdot (v_k - v)\, dz \bigg|. \end{aligned}$$

Here the first integral goes to zero due to *weak convergence of $A_{(k)}$ to A and the second one by convergence of v_k to v in $L^2(Q)$. Thus $J_k \to 0$ for $k \to \infty$ and the left-hand side of (5.24) tends to zero when $k \to \infty$. We obtain that $v = 0$ and

$$\|v_k\|_{V(Q_{1/2})} \to 0, \quad \text{as } k \to \infty. \tag{5.25}$$

Next we put $g_k = u - v_k$. Then $g_k \to u$ in $V(Q_{1/2})$ and the functions g_k are solutions to the problem

$$(g_k)_t + (-1)^m A^{\alpha\beta}_{(k)}(t) D^{\alpha+\beta} g_k = 0, \quad z \in Q_{1/2}, \tag{5.26}$$
$$|D^s(g_k - u)|_{\Gamma_{1/2}} = 0, \quad s = 0, \dots, m-1,\ g_k - u|_{t=-2^{-2m}} = 0.$$

It means that g_k are $(A_{(k)}(t), m)$- caloric functions on $Q_{1/2}$ and they tend to u, i.e.

$$\|g_k - u\|_{V(Q_{1/2})} = \|v_k\|_{V(Q_{1/2})} \to 0, \quad \text{as } k \to \infty. \tag{5.27}$$

We denote by $\chi_k = \|v_k\|_{V(Q_{1/2})}$ and

$$\|g_k\|_{V(Q_{1/2})} \le \|u\|_{V(Q_{1/2})} + \chi_k \le \sqrt{\omega_n} + \chi_k, \quad \text{as } \chi_k \to 0,\ k \to \infty$$

which implies

$$⨍_{Q_{1/2}} \sum_{|\alpha|\le m} |D^\alpha g_k|^2\, dz \le 2^{n+2m+1} + 2\chi_k^2, \quad \text{for all } k \in \mathbb{N}. \tag{5.28}$$

There exists a number $k_0 \in \mathbb{N}$ such that $2\chi_k^2 \le 2^{n+2m+1}$ for all $k \ge k_0$. It means that

$$⨍_{Q_{1/2}} \sum_{|\alpha|\le m} |D^\alpha g_k|^2\, dz \le 2^{n+2m+2} \quad \text{for all } k \ge k_0. \tag{5.29}$$

It follows that $g_k \in H_k$ (see the notation of H_k in (5.7)). Relations (5.9) and (5.27) means that

$$\begin{aligned}\|u_k - g_k\|&_{L^2(\Lambda_{1/2}; W_2^{m-1}(B_1))}\\ &\le \|u_k - u\|_{L^2(\Lambda_{1/2}; W_2^{m-1}(B_1))} + \|u - g_k\|_{L^2(\Lambda_{1/2}; W_2^{m-1}(B_1))} \to 0,\end{aligned}$$

when $k \to \infty$, and we arrive at the contradiction. □

Further, we will use a consequence of Lemma 5.1.

Lemma 5.2. *Let the assumptions of Lemma* 5.1 *be satisfied. Then, for any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon) = C(\varepsilon, n, N, \nu, \mu, m, n, N)$ such that the following holds: for matrices $A(t) = \{A^{\alpha\beta}(t)\}$, $|\alpha| = |\beta| = m$, with entries in $L^\infty(\Lambda_R)$ satisfying the condition* (H1) *and, for any $u \in V(Q_R)$, there exist an $(A(t), m)$-caloric $h \in V(Q_{R/2})$ and $\varphi \in C_0^\infty(Q_R)$ such that $||D^m\varphi||_{L^\infty(Q_r)} \le 1$ and moreover, for $\widehat{u}(z) = u(z) - P^*_{m-1,R}(x)$ it holds*

$$⨍_{Q_{R/2}} \sum_{|\alpha|\le m} R^{2|\alpha|}\, |D^\alpha h|^2\, dz \le 2^{n+2m+2} ⨍_{Q_R} \sum\nolimits_{|\alpha|\le m} |D^\alpha \widehat{u}|^2\, R^{2|\alpha|}\, dz, \tag{5.30}$$

and

$$(5.31)\qquad \fint_{Q_{R/2}} \sum_{|\alpha|\le m-1} |D^\alpha(\widehat{u}-h)|^2\, dz \\ \le \varepsilon \fint_{Q_R} \sum\nolimits_{|\alpha|\le m} |D^\alpha \widehat{u}|^2\, R^{2|\alpha|}\, dz + C(\varepsilon) R^{2m} \mathcal{L}_R^2(\widehat{u},\varphi)$$

where

$$(5.32)\qquad \mathcal{L}_{\mathcal{R}}(\widehat{u},\varphi) = \mathcal{L}_{\mathcal{R}}(u,\varphi) = \left| \fint_{Q_R} [-u\varphi_t + (A^{\alpha\beta}(t) D^\beta u, D^\alpha \varphi)\, dz \right|.$$

Here the polynomial $P^*_{m-1,R}(x)$ *minimizes the integral* $\int_{Q_R} |u(z)-P|^2\, dz$ *among all* $P \in \mathbb{P}_{m-1}$.

First of all we remark that such reformulation was proposed in elliptic case by M. Giaquinta (see Appendix in [15]). For parabolic case and for the proof of Lemma 5.2 in the case $m = 1$ see [2].

PROOF. As it was noted in the proof of Lemma 5.1, it is enough to consider the case $R = 1$ and to make the dilatation of the independent variables. Thus let $\varepsilon > 0$ and $u \in V(Q_1)$ be fixed. We denote by $\delta = \delta(\varepsilon)$ the number which is guaranteed by Lemma 5.1. We put

$$v(z) = \frac{\widehat{u}(z)\sqrt{\omega_n}}{\|\widehat{u}\|_{V(Q_1)}}, \qquad \widehat{u}(z) = u(z) - P^*_{m-1,1}(x).$$

Then v satisfies (5.1).

There are two possibilities:

(a) For all $\varphi \in C_0^\infty(Q_1)$ the inequality

$$(5.33)\qquad \left| \fint_{Q_1} [-v\ \cdot \varphi_t + A^{\alpha\beta} D^\beta v \cdot D^\alpha \varphi]\, dz \right| \le \delta \sup_{Q_1} |D^m \varphi|$$

holds (see (5.2)).

In this case by Lemma 5.1 there exists an $(A(t), m)$-caloric function $\widehat{h} \in V(Q_{1/2})$ such that

$$(5.34)\qquad \fint_{Q_{1/2}} \sum_{|\alpha|\le m} |D^\alpha \widehat{h}|^2\, dz \le 2^{n+2m+2},$$

$$(5.35)\qquad \fint_{Q_{1/2}} \sum_{|\beta|\le m-1} |D^\beta\, (v-\widehat{h})|^2\, dz \le \varepsilon.$$

We set now $h(z)$ from the relation $h(z) = \widehat{h}\, \|\widehat{u}\|_{V(Q_1)}/\sqrt{\omega_n}$. It follows from (5.34), (5.35) that

$$(5.36)\qquad \fint_{Q_{1/2}} \sum_{|\alpha|\le m} |D^\alpha\, h|^2\, dz \le 2^{n+2m+2} \fint_{Q_1} \sum_{|\alpha|\le m} |D^\alpha\, \widehat{u}|^2\, dz,$$

and

$$(5.37)\qquad \fint_{Q_{1/2}} \sum_{|\alpha|\le m-1} |D^\alpha(\widehat{u}-h)|^2\,dz \le \varepsilon \fint_{Q_1} \sum_{|\alpha|\le m} |D^\alpha\,\widehat{u}|^2\,dz.$$

It means that in the situation (a) relations (5.30), (5.31) hold.

In the case (b) there exists a function $\varphi \in C_0^\infty(Q_1)$, such that inequality (5.33) does not hold. Then we put $\varphi_0 = \varphi \Big/ \sup\limits_{Q_1} |D^m\varphi|$, $\sup\limits_{Q_1}|D^m\varphi_0| \le 1$ and obtain

$$(5.38)\qquad \bigg| \fint_{Q_1} [-v\cdot(\varphi_0)_t + A^{\alpha\beta}D^\beta v\cdot D^\alpha\varphi_0]\,dz \bigg| > \delta.$$

Then we substitute $\widehat{u}$ instead of v in (5.38) and obtain that

$$(5.39)\qquad \|\widehat{u}\|^2_{V(Q_1)} \le \frac{1}{\delta^2}\,\omega_n \bigg| \fint_{Q_1} [-u\cdot(\varphi_0)_t + A^{\alpha\beta}D^\beta u\cdot D^\alpha\varphi_0]\,dz \bigg|^2.$$

We put $h=0$ in $Q_{1/2}$. It is the $(A(t),m)$-caloric function and $\|\widehat{u}-h\|^2_{V(Q_{1/2})} = \|\widehat{u}\|^2_{V(Q_{1/2})} \le 2^{n+2m}\|\widehat{u}\|^2_{V(Q_1)}$. Estimate (5.30) in this situation is the trivial one and we obtain (5.31) with $C_\varepsilon = 2^{n+2m}\delta^{-2}$. □

6. Proofs of Theorems 2.1, 2.2 and 2.4

First of all we introduce a proposition we will use to justify local smoothness of the lower order derivatives of a weak solution u to system (1.1).

Proposition 6.1. *Let $Q_r(z^0)\subset Q$ and $u\in W_2^{m,0}(Q_r(z^0))$ be a weak solution to system* (1.1). *Then, for all $k\le m-1$,*

$$(6.1)\qquad \fint_{Q_r(z^0)} |D^k u - (D^k u)_{r,z^0}|^2\,dz \le c\,r^2 \fint_{Q_r(z^0)} |D^{k+1}u|^2\,dz + c\,r^{2(m-k)} \fint_{Q_r(z^0)} |D^m u|^2\,dz + c\mathbb{B}_F\,r^{2(m-1-k+\gamma)}.$$

Here and below

$$\mathbb{B}_F = \sum_{|\alpha|\le m} \|F^\alpha\|^2_{L^{2,n-2+2|\alpha|+2\gamma}(Q)}.$$

Proof. By the same way as in the proof of Lemma 5.1 we derive the inequality

$$(6.2)\qquad \bigg| \int_{B_r(x^0)} [u(x,\tau)-u(x,s)]\cdot\psi(x)\,dx \bigg| \le c \sup_{B_r(x^0)} |D^m\psi| \int_{Q_r(z^0)} |D^m u|\,dz + \sum_{|\alpha|\le m} \sup_{B_r(x^0)} |D^{|\alpha|}\psi| \int_{Q_r(z^0)} |F^\alpha|\,dz,$$

for $\tau, s \in \Lambda_r(t^0)$. Here $\psi \in C_0^\infty(B_r(x^0))$ satisfies the conditions

$$\int_{B_r(x^0)} \psi(x)\,dx = 1, \qquad \sup_{B_r(x^0)} |D^s\psi(x)| \le \frac{c_s}{r^{n+s}}, \tag{6.3}$$

for $s \in \mathbb{N}$.

Now we take into account the identity

$$\int_{B_r(x^0)} D^k(u(x,\tau) - u(x,s)) \cdot \psi(x)\,dx = (-1)^k \int_{B_r(x^0)} (u(x,\tau) - u(x,s)) \cdot D^k\psi(x)\,dx$$

and derive from inequality (6.2) the relation

$$\begin{aligned} &\left| \int_{B_r(x^0)} (D^k u(x,\tau) - D^k u(x,s)) \cdot \psi(x)\,dx \right| \\ &\le c \sup_{B_r(x^0)} |D^{m+k}\psi| \int_{Q_r(z^0)} |D^m u|\,dz + \sum_{|\alpha|\le m} \sup_{B_r(x^0)} |D^{|\alpha|+k}\psi| \int_{Q_r(z^0)} |F^\alpha|\,dz. \end{aligned} \tag{6.4}$$

We denote the "weighted" average of a function $v(\,\cdot\,,t) \in L^1(B_r(x^0))$ as

$$\widetilde{v_{r,x^0}}(t) = \int_{B_r(x^0)} v(x,t)\,\psi(x)\,dx,$$

where the function ψ satisfies conditions (6.3). It follows from (6.4) that

$$\begin{aligned} \left| \widetilde{D^k u_{r,x^0}}(\tau) - \widetilde{D^k u_{r,x^0}}(s) \right| &\le c\, r^{2m-(m+k)} \fint_{Q_r(z^0)} |D^m u|\,dz \\ &\quad + c \sum_{|\alpha|\le m} r^{2m-(|\alpha|+k)} \fint_{Q_r(z^0)} |F^\alpha|\,dz. \end{aligned} \tag{6.5}$$

As $F^\alpha \in L^{2,n+2|\alpha|-2+2\gamma}(Q);\delta)$ then we obtain from (6.5) that

$$\begin{aligned} &\left| \widetilde{D^k u_{r,x^0}}(\tau) - \widetilde{D^k u_{r,x^0}}(s) \right|^2 \\ &\qquad \le c\, r^{2(m-k)} \fint_{Q_r(z^0)} |D^m u|^2\,dz + c\,\mathbb{B}_F\, r^{2(m-1-k+\gamma)}, \end{aligned} \tag{6.6}$$

for $s, t \in \Lambda_r(t^0)$. Now we put

$$I_k(t) = \fint_{B_r(x^0)} |D^k u(x,t) - (D^k u)_{r,z^0}|^2\,dx, \tag{6.7}$$

$$\fint_{\Lambda_r(t^0)} I_k(t)\,dt = \fint_{Q_r(z^0)} |D^k u - (D^k u)_{r,z^0}|^2\,dz \tag{6.8}$$

and estimate $I_k(t)$ in the way:

$$I_k(t) \le 2 \fint_{B_r(x^0)} |D^k u(x,t) - (D^k u)_{r,x^0}(t)|^2\, dz + 2 \left| (D^k u)_{r,x^0}(t) - \fint_{\Lambda_r(t^0)} (D^k u)_{r,x^0}(s)\, ds \right|^2 \le c\, r^2 \fint_{B_r(x^0)} |D^{k+1} u(x,t)|^2\, dx + 2 \left| \fint_{\Lambda_r(t^0)} [(D^k u)_{r,x^0}(s) - (D^k u)_{r,x^0}(t)]\, ds \right|^2. \tag{6.9}$$

We have used the Poincaré inequality for the fixed t to estimate the first term in the right-hand side of the previous inequality.

Further,

$$\left| \fint_{\Lambda_r(t^0)} [(D^k u)_{r,x^0}(s) - (D^k u)_{r,x^0}(t)]\, ds \right|^2 \le 4 \fint_{\Lambda_r(t^0)} |(D^k u)_{r,x^0}(s) - ((\widetilde{D^k u})_{r,x^0}(s)|^2\, ds + 4 \fint_{\Lambda_r(t^0)} |((\widetilde{D^k u})_{r,x^0}(s) - ((\widetilde{D^k u})_{r,x^0}(t)|^2\, ds + 4 \left| (D^k u)_{r,x^0}(t) - ((\widetilde{D^k u})_{r,x^0}(t) \right|^2 =: l_1 + l_2 + l_3. \tag{6.10}$$

By (6.6),

$$l_2 \le c\, r^{2(m-k)} \fint_{Q_r(z^0)} |D^m u|^2\, dz + c\, \mathbb{B}_F r^{2(m-1-k+\gamma)}.$$

We estimate l_1 and l_3 in the same way:

$$\begin{aligned} l_1 &\le 4 \fint_{\Lambda_r(t^0)} \left| \fint_{B_r(x^0)} [D^k u(x,s) - (D^k u)_{r,x^0}(s)] \psi(x)\, dx \right|^2 ds \\ &\le c\, r^2 \fint_{\Lambda_r(t^0)} \fint_{B_r(x^0)} |D^{k+1} u|^2\, dx\, dt + c\, r^2 \fint_{Q_r(z^0)} |D^{k+1} u|^2\, dz, \\ l_3 &= l_3(t) \le c\, r^2 \fint_{B_r(x^0)} |D^{k+1} u(x,t)|^2\, dx. \end{aligned}$$

Taking into account the estimates of all integrals in (6.9), we obtain the relation

$$I_k(t) \le c\, r^2 \fint_{B_r(x^0)} |D^{k+1} u(x,t)|^2\, dx + c\, r^2 \fint_{Q_r(z^0)} |D^{k+1} u|^2\, dz + c\, r^{2(m-k)} \fint_{Q_r(z^0)} |D^m u|^2\, dz + c\, \mathbb{B}_F\, r^{2(m-1-k+\gamma)}.$$

We integrate this relation in $t \in \Lambda_r(t^0)$ and divide the relation by $|\Lambda_r|$. Estimate (6.1) follows. □

PROOF OF THEOREM 2.1. Let $u \in V(Q)$ be a weak solution of the system

$$u_t+(-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(A^{\alpha\beta}(z, D^{m-1}\, u) D^\beta\, u) = \sum_{|\alpha|\le m} (-1)^{|\alpha|}\, D^\alpha\, F_\alpha(z), \tag{6.11}$$

for $z \in Q$. Let $Q_{2R}(z^0) \subset Q$ be fixed and $P^*_{m-1,r}[u]$ be the polynomial which minimize the integral $⨍_{Q_r(z^0)} |u(z) - P(x)|^2\, dz$ among all polynomials of the degree less or equal to $m-1$ for a fixed $r \le 2R$. We put

$$\widehat{u}(z) = u(z) - P^*_{m-1,R}[u] \tag{6.12}$$

and, for all $r \le 2R$, we define

$$\Psi(r, z^0) := ⨍_{Q_r(z^0)} |\widehat{u} - P^*_{m-1,r}[\widehat{u}]|^2\, dz,$$
$$\Phi(r, z^0) = r^{2m} ⨍_{Q_r(z^0)} |D^m u|^2\, dz = r^{2m} ⨍_{Q_r(z^0)} |D^m \widehat{u}|^2\, dz.$$

Let ε be fixed. We put

$$A(t) = ⨍_{B_R(x^0)} A(x, t, (D^{m-1}u)_{R,z^0})\, dx. \tag{6.13}$$

Then the matrix $A(t) \in L^\infty(\Lambda_R(t^0))$, $A(t) \in \{\nu, \mu\}$ for almost all $t \in \Lambda_R(t^0)$ where $0 < \nu \le \mu$ are fixed in the assumption (H1).

For $\widehat{u} \in V(Q_R(z^0))$ there exist an $(A(t), m)$-caloric function $h \in V(Q_{R/2})$, $C_\varepsilon > 0$, and $\phi \in C_0^\infty(Q_R)$, $\sup\limits_{Q_R} |D^m\phi| \le 1$, such that

$$⨍_{Q_{R/2}(z^0)} \sum_{0\le|\alpha|\le m} R^{2|\alpha|} |D^\alpha h|^2\, dz \le 2^{n+2m+2} M_R, \tag{6.14}$$

$$M_R := ⨍_{Q_R(z^0)} \sum_{0\le|\alpha|\le m} R^{2|\alpha|} |D^\alpha \widehat{u}|^2\, dz, \tag{6.15}$$

$$⨍_{Q_{R/2}(z^0)} \sum_{0\le|\alpha|\le m-1} R^{2|\alpha|} |D^\alpha \widehat{u} - D^\alpha h|^2\, dz \le \varepsilon\, M_R + C_\varepsilon\, R^{2m} \mathcal{L}^2(\widehat{u}, \phi), \tag{6.16}$$

where

$$\mathcal{L}(\widehat{u}, \phi) = \mathcal{L}(u, \phi) = \left| ⨍_{Q_R(z^0)} (-u\, \cdot \phi_t + A(t) D^m u \cdot D^m \phi)\, dz \right| \tag{6.17}$$

and the function $\widehat{u}$ is defined in (6.12).

By the interpolation inequality (3.1)

$$\begin{aligned} R^{2|\alpha|} ⨍_{Q_R(z^0)} |D^\alpha \widehat{u}|^2\, dz &\le c\, R^{2m} ⨍_{Q_R(z^0)} |D^m u|^2\, dz + c ⨍_{Q_R(z^0)} |\widehat{u}|^2\, dz \\ &\le c\, \Phi(2r, z^0) + c\, \mathbb{B}_F\, R^{2(m-1+\gamma)}, \end{aligned} \tag{6.18}$$

where $1 \le |\alpha| \le m-1$. The last inequality is valid due to Poincaré inequality (3.14) for the integral $⨍_{Q_R(z^0)} |\widehat{u}|^2\, dz$. Here the constants c depends on ν, μ, m, $|\alpha|$ and n.

Using (6.18), we estimate the sum M_R:

$$M_R \le c\,\Phi(2R, z^0) + c\mathbb{B}_F\, R^{2(m-1+\gamma)}. \tag{6.19}$$

Thus estimates (6.14)–(6.16) and (6.19) imply the inequalities

$$\fint_{Q_{R/2}(z^0)} \sum_{0\le|\alpha|\le m} R^{2|\alpha|}|D^\alpha h|^2\,dz \le c\Phi(2R, z^0) + c\mathbb{B}_F\, R^{2(m-1+\gamma)}, \tag{6.20}$$

$$\begin{aligned}\fint_{Q_{R/2}(z^0)} \sum_{0\le|\alpha|m-1} & R^{2|\alpha|}|D^\alpha \widehat{u} - D^\alpha h|^2\,dz \\ &\le \varepsilon\,\Phi(2R, z^0) + c\mathbb{B}_F\, R^{2(m-1+\gamma)} + C_\varepsilon\, R^{2m}\mathcal{L}^2(\widehat{u}, \phi).\end{aligned} \tag{6.21}$$

The next step of the proof consists of deriving the inequality (6.30)? for the function Φ with $r = 2R$. As z^0 is fixed up to the formula (6.43) we write Q_ρ, $\Phi(\rho)$, $\Psi(\rho)$ instead of $Q_\rho(z^0)$, $\Phi(\rho, z^0)$, $\Psi(\rho, z^0)$.

By (3.4) we have the inequalities

$$\Phi(\rho/2) \le c\,\Psi(\rho) + c\,\mathbb{B}_F \rho^{2(m-1+\gamma)}, \quad \rho \le R/2. \tag{6.22}$$

As the polynomial $P^*_{m-1,\rho}[\widehat{u} - h] + P^*_{m-1,\rho}[h] \in \mathbb{P}_{m-1}$ and due to the minimality of $P^*_{m-1,\rho}[\widehat{u}]$ for the integral $\Psi(\rho)$ we get

$$\Psi(\rho) \le \fint_{Q_\rho} |(\widehat{u} - \{P^*_{m-1,\rho}[\widehat{u} - h] + P^*_{m-1,\rho}[h]\}|^2\,dz =: I_\rho.$$

Further,

$$I_\rho \le 2\fint_{Q_\rho} |(\widehat{u} - h) - P^*_{m-1,\rho}[\widehat{u} - h]|^2\,dz + 2\fint_{Q\rho} |h - P^*_{m-1,\rho}[h]|^2\,dz. \tag{6.23}$$

Using minimality of the polynomial $P^*_{m-1,\rho}[\widehat{u} - h]$ we obtain that

$$\fint_{Q_\rho} |(\widehat{u} - h) - P^*_{m-1,\rho}[\widehat{u} - h]|^2 \le \fint_{Q_\rho} |(\widehat{u} - h)|^2 dz$$

and, by Campanato inequality (4.11), also

$$\begin{aligned}\fint_{Q\rho} |h - P^*_{m-1,\rho}[h]|^2\,dz &\le c\left(\frac{\rho}{R}\right)^{2m} \fint_{Q_{R/2}} |h - P^*_{m-1,R/2}[h]|^2\,dz \\ &\le c\left(\frac{\rho}{R}\right)^{2m} \fint_{Q_{R/2}} |h|^2\,dz.\end{aligned}$$

The last inequality follows once more from the minimality of the polynomial $P^*_{m-1,R}[h]$.

From the said it follows that

$$\Phi(\rho/2) \le c\left(\frac{\rho}{R}\right)^{2m} \fint_{Q_{R/2}} |h|^2\,dz + c\left(\frac{R}{\rho}\right)^{n+2m} \fint_{Q_{R/2}} |\widehat{u} - h|^2\,dz. \tag{6.24}$$

Using the inequalities (6.20) and (6.21) we obtain

$$\Phi(\rho/2) \le c\left(\frac{R}{\rho}\right)^{n+2m}\{\varepsilon\,\Phi(2R) + C_\varepsilon\, R^{2m}\mathcal{L}_R^2(u,\phi)\} + c\left(\frac{\rho}{R}\right)^{2m}\Phi(2R) + c\,C_\varepsilon\left(\frac{R}{\rho}\right)^{n+2m}\mathbb{B}_F\rho^{2(m-1+\gamma)}. \tag{6.25}$$

The constants c in this inequality depend on ν, μ, m and n.

Now we estimate the expression $\mathcal{L}^2(\widehat{u},\phi)$:

$$\mathcal{L}^2(\widehat{u},\phi) = \mathcal{L}^2(u,\phi) \le \fint_{Q_R}|A(x,t,D^{m-1}u) - A(t)|^2\,dz \fint_{Q_R}|D^m u|^2\,dz + \sum_{|\alpha|\le m}\sup_{B_R}|D^\alpha\varphi(x)|^2\fint_{Q_R}|F_\alpha|^2\,dz, \tag{6.26}$$

where the matrix $A(t)$ is defined in (6.13). Further,

$$|A(x,t,D^{m-1}u) - A(t)| \le |A(x,t,D^{m-1}u) - A(x,t,(D^{m-1}u)_R)| + |A(x,t,(D^{m-1}u)_R - A(t)|.$$

Taking into account that $\sup\limits_{B_R}|D^\alpha\varphi| \le c\,R^{m-|\alpha|}$ and using the assumption (H2) we obtain from (6.26) the inequality

$$\mathcal{L}^2(\widehat{u},\phi) \le \left\{2\omega_0\fint_{Q_R}(\omega(|D^{m-1}u - (D^{m-1}u)_R|^2)\,dz + 2\fint_{Q_R}|A(x,t,(D^{m-1}u)_R) - A(t)|^2\,dz\right\}\fint_{Q_R}|D^m u|^2\,dz + c\mathbb{B}_F R^{-2+2\gamma}.$$

Here $\omega_0 = \sup\limits_{s\ge 0}\omega(s)$. Further we apply the assumption (H3), use the concavity of the function $\omega(\,\cdot\,)$, and derive that

$$R^{2m}\mathcal{L}^2(\widehat{u},\phi) \le 2\left[\omega_0\,\omega\left(\fint_{Q_R}|D^{m-1}(u) - (D^{m-1}u)_R)|^2\,dz\right)dz + q^2(R)\right]\Phi(R) + c\,\mathbb{B}_F(2R)^{2(\gamma+m-1)}. \tag{6.27}$$

By inequality (6.1) we obtain that

$$\fint_{Q_R}|D^{m-1}u - (D^{m-1}u)_R)|^2\,dz \le c\,R^2\fint_{Q_R}|D^m u|^2\,dz + \,c\,\mathbb{B}_F(2R)^{2\gamma}.$$

Thus,

$$\omega\left(\fint_{Q_R}|D^{m-1}u - (D^{m-1}u)_R|^2\,dz\right) \le \omega(\widehat{c}[(2R)^{-2(m-1)}\Phi(2R) + \mathbb{B}_F(2R)^{2\gamma}]) \tag{6.28}$$

where $\widehat{c}$ depends on ν, μ, n and m.

Now it follows from (6.27) and (6.28) that

$$
\begin{aligned}
(6.29)\qquad R^{2m}\mathcal{L}^2(u,\phi) \le c\,\{&\omega_0\omega(\widehat{c}(2R)^{-2(m-1)}\Phi(2R) + \widehat{c}\mathbb{B}_F(2R)^{2\gamma}) \\
&+ q^2(2R)\}\Phi(2R) + c\,\mathbb{B}_F(2R)^{2(\gamma+m-1)}.
\end{aligned}
$$

Then we obtain from (6.25) and (6.29) for $r = 2R$ the inequality

$$
\begin{aligned}
(6.30)\qquad \Phi\left(\frac{\rho}{2}\right) \le c\,C_\varepsilon\left(\frac{r}{\rho}\right)^{n+2m}&\mathbb{B}_F\, r^{2(m-1+\gamma)} + c\left\{\left(\frac{\rho}{r}\right)^{2m} + \varepsilon\left(\frac{r}{\rho}\right)^{n+2m}\right. \\
&\left. + C_\varepsilon\left(\frac{r}{\rho}\right)^{n+2m}\left[\omega(\widehat{c}\,[r^{-2(m-1)}\Phi(r) + \mathbb{B}_F\, r^{2\gamma}]) + q^2(r)\right]\right\}\Phi(r).
\end{aligned}
$$

Now let

$$
(6.31)\qquad Z(r) = r^{-2(m-1)}\Phi(r).
$$

We multiply (6.31) by $\rho^{-2(m-1)}$ and obtain

$$
\begin{aligned}
(6.32)\qquad Z\left(\frac{\rho}{2}\right) \le c&\left\{\left(\frac{\rho}{r}\right)^{2} + \varepsilon\left(\frac{r}{\rho}\right)^{n+4m-2}\right. \\
&\left. + C_\varepsilon\left(\frac{r}{\rho}\right)^{n+4m-2}\left[\omega(\widehat{c}\,[Z(r) + \mathbb{B}_F r^{2\gamma}]) + q^2(r)\right]\right\}Z(r) \\
&+ c\,C_\varepsilon\left(\frac{r}{\rho}\right)^{n+4m-2}\mathbb{B}_F r^{2\gamma}.
\end{aligned}
$$

Further we put $\rho/2 = \tau r$ in (6.32) (the parameter $\tau \le 1/8$ we'll define later):

$$
\begin{aligned}
(6.33)\qquad Z(\tau\, r) \le c_0\{&\tau^2 + \tau^{-(n+4m-2)}\varepsilon \\
&+ C_\varepsilon\tau^{-(n+4m-2)}[\omega(\widehat{c}\,(Z(r) + \mathbb{B}_F r^{2\gamma})) + q^2(r)]\}Z(r) + \mathbb{K}(\varepsilon,\tau)\mathbb{B}_F r^{2\gamma}.
\end{aligned}
$$

The constants c_0 and $\widehat{c}$ in (6.33) depend on ν, μ, n and m only, $\mathbb{K}(\varepsilon,\tau) := c\,C_\varepsilon\tau^{-(n+4m-2)}$).

Now we want to define the parameters in (6.33). First, we fix $\tau \le 1/8$ and $\beta = (1+\gamma)/2 \in (\gamma, 1)$ (here $\gamma \in (0,1)$ is the parameter from the assumption (H4) on F_α). We sharp the choice of the parameter τ:

$$
(6.34)\qquad c_0\tau^2 \le \frac{\tau^{2\beta}}{8}.
$$

Then we fix $\varepsilon > 0$ from the condition

$$
(6.35)\qquad c_0\tau^{-(n+4m-2)}\varepsilon \le \frac{\tau^{2\beta}}{8}.
$$

Note that the constant $\mathbb{K} = \mathbb{K}(\varepsilon,\tau) \ge 1$ is fixed now by the data of the problem.

The next step is to fix $\theta \in (0,1)$ such that

$$
(6.36)\qquad c_0\tau^{-(n+4m-2)}C_\varepsilon\,\omega(2\widehat{c}\,\theta) \le \frac{\tau^{2\beta}}{8}.
$$

At last, we choose number $r_0 \le 1$ from the conditions

$$c_0 C_\varepsilon\, \tau^{-(n+4m-2)} q^2(r_0) \le \frac{\tau^{2\beta}}{8}, \qquad (\mathbb{K} + \widehat{c})\mathbb{B}_F (r_0)^{2\gamma} \le \frac{\theta}{2}. \tag{6.37}$$

The constant $\widehat{c}$ in the last inequality such as in (6.33).

Now we assume that in the fixed point z^0

$$Z(r) = Z(r, z^0) < \theta \tag{6.38}$$

for some $r \le r_0$ and θ fixed in (6.36). Then (6.36) follows from (6.38) and the last inequality (6.37). In a result, the inequality

$$Z(\tau r) \le \frac{\tau^{2\beta}}{2}\, Z(r) + \mathbb{K}\,\mathbb{B}_F r^{2\gamma}, \quad \beta > \gamma, \tag{6.39}$$

is valid. From (6.37)–(6.39) we obtain that $Z(\tau\, r) < \theta$ and we can repeat all considerations with $\tau\, r$ but not r. It allows us to consider the sequence $r_j = \tau^j r$, $j \in \mathbb{N}$, and derive that

$$Z(\tau^j\, r) \le \tau^{2\beta} Z(\tau^{j-1} r) + \mathbb{K}\,\mathbb{B}_F (\tau^{j-1} r)^{2\gamma}, \quad \beta > \gamma,\ j \in \mathbb{N}. \tag{6.40}$$

The iterative process supplies us the inequality

$$Z(\tau^j\, r) \le c\, \tau^{2\gamma\, j} [Z(r) + \mathbb{K}\,\mathbb{B}_F\, r^{2\gamma}]. \tag{6.41}$$

It follows from (6.41) that

$$Z(\rho) \le c \left(\frac{\rho}{r}\right)^{2\gamma} Z(r) + c\mathbb{B}_F \rho^{2\gamma}, \quad \text{for all } \rho \le r. \tag{6.42}$$

We recall that all our considerations were justified under assumption (6.38) in the fixed point z^0 and for the fixed $r = r(z^0) \le r_0$. Thus,

$$\frac{1}{\rho^{n+2(m-1)+2\gamma}} \int_{Q_\rho(z^0)} |D^m u|^2\, dz \le c(r^{-1}) \|D^m u\|^2_{2,Q} + c\,\mathbb{B}_F. \tag{6.43}$$

It is easy to see that inequality (6.38) is also valid (for the fixed r) in some cylinder $Q_{\rho_0}(z^0)$, i.e.

$$Z(r, \xi^0) < \theta \quad \text{for all } \xi^0 \in Q_{\rho_0}(z^0). \tag{6.44}$$

We can repeat all considerations for a point $\xi^0 \in Q_{\rho_0}(z^0)$ instead of z^0 and derive estimate (6.43) for any $\xi^0 \in Q_{\rho_o(z^0)}$:

$$\rho^{-(n+2(m-1)+2\gamma)} \int_{Q_\rho(\xi^0)} |D^m u|^2\, dz \le c(r^{-1}) \|D^m u\|^2_{2,Q} + c\,\mathbb{B}_F. \tag{6.45}$$

Taking supremum in the left-hand side of (6.45) in $\xi^0 \in Q_{\rho_0}(z^0)$ we obtain

$$\|D^m u\|^2_{L^{2,n+2(m-1)+2\gamma}(Q_{\rho_0}(z^0))} \le c(r^{-1}) \|D^m u\|^2_{2,Q} + c\,\mathbb{B}_F =: H. \tag{6.46}$$

Now from estimate (6.1) with $k = m-1$ we obtain that

$$⨍_{Q_\rho(\xi^0)} |D^{m-1} u - (D^{m-1} u)_{\rho,\xi^0}|^2\, dz \le c\, H\, \rho^{2\gamma}, \quad \text{for all } \xi^0 \in Q_{\rho_0}(z^0). \tag{6.47}$$

It means that in $\mathcal{L}^{2,n+2m+2\gamma}(Q_{\rho_0}(z^0))$ we estimated locally the seminorm of $D^{m-1}u$. Due to the isomorphism of $\mathcal{L}^{2,n+2m+2\gamma}(Q_{\rho_0}(z^0))$ to the Hölder space $C^\gamma(Q_{\rho_0}(z^0))$, $\gamma \in (0,1)$, we have the estimate

$$\|D^{m-1}u\|^2_{C^\gamma(\overline{Q_{\rho_0}(z^0)})} \le c\,H. \tag{6.48}$$

Let now $k = m-2$ in (6.1) then it follows from (6.46) and (6.48) that

$$\begin{aligned}&\fint_{Q_\rho(\xi^0)} |D^{m-2}u - (D^{m-2}u)_\rho|^2\,dz \\ &\qquad \le c\,\rho^2 \fint_{Q_\rho(\xi^0)} |D^{m-1}u|^2\,dz + c\rho^4 \fint_{Q_\rho(\xi^0)} |D^m u|^2\,dz \le c(\rho^2 + \rho^{2\gamma}) \le c\,\rho^{2\gamma},\end{aligned}$$

for all $Q_\rho(\xi^0) \subset Q_{\rho_0}(z^0)$. Thus, $D^{m-2}u \in C^\gamma(Q_{\rho_0}(z^0))$. Using inequality (6.1) one can prove that all derivatives $D^j u$, $j \le m-1$, are Hölder continuous in $Q_{\rho_0}(z^0)$ in the parabolic metric δ_m.

As the assumption (6.38) is equivalent to the condition

$$\frac{1}{r^{n+2(m-1)}} \int_{Q_r(z^0)} |D^m u|^2\,dz < \theta \tag{6.49}$$

for some $r \le r_0$, we have proved that condition (6.49) supplies estimate (6.48).□

Proof of Theorem 2.2. We define the set

$$\Sigma = \biggl\{\widehat{z} \in Q : \liminf_{r\to 0} \frac{1}{r^{n+2(m-1)}} \int_{Q_r(\widehat{z})} |D^m u|^2\,dz > 0\biggr\}. \tag{6.50}$$

Then, for $z^0 \in Q \setminus \Sigma$, the relation

$$\liminf_{r\to 0} \frac{1}{r^{n+2(m-1)}} \int_{Q_r(z^0)} |D^m u|^2\,dz = 0$$

holds. It means that for such points z^0 condition (6.49) with some $r \le r_0$ is valid and the assertion of Theorem 2.1 holds. We will say that the set $Q_0 = Q \setminus \Sigma$ is the set of regular points of the solution u. It is easy to see that Σ is the closed set and $D^\alpha u$, $|\alpha| \le m-1$, are the Hölder continuous functions on the open set Q_0. The relation $\mathcal{H}_{n+2(m-1)}(\Sigma;\delta_m) = 0$ follows from the well known results and the definition (6.50) (see, for example, [16] or [17, Section 9.25]). □

Proof of Theorem 2.4. In the linear case we can repeat the proof of Theorem 2.1 and note that the estimates of the function $\mathcal{L}^2(\widehat{u},\phi)$ does not depend on the function $\omega(\,\cdot\,)$. In this case we have not restriction (6.38) and obtain the following inequality for Z (compare with (6.33)):

$$Z(\tau\,r, z^0) \le c_0\{\tau^2 + \tau^{-(n+4m-2)}\varepsilon + \tau^{-(n+4m-2)}C_\varepsilon\,q^2(r)\}Z(r,z^0) + \mathbb{K}\,r^{2\gamma}\mathbb{B}_F,$$

for all $z^0 \in Q$. It allows us to fix the same parameters τ, ε and r_0 for all points $z^0 \in Q$, $Q_{r_0}(z^0) \subset\subset Q$. Estimate (6.43) is valid for all points $z^0 \in Q$. □

REFERENCES

[1] A.A. ARKHIPOVA, *Regularity of weak solutions to the model Venttsel problem for linear parabolic systems with non smooth in time principal matrix. $A(t)$-caloric approximation method*, Manuscripta Math. **151** (2016), no. 3, 519–548.

[2] A.A. ARKHIPOVA AND J. STARÁ, *A priori estimates for quasilinear parabolic systems with quadratic nonlinearities in the gradient*, Comment. Math Univ. Carolin. **51**, (2010), 2–16.

[3] A.A. ARKHIPOVA AND J. STARÁ, *Boundary partial regularity for solutions of quasilinear parabolic systems with non smooth in time principal matrix*, Nonlinear Anal. **120** (2015), 236–261.

[4] A.A. ARKHIPOVA AND J. STARÁ, *Regularity of weak solutions to linear and quasilinear parabolic systems of non divergence type with non smooth in time principal matrix: $A(t)$-caloric method*, Forum Math. (2016), DOI: 101515/forum-2015-0222.

[5] A.A. ARKHIPOVA, J. STARÁ AND O. JOHN, *Partial regularity for solutions of quasilinear parabolic systems with nonsmooth in time principal matrix*, Nonlinear Anal. **95** (2014), 421–435.

[6] Bögelein V.: *Partial regularity and singular sets of solutions of higher order parabolic systems.* Annali Mat. Pura Appl.(4), **188**, (1), (2009), 61-122.

[7] V. BÖGELEIN, *Partial regularity and singular sets of solutions of higher order parabolic systems*, Ph.D. Theses Universität Erlangen–Nürnberg, 2007.

[8] V. BÖGELEIN, F. DUZAAR AND G. MINGIONE, *The boundary regularity of non-linear parabolic systems* I, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 201–255.

[9] S. CAMPANATO, *Equazioni paraboliche del secondo ordine e spazi* $L^{2,\theta}(\Omega,\delta)$ Ann. Mat. Pura Appl. (4) **137** (1966), 55–102.

[10] E. DE GIORGI, *Frontiere orientate di misura minima*, Ann. Scuola Norm. Sup. Pisa **8**, 285–309.

[11] H. DONG AND D. KIM, *Parabolic and elliptic systems with VMO coefficients*, Methods Appl. Math. **16** (2009), 365–388.

[12] H. DONG AND D. KIM, *On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients*,h Arch. Ration. Mech. Anal. **199** (2011), 889–941.

[13] H. DONG AND D. KIM, *Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth*, Comm. Partial Differential Equations **36** (2011), 1750–1777.

[14] F. DUZAAR AND J.F. GROTOWSKI, *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation*, Manuscripta Math. **103**, (2000), 267–298.

[15] F. DUZAAR AND G. MINGIONE, *Second order parabolic systems, optimal regularity and singular sets of solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 6, 705–751.

[16] M. GIAQUINTA AND E. GIUSTI, *Partial regularity for solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. **97** (1973), 253–261.

[17] M. GIAQUINTA AND L. MARTINAZZI, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Appunti. Sc. Norm. Super. Pisa, 2012, ISBN: 978-88-7642-442-7.

[18] N.V. KRYLOV, *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations **32** (2007), 453–475.

[19] N.V. KRYLOV *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Graduate Studies in Mathematics, Vol. 96, Amer. Math. Soc., Providence, 2008.

[20] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*. Transl. Math. Monogr. **23** (1967).

[21] J. Simon, *Compact sets in* $L^p(0,T;B)$, Ann.i Mat. Pura Appl. (4) **146** (1987), 65–96.

[22] J. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Lectures in Math., ETH Zürich, Birkhäuser, Basel, 1996.

Manuscript received May 4, 2017

accepted November 1, 2017

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