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CONCENTRATION–COMPACTNESS FOR SINGULAR NONLOCAL SCHRÖDINGER EQUATIONS WITH OSCILLATORY NONLINEARITIES

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Abstract. The paper is dedicated to the theory of concentration–compactness principles for inhomogeneous fractional Sobolev spaces. This subject for the local case has been studied since the publication of the celebrated works due to P.-L. Lions, which laid the broad foundations of the method and outlined a wide scope of its applications. Our study is based on the analysis of the profile decomposition for the weak convergence following the approach of dislocation spaces, introduced by K. Tintarev and K.-H. Fieseler. As an application, we obtain existence of nontrivial and nonnegative solutions and ground states for fractional Schrödinger equations for a wide class of possible singular potentials, not necessarily bounded away from zero. We consider possible oscillatory nonlinearities for both cases, subcritical and critical which are superlinear at the origin, without the classical Ambrosetti and Rabinowitz growth condition. In some of our results we prove existence of solutions by means of compactness of Palais–Smale sequences of the associated functional at the mountain pass level. To this end we study and provide the behavior of the weak profile decomposition convergence under the related functionals. Moreover, we use a Pohozaev type identity in our argument to compare the minimax levels of the energy functional with the ones of the associated limit problem. Motivated by this fact, in our work we also prove that this kind of identities hold for a larger class of potentials and nonlinearities for the fractional framework.

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1. Introduction

The main goal of the present work is to analyze concentration–compactness principles for inhomogeneous fractional Sobolev spaces. As an application we address questions on the existence of solutions for the following nonlocal Schrödinger equation:

$$(\mathcal{P}_s) \qquad (-\Delta)^s u + a(x)u = f(x,u) \quad \text{in } \mathbb{R}^N,\ N \geq 3,$$

where $0 < s < 1$ and $(-\Delta)^s$ is the fractional Laplacian (see [40, 12] for more details).

During the past years there has been a considerable amount of research on nonlinear elliptic problems involving the fractional Laplacian operator, motivated from the fact that this class of problems arise naturally in several branches of mathematical physics. For instance, solutions of $(\mathcal{P}_s)$ can be seen as stationary states (corresponding to solitary waves) of nonlinear Schrödinger equations of the form

$$i\phi_t - (-\Delta)^s\phi + a(x)\phi + f(x,\phi) = 0 \quad \text{in } \mathbb{R}^N.$$

For more details we refer to [2].

This paper is motivated by recent advances in the study of existence of solutions for nonlinear and nonlocal Schrödinger field equations. In [37] Secchi investigated the existence of ground state solutions for fractional Schrödinger equations by using a minimization argument on the Nehari manifold. He proved existence results under suitable assumptions on the behavior of the potential $a(x)$ and superlinear growth conditions on the nonlinearity. See also [22], where Feng proved the existence of ground state solutions of $(\mathcal{P}_s)$, in the particular case that $f(x,t) = |t|^{p-2}t$, $2 < p < 2(N+2s)/N$, $N \geq 2$, by using the Lions concentration–compactness principle (see [29]). Lehrer et al. [25] studied existence of solutions by means of projection over an appropriate Pohozaev manifold, assuming that $f(x,t) = a(x)f_0(t)$, where $f_0(t)$ is asymptotically linear, that is, $\lim\limits_{t\to\infty} f_0(t)/t = 1$ and $\lim\limits_{|x|\to\infty} a(x) = a_\infty > 0$. For the local case ($s = 1$), de Marchi [9] studied existence of nontrivial solutions for $(\mathcal{P}_s)$ assuming that $a(x)$ and $f(x,t)$ are asymptotically $\mathbb{Z}^N$-periodic, combining variational methods and the concentration–compactness principle. Also, in [9] existence of ground states, without assuming that $t \mapsto f(x,t)t^{-1}$ is an increasing function, was established. By using a similar approach, Zhang et al. [51] studied existence of ground states and infinitely many geometrically distinct solutions of equation $(\mathcal{P}_s)$, based on the method of Nehari manifold and Lusternik–Schnirelmann category theory. For recent works on nonlinear Schrödinger equations where the Ambrosetti–Rabinowitz condition is not required, we refer to [9, 25, 51]. See also the recent work due to Ambrosetti et al. [1], where existence of ground states with potentials $a(x)$ and $K(x)$ vanishing at infinity, where $f(x,t) = K(x)t^{p-1}$ and $2 < p < 2N/(N-2)$, was studied.

Let us mention [26, 39, 38], and references given there, for problems involving potentials bounded away from zero and *critical Sobolev exponent*, precisely, when $f(x,t) = g(x,t) + |t|^{2_s^*-2}t$, $2_s^* = 2N/(N-2s)$, where $g(x,t)$ has subcritical growth. In these works, the presence of perturbation $g(x,t)$ of the critical power $|t|^{2_s^*-2}t$, was crucial. Moreover, the following condition on the potential was assumed: $0 < \inf_{x\in\mathbb{R}^N} a(x) < \liminf_{|x|\to\infty} a(x)$ which was introduced by Rabinowitz in [35] to study the local case of equation $(\mathcal{P}_s)$ (see also the critical case in [32]). We cite [7, 11, 42] for works on local Schrödinger equations with nonlinearities of the pure critical power type (without subcritical perturbation term) and inverse square type potentials. For the fractional case, see [14], where existence and qualitative properties of positive solutions were studied.

Motivated by the above works, we study existence of nontrival solutions for $(\mathcal{P}_s)$ in several cases, which were not considered in the aforementioned papers. Our potential $a(x)$ may change sign, can have singular points of blow-up and even vanish, and the nonlinearity can be considered with subcritical or critical oscillatory growth. We prove some of our existence results by means of compactness of Palais–Smale sequences (PS sequences, for short) of the associated functional at the mountain pass level.

In the subcritical case we assume a condition on $a(x)$ which ensures the continuous embedding of the associated space of functions similar to [41]. Nevertheless, differently from [41], we do not impose assumption on $a(x)$ to guarantee the compactness of the Sobolev embedding. To compensate it, we require that the limit of $a(x)$, as $|x|$ goes to infinity, exists and is positive, or alternatively, that $a(x)$ is $\mathbb{Z}^N$-periodic. Moreover, by considering assumptions similar to those in [10], the potential does not need to be bounded from below by a constant. We also take into account the case where the nonlinearity has oscillatory behavior and does not satisfy the typical assumption of Ambrosetti–Rabinowitz. Similarly to [9], the nonlinearity $f(x,t)$ is supposed to have a periodic asymptote $f_{\mathcal{P}}(x,t)$, which allows us to "transfer" the usual assumptions to it. Also let us mention that we complement and improve some results from [9], since we consider the fractional case and we do not require the monotonicity of $t \mapsto f_{\mathcal{P}}(x,t)t^{-1}$.

In the critical case, inspired by some ideas contained in [7], we treat in this work a class of potentials somehow different, since we consider a general class that includes, as a particular case, the inverse fractional square potential $a(x) = -\lambda|x|^{-2s}$, where $0 < \lambda < \Lambda_{N,s}$ and $\Lambda_{N,s}$ is the sharp constant of the Hardy–Sobolev inequality

$$(1.1)\qquad \Lambda_{N,s}\int_{\mathbb{R}^N} |x|^{-2s}u^2\,dx \le \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2\,d\xi, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N).$$

Further details on (1.1) can be found in [50]. Here we consider self-similar nonlinearities which generalize the idea of oscillations about the critical power $|t|^{2_s^*-2}t$, turning the approach by variational methods more involved. This class of functions was introduced in [16] and for the local case in [47]–[49].

We are able to avoid the monotonicity of $t \mapsto f_{\mathcal{P}}(x,t)t^{-1}$ by comparing the minimax level of the associated energy functional of $(\mathcal{P}_s)$ with the one of the associated limit problem. To this end, we use a Pohozaev-type identity and an appropriate concentration–compactness principle. The proof of this identity is based in a truncation argument, and for that we use the so called s-harmonic extension introduced by Caffarelli and Silvestre [4], and in remarks contained in [17] and [24], which allow us to "transform" the nonlocal problem $(\mathcal{P}_s)$ into a local one. Our method of proof is more general than the usual one, in the sense that in our argument we do not have to study behavior of solutions in the whole space $\mathbb{R}^N$; and we also can consider singular potentials (see Proposition 6.11).

It is worth to mention that the main difficulty to approach the problem $(\mathcal{P}_s)$ using variational methods lies in the lack of compactness, which, roughly speaking, originates from the invariance of $\mathbb{R}^N$ with respect to translation and dilation and, analytically, appears because of noncompactness of the Sobolev embedding. We are able to overcome this difficulty by referring to a concentration–compactness principle by means of profile decomposition for weak convergence in inhomogeneous fractional Sobolev spaces, which can be considered as extension of the Banach–Alaoglu theorem (see Theorem 2.3). This kind of results were considered in various settings, see for instance [23], [34], [44]. It describes how the convergence of a bounded sequence fails in the considered space. Our approach in this matter was motivated by [8] and based on the abstract version of profile decomposition in Hilbert spaces due to Tintarev and Fieseler [49]. It seems for us that this approach is more appropriate to study existence of nontrivial solutions for problems like $(\mathcal{P}_s)$, in our setting, rather than the standard ones using the Lions concentration–compactness principle (see [21, Lemma 2.2]).

Another important goal here is the study of existence of ground states for $(\mathcal{P}_s)$, i.e. nontrivial solutions with least possible energy. We consider three cases: First when $(\mathcal{P}_s)$ is invariant under the action of translations in $\mathbb{Z}^N$ (subcritical growth), second when $(\mathcal{P}_s)$ is invariant under dilations (critical growth), and third when the monotonicity of $t \mapsto f(x,t)t^{-1}$ is considered.

This paper is organized as follows. In Section 2, we provide a description of the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$. In Section 3, we give some applications of the profile decomposition to study existence of mountain pass type solutions of $(\mathcal{P}_s)$ for autonomous and nonautonomous cases. Next, Section 4 recalls some basic results on fractional Sobolev spaces. In Section 5, we prove the abstract result stated in Section 2. Section 6 is devoted to provide a suitable variational setting to prove our main results. More precisely, we describe the limit under the profile decomposition of the PS sequences at the mountain pass level of the Lagrangian of $(\mathcal{P}_s)$. We also prove that solutions for $(\mathcal{P}_s)$ in the autonomous case satisfy a Pohozaev-type identity. Sections 7–10 are dedicated to the proof of our main results concerning existence of mountain pass solutions of equation $(\mathcal{P}_s)$.

2. Profile decomposition for weak convergence in fractional Sobolev spaces

Assume $0 < s < N/2$ and let $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be the homogeneous fractional Sobolev space, which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $[u]_s^2 := \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2\,d\xi$. It is well known that $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded in the Lebesgue space $L^{2_s^*}(\mathbb{R}^N)$. The following result is a refined version of the concentration–compactness method introduced by M. Struwe [44] for Palais–Smale sequences for some semilinear elliptic functionals. It was extended to general bounded sequences in $\dot{H}^{1,p}(\mathbb{R}^N)$ by Solimini [43], and was studied in [16], [23], [34] in the fractional framework.

THEOREM 2.1 ([16, Theorem 2.1]). *Let* $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ *be a bounded sequence,* $\gamma > 1$ *and* $0 < s < \min\{1, N/2\}$. *Then there exist* $\mathbb{N}_* \subset \mathbb{N}$, *disjoint sets* (*if non-empty*) $\mathbb{N}_0, \mathbb{N}_-, \mathbb{N}_+ \subset \mathbb{N}$, *with* $\mathbb{N}_* = \mathbb{N}_0 \cup \mathbb{N}_+ \cup \mathbb{N}_-$, *and sequences* $(w^{(n)})_{n\in\mathbb{N}_*} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}^N$, $(j_k^{(n)})_{k\in\mathbb{N}} \subset \mathbb{Z}$, $n \in \mathbb{N}_*$, *such that, up to a subsequence of* (u_k),

$$\gamma^{-(N-2s)j_k^{(n)}/2} u_k\big(\gamma^{-j_k^{(n)}}\cdot + y_k^{(n)}\big) \rightharpoonup w^{(n)} \quad \text{as } k \to \infty, \ \text{in } \mathcal{D}^{s,2}(\mathbb{R}^N),$$

$$\big|j_k^{(n)} - j_k^{(m)}\big| + \big|\gamma^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})\big| \to \infty, \qquad \text{as } k \to \infty, \ \text{for } m \neq n,$$

$$\sum_{n\in\mathbb{N}_*} \big[w^{(n)}\big]_s^2 \le \limsup_k [u_k]_s^2,$$

$$u_k - \sum_{n\in\mathbb{N}_*} \gamma^{(N-2s)j_k^{(n)}/2} w^{(n)}\big(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)})\big) \to 0, \quad \text{as } k \to \infty, \ \text{in } L^{2_s^*}(\mathbb{R}^N), \tag{2.1}$$

and the series in (2.1) *converges uniformly in* k. *Furthermore,* $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$, $j_k^{(n)} = 0$ *whenever* $n \in \mathbb{N}_0$, $j_k^{(n)} \to -\infty$ *whenever* $n \in \mathbb{N}_-$, *and* $j_k^{(n)} \to +\infty$ *whenever* $n \in \mathbb{N}_+$.

Theorem 2.1 can be used to prove the fractional version of Lions concentration–compactness principle due to G. Palatucci and A. Pisante [34, Theorem 5]. Indeed, Theorem 2.1 improves [34, Theorem 5] for the case $\Omega = \mathbb{R}^N$, since the sums of Dirac masses that appear in this result come from the profiles given in (2.1). The new notion of criticality introduced in [16] together with the concentration–compactness given in Theorem 2.1 can lead to a new way to approach nonlocal elliptic problems involving critical growth. For instance, it is usual to apply [34, Theorem 5] to study $(\mathcal{P}_s)$ considering nonlinearities of the type $f(x,t) = K(x)|t|^{2^*_s-2}t$. With the aid of Theorem 2.1, it is possible to consider more general self-similar critical nonlinearities (for more details see [16, Section 3.1]).

Remark 2.2. We can consider the closed subspace consisting of radial functions $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : u(x) = u(y) \text{ if } |x| = |y|\}$ to get more compactness. In this case, by the same argument as in [49, Proposition 5.1], we have $w^{(n)} \in \mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$ with $w^{(n)} = 0$, for all $n \in \mathbb{N}_0$.

In this paper, we prove the inhomogeneous case of Theorem 2.1, that is, for the space $H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\cdot|^{2s}\mathscr{F}u \in L^2(\mathbb{R}^N)\}$, $0 < s \leq N/2$, with the norm $\|u\|^2 := \int_{\mathbb{R}^N} |\xi|^{2s}|\mathscr{F}u|^2 + u^2\, d\xi$. It is known that $H^s(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$, for $2 \leq p \leq 2^*_s$, in the case where $N > 2s$, and in $L^p(\mathbb{R}^N)$, for $2 \leq p < \infty$, in the case where $N = 2s$. The following version of Theorem 2.1 will be used to study the existence of solutions of $(\mathcal{P}_s)$ for the case where $f(x,t)$ has subcritical growth. We set $2^*_s = \infty$, when $N = 2s$.

Theorem 2.3. *Let $(u_k) \subset H^s(\mathbb{R}^N)$ be a bounded sequence and $0 < s \leq N/2$. Then there exist $\mathbb{N}_0 \subset \mathbb{N}$, $\big(w^{(n)}\big)_{n\in\mathbb{N}_0} \subset H^s(\mathbb{R}^N)$, $\big(y_k^{(n)}\big)_{k\in\mathbb{N}} \subset \mathbb{Z}^N$, $n \in \mathbb{N}_0$, such that, up to a subsequence of (u_k),*

$$u_k\big(\cdot + y_k^{(n)}\big) \rightharpoonup w^{(n)}, \quad \text{as } k \to \infty, \text{ in } H^s(\mathbb{R}^N), \tag{2.2}$$

$$\big|y_k^{(n)} - y_k^{(m)}\big| \to \infty, \qquad \text{as } k \to \infty, \text{ for } m \neq n, \tag{2.3}$$

$$\sum_{n\in\mathbb{N}_0} \big\|w^{(n)}\big\|^2 \leq \limsup_k \|u_k\|^2, \tag{2.4}$$

$$u_k - \sum_{n\in\mathbb{N}_0} w^{(n)}\big(\cdot + y_k^{(n)}\big) \to 0, \quad \text{as } k \to \infty, \text{ in } L^p(\mathbb{R}^N), \tag{2.5}$$

for any $p \in (2, 2^*_s)$. Moreover, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$, and the series in (2.5) converges uniformly in k.

These profile decompositions for bounded sequence are unique up to a permutation of index and constant operator (see [49, Proposition 3.4]). Theorem 2.3 is the fractional counterpart of [49, Corollary 3.3] and it describes how bounded sequences in $H^s(\mathbb{R}^N)$ fail to converge in $L^p(\mathbb{R}^N)$, $2 < p < 2^*_s$. This "error" of convergence is generated by the invariance of action of translations in $H^s(\mathbb{R}^N)$. Moreover, it can be seen as an alternative result to a version of Lions' compactness lemma, for $H^s(\mathbb{R}^N)$, proved in [21, Lemma 2.2]. Also, we point out that the profile decomposition of Theorem 2.3 is given by translations of the form $u \mapsto u(\cdot - y)$, $y \in \mathbb{Z}^N$, and, differently from [34, Theorems 4 and 8], we also consider the limit case where $s = N/2$. Theorem 2.3 holds thanks to a co-compactness result contained in [8], and using the abstract approach of [49], considering $H^s(\mathbb{R}^N)$, $0 < s \le N/2$, as the starting Hilbert space.

3. Nonlinear fractional Schrödinger equation

3.1. Hypotheses. In order to describe our results in a more precise way, next we state the main assumptions on the potential $a(x)$ and the nonlinearity $f(x,t)$. We always assume that $0 < s < \min\{1, N/2\}$. We denote by $\|\cdot\|_p$ and $\|\cdot\|_\infty$ the norms of the spaces $L^p(\mathbb{R}^N)$, $1 \le p < \infty$, and $L^\infty(\mathbb{R}^N)$, respectively. $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}^N$.

3.1.1. Subcritical case. Let us first introduce the assumptions on $a(x) = V(x) - b(x)$.

- (V_1) $V(x)$ is a $\mathbb{Z}^N$-periodic function in the space $L^\sigma_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\sigma > 2N/(N+2s)$.
- (V_2) There exists $\mathcal{B} > 0$ such that $V(x) \ge -\mathcal{B}$ for almost every $x \in \mathbb{R}^N$ and
$$\mathcal{C}_V := \inf_{u \in C_0^\infty(\mathbb{R}^N),\, \|u\|_2 = 1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \, dx > 0.$$
- (V_3) There exists $\beta > N/2s$ such that $0 \le b(x) \in L^\beta(\mathbb{R}^N)$ and, for $\beta' = \beta/(\beta-1)$,
$$\|b(x)\|_\beta < \mathcal{C}_V^{(\beta)} := \inf_{u \in H^s_V(\mathbb{R}^N),\, \|u\|_{2\beta'} = 1} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + V(x) u^2 \, dx.$$
- (V_4) There exists $\sigma > N/2s$ such that $V(x) \in L^\sigma_{\mathrm{loc}}(\mathbb{R}^N)$ and there exists $V_\infty := \lim_{|x| \to \infty} V(x) > 0$.

We also assume the following conditions on the nonlinear function $f(x,t)$.

- (f_1) $f \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$, $p_\varepsilon \in (2, 2_s^*)$, such that
$$|f(x,t)| \le \varepsilon(|t| + |t|^{2_s^* - 1}) + C_\varepsilon |t|^{p_\varepsilon - 1},$$
for almost every $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$.
- (f_2) There exists $\mu > 2$ such that
$$\mu F(x,t) := \mu \int_0^t f(x,\tau)\, d\tau \le f(x,t) t,$$
for almost every $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$.
- (f_3) There exists $R > 0$, $t_0 > 0$, $x_0 \in \mathbb{R}^N$, such that setting $C_R(x_0, t_0) = (B_{R+1}(x_0) \setminus B_R(x_0)) \times [0, t_0]$,
$$|B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \setminus B_R| \inf_{(x,t) \in C_R(x_0,t_0)} F(x,t) > 0.$$

(f$_4$) $\lim_{t\to 0} f(x,t)/t = 0$ and $\lim_{|t|\to\infty} F(x,t)/t^2 = \infty$ uniformly in x and for all compact $K \subset \mathbb{R}$, there exists $C = C(K) > 0$ such that $|f(x,t)| \leq C$, for almost every $x \in \mathbb{R}^N$ and for all $t \in K$.

(f$_5$) For all $0 < a < b$,

$$\inf_{x\in\mathbb{R}^N} \inf_{a\leq|t|\leq b} \mathcal{F}(x,t) > 0,$$

where $\mathcal{F}(x,t) := f(x,t)t/2 - F(x,t)$.

(f$_6$) There exist $p_0 > \max\{1, N/2s\}$, a_0, $R_0 > 0$ such that

$$|f(x,t)|^{p_0} \leq a_0|t|^{p_0}\mathcal{F}(x,t),$$

for almost every $x \in \mathbb{R}^N$ and for all $|t| > R_0$.

(f$_7$) There exists $\mathbb{Z}^N$-periodic function $f_{\mathcal{P}}(x,t)$, satisfying (f$_1$) and either (f$_2$)–(f$_3$) or (f$_4$), such that

$$\lim_{|x|\to\infty} |f(x,t) - f_{\mathcal{P}}(x,t)| = 0,$$

uniformly in compact subsets of $\mathbb{R}$.

(f$_8$) For almost every $x \in \mathbb{R}^N$, the function $t \mapsto f_{\mathcal{P}}(x,t)/|t|$, is strictly increasing in $\mathbb{R}$.

Next we assume that $f_{\mathcal{P}}(x,t)$ is independent of x and set $f_\infty(t) = f_{\mathcal{P}}(t)$.

(f$_9$) $f_\infty(t) \in C^1(\mathbb{R})$, there exists $t_0 > 0$ such that

$$F_\infty(t_0) - \frac{V_\infty}{2}t_0^2 > 0, \quad \text{where } F_\infty(t) = \int_0^t f_\infty(\tau)\,d\tau.$$

We look for solutions in the Hilbert space $H^s_V(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm and scalar product

$$\|u\|_V^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(x)u^2\,dx,$$

$$(u,v)_V := \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + V(x)uv\,dx,$$

see Proposition 6.1. Writing $a(x) = V(x) - b(x)$ and assuming (V$_3$) and (f$_1$) we can see that the functional associated with $(\mathcal{P}_s)$, $I\colon H^s_V(\mathbb{R}^N) \to \mathbb{R}$, given by

$$I(u) = \frac{1}{2}\|u\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

is well defined, belongs to $C^1(H^s_V(\mathbb{R}^N))$, with

$$I'(u)\cdot v = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + (V(x) - b(x))uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx,$$

for $u, v \in H^{s,2}_V(\mathbb{R}^N)$. Thus critical points of I correspond to weak solutions of $(\mathcal{P}_s)$ and conversely. Consider the minimax level

$$c(I) = \inf_{\gamma\in\Gamma_I} \sup_{t\geq 0} I(\gamma(t)), \tag{3.1}$$

where

$$\Gamma_I = \Big\{\gamma \in C([0,\infty), H^s_V(\mathbb{R}^N)) : \gamma(0) = 0,\ \lim_{t\to\infty} I(\gamma(t)) = -\infty\Big\}. \tag{3.2}$$

Associated with the limit functions given in (V_4), (f_7), (f_9), we consider the C^1 functionals

$$I_{\mathcal{P}}(u) := \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F_{\mathcal{P}}(x,u)\,dx, \quad u \in H^s_V(\mathbb{R}^N),$$
$$I_\infty(u) := \frac{1}{2}\|u\|_{V_\infty}^2 - \int_{\mathbb{R}^N} F_\infty(u)\,dx, \quad u \in H^s_V(\mathbb{R}^N),$$

where $F_{\mathcal{P}}(x,t) = \int_0^t f_{\mathcal{P}}(x,\tau)\,d\tau$. Similarly, as in (3.1) and (3.2), we can define $c(I_{\mathcal{P}})$, $c(I_\infty), \Gamma_{I_{\mathcal{P}}}$ and Γ_{I_∞}. Next we state the assumption on the minimax levels to guarantee compactness of the PS sequences at the mountain pass level of I.

(f_{10}) $c(I) < c(I_{\mathcal{P}})$,

(f'_{10}) $c(I) < c(I_\infty)$.

In the autonomous case, $f(x,t) = f(t)$, we consider the following variant of (f_3).

(f'_3) There exists $t_0 > 0$ such that $F(t_0) > 0$.

3.1.2. Critical case. Next we state our hypothesis on $a(x) \equiv V(x)$, assuming that $b(x) \equiv 0$.

(V^*_1) There exists a finite set $\mathcal{O} \subset \mathbb{R}^N$ such that $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N \setminus \mathcal{O})$, $V(x) \le 0$ almost everywhere in $\mathbb{R}^N$ and

$$\mathcal{C}^*_V := \inf_{u \in C_0^\infty(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 + V(x)u^2\,dx}{\displaystyle\int_{\mathbb{R}^N} |V(x)|u^2\,dx} > 0.$$

(V^*_2) There exists $a_* \in \mathbb{R}^N$ such that the following limits exist and are uniform in every compact subset of $\mathbb{R}^N$,

$$V_+(x) := \lim_{\lambda\to\infty} \lambda^{-2s} V(\lambda^{-1}x + a_*) \quad \text{and} \quad V_-(x) := \lim_{\lambda\to 0} \lambda^{-2s} V(\lambda^{-1}x + a_*).$$

Moreover, $V_\kappa(x)$ satisfies (V^*_1) if $V_\kappa(x) \not\equiv 0$ for $\kappa = +,-$. Also,

$$\lim_{|x|\to\infty} V(x) = 0.$$

(V^*_3) For all $(\lambda_k) \subset \mathbb{R}^+$ such that either $|\lambda_k| \to \infty$ or $|\lambda_k| \to 0$, and $(y_k) \subset \mathbb{R}^N$ such that $|\lambda_k(y_k - a_*)| \to \infty$, $\lim_{k\to\infty} \lambda_k^{-2s} V(\lambda_k^{-1}x + y_k) = 0$ uniformly in every compact subset of $\mathbb{R}^N$.

We assume the following conditions on the nonlinearity $f(x,t)$.

(f^*_1) $f \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition, there exists $C > 0$ such that $|f(x,t)| \le C|t|^{2^*_s - 1}$ for almost every $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$.

(f_2^*) For all $a_1, \ldots, a_M \in \mathbb{R}$, there exists $C = C(M) > 0$ such that

$$\bigg|F\bigg(x, \sum_{n=1}^{M} a_n\bigg) - \sum_{n=1}^{M} F(x, a_n)\bigg| \leq C(M) \sum_{m \neq n \in \{1,\ldots,M\}} |a_n|^{2_s^*-1}|a_m|$$

for almost every $x \in \mathbb{R}^N$.

(f_3^*) There exists $\gamma > 1$ such that

$$\begin{aligned} f_0(t) &:= \lim_{|x|\to\infty} f(x,t), \\ f_+(t) &:= \lim_{j\in\mathbb{Z},\, j\to+\infty} \gamma^{-(N+2s)j/2} f\big(\gamma^{-j}x, \gamma^{(N-2sj/2}t\big), \\ f_-(t) &:= \lim_{j\in\mathbb{Z},\, j\to-\infty} \gamma^{-(N+2s)j/2} f\big(\gamma^{-j}x, \gamma^{(N-2s)j/2}t\big), \end{aligned}$$

uniformly in every compact subset of $\mathbb{R}^N$. Moreover, the primitive $F_\kappa(t)$ satisfies (f_3') for $\kappa = 0, +, -$.

(f_4^*) The function $t \mapsto f_\kappa(t)/|t|$ is strictly increasing for $\kappa = 0, +, -$.

From (V_1^*) we can see that $\|\cdot\|_V$ defines a norm in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ which is equivalent to the standard one (see Proposition 6.1). Thus, the energy functional $I_*\colon \mathcal{D}^{s,2}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_*(u) = \frac{1}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(x,u)\,dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$$

is well defined and is continuously differentiable provided that (f_1^*) holds. We can define $c(I_*)$ and Γ_{I_*} similarly as in (3.1) and (3.2), just by replacing $H_V^s(\mathbb{R}^N)$ with $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

We use the next assumptions to compare certain minimax levels.

($\mathscr{H}^*$) $V(x) \leq V_\pm(x)$ for almost every $x \in \mathbb{R}^N$. $F_\kappa(t) \leq F(x,t)$ for almost every $x \in \mathbb{R}^N$, for all $t \in \mathbb{R}$, and for any $\kappa = 0, +, -$.

($\mathscr{H}_0^*$) Assume (3.1.2). Either the first inequality in (3.1.2) is strict over a set of positive measure or there exists $\delta > 0$ such that the second inequality in (3.1.2) is strict for almost every $x \in \mathbb{R}^N$, for all $t \in (-\delta, \delta)$.

In order to study the autonomous case $f(x,t) = f(t)$ we assume that the nonlinearity is self-similar.

(f_5^*) There exists $\gamma > 1$ such that

$$F(t) = \gamma^{-Nj} F\big(\gamma^{(N-2s)j/2}t\big) \quad \text{for all } t \in \mathbb{R},\ j \in \mathbb{Z}.$$

3.2. Statement of the main existence results. We first state our results on the existence of ground states for equation ($\mathcal{P}_s$) for subcritical and critical growth.

THEOREM 3.1.

(a) *Suppose that* $f(x,t)$ *and* $a(x) \equiv V(x)$ *are* $\mathbb{Z}^N$*-periodic and satisfy* (f_1)–(f_3) *or* (f_3)–(f_6) *and* (V_1)–(V_2), *respectively. Then the equation* $(\mathcal{P}_s)$ *has a ground state.*

(b) *Suppose that* $f(t) \in C^1(\mathbb{R}^N)$ *satisfies* (f_3') *and* (f_5^*). *Let* $0 < \lambda < \Lambda_{N,s}$, *given in* (1.1), *and*

$$\mathcal{G} = \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} F(u)\,dx = 1 \right\}.$$

Then, there is a radial minimizer w *for*

$$\mathcal{I}_\lambda = \inf_{u\in\mathcal{G}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 - \lambda|x|^{-2s}u^2\,dx. \tag{3.3}$$

Furthermore, for any w *minimizer of* (3.3), *there exists* $\alpha > 0$ *such that* $u = w(\,\cdot\,/\alpha)$ *is a ground state of* $(\mathcal{P}_s)$ *for* $a(x) = -\lambda|x|^{-2s}$.

Theorem 3.1 takes into account the invariance of I under the action of translations and dilations in $H^s(\mathbb{R}^N)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to obtain the concentration–compactness of Palais–Smale and minimizing sequences in each case, respectively. These properties are sufficient to ensure existence of ground states of $(\mathcal{P}_s)$. Our results improve and complement [9] in the fractional framework, since we consider potential $a(x)$ and nonlinearity $F(x,t)$ which can change sign. In Theorem 3.1 (b), we do not require the Ambrosetti–Rabinowitz condition (f_2). Our argument to prove Theorem 3.1 (b) involves a Pohozaev-type identity, and as usual we require C^1-regularity of $f(t)$.

THEOREM 3.2. *Nontrivial weak solutions in* $H^s_V(\mathbb{R}^N)$ *of* $(\mathcal{P}_s)$ *at the mountain pass level are ground states. Precisely, for the Nehari manifold* $\mathcal{N} = \{u \in H^s_V(\mathbb{R}^N) \setminus \{0\} : I'(u)\cdot u = 0\}$, *consider*

$$\overline{c}(I) := \inf_{u\in H^s_V(\mathbb{R}^N)\setminus\{0\}} \sup_{t\geq 0} I(tu) \quad \text{and} \quad c_{\mathcal{N}}(I) := \inf_{u\in\mathcal{N}} I(u).$$

Assume that $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, $a(x) = V(x) - b(x)$ *satisfies* (V_2)–(V_3) *and* $f(x,t)$ *fulfils* (f_1)–(f_2). *Moreover, suppose that*

$$t \mapsto \frac{f(x,t)}{|t|} \text{ is strictly increasing in } \mathbb{R}, \text{ a.e. } x \in \mathbb{R}^N. \tag{3.4}$$

Then $c(I) = \overline{c}(I) = c_{\mathcal{N}}(I)$.

Theorem 3.2 improves some results of [37] since we deal with the case where $a(x)$ may change sign and is not necessarily bounded from below, also with nonlinearity having the behavior at 0 described by (f_1') Moreover, Theorem 3.2 proves the existence of ground state by replacing the aforementioned invariance by (3.4). In fact, our results below give some conditions that guarantee existence of nontrivial weak solutions in $H^s_V(\mathbb{R}^N)$ at the mountain pass level.

Our next results are on the existence of weak solutions of $(\mathcal{P}_s)$ at the mountain-pass level by using the concentration–compactness principle.

Theorem 3.3. *Assume that* (f_1)–(f_3) *or* (f_3)–(f_6) *hold, and additionally* (f_7). *Suppose also that* $a(x)$ *and* $f(x,t)$ *satisfy either one of the following conditions:*

(a) $b(x) \equiv 0$, (V_1)–(V_2), (f_8) *and* (f_{10}); *or*

(b) $V(x) \geq 0$, $b(x)$ *has compact support,* (V_2)–(V_4), (f_9) *and* (f'_{10}); *or*

(c) *replace condition* (f_{10}) (*respectively,* (f'_{10})) *in* (a) (*respectively,* (b)) *by*

$$I(u) \leq I_{\mathcal{P}}(u) \quad (\textit{respectively, } I(u) \leq I_\infty(u)), \quad \textit{for all } u \in H^s_V(\mathbb{R}^N). \tag{3.5}$$

Then equation $(\mathcal{P}_s)$ *possesses a nontrivial weak solution* u *in* $H^s_V(\mathbb{R}^N)$ *at the mountain pass level, that is,* $I(u) = c(I)$. *Moreover, under the assumptions of items* (a) *and* (b), *any sequence* (u_k) *in* $H^s_V(\mathbb{R}^N)$ *such that* $I(u_k) \to c(I)$ *and* $I'(u_k) \to 0$ *has a convergent subsequence.*

Theorems 3.1 (a) and 3.3 extend and complement some results of [9], [37], [48] in the fractional framework. In Theorem 3.3 the potential $a(x) = V(x) - b(x)$ is not necessarily bounded from below and in Theorem 3.3 (b) we do not ask for (f_8) as it was made in those works.

Theorem 3.4. *Assume that* $f(x,t)$ *and* $a(x) \equiv V(x)$ *satisfy* (f_1^*)–(f_4^*), $(\mathscr{H}^*)$, (f_2)–(f_3) *and* (V_1^*)–(V_3^*), *respectively. Then* $(\mathcal{P}_s)$ *has a nontrivial weak solution in* $\mathcal{D}^{s,2}(\mathbb{R}^N)$ *at the mountain pass level. If we assume additionally that* $(\mathscr{H}_0^*)$ *holds, then any sequence* (u_k) *in* $\mathcal{D}^{s,2}(\mathbb{R}^N)$ *such that* $I_*(u_k) \to c(I_*)$ *and* $I'_*(u_k) \to 0$ *has a convergent subsequence.*

Theorems 3.1 (b) and 3.4 complement the study made in [7, 14]. Theorem 3.4 can be seen as a nonlocal version of [7, Theorem 5.2], since we take into account that the critical nonlinearity is not autonomous. It also can be seen as a complement for many results from the literature on the existence of nontrivial weak solutions for Schrödinger equation involving critical nonlinearities and singular potentials (cf. [19], [20], [42, 46] and references therein), because we consider a general class that includes, as a particular case, the inverse fractional square potential given in (1.1).

3.2.1. Some remarks on the hypotheses.

Remark 3.5. (a) Assumption (f_1) can be seen as a subcritical version of (f_5^*) in the sense that it is oscillating around a subcritical power $|t|^{p-2}t$, $2 < p < 2^*_s$ (see [48] for the local case). We can see that (f_1) holds if $f(x,t)$ satisfies the following conditions:

(f'_1) $\displaystyle\lim_{t\to 0} \frac{f(x,t)}{|t| + |t|^{2^*_s - 1}} = 0$ uniformly in x.

(f_1'') There exists $\varrho(t) \in C(\mathbb{R} \setminus \{0\}) \cap L^\infty(\mathbb{R})$ such that

$$2 < \inf_{t\in\mathbb{R}} \varrho(t) \leq \sup_{t\in\mathbb{R}} \varrho(t) < 2_s^*,$$
$$|f(x,t)| \leq C(1 + |t|^{\varrho(t)-1}) \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ for all } t \in \mathbb{R}.$$

Notice that $f(x,t) = k(x)[\varrho'(t)(\ln|t|t) + \varrho(t)]|t|^{\varrho(t)-2}t$, $f(x,0) \equiv 0$, satisfies (f_1')–(f_1''), where

$$\varrho(t) = \frac{2_s^* - 2}{16} \sin(\ln(|\ln|t||)) + \frac{52_s^* + 6}{8} \qquad \text{and} \qquad 0 \leq k(x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}^N).$$

(b) Conditions (f_4) and (f_6) imply that (see [9, Lemma 2.1]) there exists $p \in (2, 2_s^*)$ such that for all ε, there exists $C_\varepsilon > 0$ such that $|f(x,t)| \leq \varepsilon|t| + C_\varepsilon|t|^{p-1}$, for almost every $x \in \mathbb{R}^N$, for all $t \in \mathbb{R}$. Note that this is a special case of (f_1), precisely when $p_\varepsilon = p$.

(c) Assumption (f_2) is the Ambrosetti–Rabinowitz condition which implies the mountain pass geometry and the boundedness of PS sequences for the associated functional (see for instance [35]). Conditions (f_4)–(f_6) are an alternative for (f_2), and they were first introduced in [13] for the local case. By an argument similar to that in [13], (f_6) holds if we assume (f_4), (f_6) and that there exist $p \in (2, 2_s^*)$ and $c_1, c_2, r_1 > 0$ such that

$$|f(x,t)| \leq c_1|t|^{p-1} \quad \text{and} \quad F(x,t) \leq \left(\frac{1}{2} - \frac{1}{c_2|t|^\nu}\right) f(x,t)t, \quad \text{for all } |t| \geq r_1,$$

where $1 < \nu < 2$ if $N = 1$, and $1 < \nu < N + p - pN/2s$ if $N \geq 2$.

(d) In view of the boundedness of PS sequences, we separate our analysis for the subcritical case into two distinct situations: $f(x,t)$ satisfies (f_1)–(f_3) or (f_3)–(f_6). The first one is associated to the case where $f(x,t)$ has an oscillatory behavior around the subcritical power and the second one refers to the case where $f(x,t)$ does not satisfy the Ambrosetti–Rabinowitz condition.

(e) In [9], considering the local case of Schrödinger equations with asymptotically periodic terms, the mountain pass geometry was proved assuming $F(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and (f_4) instead of the classical Ambrosetti–Rabinowitz condition. Here, in this work, we have an improvement even to the local case because we assume (f_3) instead of requiring that $F(x,t) > 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

(f) The assumption (f_5) was used to prove the boundedness of PS sequences at the mountain pass level for the functional of equation ($\mathcal{P}_s$). In [9], to prove a similar result, the author used a more restrictive condition,

$$\mathscr{F}(x,t) = \frac{1}{2} f(x,t)t - F(x,t) \geq b(t)t^2,$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and for some $b(t) \in C(\mathbb{R} \setminus \{0\}, \mathbb{R}^+)$.

(g) To study the existence of weak solutions of equation $(\mathcal{P}_s)$ we use (f_7), unlike in the aforementioned papers, where the authors impose a more tight condition,

$$|f(x,t) - f_{\mathcal{P}}(x,t)| \leq h(x)|t|^{q-1},$$

for almost every x in $\mathbb{R}^N$ and for all t in $\mathbb{R}$, for some $h(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that for any $\varepsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \geq \varepsilon\}$ has finite Lebesgue measure.

(h) The condition (f_9) is used in the literature to prove that weak solutions of equation $(\mathcal{P}_s)$ satisfy a Pohozaev-type identity.

(i) We prove in Proposition 6.1 that $H^s_V(\mathbb{R}^N)$ is well defined and it is continuously embedded in $H^s(\mathbb{R}^N)$, and consequently, the infimum $\mathcal{C}_V^{(\beta)}$ defined in (V_3) is strictly positive.

(j) Functions satisfying (f_5^*) can be seen as nonlinearities asymptotically oscillating around the critical power $|t|^{2_s^*-2}t$; they were introduced in [16], [47].

(k) The asymptotic additivity in (3.1.2) ensures the convergence of I_* under the weak profile decomposition for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ described in Theorem 2.1 (see also [16]).

(l) As already mentioned, (V_1^*)–(V_3^*) define a class of singular potentials that vanishes at infinity, see Example 3.7 (d).

(m) Provided the limits in (V_4), (f_7), (f_9) or (f_3^*) exist, in order to obtain compactness of PS sequences at the minimax levels we need to require the additional conditions over the minimax levels $c_{\mathcal{P}}$, c_∞, c_0, c_+, c_- given in (f_{10}), (f'_{10}), $(\mathscr{H}^*)$–$(\mathscr{H}_0^*)$. In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorems 3.3 and 3.4 without these conditions. This approach was introduced by P.-L. Lions in [28]–[31].

(n) We also consider the case when (f_{10}), (f'_{10}), $(\mathscr{H}^*)$–$(\mathscr{H}_0^*)$ do not hold. Precisely, when $c(I) = c(I_{\mathcal{P}})$ or $c(I) = c(I_\infty)$. In this case, we cannot use the concentration–compactness argument at the mountain pass level. We apply [27, Theorem 2.3] to overcome this difficulty and prove existence of solution at the mountain pass level.

(o) For the problem $(\mathcal{P}_s)$ involving critical growth we require (V_1^*)–(V_3^*) and (f_3^*)–$(\mathscr{H}_0^*)$. These assumptions are suitable for our argument, unlike (f_{10})–(f'_{10}), because the potential that appears in the associated limiting equation depends on the profile decomposition of Theorem 2.1 for a given PS sequence at the mountain pass level (for more details see the estimate (10.1)).

Remark 3.6. Under (V_4) and (f_7) the following conditions imply that (f_{10}) and (f'_{10}) hold:

($\mathscr{H}$) $F_{\mathcal{P}}(x,t) \leq F(x,t)$, for almost every $x \in \mathbb{R}^N$, for all $t \in \mathbb{R}$ and $V(x) \leq V_\infty$, for almost every $x \in \mathbb{R}^N$. Moreover, either the first inequality holds strictly in some open interval containing the origin or the second one holds in a set of positive measure.

In Proposition 9.1 we proved the following estimates for the minimax levels: $c(I) \leq c(I_{\mathcal{P}})$ and $c(I) \leq c(I_\infty)$. Moreover, we proved that under ($\mathscr{H}$), (f_{10}) and (f'_{10}) hold. We observe that with the corresponding assumption of Theorem 3.3, it is easy to see that ($\mathscr{H}$) implies that (3.5) is satisfied.

Example 3.7. Our approach includes the following classes of potentials:

(a) For a potential satisfying assumption (V_2), that is not bounded away from zero, consider $0 \leq a(x) \equiv V_0(x) \in L^p_{\mathrm{loc}}(\mathbb{R}^N) \cap (C(\mathbb{R}^N \setminus \mathcal{O})$, where $p \geq 1$ and $\mathcal{O}$ is a countable set, and suppose that $Z = \{x \in \mathbb{R}^N : V(x) = 0\} \neq \emptyset$ is a countable discrete set.

(b) Let $V_0(x)$ be the potential given above. For a potential that changes sign and satisfies (V_2), consider $a(x) \equiv V_0(x) - \varepsilon$, where $0 < \varepsilon < \mathcal{C}_{V_0}/2$.

(c) Let $0 < \delta < N/\beta$, $p > N/s$ and

$$V(x) = 2 - \frac{1}{1+|x|^2}, \qquad V_\infty = 2, \qquad b(x) = \begin{cases} \mathcal{C}_b |x|^{-\delta} & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Then $a(x) = V(x) - b(x)$ satisfies (V_2)–(V_4), where $\mathcal{C}_b > 0$ is a normalization constant.

(d) For the potential $a(x) \equiv V(x)$, satisfying (V_1^*)–(V_3^*), in view of (1.1), we can consider

$$V(x) = -\frac{1}{L}\sum_{j=1}^{L} \frac{\lambda_j}{|x-a_*|^{2s}}, \quad \text{with } \ 0 < \lambda_j < \frac{\Gamma_{N,s}}{2}, \quad j = 1, \ldots, L.$$

Example 3.8. Hypotheses of Theorems 3.1–3.4 are satisfied in the following situations:

(a) Taking $\varrho(t)$ as in Remark 3.5 (a) and $k(x) = |x|^2/(1+|x|^2)$, one can see that $f(x,t) = k(x)\big[\varrho'(t)(\ln|t|t) + \varrho(t)\big]|t|^{\varrho(t)-2}t$, $f(x,0) \equiv 0$, satisfies (f_1)–(f_3), (f_9) and (f'_{10}).

(b) For a nonlinearity satisfying conditions (f_3)–(f_8) and (f_{10}) we can define

$$f(x,t) = h(x,t) \quad \text{for } t \geq 0 \quad \text{and} \quad f(x,t) = -h(x,-t) \quad \text{for } t < 0,$$

where $h(x,t) = k(x)t\ln(1+t) + k_1(x)\big[(1+\cos(t))t^2 + 2(t+\sin(t))t\big]$, for $t \geq 0$, $s > N/6$; $k(x) = |x|^2/(1+|x|^2)$ and $0 \leq k_1(x) \in C(\mathbb{R}^N)$ satisfies $\lim\limits_{|x|\to\infty} k_1(x) = 0$.

(c) Let $c(x)$ be a continuous nonnegative $\mathbb{Z}^N$-periodic function and $f(x,t) = c(x)[ph_\varepsilon(t) + h'_\varepsilon(t)t]|t|^{p-1}$, $2 < p < 2^*_s$, with $h_\varepsilon(t) \in C^\infty(\mathbb{R})$ a nondecreasing cut-off function satisfying $|h'_\varepsilon(t)| \leq C/t$, $|h_\varepsilon(t)| \leq C$, for all $t \in \mathbb{R}$, $h_\varepsilon(t) = -\varepsilon$, for $t \leq 1/4$, $h_\varepsilon(t) = \varepsilon$, for $t \geq 1/4$, with ε small enough. In this case $F(x,t)$ may change sign.

(d) The nonlinearity

$$f(x,t) = \exp\{k_0(x)(\sin(\ln|t|) + 2)\}[k_0(x)\cos(\ln|t|) + 2^*_s]|t|^{2^*_s-2}t, \quad f(x,0) \equiv 0,$$

satisfies the hypothesis of Theorem 3.4, if $k_0(x)$ is continuous and

$$2^*_s - \mu > \sup_{x\in\mathbb{R}^N} k_0(x) \geq k_0(x) > k_0(0) = \inf_{x\in\mathbb{R}^N} k_0(x) = \lim_{|x|\to\infty} k_0(x) = 0.$$

4. Preliminaries

4.1. Fractional Sobolev spaces. Let $0 < s < N/2$. By the Plancherel Theorem,

$$[u]^2_s = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx, \quad \text{for all } u \in C^\infty_0(\mathbb{R}^N). \tag{4.1}$$

Thus $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is a Hilbert space when endowed with the inner product

$$[u,v]_s = \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v\,dx,$$

and the following characterization holds:

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \big\{u \in L^{2^*_s}(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\big\}.$$

By (4.1) we also have that $H^s(\mathbb{R}^N)$ is a Hilbert space with norm and inner product

$$\|u\|^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx + u^2\,dx, \quad (u,v) := \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(-\Delta)^{s/2}v + uv\,dx.$$

Thus $H^s(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\big\}$. For $\Omega \subset \mathbb{R}^N$ a $C^{0,1}$ domain with bounded boundary and $0 < s < 1$, the fractional Sobolev space is defined as

$$H^s(\Omega) = \bigg\{u \in L^2(\Omega) : \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy < \infty\bigg\},$$

with the norm

$$\|u\|^2_{H^s(\Omega)} := \int_\Omega u^2\,dx + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy.$$

By [12, Proposition 3.4], we have

$$[u]^2_s = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy, \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Moreover, we have the continuous embedding

$$H^s(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq 2^*_s, \quad \text{for } 0 < s < N/2, \tag{4.2}$$

and the following compact embedding (see [12, Section 7]):

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N), \quad 1 \le p < 2_s^*, \quad \text{for } 0 < s < \min\{1, N/2\}. \tag{4.3}$$

Thus, any bounded sequence in $H^s(\mathbb{R}^N)$ has a subsequence that converges strongly in $L^p(\Omega)$, $1 \le p < 2_s^*$, for any compact set Ω of $\mathbb{R}^N$. The Plancherel Theorem also gives the next identity,

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v \, dx = \int_{\mathbb{R}^N} (-\Delta)^s u v \, dx, \tag{4.4}$$

for all $u \in H^{2s}(\mathbb{R}^N)$, $v \in H^s(\mathbb{R}^N)$.

4.2. The s-harmonic extension. Next we introduce the harmonic extension following [24, Section 2]. Let

$$P_s(x,y) = \beta(N,s) \frac{y^{2s}}{(|x|^2 + y^2)^{(N+2s)/2}},$$

where $\beta(N,s)$ is such that $\int_{\mathbb{R}^{\mathbb{N}}} P_s(x,1)\, dx = 1$ and $0 < s < 1$. With the standard notation $\mathbb{R}^{N+1}_+ = \{(x,y) \in \mathbb{R}^{N+1} : y > 0\}$, for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ let us set the s-harmonic extension of u as

$$w(x,y) = E_s(u)(x,y) := \int_{\mathbb{R}^N} P_s(x - \xi, y) u(\xi)\, d\xi, \quad (x,y) \in \mathbb{R}^{N+1}_+.$$

Then, for $K \subset \overline{\mathbb{R}^{N+1}_+}$ compact we have $w \in L^2(K, y^{1-2s})$, $\nabla w \in L^2(\mathbb{R}^{N+1}_+, y^{1-2s})$ and $w \in C^\infty(\mathbb{R}^{N+1}_+)$. Moreover, w satisfies, in the distribution sense, the following:

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\lim\limits_{y \to 0^+} y^{1-2s} w_y(x,y) = \kappa_s (-\Delta)^s u(x) & \text{in } \mathbb{R}^N, \\ \|\nabla w\|^2_{L^2(\mathbb{R}^{N+1}_+, y^{1-2s})} = \kappa_s \|u\|^2, \end{cases} \tag{4.5}$$

where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, and Γ is the gamma function. Precisely, for $R > 0$,

$$\int_{B_R^+} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx\, dy = \kappa_s \int_{B_R^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx, \tag{4.6}$$

for all $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$, where $B_R := \{z = (x,y) \in \mathbb{R}^{N+1} : |z|^2 < R^2\}$, $B_R^+ := B_R \cap \mathbb{R}^{N+1}_+$, $B_R^N := \{z = (x,y) \in \mathbb{R}^{N+1}_+ : |z|^2 < R^2,\ y = 0\}$ and the right-hand side of (4.6) is in the trace sense for φ (for details on the trace operator see [33]). More generally, given $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $v \in H^1(B_R^+, y^{1-2s})$ is a weak solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s} \nabla v) = 0 & \text{in } B_R^+, \\ -\lim\limits_{y \to 0^+} y^{1-2s} v_y(x,y) = \kappa_s g(x, v(x)) & \text{in } B_R^N, \end{cases} \tag{4.7}$$

if we have

$$\int_{B_R^+} y^{1-2s}\langle\nabla v,\nabla\varphi\rangle\,dx\,dy=\kappa_s\int_{B_R^N} g(x,v)\varphi\,dx \tag{4.8}$$

for all $\varphi\in C_0^\infty(B_R^+\cup B_R^N)$.

Let $g(x,t)=f(x,t)-a(x)t$ and $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $f(u),F(u)\in L^1(\mathbb{R}^N)$. Let $V(x)\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfy (V_2) and $b(x)$ verify (V_3). Then $w=E_s(u)$ is a weak solution of (4.7) for all R iff u is a weak solution of ($\mathcal{P}_s$).

4.3. Regularity results. In order to make our discussion clear, we follow the approach of [24] to describe how the s-harmonic extension can be used to obtain regularity for solutions of elliptic problems involving the fractional Laplacian. Next, we consider $Q_R=B_R^N\times(0,R)$ and $C^\alpha(\Omega)$ to denote $C^{[\alpha],\alpha-[\alpha]}(\Omega)$, where $[\alpha]$ is the integer part of the number $\alpha>0$. We always assume that $0<s<1$.

Proposition 4.1. *Let $v\in H^1(B_R^+,y^{1-2s})$ be a weak solution of* (4.7)*, where $g(x,t)=f(t)-a(x)t$. Suppose that for $f(t)\in C^1(\mathbb{R})$ there exist $\mathcal{C}_1,\mathcal{C}_2>0$, $2<p<2_s^*$, such that*

$$|f(t)|\le\mathcal{C}_1|t|^{p-1}+\mathcal{C}_2(|t|+|t|^{2_s^*-1}),$$

for all $t\in\mathbb{R}$ and that $a(x)\in C^1(\mathbb{R}^N)$. If $t_r(v)\in L^{p_0}_{\mathrm{loc}}(\mathbb{R}^N)$, for some $p_0>2_s^*$, then for any $R>0$ there exist $y_0>0$, $r<R$ with $B_r^N\times(0,y_0)\subset B_R^+$, and $\alpha\in(0,1)$, such that

$$v,\nabla_x v,y^{1-2s}v_y\in C^{0,\alpha}(B_r^N\times[0,y_0]). \tag{4.9}$$

Proof. (1) In fact, since

$$\begin{gathered}\frac{g(t_rv)}{1+|t_rv|}\in L^q_{\mathrm{loc}}(\mathbb{R}^N),\quad\text{for all } N/2s<q\le p_0/(2_s^*-2),\quad\text{and}\\ g(t_rv)=\bigg[\frac{g(t_rv)}{1+|t_rv|}\,\mathrm{sgn}\,(t_rv)\bigg]t_rv+\frac{g(t_rv)}{1+|t_rv|},\end{gathered}$$

we can use [24, Proposition 2.6] to get that v belongs to $C^\alpha(\overline{Q}_{R/2})$, for some $\alpha\in(0,1)$.

(2) Since $h(t)\in C^1(\mathbb{R})$, thanks to [24, Theorem 2.14] we can apply a bootstrap argument to obtain that $t_r(v)\in C^{\alpha_1}(B_{R/4k})$, $\alpha_1\in(1,2)$, for some positive integer k.

(3) To get that $\nabla_x v\in H^1(Q_R,y^{1-2s})\cap C^{\alpha_2}(\overline{Q}_{R/6k})$, for some $\alpha_2\in(0,1)$, we apply [24, Proposition 2.13] with $A(x)=0$ and $B(x)=h(v)\in C^1(\mathbb{R})$.

(4) Finally, the fact that $y^{1-2s}v_y(x,y)\in C^{\alpha_3}(\overline{Q}_{R/2})$, $\alpha_3\in(0,1)$, follows by using [24, Lemma 2.18] in (1) of this proof. $\square$

REMARK 4.2. Let $v \in H^1(Q_R, y^{1-2s})$ be a weak solution of (4.7). If v possesses the regularity described in (4.9), then v satisfies the conditions in (4.7) for each point of $B_R^+ \cup B_R^N$ (classical sense). Moreover, denoting $\mathcal{N}_v(x,y) = y^{1-2s}v(x,y)$, we have that

$$\mathcal{N}_v(x,0) = \kappa_s h(v(x,0)), \quad \text{for all } x \in B_R^N. \tag{4.10}$$

Indeed, the fact that v satisfies the first equation in (4.7) for each point in B_R^+ follows by standard elliptic interior regularity arguments using the difference quotient technique. To prove that condition (4.10) holds, we take $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$ and use integration by parts formula to get

$$0 = \int_{B_{R,\delta}} \operatorname{div}(y^{1-2s}\nabla v)\,dx\,dy = \int_{B_{R,\delta}} y^{1-2s}\langle \nabla v, \nabla\varphi\rangle\,dx\,dy - \int_{F^1_{R,\delta}} y^{1-2s}v_y\varphi\,dx,$$

where the fact that $\varphi = 0$ over $F^2_{R,\delta}$ and that $\eta = (0,\dots,0,-1)$ is the normal vector of $F^1_{R,\delta}$, is used. Now notice that

$$\begin{aligned}\int_{F^1_{R,\delta}} y^{1-2s}v_y\varphi\,dx &= \int_{B^N_{\sqrt{R^2-\delta^2}}} \delta^{1-2s}v_y(x,\delta)\varphi(x,\delta)\,dx \\ &= \int_{B_R^N} \delta^{1-2s}v_y(x,\delta)\mathcal{X}B^N_{\sqrt{R^2-\delta^2}}(x)\varphi(x,\delta)\,dx,\end{aligned}$$

where $\mathcal{X}_A$ denotes the characteristic function of the set A. Thus, by the Dominated Convergence Theorem, we obtain that

$$\lim_{\delta\to 0}\int_{F^1_{R,\delta}} y^{1-2s}v_y\varphi\,dx = \int_{B_R^N} \mathcal{N}_v(x,0)\varphi(x,0)\,dx.$$

Consequently, from definition (4.8), we have

$$\begin{aligned}\kappa_s\int_{B_R^N} h(v(x,0))\varphi(x,0)\,dx &= \kappa_s\int_{B_R^N} h(t_r(v))t_r(\varphi)\,dx \\ &= \int_{B_R^+} y^{1-2s}\langle\nabla v,\nabla\varphi\rangle\,dx\,dy = \int_{B_R^N}\mathcal{N}_v(x,0)\varphi(x,0)\,dx.\end{aligned}$$

Since $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$ is arbitrary, condition (4.10) follows.

REMARK 4.3. Using the s-harmonic extension, the existence of nonnegative weak solutions of $(\mathcal{P}_s)$ if $f(x,t) \ge 0$ for all $t \ge 0$ and almost every x in $\mathbb{R}^N$ can be proved. For that, one can consider the truncation $\overline{f}(x,t) = f(x,t)$, if $t \ge 0$, $\overline{f}(x,t) = 0$, if $t < 0$. Assume that $a(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and that (f_1) and (V_2) hold true with $b(x) \equiv 0$. Thus for u a weak solution of $(\mathcal{P}_s)$, with $f(x,t)$ replaced by $\overline{f}(x,t)$, we have that u is also a weak nonnegative solution for $(\mathcal{P}_s)$. To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0,1])$ be such that $\xi(t) = 1$, if $t \in [-1,1]$ and $\xi(t) = 0$, if $|t| \ge 2$, with $|\xi'(t)| \le C$ for all $t \in \mathbb{R}$. For each $n \in \mathbb{N}$, define $\xi_n \colon \mathbb{R}^{N+1} \to \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and it verifies $|\nabla\xi_n(z)| \le C$

and $|z||\nabla \xi_n(z)| \leq C$, for all $z \in \mathbb{R}^{N+1}$. By a density argument, we can take $\varphi = \xi_n w_-$ in (4.8), where $w_-(z) = \min\{w(z), 0\}$.

Since $w_-(z) = E_s(u_-)$, we have

$$\int_{\mathbb{R}^{N+1}_+} y^{1-2s}\xi_n|\nabla w_-|^2 + y^{1-2s}\xi_n\langle \nabla w_+, \nabla w_-\rangle \\ + y^{1-2s}\langle \nabla w_+ + \nabla w_-, w_-\nabla \xi_n\rangle \, dx\, dy = \kappa_s \int_{\mathbb{R}^N} (\overline{f}(x,u) - a(x)u)\xi_n u_- \, dx.$$

Applying the Lebesgue Theorem and (4.5) we get $\|u_-\|_V^2 = \int_{\mathbb{R}^N} \overline{f}(x,u)u_- \, dx = 0$, thus $u_- = 0$. If u has sufficient regularity, one can show that u is positive by applying the maximum principle as described in [40]. In order to regularize solutions of equation $(\mathcal{P}_s)$, we can follow [37, Section 6].

5. Proof of Theorem 2.3

We shall prove the profile decomposition for bounded sequences in $H^s(\mathbb{R}^N)$, $0 < s \leq N/2$. To achieve that, we start by considering

$$D = D_{\mathbb{Z}^N} := \{g_y \colon H^s(\mathbb{R}^N) \to H^s(\mathbb{R}^N) \mid g_y u(x) = u(x-y),\ y \in \mathbb{Z}^N\},$$

which turns to be a group of unitary operators in $H^s(\mathbb{R}^N)$. The idea is to use [49, Theorem 3.1] to obtain Theorem 2.3. We need first to determine how elements of $H^s(\mathbb{R}^N)$ become asymptotically orthogonal in $H^s(\mathbb{R}^N)$ with respect to any fixed function under a sequence of dislocations.

Lemma 5.1. *Let (y_k) be a sequence in $\mathbb{R}^N$ and $u \in H^s(\mathbb{R}^N) \setminus \{0\}$. The sequence $(u(\,\cdot - y_k))$ converges weakly to zero in $H^s(\mathbb{R}^N)$ if, and only if $|y_k| \to \infty$.*

Proof. Suppose that $u(\,\cdot\, - y_k) \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$, and assume by contradiction, that $y_k \to y$ up to a subsequence. By density we may assume that $u \in C_0^\infty(\mathbb{R}^N)$, also by [16, Lemma 5.1] we have that $u(\,\cdot\, - y_k) \to u(\,\cdot\, - y)$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, consequently by the Dominated Convergence Theorem,

$$0 = \lim_{k\to\infty}\bigg[\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(\,\cdot - y_k)(-\Delta)^{s/2}u(\,\cdot - y) + u(\,\cdot - y_k)u(\,\cdot - y)\, dx\bigg] = \|u\|^2,$$

which leads to a contradiction with the assumption that $u \neq 0$. Conversely, assume that $|y_k| \to \infty$. Again, by a density argument, we may assume $u \in C_0^\infty(\mathbb{R}^N)$, and using [16, Lemma 5.2] we obtain that $u(\,\cdot - y_k) \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Since $\operatorname{supp}(u(\,\cdot\, - y_k)) \cap \operatorname{supp} v = \emptyset$, for k large enough, we have

$$\lim_{k\to\infty}\bigg[\int_{\mathbb{R}^N} (-\Delta)^{s/2}u(\,\cdot\, - y_k)(-\Delta)^{s/2}v + u(\,\cdot\, - y_k)v\, dx\bigg] = 0,$$

for all $v \in C_0^\infty(\mathbb{R}^N)$. $\square$

We complement [8] by establishing equivalence between the L^p-convergence and $D_{\mathbb{Z}^N}$-convergence (see [49, Definition 3.1] or [8, Definition 1.1]) in $H^s(\mathbb{R}^N)$. Thus, Theorem 2.3 follows by an argument from [49, Corollary 3.3].

Proposition 5.2. *Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$. Then $u_k \overset{D_{\mathbb{Z}^N}}{\rightharpoonup} 0$ in $H^s(\mathbb{R}^N)$, if and only if $u_k \to 0$ in $L^p(\mathbb{R}^N)$, for all $2 < p < 2_s^*$.*

Proof. Suppose that $u_k \to 0$ in $L^p(\mathbb{R}^N)$, $2 < p < 2_s^*$. Take an arbitrary sequence (g_{y_k}) in $D_{\mathbb{Z}^N}$ and let $\varphi \in C_0^\infty(\mathbb{R}^N)$. Using (4.4) we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N} (-\Delta)^{s/2}(g_{y_k}^* u_k)(-\Delta)^{s/2}\varphi\,dx\bigg| \\ \leq \bigg(\int_{\mathbb{R}^N} |u_k|^p\,dx\bigg)^{1/p}\bigg(\int_{\mathbb{R}^N} |(-\Delta)^s\varphi(\,\cdot\, - y_k)|^{p/(p-1)}\,dx\bigg)^{(p-1)/p}.\end{aligned}$$

Thus, applying the Hölder inequality in L^2 to the inner product of $H^s(\mathbb{R}^N)$, we conclude that $g_{y_k}^* u_k \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. For the rest of the proof we refer the reader to [8, Theorem 2.4]. □

Proof of Theorem 2.3 completed. We prove it by applying [49, Theorem 3.1]. In fact, let (g_{y_k}) in $D_{\mathbb{Z}^N}$ such that $g_{y_k} \not\rightharpoonup 0$ in $H^s(\mathbb{R}^N)$. By Lemma 5.1, $y_k \to y$ up to a subsequence and, by [16, Lemma 5.2], $g_{y_k} \to g_y$. Thus, in view of [49, Proposition 3.1], $(H^s(\mathbb{R}^N), D_{\mathbb{Z}^N})$ is a dislocation space. Assertions (2.3) and (2.5) follow by Lemma 5.1 and Proposition 5.2, respectively. □

6. Variational settings

In this section we set the framework in which the variational argument for the study of $(\mathcal{P}_s)$ is applied.

Proposition 6.1. *Let $V(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfy* (V_2). *Then $H_V^s(\mathbb{R}^N)$ is a Hilbert space continuously embedded in $H^s(\mathbb{R}^N)$. If $V(x)$ satisfies* (V_1^*), *then $\|\cdot\|_V$ is equivalent to the norm of $\mathcal{D}^{s,2}(\mathbb{R}^N)$.*

Proof. Let us prove first that there exists a positive constant C such that

$$C[\varphi]_s^2 \leq \|\varphi\|_V^2, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \tag{6.1}$$

Indeed, on the contrary, there would exist a sequence (φ_n) in $C_0^\infty(\mathbb{R}^N)$, such that

$$[\varphi_n]_s^2 > n\|\varphi_n\|_V^2, \quad \text{for all } n \in \mathbb{N}.$$

Taking $v_n = \varphi_n/[\varphi_n]_s$, we would have $1/n > \|v_n\|_V^2$ and $\mathcal{C}_V\|v_n\|_2^2 \leq \|v_n\|_V^2$, for all $n \in \mathbb{N}$, and consequently $\lim_{n\to\infty} \|v_n\|_V^2 = \lim_{n\to\infty} \|v_n\|_2^2 = 0$. This would lead to a contradiction with the fact that $1 - \mathcal{B}\|v_n\|_2^2 \leq \|v_n\|_V^2$, $n \in \mathbb{N}$.

Now consider any sequence (φ_n) in $C_0^\infty(\mathbb{R}^N)$. Using the inequality (6.1) we have $C[\varphi_m - \varphi_n]_s^2 \leq \|\varphi_m - \varphi_n\|_V^2$, for all $m \neq n$. Consequently,

$$\|\varphi_m - \varphi_n\|^2 \leq \min\{1, C\}^{-1}(1 + \mathcal{C}_V^{-1})\|\varphi_m - \varphi_n\|_V^2,$$

for all $m \neq n$. Thus $H^s_V(\mathbb{R}^N)$ is well defined. Moreover, the Fatou Lemma and embedding (4.2) imply

$$H^s_V(\mathbb{R}^N) \subset \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty \right\},$$

with the continuous embedding $H^s_V(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N)$. Assuming (V_1^*),

$$[u]_s^2 + \int_{\mathbb{R}^N} V(x)u^2\,dx \geq \mathcal{C}_V^* \int_{\mathbb{R}^N} |V(x)|u^2\,dx,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. From this we derive

$$\mathcal{C}_V^*[u]_s^2 \leq (\mathcal{C}_V^*+1)[u]_s^2 + \int_{\mathbb{R}^N} (V(x) - \mathcal{C}_V^*|V(x)|)u^2\,dx \leq (\mathcal{C}_V^*+1)\|u\|_V^2,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. Since $V(x) \leq 0$ almost everywhere in $\mathbb{R}^N$, the norms $[\,\cdot\,]_s$ and $\|\cdot\|_V$ are equivalent in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. $\square$

Remark 6.2. (a) If $V(x)$ fulfills (V_2) and (V_4), then $H^s_V(\mathbb{R}^N) = H^s(\mathbb{R}^N)$. Moreover, the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent. Consequently, the path $\lambda_u(t) := u(\,\cdot\,/t)$, $t \geq 0$, belongs to $C([0,\infty), H^s_V(\mathbb{R}^N))$ and $u(\,\cdot\, - y) \in H^s_V(\mathbb{R}^N)$ for all $u \in H^s_V(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$. Indeed, there is a ball B_{R_1} with center at the origin such that

$$\begin{aligned}\int_{\mathbb{R}^N} V(x)u^2\,dx &= \int_{B_{R_1}} V(x)u^2\,dx + \int_{\mathbb{R}^N\setminus B_{R_1}} V(x)u^2\,dx \\ &\leq \bigg(\int_{B_{R_1}} |V(x)|^\sigma\,dx\bigg)^{1/\sigma}\bigg(\int_{B_{R_1}} |u|^{2\sigma/(\sigma-1)}\,dx\bigg)^{(\sigma-1)/\sigma} \\ &\quad + (V_\infty+1)\int_{\mathbb{R}^N\setminus B_{R_1}} u^2\,dx,\end{aligned}$$

for all $u \in H^s_V(\mathbb{R}^N)$, where $2 \leq 2\sigma/(\sigma-1) \leq 2_s^*$. So we can apply (4.2) to conclude. To obtain that λ_u belongs to $H^s_V(\mathbb{R}^N)$ we use [16, Lemma 8.3].

(b) If (V_1)–(V_2) hold, then Theorem 2.3 holds with $H^s(\mathbb{R}^N)$ replaced by $H^s_V(\mathbb{R}^N)$ and $\|\cdot\|$ by $\|\cdot\|_V$. In fact, (V_1) implies that $D_{\mathbb{Z}^N}$ is a group of unitary operators in $H^s_V(\mathbb{R}^N)$.

Lemma 6.3. *Suppose that $f(x,t)$ satisfies* (f_1) *and either* (f_2)–(f_3) *or* (f_4). *If $a(x) = V(x) - b(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ fulfills* (V_2)–(V_3), *then I possesses the mountain pass geometry. Precisely,*

(a) $I(0) = 0$;
(b) *there exist $r, b > 0$ such that $I(u) \geq b$, whenever $\|u\|_V = r$;*
(c) *there is $e \in H^s_V(\mathbb{R}^N)$ with $\|e\|_V > r$ and $I(e) < 0$.*

In particular, $0 < c(I) < \infty$.

Proof. Let $\xi_R \in C_0^\infty(\mathbb{R})$, $R > 0$, be such that $0 \le \xi_R(t) \le t_0$, $\xi_R(t) = t_0$ if $|t| \le R$, and $\xi_R(t) = 0$ if $|t| > R+1$. Setting $v(x) := \xi_R(|x - x_0|)$, we have $v \in H_V^s(\mathbb{R}^N)$ and by assumption (f_3),

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,v)\,dx &= \int_{B_R(x_0)} F(x,t_0)\,dx + \int_{B_{R+1}(x_0)\setminus B_R(x_0)} F(x,v)\,dx \\ &\ge |B_R| \inf_{B_R(x_0)} F(x,t_0) + |B_{R+1}\setminus B_R| \inf_{(x,t)\in C_R(x_0,t_0)} F(x,t) > 0.\end{aligned}$$

First assume that (f_2) holds. Since $b(x) \in L^\beta(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} b(x)u^2\,dx \le \bigg(\int_{\mathbb{R}^N} |b(x)|^\beta\,dx\bigg)^{1/\beta}\bigg(\int_{\mathbb{R}^N} |u|^{2\beta/(\beta-1)}\,dx\bigg)^{(\beta-1)/\beta},$$

for all $u \in H_V^s(\mathbb{R}^N)$, with $2 < 2\beta/(\beta-1) < 2_s^*$, by ($\mathrm{f}_1$) and ($\mathrm{V}_3$), for any ε we get

$$I(u) \ge \bigg[\frac{1}{2}\bigg(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} - 2\varepsilon\mathcal{C}_2\bigg) - \varepsilon\mathcal{C}_{2_s^*}\|u\|_V^{2_s^*-2} - C_\varepsilon\mathcal{C}_{p_\varepsilon}\|u\|_V^{p_\varepsilon-2}\bigg]\|u\|_V^2, \tag{6.2}$$

for all $u \in H_V^s(\mathbb{R}^N)$, where $\mathcal{C}_2$, $\mathcal{C}_{2_s^*}$ and $\mathcal{C}_{p_\varepsilon}$ are positive constants given in Proposition 6.1. This allows to consider ε such that the first term in the right-hand side of (6.2) is positive, once $\|u\|_V$ is taken small enough. Hence there exists $r > 0$ such that $I(u) > 0$ provided that $\|u\|_V = r$. Since (f_2) is equivalent to $d/dt(F(x,t)t^{-\mu}) \ge 0$, for $t > 0$, we have

$$\int_{\mathbb{R}^N} F(x,tv)\,dx \ge t^\mu \int_{\mathbb{R}^N} F(x,v)\,dx, \quad \text{whenever } t > 1.$$

Hence, as $t \to \infty$,

$$\begin{aligned}I(tv) &= \frac{t^2}{2}\|v\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2\,dx - \int_{\mathbb{R}^N} F(x,tv)\,dx \\ &\le \frac{t^2}{2}\|v\|_V^2 - t^\mu \int_{\mathbb{R}^N} F(x,v)\,dx \to -\infty,\end{aligned}$$

Now suppose that (f_4) holds. By Remark 3.5 (b) we can argue as above to conclude the existence of $r > 0$ such that $I(u) > 0$ whenever $\|u\|_V < r$. Given $R > 0$, there exists $t_R > 0$ such that $F(x,t) > Rt^2$, for all $|t| > t_R$. Let $A(R,t) := \{x \in \mathbb{R}^N : t|v(x)| > t_R\}$, for $t > 0$. We have that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,tv)\,dx &= \int_{K_t} F(x,tv)\,dx + \int_{A(R,t)} F(x,tv)\,dx \\ &\ge \int_{K_t} F(x,tv)\,dx + Rt^2 \int_{A(R,t)} v^2\,dx,\end{aligned} \tag{6.3}$$

where $K_t = (\mathbb{R}^N \setminus A(R,t)) \cap \operatorname{supp} v$. Using Remark 3.5 (b), for each $t > 0$, we get that

$$|F(x,tv)| \le C, \quad \text{for a.e. } x \in K_t,$$

where $C > 0$ does not depend on x and t. Consequently, for any $x \in \operatorname{supp} v$, $F(x,tv)\mathcal{X}_{K_t}(x) \to 0$ as $t \to \infty$, where we have used that, for any $x \in \operatorname{supp} v$, $\mathcal{X}_{\mathbb{R}^N \setminus A(R,t)}(x) \to \mathcal{X}_{\mathbb{R}^N \setminus \operatorname{supp} v}(x) = 0$, as $t \to \infty$. Thus the Dominated Convergence Theorem implies that the first integral in the right-hand side of inequality (6.3) goes to zero as t goes to infinity. By the same reason, we also have

$$\lim_{t\to\infty} \int_{A(R,t)} v^2\,dx = \lim_{t\to\infty} \int_{\mathbb{R}^N} v^2 \mathcal{X}_{A(R,t)}\,dx = \int_{\mathbb{R}^N} v^2 \mathcal{X}_{\{v\neq 0\}}\,dx = \int_{\mathbb{R}^N} v^2\,dx.$$

In particular, there exists a positive number $t_{0,R}$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} v^2\,dx < \int_{A(R,t)} v^2\,dx, \quad \text{for all } t > t_{0,R}. \tag{6.4}$$

Replacing (6.4) in (6.3) we have, for R sufficiently large,

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\|v\|_V^2 - \frac{t^2}{2}\int_{\mathbb{R}^N} b(x)v^2\,dx - \int_{\mathbb{R}^N} F(x,tv)\,dx \\ &\le \frac{1}{2}(\|v\|_V^2 - R\|v\|_2^2)t^2 - \int_{K_t} F(x,tv)\,dx < 0, \end{aligned}$$

for $t > t_{0,R}$. □

Remark 6.4. (a) In view of Lemma 6.3, we define the set $\Gamma_I^1 = \{\gamma \in C([0,1], H_V^s(\mathbb{R}^N)) : \gamma(0) = 0,\ \|\gamma(1)\|_V > r,\ I(\gamma(1)) < 0\}$, and

$$c_1(I) = \inf_{\gamma\in\Gamma_I} \sup_{t\in[0,1]} I(\gamma(t)),$$

the usual minimax level. Thus we have $c_1(I) = c(I)$.

(b) If $f(x,t) \equiv f(t)$, the mountain pass geometry can be proved by replacing (f_3) by (f_3'). In fact, let ξ_R as in the proof of Lemma 6.3 and define $\eta_R(x) = \xi_R(|x|)$. Then, as in [15, Remark 2.8], we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(\eta_R)\,dx &= \int_{B_R(x_0)} F(t_0)\,dx + \int_{B_{R+1}(x_0)\setminus B_R(x_0)} F(\eta_R)\,dx \\ &\ge F(t_0)|B_R| - |B_{R+1}\setminus B_R| \max_{t\in[0,t_0]} |F(t)|. \end{aligned}$$

Thus, there exist positive constants C_1 and C_2 such that for R large,

$$\int_{\mathbb{R}^N} F(\eta_R)\,dx \ge C_1 R^N - C_2 R^{N-1} > 0.$$

The mountain pass geometry now follows as in the proof of Lemma 6.3.

(c) Let $f(x,t)$ satisfy (f_1) and either (f_2)–(f_3) or (f_4); and additionally (f_7). Suppose also that $a(x)$ and $f(x,t)$ fulfill (V_2)–(V_4) and (f_9), respectively. Then the limiting functional I_∞ has the mountain pass geometry. In fact, (f_9) together with [16, Lemma 8.3] implies that $\lambda_u(t) := u(\,\cdot\,/t)$, $t \ge 0$, belongs to Γ_{I_∞}, where $u \in H^s(\mathbb{R}^N)$ is such that

$$\int_{\mathbb{R}^N} F_\infty(u) - \frac{V_\infty}{2} u^2\,dx > 0. \tag{6.5}$$

As in Remark 6.4 (b), we can see that there exists $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ satisfying (6.5) and

$$I_\infty(\lambda_{\varphi_0}(t)) = \frac{1}{2}\, t^{N-2s}[\varphi_0]_s^2 - t^N \bigg[\int_{\mathbb{R}^N} F_\infty(\varphi_0) - \frac{V_\infty}{2}\, \varphi_0^2\, dx\bigg] \to -\infty, \quad \text{as } t \to \infty.$$

Moreover, $I(u) > 0$ if $\|u\|_V = r$, for $r > 0$ small enough (see proof of Lemma 6.3).

(d) Under the assumptions of Lemma 6.3, and if $F(x,t) > 0$ for almost every $x \in \mathbb{R}^N$ and $t \neq 0$, then, for any $u \in H_V^s(\mathbb{R}^N) \setminus \{0\}$, $\zeta(t) = tu$ belongs to Γ_I. In fact, in the proof of Lemma 6.3, replacing v by u and considering the same notations, we get

$$\int_{\mathbb{R}^N} F(x,tu)\, dx \geq Rt^2 \int_{A(R,t)} u^2\, dx,$$
$$\lim_{t\to\infty} \int_{A(R,t)} u^2\, dx = \lim_{t\to\infty} \int_{\mathbb{R}^N} u^2 \mathcal{X}_{A(R,t)}\, dx = \int_{\mathbb{R}^N} u^2 \mathcal{X}_{\{u\neq 0\}}\, dx = \int_{\mathbb{R}^N} u^2\, dx,$$

which enables us to proceed as in (6.4) to get (for R is large)

$$\varphi(t) := I(tu) \leq \frac{1}{2}(\|u\|_V^2 - R\|u\|_2^2)t^2 \to -\infty, \quad \text{as } t \to \infty.$$

Moreover, assuming (3.4) we can infer that $\zeta(t)$ has a unique critical point.

From the previous results, the existence of bounded PS sequence at the mountain pass level is obtained.

Proposition 6.5. *Assume $a(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ fulfills* (V_2)–(V_3) *and $f(x,t)$ satisfies either*

(a) (f_1)–(f_3), *or*
(b) (f_3)–(f_6).

Then there is a bounded sequence (u_k) such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$.

Proof. (a) In view of Lemma 6.3, the standard Mountain Pass Theorem implies the existence of $(u_k) \subset H_V^s(\mathbb{R}^N)$ such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$. For large k, we have

$$\begin{aligned} c(I) + 1 + \|u_k\|_V &\geq I(u_k) - \frac{1}{\mu}\, I'(u_k)\cdot u_k \\ &= \bigg(\frac{1}{2} - \frac{1}{\mu}\bigg)\bigg(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\bigg)\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x,u_k) - \frac{1}{\mu} f(x,u_k)u_k\, dx \\ &\geq \bigg(\frac{1}{2} - \frac{1}{\mu}\bigg)\bigg(1 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\bigg)\|u_k\|_V^2, \end{aligned}$$

which implies that (u_k) is bounded in $H_V^s(\mathbb{R}^N)$.

(b) The proof follows as in [9, Lemma 2.5] and [13, Lemma 4.1]. By Lemma 6.3, applying a variant of the Mountain Pass Theorem, we obtain a Cerami sequence (u_k) for I at the level $c(I)$, precisely, $I(u_k) \to c(I)$ and $(1 +$

$\|u_k\|_V)\|I'(u_k)\|_* \to 0$, where $\|\cdot\|_*$ denotes the usual norm of the dual of $H^s_V(\mathbb{R}^N)$. We claim that (u_k) is bounded in $H^s_V(\mathbb{R}^N)$. Assume by contradiction that, up to a subsequence, $\|u_k\|_V \to \infty$. Let $v_k = u_k/\|u_k\|_V$. Thus

$$\lim_{k\to\infty}\bigg[1 - \int_{\mathbb{R}^N} \frac{f(x,u_k)}{\|u_k\|_V}\, v_k\,dx - \frac{1}{\|u_k\|_V^2}\int_{\mathbb{R}^N} b(x)u_k^2\,dx\bigg] = \lim_{k\to\infty}\bigg[\frac{1}{\|u_k\|_V^2} I'(u_k)\cdot u_k\bigg] = 0.$$

We can use an indirect argument to prove that

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} f(x,u_k)v_k\|u_k\|_V^{-1}\,dx = 0,$$

which by (V_3) leads to the following contradiction:

$$1 = \lim_{k\to\infty}\frac{1}{\|u_k\|_V^2}\int_{\mathbb{R}^N} b(x)u_k^2\,dx < \frac{1}{2}.$$

For $0 \le a < b \le \infty$, defining $\Omega_k(a,b) = \{x \in \mathbb{R}^N : a \le |u_k(x)| \le b\}$, we are going to prove that for $0 < \varepsilon < 1$ there exist $k_\varepsilon, a_\varepsilon, b_\varepsilon$ such that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{f(x,u_k)}{\|u_k\|_V}\, v_k\,dx &= \int_{\Omega_k(0,a_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V}\, v_k\,dx \\ &+ \int_{\Omega_k(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_k)}{\|u_k\|_V}\, v_k\,dx + \int_{\Omega_k(b_\varepsilon,\infty)} \frac{f(x,u_k)}{\|u_k\|_V}\, v_k\,dx < \varepsilon,\end{aligned} \tag{6.6}$$

for all $k > k_\varepsilon$. In order to do that, we first make some estimates involving $\mathcal{F}(x,t)$. Define $g(r) = \inf\{\mathcal{F}(x,t) : x \in \mathbb{R}^N,\ |t| > r\}$, which is positive and goes to infinity as $r \to \infty$. Indeed, thanks to (f_5) and (f_6), we have

$$a_0\mathcal{F}(x,t) \ge |f(x,t)t|^{p_0} > \bigg|2\,\frac{F(x,t)}{t^2}\bigg|^{p_0}, \quad \text{for all } |t| > R_0.$$

Consequently, by (f_4), we obtain that $\mathcal{F}(x,t) \to \infty$, as $|t| \to \infty$, uniformly in x. Due to (f_5), we also can define a positive number $m_a^b = \inf\{\mathcal{F}(x,t)/t^2 : x \in \mathbb{R}^N,\ a \le |t| \le b\}$. Using these notations, we see that there exists k_0 such that

$$\begin{aligned} c(I) + 1 &\ge I(u_k) - \frac{1}{2}I'(u_k)\cdot u_k \\ &= \int_{\Omega_k(0,a)} \mathcal{F}(x,u_k)\,dx + \int_{\Omega_k(a,b)} \mathcal{F}(x,u_k)\,dx + \int_{\Omega_k(b,\infty)} \mathcal{F}(x,u_k)\,dx \\ &\ge \int_{\Omega_k(0,a)} \mathcal{F}(x,u_k)\,dx + m_a^b\int_{\Omega_k(a,b)} u_k^2\,dx + g(b)|\Omega_k(b,\infty)|,\end{aligned} \tag{6.7}$$

for all $k > k_0$. Inequality (6.7) implies $\lim\limits_{b\to\infty}|\Omega_k(b,\infty)| = 0$, uniformly in $k > k_0$. Moreover, for a fixed $2 < q \le 2_s^*$,

$$\int_{\Omega_k(a,b)} |v_k|^q\,dx \le \bigg(\int_{\Omega_k(a,b)} |v_k|^{2_s^*}\bigg)^{q/2_s^*} |\Omega_k(a,b)|^{(2_s^*-q)/2_s^*},$$

in particular,

$$\lim_{b\to\infty}\int_{\Omega_k(a,b)}|v_k|^q\,dx=0,\quad \text{uniformly in } k>k_0. \tag{6.8}$$

On the other hand, it follows that

$$\begin{aligned}\int_{\Omega_k(a,b)}v_k^2\,dx&=\frac{1}{\|u_k\|_V^2}\int_{\Omega_k(a,b)}u_k^2\,dx\\&\le\left(\frac{1}{\|u_k\|_V^2}\right)\left(\frac{1}{(c(I)+1)m_a^b}\right)\to 0,\quad \text{as } k\to\infty.\end{aligned}\tag{6.9}$$

We now pass to prove the estimate (6.6). By (f$_4$), there exists $a_\varepsilon>0$ such that

$$|f(x,t)|<\varepsilon|t|,\quad \text{a.e. } x\in\mathbb{R}^N,\ \text{provided that } |t|<a_\varepsilon.$$

Thus, using (6.9), we have

$$\int_{\Omega_k(0,a_\varepsilon)}\frac{f(x,u_k)}{\|u_k\|_V}\,v_k\,dx\le\int_{\Omega_k(0,a_\varepsilon)\cap\{|u_k|>0\}}\frac{f(x,u_k)}{|u_k|}\,v_k^2\,dx<\frac{\varepsilon}{3},$$

for all $k>k_\varepsilon^{(1)}$. Taking $2q_0:=2p_0/(p_0-1)$, using (f$_6$) and (6.8), we get

$$\begin{aligned}\int_{\Omega_k(b_\varepsilon,\infty)}\frac{f(x,u_k)}{\|u_k\|_V}\,v_k\,dx&\le\int_{\Omega_k(b_\varepsilon,\infty)}\frac{f(x,u_k)}{|u_k|}\,v_k^2\,dx\\&\le(a_0(c(I)+1))^{1/p_0}\left(\int_{\Omega_k(b_\varepsilon,\infty)}|v_k|^{2q_0}\,dx\right)^{1/q_0}<\frac{\varepsilon}{3},\end{aligned}$$

for all $k>k_\varepsilon^{(2)}$. Using (f$_4$) we get that $|f(x,u_k)|\le C_\varepsilon|u_k|$, for almost every $x\in\Omega_k(a_\varepsilon,b_\varepsilon)$, for $C_\varepsilon>0$ which does not depend on k and x. Thus,

$$\int_{\Omega_k(a_\varepsilon,b_\varepsilon)}\frac{f(x,u_k)}{\|u_k\|_V}v_k\,dx\le\int_{\Omega_k(a_\varepsilon,b_\varepsilon)}\frac{f(x,u_k)}{|u_k|}v_k^2\,dx\le C_\varepsilon\int_{\Omega_k(a_\varepsilon,b_\varepsilon)}v_k^2\,dx<\frac{\varepsilon}{3},$$

for all $k>k_\varepsilon^{(3)}$, where $k_\varepsilon^{(3)}>k_0$ is obtained from (6.9). $\square$

6.1. Behavior of weak profile decomposition convergence under nonlinearities. We now pass to describe the limit of the profile decomposition (Theorem 2.3) for bounded sequences of the associated functional.

Proposition 6.6. *If $a(x)\equiv V(x)\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, (f$_1$) and (V$_2$) hold, then for $(u_k)\subset H^s_V(\mathbb{R}^N)$, a bounded sequence such that $u_k\to u$ in $L^p(\mathbb{R}^N)$, for some $p\in(2,2_s^*)$, up to subsequence, we have*

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}f(x,u_k)u_k\,dx=\int_{\mathbb{R}^N}f(x,u)u\,dx. \tag{6.10}$$

Moreover, if (v_k) is a bounded sequence in $H^s_V(\mathbb{R}^N)$ with $u_k-v_k\to 0$ in $L^p(\mathbb{R}^N)$, for some $2<p<2_s^*$, then, up to a subsequence,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}F(x,u_k)-F(x,v_k)\,dx=0. \tag{6.11}$$

Proof. Note that $u_k \to u$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2_s^*)$. This follows by an interpolation inequality, if $q < p$ then $\|u_k - u\|_q \leq \|u_k - u\|_2^\theta \|u_k - u\|_p^{1-\theta}$ where $1/q = \theta/2 + (1-\theta)/p$, and if $q > p$ then $\|u_k - u\|_q \leq \|u_k - u\|_p^\theta \|u_k - u\|_{2_s^*}^{1-\theta}$ for $1/q = \theta/p + (1-\theta)/2_s^*$. On the other hand, by (4.3) and Proposition 6.1, $u \in H_V^s(\mathbb{R}^N)$ and

$$u_k(x) \to u(x) \quad \text{as } k \to \infty, \text{ for a.e. } x \in \mathbb{R}^N \\ \text{and} \quad |u_k(x)|, |u(x)| \leq h_\varepsilon(x) \quad \text{for a.e. } x \in \mathbb{R}^N,\ k \in \mathbb{N},$$

for some $h_\varepsilon \in L^{p_\varepsilon}(\mathbb{R}^N)$. Now note that

$$\int_{\mathbb{R}^N} |f(x,u_k)u_k - f(x,u)u|\,dx \\ \leq \int_{\mathbb{R}^N} |f(x,u_k)(u_k - u)|\,dx + \int_{\mathbb{R}^N} |(f(x,u_k) - f(x,u))u|\,dx.$$

The first integral can be estimated by Hölder inequality as follows:

$$\int_{\mathbb{R}^N} |f(x,u_k)(u_k - u)|\,dx \\ \leq \varepsilon\big(\|u_k\|_2 \|u_k - u\|_2 + \|u_k\|_{2_s^*}^{2_s^*-1} \|u_k - u\|_{2_s^*}\big) + C_\varepsilon \|u_k\|_{p_\varepsilon}^{p_\varepsilon - 1} \|u_k - u\|_{p_\varepsilon}.$$

For the second one, consider

$$X_k^\varepsilon := \big\{x \in \mathbb{R}^N : \varepsilon(|u_k(x)| + |u_k(x)|^{2_s^*-1}) \leq C_\varepsilon |u_k(x)|^{p_\varepsilon - 1}\big\}, \\ X^\varepsilon := \big\{x \in \mathbb{R}^N : \varepsilon(|u(x)| + |u(x)|^{2_s^*-1}) \leq C_\varepsilon |u(x)|^{p_\varepsilon - 1}\big\}.$$

Thus

$$\int_{X_k^\varepsilon} |(f(x,u_k) - f(x,u))u|\,dx = \int_{\mathbb{R}^N} |(f(x,u_k) - f(x,u))u| \mathcal{X}_{X_k^\varepsilon}\,dx.$$

Since $\mathcal{X}_{X_k^\varepsilon}(x) \to \mathcal{X}_{X^\varepsilon}(x)$ in $\mathbb{R}^N$ and $\big|(f(x,u_k) - f(x,u))u\mathcal{X}_{X_k^\varepsilon}\big| \leq 2C_\varepsilon h_\varepsilon^{p_\varepsilon} \in L^1(\mathbb{R}^N)$, we may apply the Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \int_{X_k^\varepsilon} |(f(x,u_k) - f(x,u))u|\,dx = 0.$$

On the other hand,

$$\limsup_{k\to\infty} \int_{\mathbb{R}^N \setminus X_k^\varepsilon} |(f(x,u_k) - f(x,u))u|\,dx \leq C\varepsilon,$$

where $C > 0$ does not depend on ε and k. Since ε is arbitrary, (6.10) holds.

Now, let us prove (6.11). Choose $(\overline{u}_k), (\overline{v}_k)$ in $C_0^\infty(\mathbb{R}^{\mathrm{N}})$ such that

$$\lim_{k\to\infty} \|\overline{u_k} - u_k\|_V = \lim_{k\to\infty} \|\overline{v_k} - v_k\|_V = 0.$$

Thus it suffices to prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x,\overline{u_k}) - F(x,\overline{v_k})\,dx = 0. \tag{6.12}$$

Consider $E := (C_0(\mathbb{R}^N), \|\cdot\|_{p_\varepsilon})$ and $\beta\colon E \to \mathbb{R}$, given by $\beta(u) = \int_{\mathbb{R}^N} F(x,u)\,dx$ with Gâteaux derivative $\beta'_G(u)\cdot v = \int_{\mathbb{R}^N} f(x,u)v\,dx$. Thus, we may apply the Mean Value Theorem to get

$$|\beta(u)-\beta(v)| \le \sup_{w\in E,\, w\in[u,v]} \|\beta'_G(w)\|\,\|u-v\|_{p_\varepsilon}, \quad \text{for all } u,v\in E, \tag{6.13}$$

where $[u,v] = \{tu+(1-t)v : t\in[0,1]\}$. Since $(u_k),(v_k),(\overline{u}_k)$ and $(\overline{v}_k)$ belong to a bounded set B in $H^s_V(\mathbb{R}^N)$, and using $H^s_V(\mathbb{R}^N)\hookrightarrow L^{p_\varepsilon}(\mathbb{R}^N)$, we have that $B\cap E$ is bounded in E. Thus β'_G is bounded in $B\cap E$, which allows us to take $u=\overline{u}_k$ and $v=\overline{v}_k$ in (6.13) to get (6.12). $\square$

The next result is the nonlocal version of [49, Lemma 5.1] and it is a generalization of the Brezis–Lieb Lemma.

PROPOSITION 6.7. *Assume $f(x,t)$ satisfies* (f_1) *and* (f_7). *Let $(u_k)\subset H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n\in\mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$, given by Theorem* 2.3. *Then*

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} F(x,u_k)\,dx = \int_{\mathbb{R}^N} F(x,w^{(1)})\,dx + \sum_{n\in\mathbb{N}_0,\, n>1}\int_{\mathbb{R}^N} F_{\mathcal{P}}(x,w^{(n)})\,dx.$$

PROOF. By Proposition 6.6 the functional

$$\Phi(u) := \int_{\mathbb{R}^N} F(x,u)\,dx, \quad u\in H^s(\mathbb{R}^N),$$

is uniformly continuous in bounded sets of $L^p(\mathbb{R}^N)$, for any $2<p<2^*_s$. Thus, by (2.4) and (2.5),

$$\lim_{k\to\infty}\bigg[\Phi(u_k) - \Phi\bigg(\sum_{n\in\mathbb{N}_0} w^{(n)}(\,\cdot\,-y_k^{(n)})\bigg)\bigg] = 0.$$

By the uniform convergence in (2.5) we can reduce to the case $\mathbb{N}_0=\{1,\ldots,M\}$. Thus taking

$$\Phi_{\mathcal{P}}(u) := \int_{\mathbb{R}^N} F_{\mathcal{P}}(x,u)\,dx, \quad u\in H^s(\mathbb{R}^N),$$

it follows from (f_7) and the Dominated Convergence Theorem that

$$\lim_{k\to\infty}\bigg[\sum_{n\in\mathbb{N}_0}\Phi(w^{(n)}(\,\cdot\,-y_k^{(n)})) - \Phi(w^{(1)}) - \sum_{n\in\mathbb{N}_0,\, n>1}\Phi_{\mathcal{P}}(w^{(n)})\bigg] = 0.$$

It remains to prove that

$$\lim_{k\to\infty}\bigg[\Phi\bigg(\sum_{n\in\mathbb{N}_0} w^{(n)}(\,\cdot\,-y_k^{(n)})\bigg) - \sum_{n\in\mathbb{N}_0}\Phi(w^{(n)}(\,\cdot\,-y_k^{(n)}))\bigg] = 0. \tag{6.14}$$

Since Φ is locally Lipschitz, using a density argument, we can assume that $w^{(n)}\in C_0^\infty(\mathbb{R}^N)$, for $n=1,\ldots,M$. Consequently, from (2.3),

$$\operatorname{supp}\big(w^{(n)}(\,\cdot\,-y_k^{(n)})\big)\cap\operatorname{supp}\big(w^{(m)}(\,\cdot\,-y_k^{(m)})\big)=\emptyset,$$

for $m \neq n$ and k large enough, which implies that (6.14) holds, since for k large enough,

$$\begin{aligned}\int_{\mathbb{R}^N} F\bigg(x, \sum_{n\in\mathbb{N}_0} w^{(n)}\big(x-y_k^{(n)}\big)\bigg)\,dx &= \int_{\bigcup_{n=1}^{M} \operatorname{supp}(w^{(n)}(\,\cdot\,-y_k^{(n)}))} F\bigg(x, \sum_{m=1}^{M} w^{(m)}\big(\,\cdot\,-y_k^{(m)}\big)\bigg)\,dx \\ &= \sum_{n=1}^{M}\int_{\operatorname{supp} w^{(n)}} F(x+y_k^{(n)}, w^{(n)})\,dx. \qquad \square\end{aligned}$$

Corollary 6.8. *Let $(u_k) \subset H^s(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n\in\mathbb{N}_0}$ in $H^s(\mathbb{R}^N)$ given by Theorem 2.3. If $f(x,t)$ is $\mathbb{Z}^N$-periodic and satisfies* (f$_1$),

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} F(x,u_k)\,dx = \sum_{n\in\mathbb{N}_0}\int_{\mathbb{R}^N} F(x,w^{(n)})\,dx. \tag{6.15}$$

Corollary 6.9. *Let $u_k \rightharpoonup u$ in $H^s(\mathbb{R}^N)$ and $F(x,t)$ be as in Corollary 6.8. Then, up to a subsequence,*

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} F(u_k) - F(u-u_k) - F(u)\,dx = 0.$$

Proof. Since $w^{(1)} = u$, following the proof of Proposition 6.7, we obtain

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} F(u_k-u)\,dx = \sum_{n\in\mathbb{N}_*,\, n>1}\int_{\mathbb{R}^N} F(w^{(n)})\,dx. \tag{6.16}$$

Subtracting (6.16) from (6.15), we get the desired convergence. $\square$

The following result is a generalization of Fatou Lemma, or alternatively, the fact that the functional $u \mapsto \int_{\mathbb{R}^N} V(x)u^2\,dx$ is sequentially weakly lower semicontinuous with respect to the profile decomposition of Theorem 2.3. Moreover, it is a complement to Proposition 6.7.

Proposition 6.10. *Suppose that $a(x) \equiv V(x) \geq 0$ and* (V$_2$) *holds true. Let (u_k) be a bounded sequence in $H^s(\mathbb{R}^N)$ and $(w^{(n)})_{n\in\mathbb{N}_0}$ given in Theorem 2.3.*

(a) *If* (V$_1$) *holds, we have*

$$\liminf_{k\to\infty}\int_{\mathbb{R}^N} V(x)u_k^2\,dx \geq \sum_{n\in\mathbb{N}_0}\int_{\mathbb{R}^N} V(x)|w^{(n)}|^2\,dx.$$

(b) *Under* (V$_4$) *we obtain*

$$\liminf_{k\to\infty}\int_{\mathbb{R}^N} V(x)u_k^2\,dx \geq \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2\,dx + \sum_{n\in\mathbb{N}_0,\, n>1}\int_{\mathbb{R}^N} V_\infty|w^{(n)}|^2\,dx.$$

PROOF. We just prove the second inequality. The first one follows by a similar argument.

$$
\begin{aligned}
(6.17)\qquad &\int_{\mathbb{R}^N} V(x)u_k^2\,dx \\
&= \int_{\mathbb{R}^N} \bigg| |V(x)|^{1/2}(u_k - w^{(1)}) - |V_\infty|^{1/2}\sum_{n=2}^{m} w^{(n)}\big(\,\cdot\, - y_k^{(n)}\big)\bigg|^2\,dx, \\
&\quad + \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2\,dx + \sum_{n=2}^{m}\int_{\mathbb{R}^N} V_\infty |w^{(n)}|^2\,dx + o(1),
\end{aligned}
$$

for all m, where with the notation $a_k = o(b_k)$ we mean that $a_k/b_k \to 0$. We proceed as in the proof of the iterated Brezis–Lieb Lemma [8, Proposition 6.7]. We start by checking that (6.17) holds for $m = 2$. In fact, by Proposition 6.1, up to a subsequence, the classical Brezis–Lieb Lemma and assertion (2.3) imply that

$$
(6.18)\qquad \int_{\mathbb{R}^N} V(x)u_k^2\,dx = \int_{\mathbb{R}^N} V(x)|w^{(1)}|^2\,dx + \int_{\mathbb{R}^N} V(x)|u_k - w^{(1)}|^2\,dx + o(1).
$$

Consequently and by the same reason,

$$
\begin{aligned}
(6.19)\qquad &\int_{\mathbb{R}^N} V(x)\big|u_k - w^{(1)}\big|^2\,dx \\
&= \int_{\mathbb{R}^N} V\big(x + y_k^{(2)}\big)\big|u_k(\,\cdot\, + y_k^{(2)}) - w^{(1)}\big(\,\cdot\, + y_k^{(2)}\big)\big|^2\,dx \\
&\quad + \int_{\mathbb{R}^N} \bigg| |V(x + y_k^{(2)})|^{1/2}\big(u_k(\,\cdot\, + y_k^{(2)}) \\
&\quad - w^{(1)}\big(\,\cdot\, + y_k^{(2)}\big)\big) - \big|V_\infty w^{(2)}\big|^{1/2}\bigg|^2\,dx + \int_{\mathbb{R}^N} V_\infty |w^{(2)}|^2\,dx + o(1).
\end{aligned}
$$

Replacing identity (6.19) in (6.18), we obtain (6.17) for $m = 2$. We shall now prove that (6.17) holds for $m+1$ provided that it is true for m. Indeed, arguing as above,

$$
\begin{aligned}
(6.20)\qquad &\int_{\mathbb{R}^N} \bigg| |V(x)|^{1/2}(u_k - w^{(1)}) - V_\infty^{1/2}\sum_{n=2}^{m} w^{(n)}(\,\cdot\, - y_k^{(n)})\bigg|^2\,dx \\
&\quad - \int_{\mathbb{R}^N} V_\infty |w^{(m+1)}|^2\,dx \\
&= \int_{\mathbb{R}^N} \bigg| |V(x)|^{1/2}\big(u_k - w^{(1)}\big) - V_\infty^{1/2}\sum_{n=2}^{m+1} w^{(n)}\big(\,\cdot\, - y_k^{(n)}\big)\bigg|^2\,dx + o(1).
\end{aligned}
$$

Applying the induction hypothesis in (6.20) we obtain (6.17). □

6.2. Pohozaev identity. We prove a Pohozaev-type identity following the same argument as in [16, Section 4] with some appropriate modifications. It complements some results in the present literature, namely: [5, Theorem 2.3], [6, Proposition 4.1], [16, Proposition 4.3] and [36, Theorem 1.1].

Proposition 6.11. *Assume that $f(x,t)$ satisfies the same assumptions of Proposition* 4.1 *and $a(x)\in C^1(\mathbb{R}^N\setminus\mathcal{O})$, where $\mathcal{O}$ is a finite set. Let $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of $(\mathcal{P}_s)$ such that $f(u)/(1+|u|)$ belongs to $L^{N/2s}_{\rm loc}(\mathbb{R}^N)$. If $F(u), f(u)u, a(x)u^2$ and $\langle\nabla a(x),x\rangle u^2$ belong to $L^1(\mathbb{R}^N)$, then $u\in C^1(\mathbb{R}^N\setminus\mathcal{O})$ and*

$$\frac{N-2s}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|\,dx+\frac{N}{2}\int_{\mathbb{R}^N}a(x)u^2\,dx \\ +\frac{1}{2}\int_{\mathbb{R}^N}\langle\nabla a(x),x\rangle u^2\,dx=N\int_{\mathbb{R}^N}F(u)\,dx.$$

Proof. We first prove the local regularity of u. For $x_0\in\mathbb{R}^N\setminus\mathcal{O}$, $\overline{u}=u(\cdot+x_0)$ is a weak solution to $(-\Delta)^s\overline{u}+\overline{a}(x)\overline{u}=f(\overline{u})$ in $\mathbb{R}^N$, where $\overline{a}(x)=a(x+x_0)$. Taking r small enough, the ball B_r^N does not contain any point of discontinuity of $\overline{a}(x)$ and so,

$$\frac{|g(\overline{u})|}{1+|\overline{u}|}\in L^{N/2s}(B_r^N),\quad \text{where } g(\overline{u}):=f(\overline{u})-\overline{a}(x)\overline{u}.$$

This enables us to use the same argument as in the proof of [16, Proposition 4.2], to conclude that $u\in L^p(B_r^N)$, for all $p\geq 1$. Moreover, since

$$g(\overline{u})=f(\overline{u})-\overline{a}(x)\overline{u}=\bigg[\frac{f(\overline{u})}{1+|\overline{u}|}\operatorname{sgn}\overline{u}-\overline{a}(x)\bigg]\overline{u}+\frac{f(\overline{u})}{1+|\overline{u}|},$$

we can proceed as in Proposition 4.1 to obtain the existence of $y_0>0$, $r_0<r$ with $B_r^N\times(0,y_0)\subset B_r^+$, and $\alpha\in(0,1)$, such that $\overline{w},\nabla_x\overline{w},y^{1-2s}\overline{w}_y\in C^{0,\alpha}(B_{r_0}^N\times[0,y_0])$, where $\overline{w}$ is the s-harmonic extension of $\overline{u}$ and $\nabla_x\overline{w}=(\overline{w}_{x_1},\ldots,\overline{w}_{x_N})$. Since x_0 is arbitrary, $w,\nabla_x w,y^{1-2s}w_y\in C(B_r^N\setminus\mathcal{O}\times[0,y_0])$, for all $r,y_0>0$.

Consider $\xi\in C_0^\infty(\mathbb{R},[0,1])$ such that $\xi(t)=1$, if $|t|\leq 1$, $\xi(t)=0$, if $|t|\geq 2$, and $|\xi'(t)|\leq C$, for all $t\in\mathbb{R}$, $C>0$. Let $\mathcal{O}=\{x^{(1)},\ldots,x^{(l)}\}$ and $z^{(i)}=(x^{(i)},0)$, $i=1,\ldots,l$. For each $n\in\mathbb{N}$ define $\xi_n\colon\mathbb{R}^{N+1}\to\mathbb{R}$ by

$$\xi_n(z)=\begin{cases}\xi(|z|^2/n^2) & \text{if } |z-z^{(i)}|^2>2/n^2,\\ 1-\xi(n^2|z-z^{(i)}|^2) & \text{if } |z-z^{(i)}|^2\leq 2/n^2.\end{cases}$$

Then, for n large enough, $\xi_n\in C_0^\infty(\mathbb{R}^N)$ and it verifies $|z||\nabla\xi_n(z)|\leq C$, for all $z\in\mathbb{R}^{N+1}$ and some $C>0$. Now observe that

$$\begin{aligned}(6.21)\quad &\operatorname{div}(y^{1-2s}\nabla w)\langle z,\nabla w\rangle\xi_n\\ &=\operatorname{div}\bigg[y^{1-2s}\xi_n\bigg(\langle z,\nabla w\rangle\nabla w-\frac{|\nabla w|^2}{2}z\bigg)\bigg]+\frac{N-2s}{2}y^{1-2s}|\nabla w|^2\xi_n\\ &\qquad+y^{1-2s}\frac{|\nabla w|^2}{2}\langle z,\nabla\xi_n\rangle-y^{1-2s}\langle\nabla w,z\rangle\langle\nabla w,\nabla\xi_n\rangle.\end{aligned}$$

Given $R,\delta>0$ we set $B_{R,\delta}=\{z=(x,y)\in\mathbb{R}^{N+1}_+ : |z|^2<R^2,\ y>\delta\}$, $F^1_{R,\delta}=\{z=(x,y)\in\mathbb{R}^{N+1}_+ : |z|^2<R^2,\ y=\delta\}$, and $F^2_{R,\delta}=\{z=(x,y)\in$

$\mathbb{R}^{N+1}_+ : |x|^2 + y^2 = R^2,\ y > \delta\}$. Note that $\partial B_{\sqrt{2}n,\delta} = F^1_{\sqrt{2}n,\delta} \cup F^2_{\sqrt{2}n,\delta}$. Let $\eta(z) = (0, \dots, -1)$ be the unit outward normal vector of $B_{\sqrt{2}n,\delta}$ on $F^1_{\sqrt{2}n,\delta}$. Since $\xi_n = 0$ on $F^2_{\sqrt{2}n,\delta}$, by condition (4.5), identity (6.21) and the Divergence Theorem we get

$$
\begin{aligned}
0 &= \int_{B_{\sqrt{2}n,\delta}} \operatorname{div}(y^{1-2s}\nabla w)\langle z, \nabla w\rangle \xi_n \, dx\, dy \\
&= \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n \left[\langle z, \nabla w\rangle\langle \nabla w, \eta\rangle - \frac{|\nabla w|^2}{2}\langle z, \eta\rangle\right] dx\, dy + \theta_{n,\delta} \\
&= \int_{F^1_{\sqrt{2}n,\delta}} \xi_n \langle x, \nabla_x w\rangle(-y^{1-2s} w_y)\, dx \\
&\quad - \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n w_y^2 y\, dx + \int_{F^1_{\sqrt{2}n,\delta}} y^{1-2s}\xi_n \frac{|\nabla w|^2}{2} y\, dx + \theta_{n,\delta} \\
&= I^1_{n,\delta} + I^2_{n,\delta} + I^3_{n,\delta} + \theta_{n,\delta},
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_{n,\delta} &= \int_{B_{\sqrt{2}n,\delta}} \frac{N-2s}{2} y^{1-2s}|\nabla w|^2 \xi_n \, dx\, dy \\
&\quad + \int_{B_{\sqrt{2}n,\delta}} y^{1-2s}\frac{|\nabla w|^2}{2}\langle z, \nabla \xi_n\rangle - y^{1-2s}\langle \nabla w, z\rangle\langle \nabla w, \nabla \xi_n\rangle \, dx\, dy.
\end{aligned}
$$

Following the argument of [17, proof of Theorem 3.7] we claim that there exists $\delta_k \to 0$ such that $I^2_{n,\delta_k} + I^3_{n,\delta_k} \to 0$ as $k \to \infty$. In fact, on the contrary we would get $\delta_0 > 0$ and $C > 0$ such that

$$
-\int_{F^1_{\sqrt{2}n,\delta}} \delta^{1-2s}\xi_n w_y^2 \, dx + \int_{F^1_{\sqrt{2}n,\delta}} \delta^{1-2s}\xi_n \frac{|\nabla w|^2}{2}\, dx \geq \frac{C}{\delta}, \quad \text{for all } \delta \in (0, \delta_0).
$$

Integrating the above inequality over $(0, \delta_0)$ and using the Fubini Theorem, we would reach a contradiction with $w \in H^1(B^+_R, y^{1-2s})$. Some computations lead to

$$
\begin{aligned}
\xi_n(x,0)\langle x, \nabla u\rangle(f(u) - a(x)u) &= \operatorname{div}\left[\xi_n(x,0)\left(F(u) - \frac{1}{2}a(x)u^2\right)x\right] \\
&\quad - \langle \nabla \xi_n(x,0), x\rangle F(u) - N\xi_n(x,0)F(u) + \frac{1}{2}\langle \nabla \xi_n(x,0), x\rangle a(x)u^2 \\
&\quad + \frac{1}{2}\xi_n(x,0)\langle \nabla a(x), x\rangle u^2 + \frac{N}{2}\xi_n(x,0)a(x)u^2.
\end{aligned}
$$

Thus, by Remark 4.2, condition (4.9) and the Divergence Theorem we have

$$
\begin{aligned}
\lim_{k\to\infty} I^1_{n,\delta_k} &= \kappa_s \int_{B^N_{\sqrt{2}n}} \xi_n(x,0)\langle x, \nabla u\rangle(f(u) - a(x)u)\, dx \\
&= -\kappa_s \int_{B^N_{\sqrt{2}n}} \langle \nabla \xi_n(x,0), x\rangle F(u) + N\xi_n(x,0)F(u)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\kappa_s}{2}\int_{B^N_{\sqrt{2}n}}\langle\nabla\xi_n(x,0),x\rangle a(x)u^2\,dx\\
&+\frac{\kappa_s}{2}\int_{B^N_{\sqrt{2}n}}\xi_n(x,0)\langle\nabla a(x),x\rangle u^2+\frac{N}{2}\,\xi_n(x,0)a(x)u^2\,dx.
\end{aligned}
$$

Summing up, we get

$$
\begin{aligned}
0 &= \lim_{k\to\infty}\big[I^1_{n,\delta_k}+I^2_{n,\delta_k}+I^3_{n,\delta_k}+\theta_{n,\delta_k}\big]\\
&= -\,\kappa_s\int_{B^N_{\sqrt{2}n}}\langle\nabla\xi_n,x\rangle F(u)+N\xi_nF(u)\,dx\\
&\quad+\kappa_s\int_{B^N_{\sqrt{2}n}}\frac{1}{2}\langle\nabla\xi_n,x\rangle a(x)u^2-\frac{1}{2}\xi_n\langle\nabla a(x),x\rangle u^2-\frac{N}{2}\,\xi_n a(x)u^2\,dx\\
&\quad+\int_{B_{\sqrt{2}n}}\frac{N-2s}{2}\,y^{1-2s}|\nabla w|^2\,\xi_n\,dx\,dy\\
&\quad+\int_{B_{\sqrt{2}n}}y^{1-2s}\frac{|\nabla w|^2}{2}\langle z,\nabla\xi_n\rangle-y^{1-2s}\langle\nabla w,z\rangle\langle\nabla w,\nabla\xi_n\rangle\,dx\,dy.
\end{aligned}
$$

Using the Dominated Convergence Theorem with $n\to\infty$, we conclude that

$$
\begin{aligned}
\frac{N-2s}{2}\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u\big|\,dx &= \frac{N-2s}{2\kappa_s}\int_{\mathbb{R}^N}y^{1-2s}|\nabla w|^2\,dx\,dy\\
&= N\int_{\mathbb{R}^N}F(u)\,dx-\frac{N}{2}\int_{\mathbb{R}^N}a(x)u^2-\frac{1}{2}\langle\nabla a(x),x\rangle u^2\,dx,
\end{aligned}
$$

where in the first equality we used condition (4.5). $\square$

Remark 6.12. In the previous proof we have applied [24, Theorem 2.15] and for that it was crucial that $a(x)$ is a continuously differentiable function in $\mathbb{R}^N\setminus\mathcal{O}$.

Corollary 6.13. *Assume* (f_1) *and* $f(x,t)\equiv f(t)\in C^1(\mathbb{R})$. *Moreover, let* $a(x)\equiv a_0$, *where* a_0 *is a positive constant. If* $u\in H^s(\mathbb{R}^N)$ *is a weak solution to* $(\mathcal{P}_s)$, *then*

$$
\int_{\mathbb{R}^N}F(u)-\frac{a_0}{2}\,u^2\,dx=\frac{N-2s}{2N}\int_{\mathbb{R}^N}\big|(-\Delta)^{s/2}u\big|^2\,dx.
$$

Corollary 6.14. *Assume* (3.1.2) *and let* $f(x,t)\equiv f(t)\in C^1(\mathbb{R})$. *If* $u\in\mathcal{D}^{s,2}(\mathbb{R}^N)$ *is a weak solution to* $(\mathcal{P}_s)$, *then for* $0<\lambda<\Lambda_{N,s}$ *given by* (1.1),

$$
\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2-\lambda|x|^{-2s}u^2\,dx=\frac{2N}{N-2s}\int_{\mathbb{R}^N}F(u)\,dx.
$$

Next we have nonexistence results, complementing the discussion from [18].

Corollary 6.15. *Assume* $f(x,t)\equiv f(t)\in C^1(\mathbb{R}^N)$ *and that either one of the following conditions is satisfied*:

(a) *$a(x) \in C^1(\mathbb{R}^N \setminus \mathcal{O})$, where $\mathcal{O}$ is a finite set, $2sa(x) + \langle \nabla a(x), x\rangle > 0$ for almost every $x \in \mathbb{R}^N$ and $2_s^* F(t) \le f(t)t$, for all $t \in \mathbb{R}$; or*

(b) *$a(x) \in C^1(\mathbb{R}^N \setminus \mathcal{O})$, where $\mathcal{O}$ is a finite set, $a(x) > 0$, $\langle \nabla a(x), x\rangle > 0$ for almost every $x \in \mathbb{R}^N$ and there exists $0 < \delta \le 2$ such that $\delta F(t) \ge f(t)t$, for all $t \in \mathbb{R}$; or*

(c) *$a(x) \equiv a_0 > 0$ constant and there exists $0 \le \delta \le 2s/(N-2s)$, in such a way that $2_s^* F(t) \le f(t)t + \delta a_0 t^2$, for all $t \in \mathbb{R}$; or*

(d) *$a(x) \equiv 0$ and there exists $0 < p < 2_s^*$ such that $pF(t) \ge f(t)t$ for all $t \in \mathbb{R}$.*

If $u \in H^s(\mathbb{R}^N)$ is a weak solution to $(\mathcal{P}_s)$ such that $F(u)$, $f(u)u$, $a(x)u^2$, and $\langle \nabla a(x), x\rangle u^2$ are in $L^1(\mathbb{R}^N)$, and $f(u)/(1+|u|)$ belongs to $L^{N/2s}_{\mathrm{loc}}(\mathbb{R}^N)$, then $u \equiv 0$.

PROOF. (a) Applying Proposition 6.11, we get

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2\,dx + \frac{N}{N-2s}\int_{\mathbb{R}^N} a(x)u^2\,dx + \frac{1}{N-2s}\int_{\mathbb{R}^N}\langle \nabla a(x), x\rangle u^2\,dx \le \int_{\mathbb{R}^N} f(u)u\,dx.$$

Using the fact that $I'(u)\cdot u = 0$ we obtain $u \equiv 0$, since

$$\int_{\mathbb{R}^N} (2sa(x) + \langle \nabla a(x), x\rangle) u^2\,dx \le 0.$$

(b) Using again Proposition 6.11 we obtain that

$$\frac{N-2s}{2N}\delta\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2} u\big|^2\,dx + \frac{\delta}{2}\int_{\mathbb{R}^N} a(x)u^2\,dx + \frac{\delta}{2N}\int_{\mathbb{R}^N}\langle \nabla a(x), x\rangle u^2\,dx \ge \int_{\mathbb{R}^N} f(u)u\,dx,$$

which implies that

$$\bigg(1 - \frac{N-2s}{2N}\,\delta\bigg)\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2} u\big|^2\,dx + \bigg(1-\frac{\delta}{2}\bigg)\int_{\mathbb{R}^N} a(x)u^2\,dx - \frac{\delta}{2N}\int_{\mathbb{R}^N}\langle \nabla a(x), x\rangle u^2\,dx \le 0.$$

From this we get $u \equiv 0$.

(c) Once more we can use Proposition 6.11 to get

$$\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2} u\big|^2\,dx + \frac{N}{N-2s}a_0\int_{\mathbb{R}^N} u^2\,dx \ge \int_{\mathbb{R}^N} f(u)u\,dx,$$

which yields

$$\frac{N-(1+\delta)(N-2s)}{N-2s}\,a_0\int_{\mathbb{R}^N} u^2\,dx \le 0.$$

In particular, $u \equiv 0$.

(d) Proposition 6.11 implies that $u \equiv 0$, since

$$\begin{aligned}\int_{\mathbb{R}^N} \big|(-\Delta)^{s/2} u\big|^2\,dx &= 2_s^* \int_{\mathbb{R}^N} F(u)\,dx \\ &\geq \frac{2_s^*}{p} \int_{\mathbb{R}^N} f(u)u\,dx = \frac{2_s^*}{p} \int_{\mathbb{R}^N} \big|(-\Delta)^{s/2} u\big|^2\,dx. \qquad \square\end{aligned}$$

7. Proof of Theorem 3.1

(a) We use Theorem 2.3, which makes our argument easier than the one found in [9, Theorem 2.1]. By Proposition 6.5, there exists a bounded sequence (u_k) such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$. Using the profile decomposition provided by Theorem 2.3, if we have $w^{(n)} = 0$ for all $n \in \mathbb{N}_0$, then by assertion (2.5), $u_k \to 0$ in $L^p(\mathbb{R}^N)$, for any $2 < p < 2_s^*$ and by (2.2), $u_k \rightharpoonup 0$ in $H_V^s(\mathbb{R}^N)$, up to a subsequence. Thus, by Proposition 6.6,

$$\begin{aligned} o(1) + c(I) = I(u_k) &= \frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x,u_k)\,dx = \frac{1}{2}\|u_k\|_V^2 + o(1), \\ o(1) = I'(u_k)\cdot u_k &= \|u_k\|_V^2 - \int_{\mathbb{R}^N} f(x,u_k)u_k\,dx = \|u_k\|_V^2 + o(1), \end{aligned} \tag{7.1}$$

which is a contradiction with $c(I) > 0$. Thus, there must be at least one nonzero $w^{(n)}$. Moreover, we have that each $w^{(n)}$ is a critical point of I. In fact, up to a subsequence, we can take $h^{(n)} \in L^{\sigma'}(\operatorname{supp}\varphi)$, $n \in \mathbb{N}_0$, such that

$$\big|u_k\big(x + y_k^{(n)}\big)\big| \leq h^{(n)}(x), \quad \text{a.e. } x \in \operatorname{supp}\varphi, \tag{7.2}$$

where $\sigma' = \sigma/(\sigma-1)$ and $\varphi \in C_0^\infty(\mathbb{R}^N)$, which can be done thanks to Proposition 6.1. Thus

$$\begin{aligned} \big|V\big(x + y_k^{(n)}\big)u_k\big(x + y_k^{(n)}\big)\varphi(x)\big| &= \big|V(x)u_k\big(x + y_k^{(n)}\big)\varphi(x)\big| \\ &\leq h^{(n)}(x)|V(x)\varphi(x)| \in L^1(\operatorname{supp}\varphi) \\ V\big(x + y_k^{(n)}\big)u_k\big(x + y_k^{(n)}\big)\varphi(x) &= V(x)u_k\big(x + y_k^{(n)}\big)\varphi(x) \to V(x)w^{(n)}(x)\varphi(x), \end{aligned}$$

almost everywhere in $\mathbb{R}^N$, which, together with the Dominated Convergence Theorem, implies

$$\begin{aligned} &\lim_{k\to\infty} \big(u_k, \varphi\big(\cdot - y_k^{(n)}\big)\big)_V \\ &\quad = \lim_{k\to\infty} \bigg[\big[u_k\big(\cdot + y_k^{(n)}\big), \varphi\big]_s + \int_{\mathbb{R}^N} V\big(x + y_k^{(n)}\big)u_k\big(\cdot + y_k^{(n)}\big)\varphi(x)\,dx\bigg] \\ &\quad = [w^{(n)}, \varphi]_s + \int_{\mathbb{R}^N} V(x)w^{(n)}\varphi\,dx. \end{aligned}$$

By the same reason and (f_1), up to a subsequence, we have

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} f\big(x + y_k^{(n)}, u_k\big(\cdot + y_k^{(n)}\big)\big)\varphi\,dx = \int_{\mathbb{R}^N} f(x, w^{(n)})\varphi\,dx.$$

Consequently, we may pass to the limit in

$$I'(u_k)\cdot\varphi\big(\,\cdot\,-y_k^{(n)}\big)=\big(u_k,\varphi\big(\,\cdot\,-y_k^{(n)}\big)\big)_V-\int_{\mathbb{R}^N}f\big(x+y_k^{(n)},u_k\big(\,\cdot\,+y_k^{(n)}\big)\big)\varphi\,dx,$$

to conclude that $I'(w^{(n)})=0$, for all $n\in\mathbb{N}_0$. In particular, we get that $\mathcal{G}_{\mathcal{S}}:=\inf\{I(u):u\in H_V^s(\mathbb{R}^N)\setminus\{0\},\ I'(u)=0\}$ is nonnegative. We are going to prove that is $\mathcal{G}_{\mathcal{S}}$ is attained and is positive. Let (u_k) be a minimizing sequence for $\mathcal{G}_{\mathcal{S}}$, that is, $I(u_k)\to\mathcal{G}_{\mathcal{S}}$ and $I'(u_k)=0$. Arguing as in Proposition 6.5, we obtain that (u_k) is bounded. Suppose by contradiction that $w^{(n)}=0$ for all $n\in\mathbb{N}_0$. In this case we have $\mathcal{G}_{\mathcal{S}}>0$, otherwise, if $\mathcal{G}_{\mathcal{S}}=0$, then using (7.1) we would conclude that $\|u_k\|_V=o(1)$, and at the same time,

$$\|u_k\|_V^2=\int_{\mathbb{R}^N}f(u_k)u_k\,dx\le\varepsilon\big(C_2\|u_k\|_V^2+C_*\|u_k\|_V^{2_s^*}\big)+C_\varepsilon\|u_k\|_V^{p_\varepsilon},$$

where $\mathcal{C}_2,\mathcal{C}_{2_s^*},\mathcal{C}_{p_\varepsilon}>0$ would be the constants given in Proposition 6.1. In particular, $(1-\varepsilon\,\mathcal{C}_2)\le\varepsilon\,\mathcal{C}_{2_s^*}\|u_k\|_V^{2_s^*-2}+\mathcal{C}_{p_\varepsilon}\|u_k\|_V^{p_\varepsilon-2}$, for all $k\in\mathbb{N}$, which, by taking ε small enough, would lead to a contradiction with the fact that $\|u_k\|_V=o(1)$. In view of that, in any case, we can argue as above to conclude that there must be a nonzero $w^{(n_0)}$ which is a critical point of I. From (2.2), $u_k\big(x+y_k^{(n_0)}\big)\to w^{(n_0)}(x)$ almost everywhere in $\mathbb{R}^N$. Thus

$$\begin{aligned}\mathcal{G}_{\mathcal{S}}&=\lim_{k\to\infty}I(u_k)=\lim_{k\to\infty}\int_{\mathbb{R}^N}\mathcal{F}\big(x,u_k\big(\,\cdot\,+y_k^{(n_0)}\big)\big)\,dx\\&=\liminf_{k\to\infty}\int_{\mathbb{R}^N}\mathcal{F}\big(x,u_k\big(\,\cdot\,+y_k^{(n_0)}\big)\big)\,dx\ge\int_{\mathbb{R}^N}\mathcal{F}(x,w^{(n_0)})\,dx=I(w^{(n_0)}),\end{aligned}$$

where we have used (f_2) or (f_5) to ensure that $\mathcal{F}\big(x,u_k\big(\,\cdot+y_k^{(n_0)}\big)\big)=\mathcal{F}(x,u_k)\ge0$ almost everywhere in $\mathbb{R}^N$. Thus, once again using (f_2) or (f_5), we can see that $\mathcal{G}_{\mathcal{S}}=I(w^{(n_0)})>0$.

(b) From Proposition 6.1, the norm

$$|\!|\!|u|\!|\!|_\lambda^2=\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u|^2-\lambda|x|^{-2s}u^2\,dx,\qquad u\in\mathcal{D}^{s,2}(\mathbb{R}^N),\quad 0<\lambda<\Lambda_{N,s},$$

is equivalent to the norm $[\,\cdot\,]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Let (u_k) be a minimizing sequence for $\mathcal{I}_\lambda$, and for each k, let u_k^* be the Schwarz symmetrization of u_k. Applying the fractional Pólya–Szegő inequality (see [3, Theorem 3]), for each k,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_k^*(x)-u_k^*(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_k(x)-u_k(y)|^2}{|x-y|^{N+2s}}\,dx\,dy,$$
$$\int_{\mathbb{R}^N}F(u_k^*)\,dx=\int_{\mathbb{R}^N}F(u_k)\,dx.$$

Thus $(u_k^*)\subset\mathcal{D}_{\mathrm{rad}}^{s,2}(\mathbb{R}^N)$ and is also a minimizing sequence for (3.3). Now observe that $|\!|\!|\cdot|\!|\!|_\lambda$ is invariant with respect to the action of dilations given in

Theorem 2.1, more precisely,

$$|||u|||_{\lambda}^{2} = \Big|\Big|\Big|\gamma^{(N-2s)/2}u(\gamma^{j}\,\cdot\,)\Big|\Big|\Big|_{\lambda}^{2}, \quad \text{for all } u \in \mathcal{D}^{s,2}(\mathbb{R}^N),\ \gamma > 1 \text{ and } j \in \mathbb{Z},$$

and satisfies the homogeneity property, $|||u(\,\cdot\,/\delta)|||_{\lambda}^{2} = \delta^{N-2s}|||u|||_{\lambda}^{2}$, $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $\delta > 0$. In view of Remark 2.2 and Corollary 6.14, the proof now follows the same argument of [16, Theorem 3.4], with $[\,\cdot\,]_s$ replaced by $|||\cdot|||_{\lambda}$. $\square$

Remark 7.1. (a) In the context of the proof of Theorem 3.1 (a), if we assume in addition that $f(x,t)$ satisfies (3.4), then $\mathcal{G}_{\mathcal{S}} = c(I) = I(w^{n_0})$ and $w^{(n_0)}$ is nonnegative. Indeed, the truncation given in Remark 4.3 satisfies the assumptions of Theorem 3.1 (a), and we can apply the same argument there, to conclude that the ground state $w^{(n_0)}$ is nonnegative. Furthermore, Remark 6.4 (d) guarantees that the path $\zeta(t) = tw^{(n_0)}$, $t \geq 0$, belongs to Γ_I and $c(I) \leq I(w^{(n_0)})$. On the other hand, considering (u_k) given in the beginning of the proof of Theorem 3.1, by Corollary 6.8, Remark 6.2 (b) and estimate (2.4), up to a subsequence, we have

$$c(I) = \lim_{k\to\infty}\bigg[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x,u_k)\,dx\bigg] \geq \sum_{n\in\mathbb{N}_0} I(w^{(n)}).$$

Consequently, using (f_2) or (f_5) to get $I(w^{(n)}) \geq 0$, we conclude that $c(I) = \mathcal{G}_{\mathcal{S}}$.

(b) If we consider the infimum (3.3) defined over $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$, by Remark 2.2 we can obtain concentration–compactness of the minimizing sequences as described in [16, Theorem 3.4]. More precisely, for any minimizing sequence (u_k) of (3.3), there exists a sequence (j_k) in $\mathbb{Z}$ such that the sequence $(\gamma^{-(N-2s)/2j_k}u_k(\gamma^{-j_k}\,\cdot\,))$ contains a convergent subsequence in $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$, whose limit is a minimizer of (3.3) in $\mathcal{D}^{s,2}_{\mathrm{rad}}(\mathbb{R}^N)$.

(c) In the context of the proof of Theorem 3.1 (ii), assume that $F(t) \geq 0$ for all $t \geq 0$. Since $|||\,|u_k|\,|||_{\lambda} \leq |||u_k|||_{\lambda}$, without loss of generality we can assume that each u_k is nonnegative. In this case, the obtained minimizer for (3.3) is nonnegative.

8. Proof of Theorem 3.2

As mentioned before, we prove Theorem 3.2 by using the Nehari manifold method (see [45]). For reader's convenience the proof will be divided into several steps.

Step 1. For each $u \in H^s_V \setminus \{0\}$ there exists a unique $\tau(u) > 0$ such that $\tau(u)u \in \mathcal{N}$ and $\max_{t\geq 0} I(tu) = I(\tau(u)u)$. In particular, $\mathcal{N} \neq \emptyset$. To see that the function $h_u(t) = I(tu)$, $t > 0$, has a maximum point t_u, we proceed in a similar way as in the Remark 6.4 (d). Moreover, $h'(t_u) = 0$ if and only if $t_u u$ belongs to $\mathcal{N}$ and

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2\,dx = \frac{1}{t_u}\int_{\mathbb{R}^N} f(x,t_u u)u\,dx. \tag{8.1}$$

By (3.4) the right-hand side of the above identity occurs at most one point. Thus there is a unique maximum point $\tau(u) = t_u$ for the function $h_u(t)$.

Step 2. The function $\tau\colon H_V^s \setminus \{0\} \to (0,\infty)$ is continuous. Thus the map $\eta\colon H_V^s \setminus \{0\} \to \mathcal{N}$, defined by $\eta(u) = \tau(u)u$ is continuous and $\eta\big|_{\mathcal{S}}$ is a homeomorphism of the unit sphere $\mathcal{S}$ of $H_V^s(\mathbb{R}^N)$ in $\mathcal{N}$. Assume that $u_n \to u$ in $H_V^s \setminus \{0\}$. From $F(x,t) > 0$ and (f_2) we get $F(x,t) \ge C_1|t|^\mu - C_2 t^2$, for almost every $x \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$. Thus, from identity (8.1) we obtain that

$$\|u_n\|_V^2 - \int_{\mathbb{R}^N} b(x)u_n^2\,dx \ge C_1|\tau(u_n)|^{\mu-2}\int_{\mathbb{R}^N}|u_n|^\mu\,dx - C_2\|u_n\|_V^2,$$

for all $n \in \mathbb{N}$, that is, $(u_n) \subset L^\mu(\mathbb{R}^N)$ with

$$\|u_n\|_V^2 \ge C|\tau(u_n)|^{\mu-2}\int_{\mathbb{R}^N}|u_n|^\mu\,dx, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, since $u \ne 0$, we get $\|u_n\| > C > 0$ for all n. Thus $(\tau(u_n))$ is bounded. We next prove that any subsequence of $(\tau(u_n))$ has a convergent subsequence with the same limit $\tau(u)$, which implies that $\tau(u_n) \to \tau(u)$. It is clear that, for a subsequence, $\tau(u_n) \to t_0 > 0$. In fact, using (f_1), (V_3), and (8.1),

$$\begin{aligned}\|u_n\|_V^2 &- \int_{\mathbb{R}^N} b(x)u_n^2\,dx \\ &\le \varepsilon C\big(\|u_n\|_V^2 + \tau(u_n)^{2_s^*-2}\|u_n\|_V^{2_s^*}\big) + C_\varepsilon \tau(u_n)^{p_\varepsilon-2}\|u_n\|_V^{p_\varepsilon},\end{aligned}$$

for all $n \in \mathbb{N}$, from which we obtain

$$\begin{aligned}&\bigg(1 - \varepsilon\mathcal{C}_2 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}}\bigg)\|u_n\|_V^2 \\ &\qquad\le \varepsilon\mathcal{C}_{2_s^*}\tau(u_n)^{2_s^*-2}\|u_n\|_V^{2_s^*} + C_\varepsilon\,\mathcal{C}_{p_\varepsilon}\tau(u_n)^{p_\varepsilon-2}\|u_n\|_V^{p_\varepsilon},\end{aligned} \tag{8.2}$$

for all $n \in \mathbb{N}$, which implies $t_0 > 0$, by taking ε small enough. Thus we may apply the Dominated Convergence Theorem in (8.1) to conclude that $t_0 = \tau(u)$ and τ is continuous. Using (8.1) to compute $\tau(u/\|u\|_V)$ we obtain that

$$\|u\|_V^2 - \int_{\mathbb{R}^N} b(x)u^2\,dx = \frac{1}{\tau(u/\|u\|_V)/\|u\|_V}\int_{\mathbb{R}^N} f\bigg(x, \frac{\tau(u/\|u\|_V)}{\|u\|_V}\bigg)u\,dx,$$

which by uniqueness gives $\tau(u/\|u\|_V) = \tau(u)u$. Consequently, the inverse of η is the retraction map given by $\varrho\colon \mathcal{N} \to \mathcal{S}$, $\varrho(u) = u/\|u\|_V$.

Step 3. $\mathcal{N}$ is away from the origin, that is, there is $R_{\mathcal{N}} > 0$ such that $\|u\|_V > R_{\mathcal{N}} > 0$ if $u \in \mathcal{N}$. Indeed, the estimate (8.2) implies that

$$1 - \varepsilon\mathcal{C}_2 - \frac{\|b(x)\|_\beta}{\mathcal{C}_V^{(\beta)}} \le \varepsilon\mathcal{C}_{2_s^*}\|u\|_V^{2_s^*-2} + C_\varepsilon\mathcal{C}_{p_\varepsilon}\|u\|_V^{p_\varepsilon-2}, \quad \text{for all } u \in \mathcal{N}.$$

Step 4. For all $\zeta \in \Gamma_I$ we have that $\zeta([0,\infty)) \cap \mathcal{N} \neq \emptyset$. Let us suppose that this assertion is false, that is, there exists $\zeta_0 \in \Gamma_I$ which does not intersect $\mathcal{N}$ at any point. Let $t_0 > 0$ be such that $I(\zeta_0(t_0)) < 0$ and $\zeta_0(t) \neq 0$, for all $(0, t_0]$. We prove now that $\tau(\zeta(t)) > 1$ for all $t \in (0, t_0]$. In fact, by continuity, there is $\delta > 0$ such that $\|\zeta_0(t)\| < R_{\mathcal{N}}$, for all $t \in [0, \delta]$. We also have that $\|\tau(\zeta_0(t))\zeta_0(t)\|_V > R_{\mathcal{N}}$, which implies $\tau(\zeta_0(t)) > 1$, for all $t \in (0, \delta]$. The continuity of $\tau(t)$ and the fact that $\zeta_0(t) \notin \mathcal{N}$, for all t, allows us to choose $\delta = t_0$. On the other hand, by (f_2) and (3.4),

$$\begin{aligned} h_{\zeta(t_0)}(t) &\geq \frac{t^2}{2}\bigg[\|\zeta_0(t_0)\|_V^2 - \int_{\mathbb{R}^N} b(x)|\zeta_0(t_0)|^2\,dx - \frac{2}{\mu}\int_{\mathbb{R}^N} \frac{f(x, t\zeta_0(t_0))}{t\zeta_0(t_0)}|\zeta_0(t_0)|^2\,dx\bigg] \\ &> \frac{t^2}{2}\bigg[\int_{\mathbb{R}^N} \frac{f(x, \tau(\zeta_0(t_0))\zeta_0(t_0))}{\tau(\zeta_0(t_0))\zeta_0(t_0)}|\zeta_0(t_0)|^2 - \frac{f(x, t\zeta_0(t_0))}{t\zeta_0(t_0)}|\zeta_0(t_0)|^2\,dx\bigg] > 0, \end{aligned}$$

for all $t \in (0, \tau(\zeta(t_0))]$. In particular, $0 < h_{\zeta(t_0)}(1) = I(\zeta_0(t_0))$, which is a contradiction with the choice of $\zeta_0(t_0)$.

Step 5. $c_{\mathcal{N}}(I) = \overline{c}(I)$. In fact, since $\eta\big|_{\mathcal{S}}$ is a homeomorphism, we have

$$\overline{c}(I) = \inf_{u \in H_V^s \setminus \{0\}} I(\tau(u)u) = \inf_{u \in \mathcal{S}} I(\tau(u)u) = c_{\mathcal{N}}(I).$$

Step 6. $\overline{c}(I) = c(I)$. Given $u \in H_V^s \setminus \{0\}$, define the path $\zeta(t) = tt_0u$, where $t_0 > 0$ is chosen in such way that $I(t_0u) < 0$. Then, by Remark 6.4 (d), it is easy to see that $\zeta \in \Gamma_I$ and $\max_{t\geq 0} I(tu) = \max_{t \geq 0} I(\zeta(t)) \geq c(I)$. Consequently, $c(I) \leq \overline{c}(I)$. On the other hand, given $\zeta \in \Gamma_I$, there exists t_0 such that $\zeta(t_0)$ belongs to $\mathcal{N}$. Thus, $\max_{t \geq 0} I(\zeta(t)) \geq I(\zeta(t_0)) \geq c_{\mathcal{N}}(I) = \overline{c}(I)$. Since $\zeta \in \Gamma_I$ is arbitrary, $c(I) \geq \overline{c}(I)$. □

Remark 8.1. In view of Remark 4.3, if $b(x) \equiv 0$, then the radial ground state u obtained in Theorem 3.2 can be considered as being nonnegative.

9. Proof of Theorem 3.3

Before the proof of Theorem 3.3, to complement our discussion, we are going to compare the minimax level of limit functionals $I_{\mathcal{P}}$ and I_∞ with the minimax level of the energy functional I associated with equation $(\mathcal{P}_s)$. Some arguments used to prove this result of comparison are used in the proof of Theorem 3.3.

Proposition 9.1. *Assume that $f(x,t)$ satisfies either* (f_1)–(f_3), (f_7) *or* (f_3)–(f_6), (f_7). *Moreover, suppose that $b(x) \equiv 0$,* (V_1)–(V_2) *and* (f_8) *hold. Then $c(I) \leq c(I_{\mathcal{P}})$. Alternatively, if instead of the last set of hypotheses we assume that $V(x) \geq 0$, $b(x)$ has compact support,* (V_2)–(V_4) *and* (f_9) *hold, then $c(I) \leq c(I_\infty)$. Moreover, under these conditions, if we assume* (3.6), *then* (f_{10}) *and* (f'_{10}) *hold true respectively for each considered case.*

Proof. (a) Let $u \in H^s_V(\mathbb{R}^N)$ be a nonnegative (see Remark 4.3) nontrivial weak solution to $(-\Delta)^s u + V(x)u = f_{\mathcal{P}}(x,u)$, at the mountain pass level for $I_{\mathcal{P}}$, that is, $I_{\mathcal{P}}(u) = c(I_{\mathcal{P}})$. For each k, we define the path $\zeta_k(t) = tu(\,\cdot\, - y_k)$, $t \geq 0$, where (y_k) is taken so that $|y_k| \to \infty$. The idea is to prove that

$$c(I) \leq \limsup_{k\to\infty} \max_{t\geq 0} I(\zeta_k(t)) \leq \max_{t\geq 0} I_{\mathcal{P}}(tu) = c(I_{\mathcal{P}}). \tag{9.1}$$

In fact, taking into account that I and $I_{\mathcal{P}}$ are locally Lipschitz sets of $H^s_V(\mathbb{R}^N)$ (they are C^1 in $H^s_V(\mathbb{R}^N)$) and the following estimate,

$$|I(\zeta_k(t)) - I_{\mathcal{P}}(tu)| \leq \int_{\mathbb{R}^N} |F(x+y_k, tu) - F_{\mathcal{P}}(x+y_k, tu)|\,dx,$$

by using a density argument we get that $\lim_{k\to\infty} I(\zeta_k(t)) = I_{\mathcal{P}}(tu)$, uniformly in compact sets of $\mathbb{R}$. Consequently, we may proceed as in [16, Proposition 9.1]. First note that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} F(x+y_k, tu)\,dx = \int_{\mathbb{R}^N} F_{\mathcal{P}}(x, tu)\,dx, \quad \text{for each } t > 0.$$

In particular,

$$\int_{\mathbb{R}^N} F(x+y_k, u)\,dx > 0, \quad \text{for } k \text{ large enough.}$$

Thus, using either (f_1)–(f_3) or (f_3)–(f_6) and the arguments of Remark 6.4 (d), we see that, for k large enough, ζ_k belongs to Γ_I. As a consequence, there exists $t_k > 0$ such that $I(\zeta_k(t_k)) = \max_{t\geq 0} I(\zeta_k(t)) > 0$. We claim that (t_k) is bounded. In fact, assume by contradiction that, up to a subsequence, $t_k \to \infty$. Thus, by the arguments of Remark 6.4 (d) we get

$$I(\zeta_k(t_k)) = \frac{t_k^2}{2}\|u\|_V^2 - \int_{\mathbb{R}^N} F(x+y_k, t_k u)\,dx \to -\infty, \quad \text{as } t \to \infty,$$

which leads to a contradiction with the fact that $I(\zeta_k(t_k)) > 0$ for all k. Therefore, up to a subsequence, $t_k \to t_0$, and thus $\lim\limits_{k\to\infty} \max_{t\geq 0} I(\zeta_k(t_k)) = I_{\mathcal{P}}(t_0 u)$, which gives us (9.1).

(b) The second case is proved in a similar way. Let $w \in H^s_V(\mathbb{R}^N) = H^s(\mathbb{R}^N)$ be a nontrivial weak solution to the equation $(-\Delta)^s w + V_\infty w = f_\infty(w)$, at the mountain pass level, more precisely, $I_\infty(w) = c(I_\infty)$. For each k, define the path $\lambda_k(t) = w((\,\cdot\, - y_k)/t)$, $t \geq 0$. where (y_k) is chosen in a such way that $|y_k| \to \infty$. As before, we consider the estimate

$$\begin{aligned} |I(\lambda_k(t)) - I_\infty(w(\,\cdot\,/t))| \leq{}& \frac{t^N}{2}\int_{\mathbb{R}^N} |(V(tx+y_k) - b(tx+y_k)) - V_\infty| w^2\,dx \\ &+ t^N \int_{\mathbb{R}^N} |F(tx+y_k, w) - F_\infty(w)|\,dx, \end{aligned}$$

and the fact that I and I_∞ are Lipschitz in bounded sets of $H^s(\mathbb{R}^N)$ to obtain, by a density argument, that $\lim\limits_{k\to\infty} I(\lambda_k(t)) = I_\infty(w(\,\cdot\,/t))$, uniformly in compact sets of $\mathbb{R}$. We also have that the path λ_k belongs to Γ_I, for k large enough. In fact, assuming the contrary, we would obtain k_0 and a sequence $t_n \to \infty$ such that $I(\lambda_{k_0}(t_n)) > 0$, for all n. On the other hand, we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(t_n x + y_{k_0}, w) - \frac{1}{2}\,[V(t_n x + y_{k_0}) - b(t_n x + y_{k_0})]\, w^2\, dx = \int_{\mathbb{R}^N} F_\infty(w) - \frac{1}{2}\, V_\infty w^2\, dx,$$

which would lead to the contradiction $I(\lambda_{k_0}(t_n)) < 0$, if n is large enough. Let $t_k > 0$ be such that $I(\lambda_k(t_k)) = \max_{t\geq 0} I(\lambda_k(t)) > 0$. We claim that (t_k) is bounded. On the contrary,

$$\begin{aligned} 0 < I(\lambda_k(t_{n_k})) = \frac{1}{2}\, t_{n_k}^{N-2s}[w]_s^2 - t_{n_k}^N \bigg[\int_{\mathbb{R}^N} & F(t_{n_k} x + y_k, w) \\ & - \frac{1}{2}(V(t_{n_k} x + y_k) - b(t_{n_k} x + y_k) w^2\, dx\bigg] \to -\infty, \end{aligned}$$

as $k \to \infty$, which is impossible. Thus, up to a subsequence, $t_k \to t_0$ and

$$\lim_{k\to\infty} \max_{t\geq 0} I(\lambda_k(t)) = I_\infty(w(\,\cdot\,/t_0)).$$

As a consequence,

$$c(I) \leq \lim_{k\to\infty} \max_{t\geq 0} I(\lambda_k(t_k)) \leq \max_{t\geq 0} I_\infty(w(\,\cdot\,/t)) = c(I_\infty),$$

where we have used Corollary 6.13 to conclude that $t = 1$ is the unique critical point of $I_\infty(w(\,\cdot\,/t))$. Now assume (3.6). Considering the above discussion, for each case respectively, we have for k large enough:

$$\begin{aligned} c(I) &\leq \max_{t\geq 0} I(\zeta_k(t)) = I(t_k u(\,\cdot\, - y_k)) < I_{\mathcal{P}}(t_k u) \leq \max_{t\geq 0} I_{\mathcal{P}}(t_k u) = c(I_{\mathcal{P}}), \\ c(I) &\leq \max_{t\geq 0} I(\lambda_k(t)) = I(u((\,\cdot\, - y_k\,)/t_k)) \\ &\quad < I_\infty(u(\,\cdot\,/t_k)) \leq \max_{t\geq 0} I_\infty(u(\,\cdot\,/t_k)) = c(I_\infty). \qquad \square \end{aligned}$$

In order to prove our existence result without using the compactness conditions (f_{10}) and (f'_{10}), we use the argument of [9, proof of Theorem 1.2]. Thus we evoke [27, Theorem 2.3], for the existence of a critical point of I whenever the minimax level (3.1) is attained (see Remark 6.4 (a)).

Now, we are going to complete the proof of Theorem 3.3. In view of Lemma 6.3 and Proposition 6.5, there exists a bounded sequence (u_k) such that $I(u_k) \to c(I)$ and $I'(u_k) \to 0$, for both cases of this theorem. Let $(w^{(n)})$ and $(y_k^{(n)})$ be the sequences given in Theorem 2.3 for the sequence (u_k). The underlying main idea to prove the concentration-compactness of Theorem 3.3 follows the same one of [16, Theorem 3.6] which we shall now describe: we prove that $w^{(n)} = 0$ for all $n \geq 2$, which by assertions (2.2), (2.5) and Proposition 6.6 imply that $u_k \to w^{(1)}$ in $H^s_V(\mathbb{R}^N)$, up to a subsequence. To this end, we argue by contradiction and assume the existence of at least one $w^{(n_0)} \neq 0$, $n_0 \geq 2$.

(a) In view of Remark 6.2 (b), by Proposition 6.7 and (2.4), up to a subsequence,

$$(9.2)\quad c(I) = \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} F(x,u_k)\,dx\right] \geq I(w^{(1)}) + \sum_{n\in\mathbb{N}_0,\, n>1} I_{\mathcal{P}}(w^{(n)}).$$

We claim that the terms on the right-hand side of (9.2) are nonnegative. Indeed, following the proof of Theorem 3.1, $w^{(1)}$ and $w^{(n)}$, $n \geq 2$, are critical points for I and $I_{\mathcal{P}}$, respectively. In view of that, (f_2) or (f_5) imply that $I(w^{(1)}) \geq 0$ and $I_{\mathcal{P}}(w^{(n)}) \geq 0$, $n \geq 2$, respectively. On the other hand, Remark 6.4 (d) guarantees that $\zeta^{(n_0)}(t) = tw^{(n_0)} \in \Gamma_{I_{\mathcal{P}}}$ and $c(I_{\mathcal{P}}) < I_{\mathcal{P}}(w^{(n_0)})$. This, together with (9.2) and (f_{10}), leads to a contradiction.

(b) Following the proof of Theorem 2.3, we can replace $\|\cdot\|$ by the equivalent norm $\|\cdot\|_{V_\infty}$ in assertions (2.2)–(2.5). Consequently, by (2.4), Propositions 6.7 and 6.10, up to a subsequence,

$$\begin{aligned}(9.3)\qquad c(I) &\geq \lim_{k\to\infty}\left[\frac{1}{2}\|u_k\|_V^2 - \int_{\mathbb{R}^N} b(x)u_k^2\,dx - \int_{\mathbb{R}^N} F(x,u_k)\,dx\right]\\ &\geq I(w^{(1)}) + \sum_{n\in\mathbb{N}_0,\, n>1} I_\infty(w^{(n)}).\end{aligned}$$

Thus, it suffices to prove that the right-hand side of (9.3) is nonnegative and $I_\infty(w^{(n)}) \geq c(I_\infty)$, for all $n \geq 2$. In fact, $c(I) \geq I(w^{(n_0)}) \geq c(I_\infty)$, which leads to a contradiction with (f_{10}). To do this, we prove that $w^{(1)}$ and $w^{(n)}$ are critical points for I and I_∞ respectively, $n \geq 2$. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $h^{(n)} \in L^{2_s^*-1}(\operatorname{supp}\varphi)$ as in (7.2). By (V_4) and (2.3), there exists $k_0 = k_0(\varphi)$ such that $V\big(x + y_k^{(n)}\big) < 1 + V_\infty$, for all $k > k_0$, $x \in \operatorname{supp}\varphi$ and $n \geq 2$. Thus,

$$\big|V\big(x+y_k^{(n)}\big)u_k\big(x+y_k^{(n)}\big)\varphi(x)\big| \leq (\varepsilon + V_\infty)h^{(n)}(x)|\varphi(x)| \in L^1(\operatorname{supp}\varphi)$$

for $k > k_0$, and

$$V\big(x+y_k^{(n)}\big)u_k\big(x+y_k^{(n)}\big)\varphi(x) \to V_\infty w^{(n)}(x)\varphi(x) \quad \text{a.e. in } \mathbb{R}^N,$$

which, together with the Dominated Convergence Theorem, implies

$$\begin{aligned}&\lim_{k\to\infty}\big(u_k, \varphi\big(\cdot - y_k^{(n)}\big)\big)_V\\ &= \lim_{k\to\infty}\left[\big[u_k\big(\cdot + y_k^{(n)}\big),\varphi\big]_s + \int_{\mathbb{R}^N} V\big(x+y_k^{(n)}\big)u_k\big(\cdot + y_k^{(n)}\big)\varphi(x)\,dx\right]\\ &= [w^{(n)},\varphi]_s + \int_{\mathbb{R}^N} V_\infty w^{(n)}(x)\varphi(x)\,dx\end{aligned}$$

and, for the same reason,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N} f\big(x+y_k^{(n)}, u_k\big(\cdot + y_k^{(n)}\big)\big)\varphi\,dx = \int_{\mathbb{R}^N} f_\infty(w^{(n)})\varphi\,dx.$$

Consequently, taking the limit in

$$I'(u_k)\cdot\varphi(\,\cdot\,-y_k^{(n)}) = \big(u_k,\varphi(\,\cdot\,-y_k^{(n)})\big)_V - \int_{\mathbb{R}^N} f\big(x+y_k^{(n)},u_k(\,\cdot\,+y_k^{(n)})\big)\varphi\,dx,$$

we deduce that $I'(w^{(1)})=0$ and $I'_\infty(w^{(n)})=0$, $n\geq 2$. Using (f_2) or (f_5) we get that $I(w^{(1)})\geq 0$ and $I_\infty(w^{(n)})\geq 0$, $n\geq 2$. Finally, define $\lambda^{(n_0)}(t)=w^{(n_0)}(\,\cdot\,/t)$, $t\geq 0$. By Corollary 6.13,

$$I_\infty(\lambda^{(n_0)}(t)) = \frac{1}{2}t^{N-2s}[w^{(n_0)}]_s^2 - t^N\bigg[\int_{\mathbb{R}^N} F_\infty(w^{(n_0)}) - \frac{V_\infty}{2}|w^{(n_0)}|^2\,dx\bigg] \to -\infty,$$

as $t\to\infty$, which, together with Remark 6.2, implies that $\lambda^{(n_0)}\in\Gamma_{I_\infty}$. Corollary 6.13 gives that $t=1$ is the unique critical point of $I_\infty(\lambda^{(n_0)}(t))$. Thus, $c(I_\infty)<\max\limits_{t\geq 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)})$, a contradiction.

(c) Finally, assume (3.5) instead of (f_{10}) and (f'_{10}). Consider the existence of $w^{(n_0)}\neq 0$, $n_0\in\mathbb{N}_0$, and the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ as above. Taking into account the above discussion, by estimates (9.2) and (9.3), for each case we have

$$\begin{aligned}
c(I) &\leq \max_{t\geq 0} I(\zeta^{(n_0)}(t)) \leq \max_{t\geq 0} I_{\mathcal{P}}(\zeta^{(n_0)}(t)) = I_{\mathcal{P}}(w^{(n_0)}) \leq c(I),\\
c(I) &\leq \max_{t\geq 0} I(\lambda^{(n_0)}(t)) \leq \max_{t\geq 0} I_\infty(\lambda^{(n_0)}(t)) = I_\infty(w^{(n_0)}) \leq c(I),
\end{aligned}$$

where we have used (3.5) to ensure that the paths $\zeta^{(n_0)}$ and $\lambda^{(n_0)}$ belong to Γ_I. Thus, we have that the minimax level $c(I)$ is attained and we can apply [27, Theorem 2.3] to obtain the existence of a critical point u for I_λ with $I_\lambda(u)=c(I_\lambda)$. If there is no $w^{(n)}\neq 0$, $n\in\mathbb{N}_0$, (which is the case where strict inequalities occur) we can obtain that $u_k\to w^{(1)}$, up to a subsequence. $\square$

10. Proof of Theorem 3.4

The proof will be divided into three steps. We first assume ($\mathscr{H}^*$) and ($\mathscr{H}_0^*$).

(a) We can proceed analogously as in the proof of Lemma 6.3, to see that there exists a sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I_*(u_k)\to c(I_*)>0$ and $I'_*(u_k)\to 0$. Let $(w^{(n)})$, $\big(y_k^{(n)}\big)$, $\big(j_k^{(n)}\big)$ be the sequences given by Theorem 2.1 and define the set $\mathbb{N}_\sharp = \big\{n\in\mathbb{N}_*\setminus\{1\} : \big|\gamma^{j_k^{(n)}}\big(y_k^{(n)}-a_*\big)\big| \text{ is bounded}\big\}$.

Passing to a subsequence and using a diagonal argument if necessary, we may assume that each sequence $\big(\gamma^{j_k^{(n)}}y_k^{(n)}\big)$, $n\in\mathbb{N}_\sharp$, converges with

$$a^{(n)} := \lim_{k\to\infty}\gamma^{j_k^{(n)}}\big(y_k^{(n)}-a_*\big),\quad n\in\mathbb{N}_\sharp.$$

(b) Now we shall prove the following estimate, up to a subsequence.

$$\begin{aligned}
\limsup_k \|u_k\|_V^2 \geq \big\|w^{(1)}\big\|_V^2 &+ \sum_{n\in\mathbb{N}_*\setminus\mathbb{N}_\sharp}[w^{(n)}]_s^2\\
&+ \sum_{n\in\mathbb{N}_+\cap\mathbb{N}_\sharp}\|w^{(n)}\|^2_{V_+(\,\cdot\,+a^{(n)}-a_*)} + \sum_{n\in\mathbb{N}_-\cap\mathbb{N}_\sharp}\|w^{(n)}\|^2_{V_-(\,\cdot\,+a^{(n)}-a_*)}.
\end{aligned}\tag{10.1}$$

In order to prove this, first consider the operator

$$d_k^{(n)}u = \gamma^{(N-2s)j_k^{(n)}/2} u\big(\gamma^{j_k^{(n)}}\big(\,\cdot\, - y_k^{(n)}\big)\big), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),\ n \in \mathbb{N}_*.$$

For each $n \in \mathbb{N}_*$, let $(\varphi_j^{(n)})$ in $C_0^\infty(\mathbb{R}^N)$ be such that $\varphi_j^{(n)} \to w^{(n)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Evaluating

$$\Big\| u_k - \sum_{n\in M_*} d_k^{(n)}\varphi_j^{(n)} \Big\|_V^2 \geq 0,$$

in a finite subset $M_* = \{1,\dots,M\}$ of $\mathbb{N}_*$, we have

$$\|u_k\|_V^2 \geq 2\sum_{n\in M_*}\big(u_k, d_k^{(n)}\varphi_j^{(n)}\big)_V - \sum_{n\in M_*}\big\|d_k^{(n)}\varphi_j^{(n)}\big\|_V^2. \tag{10.2}$$

We are now going to study the limit in (10.2). Taking

$$v_k^{(n)} := d_k^{(n)}u_k = \gamma^{-(N-2s)j_k^{(n)}/2} u_k\big(\gamma^{-j_k^{(n)}}\,\cdot + y_k^{(n)}\big),$$

we have

$$\begin{aligned}
\big(u_k, d_k^{(n)}\varphi_j^{(n)}\big)_V &= \big[v_k^{(n)}, \varphi_j^{(n)}\big]_s \\
&\quad + \int_{\mathbb{R}^N} \gamma^{-2sj_k^{(n)}} V\big(\gamma^{-j_k^{(n)}}\big(\big(x+\gamma^{j_k^{(n)}}(y_k^{(n)}-a_*)\big)+a_*\big)\big) v_k^{(n)}\varphi_j^{(n)}\,dx,\\
\big\|d_k^{(n)}\varphi_j^{(n)}\big\|_V^2 &= \big[\varphi_j^{(n)}\big]_s^2 \\
&\quad + \int_{\mathbb{R}^N} \gamma^{-2sj_k^{(n)}} V\big(\gamma^{-j_k^{(n)}}\big(\big(x+\gamma^{j_k^{(n)}}(y_k^{(n)}-a_*)\big)+a_*\big)\big) |\varphi_j^{(n)}|^2\,dx.
\end{aligned}$$

Fixing j and using (V_3^*), we get (up to a subsequence)

$$\lim_{k\to\infty}\big(u_k, d_k^{(n)}\varphi_j^{(n)}\big)_V = \big[w^{(n)}, \varphi_j^{(n)}\big]_s, \quad \lim_{k\to\infty}\big\|d_k^{(n)}\varphi_j^{(n)}\big\|_V^2 = \big[\varphi_j^{(n)}\big]_s^2, \tag{10.3}$$

provided that $n \notin \mathbb{N}_\sharp$ (this is the case when $n \in \mathbb{N}_0$). Similarly, up to a subsequence, by (V_2^*),

$$\begin{aligned}
\lim_{k\to\infty}\big(u_k, d_k^{(n)}\varphi_j^{(n)}\big)_V &= \big(w^{(n)}, \varphi_j^{(n)}\big)_{V_\kappa(\,\cdot\,+a^{(n)})},\\
\lim_{k\to\infty}\big\|d_k^{(n)}\varphi_j^{(n)}\big\|_V^2 &= \big\|\varphi_j^{(n)}\big\|_{V_\kappa(\,\cdot\,+a^{(n)})}^2,
\end{aligned} \tag{10.4}$$

where $\kappa = +,-$, whenever $n \in \mathbb{N}_+\cap\mathbb{N}_\sharp$ or $\mathbb{N}_-\cap\mathbb{N}_\sharp$, respectively. Since $\mathbb{N}_*\setminus\{1\} = (\mathbb{N}_*\setminus\mathbb{N}_\sharp)\,\dot\cup\,[(\mathbb{N}_+\cap\mathbb{N}_\sharp)\,\dot\cup\,(\mathbb{N}_-\cap\mathbb{N}_\sharp)]$ we can apply the limits (10.3) and (10.4) in (10.2), up to a subsequence, to get

$$\begin{aligned}
\limsup_k \|u_k\|_V^2 &\geq \big\|w^{(1)}\big\|_V^2 \\
&\quad + \sum_{n\in M_*\cap\mathbb{N}_+\cap\mathbb{N}_\sharp} 2\big(w^{(n)}, \varphi_j^{(n)}\big)_{V_+(\,\cdot\,+a^{(n)})} - \big\|\varphi_j^{(n)}\big\|_{V_+(\,\cdot\,+a^{(n)})}^2\\
&\quad + \sum_{n\in M_*\cap\mathbb{N}_-\cap\mathbb{N}_\sharp} 2\big(w^{(n)}, \varphi_j^{(n)}\big)_{V_-(\,\cdot\,+a^{(n)})} - \big\|\varphi_j^{(n)}\big\|_{V_-(\,\cdot\,+a^{(n)})}^2\\
&\quad + \sum_{n\in M_*\setminus\mathbb{N}_\sharp} 2\big[w^{(n)}, \varphi_j^{(n)}\big]_s - \big[\varphi_j^{(n)}\big]_s^2.
\end{aligned} \tag{10.5}$$

Since $\|\cdot\|_{V_+}$ and $\|\cdot\|_{V_-}$ are equivalent to $[\,\cdot\,]_s$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we can take the limit in j in (10.5) and use the arbitrariness of choice for M to obtain (10.1).

(c) If $w^{(n)} = 0$ for all $n \geq 2$, then $u_k \to w^{(1)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, with $w^{(1)}$ being a critical point of I_*. Let us argue by contradiction and assume the existence of $w^{(n_0)} \neq 0$, with $n_0 \geq 2$. By [16, Proposition 7.1] and estimate (10.1), up to a subsequence, we have

$$\begin{aligned} c(I_*) \geq I_*(w^{(1)}) &+ \sum_{n\in\mathbb{N}_*\setminus\mathbb{N}_\sharp} I_0(w^{(n)}) \\ &+ \sum_{n\in\mathbb{N}_+\cap\mathbb{N}_\sharp} I_+^{(n)}(w^{(n)}) + \sum_{n\in\mathbb{N}_-\cap\mathbb{N}_\sharp} I_-^{(n)}(w^{(n)}), \end{aligned}\tag{10.6}$$

where

$$\begin{aligned} I_\pm^{(n)}(u) &= \frac{1}{2}\|u\|^2_{V_\pm(\,\cdot\,+a^{(n)})} - \int_{\mathbb{R}^N} F_\pm(u)\,dx, \\ I_0(u) &= \frac{1}{2}[u]_s^2 - \int_{\mathbb{R}^N} F_0(u)\,dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N). \end{aligned}$$

As before, we prove that each $w^{(n)}$ is a critical point for the functionals in the respective index of the sums in (10.6), and as a consequence of (f_2), the right-hand side of (10.6) is nonnegative. In the next step we obtain that $c(I_*) < I_\kappa^{(n)}(w^{(n)})$ in the correspondent index, which leads to a contradiction with estimate (10.6). In fact, given φ in $C_0^\infty(\mathbb{R}^N)$, as in the proof of (10.1),

$$\lim_{k\to\infty}\big(u_k, d_k^{(n)}\varphi\big)_V = \big[w^{(n)},\varphi\big]_s \quad \text{and} \quad \lim_{k\to\infty}\big(u_k, d_k^{(n)}\varphi\big)_V = \big(w^{(n)},\varphi\big)_{V_\pm(\,\cdot\,+a^{(n)})},$$

provided that $n \in \mathbb{N}_*\setminus\mathbb{N}_\sharp$ and $n \in \mathbb{N}_\pm\cap\mathbb{N}_\sharp$, respectively. Since

$$\big|\gamma^{-(N+2s)j_k^{(n)}/2} f\big(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{(N-2s)j_k^{(n)}/2}t\big)\varphi\,\big| \leq C|t|^{2_s^*-1},$$

for all k,n,t, thanks to the Lebesgue Theorem, taking the limit in k, up to a subsequence, in the following identity,

$$\begin{aligned} I_*'(u_k)\cdot\big(d_k^{(n)}\varphi\big) &= \big(v_k^{(n)},\varphi\big)_V \\ &\quad - \int_{\mathbb{R}^N} \gamma^{-(N+2s)j_k^{(n)}/2} f\big(\gamma^{-j_k^{(n)}}x + y_k^{(n)}, \gamma^{(N-2s)j_k^{(n)}/2}v_k^{(n)}\big)\varphi\,dx, \end{aligned}$$

we can conclude that $I_*'(w^{(1)}) = \big(I_\pm^{(n)}\big)'(w^{(n)}) = I_0'(w^{(n)}) = 0$, in the corresponding index.

(d) To conclude the proof, we verify now that $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$ or $c(I_*) < I_\pm^{(n_0)}(w^{(n_0)})$, where n_0 belongs to $\mathbb{N}_*\setminus\mathbb{N}_\sharp$ or $\mathbb{N}_\pm\cap\mathbb{N}_\sharp$, respectively. Define the path $\zeta^{(n_0)}(t) = tw^{(n_0)}$, for $t \geq 0$ if $n_0 \in \mathbb{N}_*\setminus\mathbb{N}_\sharp$ and $\zeta^{(n_0)}(t) = tw^{(n_0)}(\,\cdot\, - a^{(n)})$, for $t \geq 0$ if $n_0 \in \mathbb{N}_\pm\cap\mathbb{N}_\sharp$. Using $(\mathscr{H}^*)$–$(\mathscr{H}_0^*)$ and Remark 6.4 (d) we have that $\zeta^{(n_0)}$ belongs to Γ_I with

$$c(I_*) \leq \max_{t\geq 0} I_*(\zeta^{(n_0)}(t)) < I_0(\zeta^{(n_0)}(\bar{t})) \leq \max_{t\geq 0} I_0(\zeta^{(n_0)}(t)) = I_0(w^{(n_0)}),$$

if $n_0 \in \mathbb{N}_* \setminus \mathbb{N}_\sharp$ and

$$c(I_*) \leq \max_{t\geq 0} I_*(\zeta^{(n_0)}(t)) < I_\pm^{(n)}(\zeta^{(n_0)}(\bar{t})) \leq \max_{t\geq 0} I_\pm^{(n)}(\zeta^{(n_0)}(t)) = I_\pm^{(n)}(w^{(n_0)}),$$

if $n_0 \in \mathbb{N}_\pm \cap \mathbb{N}_\sharp$, which, together with (10.6), leads to a contradiction ($\bar{t}$ is the maximum of $I_*(\zeta^{(n_0)}(t))$).

(e) We now assume only $(\mathscr{H}^*)$. Arguing as before, we can prove $u_k \to w^{(1)}$ in a subsequence or $c(I_*) = \max_{t\geq} I_*(\zeta^{(n_0)}(t))$. If the minimax level $c(I_*)$ is attained, we apply [27, Theorem 2.3] to obtain the existence of a critical point $u \in \zeta^{(n_0)}([0,\infty))$ such that $I_*(u) = c(I_*)$. $\square$

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