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ON A SINGULAR SEMILINEAR ELLIPTIC PROBLEM: MULTIPLE SOLUTIONS VIA CRITICAL POINT THEORY

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Abstract. We study existence and multiplicity of solutions of a semilinear elliptic problem involving a singular term. Combining various techniques from critical point theory, under different sets of assumptions, we prove the existence of k solutions ($k \in \mathbb{N}$) or infinitely many weak solutions.

1. Introduction and statement of results

In the present paper we deal with the following semilinear elliptic problem involving a singular term:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u = f(u) + u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N > 2$) with smooth boundary $\partial\Omega$, $f\colon [0,+\infty[\ \to \mathbb{R}$ is a continuous function and $0 < \gamma < 1$. The existence of multiple weak solutions is established under various assumptions on the nonlinearity f by combining different techniques from critical point theory. We remark that the energy functional associated to $(\mathcal{P})$ is not in general of class C^1 and this causes an obstacle to the application of such a theory.

The study of singular elliptic problems started with the pioneering work of Fulks and Maybee ([8]) as a mathematical model for describing the heat

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conduction in an electric medium and received a considerable attention after the seminal paper of Crandall, Rabinowitz and Tartar ([4]) where the existence of a classical solution for a pure singular problem was proved.

The existence of multiple solutions for such kind of problems has been investigated in a number of papers using different techniques. Two weak solutions are obtained for instance by Hirano, Saccon and Shioji in [11] via non-smooth critical point theory, by Perera and Silva in [17] with sub-supersolution methods and truncation techniques, by Sun, Wu and Long in [21] through minimization procedures on suitable manifolds, by Papageorgiou and Smyrlis in [16], where suitable truncation and comparison techniques are adopted and by Giacomoni, Schindler, Takáč in [10] where classical variational methods are combined with new regularity results for singular problems.

The literature is not so rich when searching for three or more solutions. As far as we know the existence of three solutions for such problems is established only in a few contributions (see [23], [6], [7], [5]). In [23], [6] and [7] three solutions for a singular elliptic problem driven by the p-Laplace operator are derived via an application of an abstract "three critical points" theorem. In contrast to our case, [23] considers only the low dimensional case, i.e. $N < p$, in order to exploit the continuity of the embedding $W_0^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. The recent works of the authors [6] and [7] do not cover the resonance case. Indeed, in that framework the energy functional associated to the problem is coercive. We mention finally the contribution [5] (see also the references therein) where the authors prove the existence of three solutions provided two pairs of ordered sub-supersolutions can be constructed. Notice that in [5] a monotonicity assumption on the nonlinearity f (stronger than condition (H_3) below) is assumed.

In this paper we prove several multiplicity results according to the value of the limit
$$l_\infty = \lim_{t\to+\infty} \frac{f(t)}{t}.$$
In particular, in Theorems 1.2–1.7 we consider the case $\lambda_k \le l_\infty \le \lambda_{k+1}$ (*double resonance* case) where λ_k is the k-th eigenvalue of the operator $(-\Delta, W_0^{1,2}(\Omega))$, $k \ge 1$. Under different sets of assumptions on f, we prove the existence of two solutions of different type: a local minimum of the energy functional obtained via sub-supersolutions techniques and a critical point of mountain pass type of suitable truncations of the energy functional. The latter is obtained via Morse theory, when resonance occurs with respect to the principal spectral interval, i.e. when $k = 1$ (Theorems 1.2, 1.4). If $k \ge 2$, we only need the classical Mountain Pass Theorem (Theorems 1.6, 1.7). To the best of our knowledge this is the first attempt to handle singular problems at resonance, especially with Morse theory.

If $l_\infty = +\infty$ and f exhibits a suitable oscillatory behaviour at infinity, the existence of infinitely many solutions is proved in Theorem 1.10. We also show in Theorem 1.8, which has been inspired by [13], that, when f oscillates near zero, the problem has an arbitrarily large number of solutions. It seems that such higher multiplicity results are new in the setting of singular problems.

As mentioned above, the coercive case, i.e. $l_\infty < \lambda_1$, has been already considered in the recent contributions [6] and [7], where topological arguments are combined with truncation methods to produce three solutions.

In the sequel we will use sub-supersolution methods and truncation techniques and we will make a deep use of the regularity results of [9] and [10]. It is worth mentioning that the novelty of our contribution relies on combining well-known techniques to deduce new results.

Notice that in some cases we will have to control more carefully the singular term, multiplying it by a positive parameter λ small enough:

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta u = f(u) + \lambda u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Throughout this paper, γ may take any value in $(0,1)$.

In order to state our results we introduce different sets of hypotheses on the reaction f and on its primitive F, i.e. $F(t) = \int_0^t f(s)\,ds$:

- (H_0) $f\colon [0,+\infty[\to \mathbb{R}$ is a locally Lipschitz function;
- (H_1) $f(0) = 0$ and there exists $\delta > 0$ such that $f(t) > 0$ for all $t \in]0,\delta]$;
- (H_2) there exists $\xi_0 > \delta$ such that $f(\xi_0) + \xi_0^{-\gamma} = 0$;
- (H_3) for every $\rho > 0$ there exists $\eta_\rho > 0$ such that the function $t \to f(t) + \eta_\rho t$ is increasing in $[0,\rho]$.

For the parameter case we will need the following:

- (H_4) $f\colon [0,+\infty[\to \mathbb{R}$ is a continuous function;
- (H_5) $f(0) = 0$, $f(t) > 0$ for all $t > 0$;
- (H_6) $\limsup_{t\to 0^+} f(t)/t < \lambda_1$.

The next two assumptions will be used to produce the second solution:

- (H_7) there exists $\sigma > 1 - \gamma$ such that
$$\liminf_{t\to+\infty} \frac{tf(t) - 2F(t)}{t^\sigma} > 0;$$
- (H_8)
$$\lambda_1 \le \liminf_{t\to+\infty} \frac{f(t)}{t} \le \limsup_{t\to+\infty} \frac{f(t)}{t} \le \lambda_2.$$

Remark 1.1. Hypothesis (H_8) says that asymptotically at $+\infty$, the quotient $f(t)/t$ reaches the principal spectral interval $[\lambda_1,\lambda_2]$, so resonance can occur with respect to both λ_1, λ_2 (double resonance case).

Our first two solutions result for changing sign reactions reads as follows:

Theorem 1.2. *Under hypotheses* (H_0)–(H_3), (H_7), (H_8) *and for any* $\gamma \in (0,1)$, *problem* $(\mathcal{P})$ *has at least two weak solutions, one of which belongs to* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Example 1.3. Define

$$f(t) = \begin{cases} t - 2t^{\vartheta-1} & \text{if } 0 \le t \le 1, \\ \lambda_1 t - (1+\lambda_1)t^{q-1} & \text{if } 1 < t, \end{cases}$$

with $1 < q < 2 < \vartheta$.

In the next result the nonlinearity f has to be of constant sign:

Theorem 1.4. *Under hypotheses* (H_4)–(H_8) *and for any* $\gamma \in (0,1)$, *there exists* $\lambda^\star > 0$ *such that, for every* $0 < \lambda < \lambda^\star$, *problem* $(\mathcal{P}_\lambda)$ *has at least two weak solutions in* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Example 1.5. Define

$$f(t) = \begin{cases} \dfrac{\lambda_1}{2}\, t - \dfrac{\lambda_1}{4}\, t^{\vartheta-1} & \text{if } 0 \le t \le 1, \\ \lambda_1 t - \dfrac{3\lambda_1}{4}\, t^{q-1} & \text{if } 1 < t, \end{cases}$$

with $1 < q < 2 < \vartheta$.

Next, we replace (H_8) by the following double resonance at a nonprinciple spectral interval $[\lambda_k, \lambda_{k+1}]$, for some $k \ge 2$:

(H_9) $\lambda_k \le \liminf\limits_{t\to+\infty} \dfrac{f(t)}{t} \le \limsup\limits_{t\to+\infty} \dfrac{f(t)}{t} \le \lambda_{k+1}$.

Assuming the above condition, we obtain in a similar but easier way the same results:

Theorem 1.6. *Under hypotheses* (H_0)–(H_3), (H_7), (H_9) *and for any* $\gamma \in (0,1)$, *problem* $(\mathcal{P})$ *has at least two weak solutions, one of which belongs to* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

Theorem 1.7. *Under hypotheses* (H_4)–(H_6), (H_7), (H_9) *and for any* $\gamma \in (0,1)$, *there exists* $\lambda^\star > 0$ *such that, for every* $0 < \lambda < \lambda^\star$, *problem* $(\mathcal{P}_\lambda)$ *has at least two weak solutions in* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

The last section of the paper is devoted to multiplicity results in the presence of an oscillatory behaviour of f. Consider the following assumptions:

(H_{10}) there exists a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to 0^+$ and $f(t_n) < 0$ for every $n \in \mathbb{N}$;

(H_{11}) $-\infty < \liminf\limits_{t\to 0^+} \dfrac{F(t)}{t^2} \le \limsup\limits_{t\to 0^+} \dfrac{F(t)}{t^2} = +\infty$.

Theorem 1.8. *Under hypotheses* (H_4), (H_{10}), (H_{11}), *for each* $k \in \mathbb{N}$, *there exists* $\lambda_k^\star > 0$ *such that, for every* $0 < \lambda < \lambda_k^\star$, *problem* ($\mathcal{P}_\lambda$) *has at least* k *essentially bounded weak solutions.*

Example 1.9. Define

$$f(t) = \begin{cases} \sqrt{t}\max\{0,\sin(1/t)\} + t^2\min\{0,\sin(1/t)\} & \text{if } t > 0,\\ 0 & \text{if } t = 0.\end{cases}$$

Our last theorem ensures the existence of infinitely many solutions if f oscillates at infinity. We will require a more general assumption than $l_\infty = +\infty$ (see (H_{13}) below).

- (H_{12}) There exist $l < 0$ and a sequence $\{t_n\} \subset \mathbb{R}^+$ such that $t_n \to +\infty$ and $f(t_n) \le l$;
- (H_{13}) $-\infty < \liminf\limits_{t\to+\infty} \dfrac{F(t)}{t^2} \le \limsup\limits_{t\to+\infty} \dfrac{F(t)}{t^2} = +\infty.$

Theorem 1.10. *Under hypotheses* (H_4), (H_{12}), (H_{13}), *there exists a sequence* $\{u_n\}$ *of essentially bounded weak solutions of* ($\mathcal{P}$) *such that* $\lim\limits_n \|u_n\|_\infty = +\infty$.

Example 1.11. Define $f(t) = t^2(1/2 + \sin t)$ for $t \ge 0$.

2. Preliminaries

In this section, for the convenience of the reader, we briefly recall some definitions and mathematical tools that we will use further.

Let us recall that, for $\lambda > 0$, a *weak solution* of

$$(\mathcal{P}_\lambda) \qquad \begin{cases} -\Delta u = f(u) + \lambda u^{-\gamma} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

is a function $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $u > 0$ almost everywhere in Ω and, for all $\varphi \in W_0^{1,2}(\Omega)$,

$$u^{-\gamma}\varphi \in L^1(\Omega), \qquad \int_\Omega \nabla u \nabla\varphi\, dx = \int_\Omega (f(u) + \lambda u^{-\gamma})\varphi\, dx.$$

A function $\overline{u}$ is called a *weak supersolution* of ($\mathcal{P}_\lambda$) if $\overline{u} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\overline{u} > 0$ in Ω, $\overline{u} \ge 0$ on $\partial\Omega$, and

$$\overline{u}^{-\gamma}\varphi \in L^1(\Omega), \qquad \int_\Omega \nabla\overline{u}\nabla\varphi\, dx \ge \int_\Omega (f(\overline{u}) + \lambda\overline{u}^{-\gamma})\varphi\, dx$$

for all $\varphi \in W_0^{1,2}(\Omega)$, $\varphi \ge 0$.

A function $\underline{u}$ is called a *weak sub solution* of $(\mathcal{P}_\lambda)$ if $\underline{u} \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\underline{u} > 0$ in Ω, $\underline{u} \leq 0$ on $\partial\Omega$, and for all $\varphi \in W_0^{1,2}(\Omega)$, $\varphi \geq 0$,

$$\underline{u}^{-\gamma}\varphi \in L^1(\Omega), \qquad \int_\Omega \nabla\underline{u}\nabla\varphi\,dx \leq \int_\Omega (f(\underline{u}) + \lambda\underline{u}^{-\gamma})\varphi\,dx.$$

Let X be a Banach space, X^* be its dual and $\mathcal{F} \in C^1(X)$. We say that $\mathcal{F}$ satisfies the *Cerami condition* if the following is true:

> "Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\mathcal{F}(u_n)\}_{n\geq 1}$ is bounded and $(1 + ||u_n||)\mathcal{F}'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence".

Although the Cerami condition is clearly weaker than the *Palais–Smale condition*, the Deformation Theorem (in particular, Mountain Pass Theorem) still holds for C^1-functionals with this property.

For each $c \in \mathbb{R}$, we introduce the following sets: $\mathcal{F}^c = \{u \in X : \mathcal{F}(x) \leq c\}$, $K_\mathcal{F} = \{u \in X : \mathcal{F}'(u) = 0\}$ and $K_\mathcal{F}^c = \{u \in K_\mathcal{F} : \mathcal{F}(u) = c\}$.

Let (Y_1, Y_2) be a topological pair with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-*relative singular homology group* with integer coefficients for the pair (Y_1, Y_2). We recall some of the basic properties of these groups.

- $H_k(Y_1, Y_2) = 0$ for $k < 0$.
- For every continuous map of topological pairs $f\colon (Y_1, Y_2) \to (E_1, E_2)$, there exists a sequence of group homomorphisms
$$f^*\colon H_k(Y_1, Y_2) \to H_k(E_1, E_2), \quad k \geq 0.$$
- There exists a sequence of group homomorphisms
$$\partial\colon H_k(Y_1, Y_2) \to H_{k-1}(Y_2, \emptyset), \quad k \geq 0$$
(we set $H_{-1}(Y_2, \emptyset) = 0$).

The above data satisfy a list of axioms. Some of them are the following:

- If $f\colon (Y_1, Y_2) \to (E_1, E_2)$ is a continuous map of topological pairs then
$$\partial \circ f^* = (f|_{Y_2})^* \circ \partial.$$
- If $f, g\colon (Y_1, Y_2) \to (E_1, E_2)$ are homotopic maps of pairs, then $f^* = g^*$.
- (Excision Property) If $Y_3, Y_2 \subseteq Y_1$ with $\overline{Y_3} \subseteq \mathrm{int}(Y_2)$, then
$$H_k(Y_1, Y_2) \simeq H_k(Y_1 \setminus Y_3, Y_2 \setminus Y_3), \quad k \geq 0.$$
- Suppose that $Y_3 \subseteq Y_2 \subseteq Y_1$. The inclusion maps

$$i\colon (Y_2, Y_3) \to (Y_1, Y_3), \quad j\colon (Y_1, Y_3) \to (Y_1, Y_2), \quad j_2\colon (Y_2, \emptyset) \to (Y_2, Y_3)$$

and the homomorphisms $\partial\colon H_k(Y_1, Y_2) \to H_{k-1}(Y_2, \emptyset)$ induce an exact sequence, i.e. the kernel of each homomorphism of the sequence coincides with the image of the predecessor homomorphism of the sequence:

$$\cdots \xrightarrow{j_2^*\circ\partial} H_k(Y_2, Y_3) \xrightarrow{i^*} H_k(Y_1, Y_3) \xrightarrow{j^*} H_k(Y_1, Y_2) \xrightarrow{j_2^*\circ\partial} H_{k-1}(Y_2, Y_3) \to \cdots$$

The *critical groups* of $\mathcal{F}$ at an isolated critical point $u \in K_{\mathcal{F}}^c$ are defined as

$$C_k(\mathcal{F}, u) = H_k\left(\mathcal{F}^c \cap U, \mathcal{F}^c \cap U \setminus \{u\}\right), \quad \text{for all } k \geq 0,$$

where U is a neighbourhood of u such that $K_{\mathcal{F}} \cap \mathcal{F}^c \cap U = \{u\}$. The *excision property* of singular homology theory implies that the above definition is independent of the choice of the neighbourhood U.

Critical groups help to distinguish between different types of critical points and are extremely useful in producing multiple critical points for a functional. For example, if u is an isolated local minimizer of $\mathcal{F}$, then $C_k(\mathcal{F}, u) \simeq \delta_{k,0}\mathbb{Z}$, where $\mathbb{Z}$ is the additive abelian group of integers and $\delta_{k,0}$ is the Kronecker δ-symbol.

Suppose that $\mathcal{F} \in C^1(X)$ satisfies the Cerami condition and $-\infty < \inf \mathcal{F}(K_{\mathcal{F}})$. Let $a < \inf \mathcal{F}(K_{\mathcal{F}})$. The *critical groups of* $\mathcal{F}$ *at infinity* are defined as

$$C_k(\mathcal{F}, \infty) = H_k(X, \mathcal{F}^a), \quad \text{for all } k \geq 0.$$

Using the deformation theorem, we see that the definition of critical groups of $\mathcal{F}$ at infinity is independent of the particular level $a < \inf \mathcal{F}(K_{\mathcal{F}})$. Critical groups at infinity help to detect critical points of $\mathcal{F}$. For example, if for some integer $k \geq 0$, $C_k(\mathcal{F}, \infty) \neq 0$, then there exists $u \in K_{\mathcal{F}}$ such that $C_k(\mathcal{F}, u) \neq 0$.

For an extensive presentation of singular homology and Morse Theory we refer to [15, Chapter 6].

In the ordered Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ the positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \Omega\}$$

has a non-empty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0 \text{ for all } x \in \partial\Omega\right\}$$

(n being the outward unit normal to $\partial\Omega$). Moreover, on the Sobolev space $W_0^{1,2}(\Omega)$, we deal with the standard norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}.$$

Given $m \in L^\infty(\Omega)_+$, $m \neq 0$, consider the nonlinear weighted eigenvalue problem

$$\begin{cases} -\Delta u = \widehat{\lambda} m(x) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The least number $\widehat{\lambda} > 0$, denoted by $\widehat{\lambda}_1(m)$, such that the above problem admits a nontrivial solution is called the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega), m)$. It is well known that $\widehat{\lambda}_1(m)$ is positive, isolated, simple and the following variational characterization holds:

$$\widehat{\lambda}_1(m) = \min\left\{\frac{\|u\|^2}{\int_\Omega m|u|^2\,dx} : u \in W_0^{1,2}(\Omega),\ u \neq 0\right\}.$$

We denote by $\varphi_{1,m}$ the normalized positive eigenfunction, which is associated to $\widehat{\lambda}_1(m)$. One has $\varphi_{1,m} \in \operatorname{int} C_+$.

As usual, if $m \equiv 1$, set $\widehat{\lambda}_1(m) \equiv \lambda_1$ and $\varphi_{1,m} \equiv \varphi_1$. The next remark contains useful information on the weighted eigenvalue problems (for the proof and further details we refer to [2]).

REMARK 2.1.

(a) If $m_1, m_2 \in L^\infty(\Omega)_+ \setminus \{0\}$ satisfy $m_1 \leq m_2$ almost everywhere in Ω, then one has $\widehat{\lambda}_1(m_2) \leq \widehat{\lambda}_1(m_1)$. If in addition $m_1 \neq m_2$, then, $\widehat{\lambda}_1(m_2) < \widehat{\lambda}_1(m_1)$.

(b) If u is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$, then $u \in C_0^1(\overline{\Omega})$ changes sign.

REMARK 2.2. Let $0 < \gamma < 1$, $v \in \operatorname{int} C_+$ be given. Then the functional $u \to \int_\Omega v^{-\gamma} u$ is of class C^1 in $W_0^{1,2}(\Omega)$. Moreover,

$$\left|\int_\Omega v(x)^{-\gamma} u(x)\,dx\right| \leq \widehat{c}\|u\|, \quad \text{for all } u \in W_0^{1,2}(\Omega),$$

where $\widehat{c}$ is a positive constant depending on v.

PROOF. Since $v \in \operatorname{int} C_+$, we know that there exists $\widetilde{c} = \widetilde{c}(v) > 0$ such that $v(x) \geq \widetilde{c} d(x)$ for every $x \in \Omega$, being $d(x) = d(x, \partial\Omega)$. The thesis follows now from Hardy's inequality. □

3. Existence of two solutions: proofs of Theorems 1.2–1.7

In the present section we propose different sets of assumptions which produce a local minimizer of the energy functional associated to the problem by sub and supersolutions techniques.

The two proposed methods differ in the construction of the supersolution: in Proposition 3.1, under sign changing conditions on f, we employ a suitable constant as a supersolution. Instead, in Proposition 3.2, f is supposed to be positive with a certain growth at zero. In this case, a supersolution is obtained via the Mountain Pass Theorem provided that the singular term is multiplied by a positive parameter λ small enough. In Proposition 3.4, following the ideas of [20], we prove, that if $\lambda_1 \leq l_\infty \leq \lambda_2$, there exists a critical point of mountain pass type. Combining the above results, we obtain our multiplicity results.

Without loss of generality, we will assume in the sequel that $f(t) = 0$ for each $t < 0$.

PROPOSITION 3.1. *Under hypotheses* (H_0)–(H_3), *assume that* $0 < \gamma < 1$. *Then there exist* $\underline{u}$, $\overline{u}$ *respectively weak subsolution and supersolution of* $(\mathcal{P})$, *and a weak solution* $u \in \operatorname{int} C_+$ *of* $(\mathcal{P})$ *such that* $u \in \operatorname{int}_{C^1}[\underline{u}, \overline{u}]$.

PROOF. *Step 1. Existence of a subsolution* $\underline{u} \in \operatorname{int} C_+$. It is well known that the problem

$$\begin{cases} -\Delta u = u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $w \in \operatorname{int} C_+$ such that

$$c_1 d(x) \le w(x) \le c_2 d(x) \quad \text{for all } x \in \Omega, \tag{3.1}$$

for some constants $0 < c_1 < c_2$ and where $d(x) = d(x, \partial\Omega)$, $x \in \Omega$. This fact follows for example from [9, Lemmas A.4 (p. 705), A.7 (p. 707), Theorem B.1 (p. 710)] and from the Strong Maximum Principle of Vázquez [22].

For $\varepsilon \le \min\{1, \delta/\|w\|_\infty\}$, $\underline{u} \equiv \varepsilon w$ turns out to be a subsolution of $(\mathcal{P})$ as, exploiting (H_1),

$$-\Delta \underline{u} = \varepsilon w^{-\gamma} \le \underline{u}^{-\gamma} < \underline{u}^{-\gamma} + f(\underline{u}).$$

Step 2. Existence of a supersolution $\overline{u}$. Put $\overline{u} = \xi_0$ where ξ_0 comes from (H_2). Without loss of generality we can assume that ξ_0 is the first zero of the function $t \mapsto f(t) + t^{-\gamma}$, which means that

$$f(t) + t^{-\gamma} > 0 \quad \text{for all } t \in \,]0, \xi_0[. \tag{3.2}$$

It is clear that $\overline{u}$ is a supersolution of $(\mathcal{P})$ and that $\underline{u} < \overline{u}$.

Step 3. Existence of a solution u. Define the following truncation of f, $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$h(x,t) = \begin{cases} f(\underline{u}(x)) + \underline{u}(x)^{-\gamma} & \text{if } t < \underline{u}(x), \\ f(t) + t^{-\gamma} & \text{if } \underline{u}(x) \le t \le \xi_0, \\ f(\xi_0) + \xi_0^{-\gamma} = 0 & \text{if } t > \xi_0. \end{cases}$$

Denote by $H\colon \Omega \times \mathbb{R} \to \mathbb{R}$ its primitive, i.e. $H(x,t) = \int_0^t h(x,s)\,ds$ and let $\mathcal{E}\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{E}(u) = \frac{1}{2}\,\|u\|^2 - \int_\Omega H(x,u)\,dx.$$

Due to Remark 2.2, it is easily checked that $\mathcal{E}$ is of class C^1, its critical points being weak solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u = h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\mathcal{E}$ is coercive and sequentially weakly lower semicontinuous, it has a global minimum u which lies in the interval $[\underline{u}, \xi_0]$. In particular, u is a weak solution of problem $(\mathcal{P})$.

Step 4. Properties of u. Since $\underline{u} = \varepsilon w \leq u \leq \xi_0$, by using (3.1), (3.2), we obtain

$$0 \leq f(u(x)) + u(x)^{-\gamma} \leq \text{const.}\, u(x)^{-\gamma} \leq \text{const.}\, d(x)^{-\gamma}, \quad \text{for all } x \in \Omega. \tag{3.3}$$

Hence,

$$\begin{cases} -\Delta u \leq \text{const.} u^{-\gamma}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and after rescaling,

$$\begin{cases} -\Delta v \leq v^{-\gamma} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $v = \text{const.}\, u$. From the Weak Comparison Principle we get that $v \leq w$ in Ω. Thus, combining the above outcomes with (3.1), we obtain two positive constants $\widetilde{c}_1 < \widetilde{c}_2$ such that

$$\widetilde{c}_1 d(x) \leq u(x) \leq \widetilde{c}_2 d(x) \quad \text{for all } x \in \Omega. \tag{3.4}$$

Now (3.3), (3.4) permit us to apply the regularity theory for singular problems developed in [10, Theorem B.1], to conclude that $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in \,]0,1[$. Since f is locally Lipschitz, the function $x \mapsto f(u(x)) + u(x)^{-\gamma}$ turns out to be locally Hölder continuous and from classical interior regularity results (see [1, p. 446]), we also have that $u \in C^2(\Omega)$. From the Strong Maximum Principle we conclude that $u \in \operatorname{int} C_+$. Notice that applying condition (H_3) with $\rho = \xi_0$ and recalling that $\underline{u} \leq u \leq \xi_0$, we have

$$-\Delta u - u^{-\gamma} + \eta_\rho u = f(u) + \eta_\rho u \geq f(\underline{u}) + \eta_\rho \underline{u} > -\Delta \underline{u} - \underline{u}^{-\gamma} + \eta_\rho \underline{u}.$$

The latter inequality implies

$$u - \underline{u} \in \operatorname{int} C_+, \tag{3.5}$$

thanks to the Strong Comparison Principle for singular problems (see [9, Theorem 1.2]).

Next, we are going to prove that $u(x) < \xi_0$ for all $x \in \Omega$. Set $z(x) = \xi_0 - u(x)$, $x \in \Omega$. Then $z \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is nonnegative in Ω and positive on $\partial\Omega$. Applying again condition (H_3) with $\rho = \xi_0$, we obtain

$$-\Delta u - u^{-\gamma} + \eta_\rho u = f(u) + \eta_\rho u \leq f(\xi_0) + \eta_\rho \xi_0 = -\xi_0^{-\gamma} + \eta_\rho \xi_0 = -\Delta\xi_0 - \xi_0^{-\gamma} + \eta_\rho \xi_0,$$

which yields

$$-\Delta z + \eta_\rho z \geq \xi_0^{-\gamma} - u^{-\gamma} \geq -\gamma u^{-\gamma-1} z.$$

(The second inequality follows from the Mean Value Theorem). Consequently,

$$\begin{cases} -\Delta z + [\eta_\rho + \text{const.}\, d(x)^{-\gamma-1}] z \geq 0 & \text{in } \Omega, \\ z \geq 0 & \text{in } \Omega, \\ z > 0 & \text{on } \partial\Omega, \end{cases}$$

(see also (3.4)).

Since the function $x \mapsto d(x)^{-\gamma-1}$ is locally bounded on Ω, the Strong Maximum Principle ([18, Theorem 2.1.2]) ensures that $z > 0$ in Ω, i.e. $u(x) < \xi_0$ for all $x \in \Omega$. Hence, $u \in \operatorname{int}_{C^1}[\underline{u}, \xi_0]$ as we claimed. □

In the next proposition we consider the parameter case.

PROPOSITION 3.2. *Under hypotheses* (H_4)–(H_6) *and* (H_8), *assume that* $0 < \gamma < 1$. *Then there exists* $\lambda^\star > 0$ *such that for every* $0 < \lambda < \lambda^\star$, *there exist* $\underline{u}_\lambda, \overline{u}_\lambda \in \operatorname{int} C_+$ *respectively* (*weak*) *subsolution and supersolution of* $(\mathcal{P}_\lambda)$, *and a weak solution* $u_\lambda \in \operatorname{int} C_+$ *of* $(\mathcal{P}_\lambda)$ *such that* $u_\lambda \in \operatorname{int}_{C^1}[\underline{u}_\lambda, \overline{u}_\lambda]$.

PROOF. *Step 1. Existence of a subsolution* $\underline{u}_\lambda \in \operatorname{int} C_+$. Let w be the function introduced in Step 1 of the proof of Proposition 3.1. For $\lambda < 1$, exploiting the positivity of f, $\underline{u}_\lambda \equiv \lambda w$ turns out to be a subsolution of $(\mathcal{P}_\lambda)$ as

$$-\Delta \underline{u}_\lambda = \lambda w^{-\gamma} < \lambda^{1-\gamma} w^{-\gamma} = \lambda \underline{u}_\lambda^{-\gamma} < \lambda \underline{u}_\lambda^{-\gamma} + f(\underline{u}_\lambda). \tag{3.6}$$

Step 2. Existence of a supersolution $\overline{u}_\lambda \in \operatorname{int} C_+$. Choose η, p such that $\limsup_{t\to 0^+} f(t)/t < \eta < \lambda_1$ (see (H_6)) and $2 < p < 2^\star$. Then, for some positive constant c_3,

$$f(t) < \eta t + c_3 t^{p-1} \quad \text{for all } t > 0. \tag{3.7}$$

To check this, one needs to combine continuity of f with (H_8) and with the properties of η.

Consider the auxiliary problem

$$\begin{cases} -\Delta u = \eta u + c_3 u^{p-1} + \lambda u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases} \tag{3.8}$$

Define $g_\lambda\colon \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$g_\lambda(x,t)=\begin{cases}\underline{u}_\lambda(x)^{-\gamma}, & \text{if } t<\underline{u}_\lambda(x),\\ t^{-\gamma} & \text{if } t\geq \underline{u}_\lambda(x),\end{cases}$$

and set

$$G_\lambda(x,t)=\int_0^t g_\lambda(x,s)\,ds,\quad x\in\Omega,\ t\in\mathbb{R}.$$

Consider also the functional $I_\lambda\colon W_0^{1,2}(\Omega)\to\mathbb{R}$ defined by

$$I_\lambda(u)=\frac{1}{2}\,\|u\|^2-\frac{\eta}{2}\,\|u\|_2^2-\frac{c_3}{p}\,\|u^+\|_p^p-\lambda\int_\Omega G_\lambda(x,u)\,dx.$$

Then, I_λ is of class C^1 and its critical points are greater than or equal to $\underline{u}_\lambda$, hence weak solutions of (3.8) (see (3.6) and also [6, proof of Proposition 2.3]). Moreover, I_λ is weakly lower semicontinuous (recall that $p<2^\star$).

Following [10, proof of Lemma 3.2], we shall prove that for $\lambda>0$ sufficiently small, I_λ has at least one nontrivial local minimizer. To this end, choose first a positive constant $\widehat{c}$ such that

$$\bigg|\int_\Omega w(x)^{-\gamma}u(x)\,dx\bigg|\leq\widehat{c}\|u\|,\quad \text{for all } u\in W_0^{1,2}(\Omega)$$

(see Remark 2.2). Then we have

$$\bigg|\int_\Omega g_\lambda(x,u)\varphi\,dx\bigg|\leq\lambda^{-\gamma}\,\widehat{c}||\varphi||,\quad \text{for all } u,\varphi\in W_0^{1,2}(\Omega),$$

and

$$\bigg|\int_\Omega G_\lambda(x,u)\,dx\bigg|\leq\lambda^{-\gamma}\,\widehat{c}||u||,\quad \text{for all } u\in W_0^{1,2}(\Omega).$$

Indeed, the definitions of g_λ and G_λ yield

$$0<g_\lambda(x,t)\leq\underline{u}_\lambda(x)^{-\gamma}=\lambda^{-\gamma}w(x)^{-\gamma},\qquad |G_\lambda(x,t)|\leq\lambda^{-\gamma}|t|w(x)^{-\gamma},$$

for all $x\in\Omega$, $t\in\mathbb{R}$.

Now, by using Rayleigh quotient for λ_1 and also the continuity of the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^p(\Omega)$ (recall that $p<2^\star$), we obtain

$$I_\lambda(u)\geq\frac{\lambda_1-\eta}{2\lambda_1}\,||u||^2-\text{const.}||u||^p-\lambda^{1-\gamma}\widehat{c}||u||,\quad \text{for all } u\in W_0^{1,2}(\Omega).$$

Let $r>0,\lambda>0$ be fixed. Since I_λ is weakly lower semicontinuous, it attains its minimum on the weakly compact set $B_r=\{u\in W_0^{1,2}(\Omega):||u||\leq r\}$. Put

$$m(r,\lambda)=\min_{B_r} I_\lambda.$$

For each $u\in\partial B_r$, we have

$$I_\lambda(u)\geq r\bigg(\frac{\lambda_1-\eta}{2\lambda_1}\,r-\text{const.}r^{p-1}-\lambda^{1-\gamma}\widehat{c}\bigg).$$

Recalling that $\eta < \lambda_1$, $p > 2$, $0 < \gamma < 1$, we may choose $r > 0$, $\lambda > 0$ both sufficiently small such that $\inf\limits_{\partial B_r} I_\lambda > 0$.

Next, choose $t > 0$ such that

$$t||\varphi_1|| < r, \qquad \frac{\lambda_1}{2}\, t < \lambda \int_\Omega \varphi_1 \underline{u}_\lambda^{-\gamma}\, dx,$$

where φ_1 is the L^2-normalized positive eigenfunction of the Laplace operator $(-\Delta, H_0^1(\Omega))$. Since $\varphi_1, \underline{u}_\lambda \in \operatorname{int} C_+$, we may choose $t > 0$ even smaller so that

$$t\varphi_1(x) < \underline{u}_\lambda(x), \quad \text{for all } x \in \Omega.$$

Then we have

$$I_\lambda(t\varphi_1) \le \frac{\lambda_1}{2}\, t^2 - \lambda \int_\Omega G_\lambda(x, t\varphi_1(x))\, dx = \frac{\lambda_1}{2}\, t^2 - \lambda t \int_\Omega \varphi_1 \underline{u}_\lambda^{-\gamma}\, dx < 0$$

and thus, $m(r, \lambda) < 0$.

The above arguments show that for $r > 0$, $\lambda > 0$ sufficiently small, we may find a point $\overline{u}_\lambda$ in the *interior* of the closed ball B_r such that

$$I_\lambda(\overline{u}_\lambda) = m(r, \lambda).$$

It turns out that $\overline{u}_\lambda$ is a local minimizer of I_λ and hence, it is a weak solution of (3.8) with $\overline{u}_\lambda \ge \underline{u}_\lambda$. Clearly, $\overline{u}_\lambda$ is a supersolution of $(\mathcal{P}_\lambda)$.

Now it follows from [9, Lemma A.7 & Theorem B.1] that $\overline{u}_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in\,]0,1[$. Then the Strong Maximum Principle of Vázquez [22] implies that $\overline{u}_\lambda \in \operatorname{int} C_+$.

Finally, we have $-\Delta\overline{u}_\lambda - \lambda\overline{u}_\lambda^{-\gamma} > 0 > -\Delta\underline{u}_\lambda - \lambda\underline{u}_\lambda^{-\gamma}$ (see (3.6), (3.8)). From the Strong Comparison Principle for singular problems (see [10, Theorem 2.3]), we infer that $\overline{u}_\lambda > \underline{u}_\lambda$.

Step 3. Existence of a solution $u_\lambda \in \operatorname{int} C_+ \cap \operatorname{int}_{C^1}[\underline{u}_\lambda, \overline{u}_\lambda]$. Consider the truncation $h_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ of the reaction term of the problem $(\mathcal{P}_\lambda)$ defined by

$$h_\lambda(x,t) = \begin{cases} f(\underline{u}_\lambda(x)) + \lambda\underline{u}_\lambda(x)^{-\gamma} & \text{if } t < \underline{u}_\lambda(x), \\ f(t) + \lambda t^{-\gamma} & \text{if } \underline{u}_\lambda(x) \le t \le \overline{u}_\lambda(x), \\ f(\overline{u}_\lambda(x)) + \lambda\overline{u}_\lambda(x)^{-\gamma} & \text{if } t > \overline{u}_\lambda(x). \end{cases}$$

Denote by $H_\lambda \colon \Omega \times \mathbb{R} \to \mathbb{R}$ its primitive, i.e.

$$H_\lambda(x,t) = \int_0^t h_\lambda(x,s)\, ds$$

and let $\mathcal{E}_\lambda \colon W_0^{1,2}(\Omega) \to \mathbb{R}$ be the functional

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\, \|u\|^2 - \int_\Omega H_\lambda(x,u)\, dx.$$

$\mathcal{E}_\lambda$ is of class C^1, sequentially weakly lower semicontinous and its critical points are the weak solutions of the semilinear elliptic problem

$$\begin{cases} -\Delta u = h_\lambda(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Moreover, $\mathcal{E}_\lambda$ is coercive so, it possesses a global minimizer u_λ which lies in the interval $[\underline{u}_\lambda, \overline{u}_\lambda]$. Thus, u_λ is a weak solution of problem $(\mathcal{P}_\lambda)$ satisfying estimates similar to (3.4).

We have $0 < f(u_\lambda(x)) + \lambda u_\lambda^{-\gamma}(x) \leq \text{const}.d(x)^{-\gamma}$ almost everywhere in Ω, so $u_\lambda \in C^{1,\beta}(\overline{\Omega})$, for some $\beta \in]0,1[$ (see [10, Theorem B.1]). Now from the Strong Maximum Principle we conclude that $u_\lambda \in \operatorname{int} C_+$.

Notice also that $-\Delta u_\lambda - \lambda u_\lambda^{-\gamma} = f(u_\lambda) > 0 \geq -\Delta \underline{u}_\lambda - \lambda \underline{u}_\lambda^{-\gamma}$, which implies $u_\lambda - \underline{u}_\lambda \in \operatorname{int} C_+$, due to the Strong Comparison Principle for singular problems ([10, Theorem 2.3].)

Also, because of (3.7) and since $\underline{u}_\lambda \leq u_\lambda \leq \overline{u}_\lambda$, one has

$$-\Delta \overline{u}_\lambda - \lambda \overline{u}_\lambda^{-\gamma} = \eta \overline{u}_\lambda + c_3 \overline{u}_\lambda^{p-1} \geq \eta u_\lambda + c_3 u_\lambda^{p-1} > f(u_\lambda) = -\Delta u_\lambda - \lambda u_\lambda^{-\gamma}$$

and from the classical Strong Comparison Principle for singular problems ([10, Theorem 2.3]), we obtain that $\overline{u}_\lambda - u_\lambda \in \operatorname{int} C_+$. Thus, $u_\lambda \in \operatorname{int}_{C^1}[\underline{u}_\lambda, \overline{u}_\lambda]$. $\square$

In the next result we are going to introduce a suitable truncation of our energy functional and to compute its critical groups at infinity (see [20]). We will need some auxiliary results.

Lemma 3.3 ([20, Proposition 1]). *Let X be a Banach space and $(t,u) \mapsto h_t(u)$ be a homotopy which belongs to $C^1([0,1] \times X)$ and it is bounded. Suppose that*

(a) *there exists $R > 0$ such that for all $t \in [0,1]$,*

$$K_{h_t} \subseteq \overline{B}_R = \{x \in X : ||x|| \leq R\};$$

(b) *the maps $u \mapsto \partial_t h_t(u)$ and $u \mapsto h_t'(u)$ are both locally Lipschitz*;
(c) *h_0 and h_1 both satisfy the Cerami condition*;
(d) *there exist $\beta \in \mathbb{R}$ and $\delta > 0$ such that*

$$h_t(u) \leq \beta \Rightarrow (1 + ||u||)||h_t'(u)||_* \geq \delta, \quad \textit{for all } t \in [0,1].$$

Then, $C_k(h_0, \infty) = C_k(h_1, \infty)$, for all $k \geq 0$ (where we are denoting by $C_k(h_t, \infty)$ the k-th critical group of h_t at infinity).

Consider the set

$$V = \left\{ u \in W_0^{1,2}(\Omega) : \int_\Omega \varphi_1 u = 0 \right\},$$

which is a closed linear subspace of $W_0^{1,2}(\Omega)$ and notice that $W_0^{1,2}(\Omega) = \mathbb{R}\varphi_1 \oplus V$.

Let $\mu \in (\lambda_1, \lambda_2)$ and consider the C^1-functional $\mathcal{G}\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{G}(u) = \frac{1}{2}\,\|u\|^2 - \frac{\mu}{2}\,\|u\|_2^2 \quad \text{for all } u \in W_0^{1,2}(\Omega).$$

By using standard arguments we may show that $\mathcal{G}$ has the following properties:

- 0 is the unique critical point of $\mathcal{G}$.
- $\mathcal{G}$ satisfies the Palais–Smale condition.
- $\mathcal{G}_{|\mathbb{R}\varphi_1}$ is anticoercive, $\mathcal{G}_{|V}$ is coercive.

The last two properties yield

$$C_1(\mathcal{G}, \infty) \neq 0 \tag{3.9}$$

(see [3, Proposition 3.8]).

PROPOSITION 3.4. *Assume that* (H$_4$), (H$_7$), (H$_8$) *hold and* $f(0) = 0$. *Let* $\lambda > 0$ *and* $\underline{u}_\lambda \in \operatorname{int} C_+$ *be a weak subsolution of the problem*

$$\begin{cases} -\Delta u = \lambda u^{-\gamma} & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Define $g_\lambda\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *by*

$$g_\lambda(x,t) = \begin{cases} \underline{u}_\lambda(x)^{-\gamma} & \textit{if } t < \underline{u}_\lambda(x), \\ t^{-\gamma} & \textit{if } t \geq \underline{u}_\lambda(x), \end{cases}$$

and set

$$G_\lambda(x,t) = \int_0^t g_\lambda(x,s)\,ds, \quad x \in \Omega,\ t \in \mathbb{R}.$$

Consider also the functional $\mathcal{F}_\lambda\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ *defined by*

$$\mathcal{F}_\lambda(u) = \frac{1}{2}\,\|u\|^2 - \int_\Omega F(u)\,dx - \lambda \int_\Omega G_\lambda(x,u)\,dx.$$

Then, $\mathcal{F}_\lambda$ *is of class* C^1 *and it possesses a critical point* v_λ *which is a weak solution of problem* $(\mathcal{P}_\lambda)$ *such that* $v_\lambda \geq \underline{u}_\lambda$, $C_1(\mathcal{F}_\lambda, v_\lambda) \neq 0$.

PROOF. Note first that for some positive constant $\widehat{c}_\lambda$, we have

$$\left|\int_\Omega g_\lambda(x,u)\varphi\,dx\right| \leq \widehat{c}_\lambda ||\varphi||, \quad \text{for all } u, \varphi \in W_0^{1,2}(\Omega), \tag{3.10}$$

and

$$\left|\int_\Omega G_\lambda(x,u)\,dx\right| \leq \widehat{c}_\lambda ||u||, \quad \text{for all } u \in W_0^{1,2}(\Omega). \tag{3.11}$$

Indeed, it follows from the definition of g_λ and G_λ that

$$0 < g_\lambda(x,t) \leq \underline{u}_\lambda(x)^{-\gamma}, \quad |G_\lambda(x,t)| \leq |t|\underline{u}_\lambda(x)^{-\gamma}, \quad \text{for all } x \in \Omega,\ t \in \mathbb{R}.$$

Now the claim follows from Remark 2.2.

Next, we remark (see [6, Proposition 2.3]) that $\mathcal{F}_\lambda$ is of class C^1 and its critical points are greater than or equal to $\underline{u}_\lambda$, hence solutions of $(\mathcal{P}_\lambda)$.

Also recall that $f(t) = 0$ for $t < 0$.

Step 1. The functional $\mathcal{F}_\lambda$ fulfills the Cerami condition.

Let $\{u_n\}$ be a sequence in $W_0^{1,2}(\Omega)$ satisfying the following conditions:

(j) $\sup_n |\mathcal{F}_\lambda(u_n)| < \infty$,

(jj) $(1 + \|u_n\|)\|\mathcal{F}'_\lambda(u_n)\| \to 0$ as $n \to \infty$.

We shall prove that $\{u_n\}$ is bounded. From (jj) we get that there exists a sequence $\varepsilon_n \to 0$ such that for every $\varphi \in W_0^{1,2}(\Omega)$

$$\left|\int_\Omega \nabla u_n \nabla \varphi - \int_\Omega f(u_n)\varphi - \lambda \int_\Omega g_\lambda(x, u_n)\varphi\right| \leq \frac{\varepsilon_n \|\varphi\|}{1 + \|u_n\|}, \quad n \in \mathbb{N}. \tag{3.12}$$

Choosing in (3.12) $\varphi = -u_n^-$ and observing that $f(u_n)u_n^- = 0$, we obtain

$$\|u_n^-\|^2 \leq -\int_\Omega \nabla u_n \nabla u_n^- + \lambda \int_\Omega g_\lambda(x, u_n)u_n^- \leq \varepsilon_n \|u_n^-\|, \quad n \in \mathbb{N},$$

which implies that

$$\lim_{n\to\infty} \|u_n^-\| = 0. \tag{3.13}$$

Let us prove now that $\{u_n^+\}$ is bounded. Assume by contradiction (and by passing to a subsequence, if necessary) that $\|u_n^+\| \to \infty$. Bearing in mind that $u_n = u_n^+ - u_n^-$ and that $f(-u_n^-) = 0, f(0) = 0$, we rewrite (3.12) in the following way:

$$\left|\int_\Omega \nabla u_n^+ \nabla \varphi - \int_\Omega \nabla u_n^- \nabla \varphi - \int_\Omega f(u_n^+)\varphi - \lambda \int_\Omega g_\lambda(x, u_n)\varphi\right| \leq \frac{\varepsilon_n \|\varphi\|}{1 + \|u_n\|}, \quad n \in \mathbb{N},$$

which implies that

$$\left|\int_\Omega \nabla u_n^+ \nabla \varphi - \int_\Omega f(u_n^+)\varphi - \lambda \int_\Omega g_\lambda(x, u_n)\varphi\right| \leq \left(\frac{\varepsilon_n}{1 + \|u_n\|} + \|u_n^-\|\right)\|\varphi\|, \tag{3.14}$$

for all $n \in \mathbb{N}, \varphi \in W_0^{1,2}(\Omega)$. Put $y_n = u_n^+/\|u_n^+\|$, $n \in \mathbb{N}$. Then,

$$y_n \geq 0 \quad \text{and} \quad \|y_n\| = 1, \quad \text{for all } n \in \mathbb{N}. \tag{3.15}$$

By passing to subsequences we may assume that

$$y_n \xrightarrow{\text{w}} y \quad \text{in } W_0^{1,2}(\Omega), \qquad y_n \to y \quad \text{in } L^2(\Omega), \qquad y_n \to y \quad \text{pointwisely in } \Omega.$$

Dividing by $\|u_n^+\|$ both members of (3.14) and taking into account (3.10) we obtain

$$\left|\int_\Omega \nabla y_n \nabla \varphi - \int_\Omega \frac{f(u_n^+)}{\|u_n^+\|}\varphi\right| \leq \left(\frac{\varepsilon_n}{1 + \|u_n\|} + \|u_n^-\| + \text{const.}\right)\frac{\|\varphi\|}{\|u_n^+\|}, \tag{3.16}$$

for all $n \in \mathbb{N}$, $\varphi \in W_0^{1,2}(\Omega)$.

Hypotheses (H_4), (H_8) imply that the sequence

$$\left\{\frac{f(u_n^+(\,\cdot\,))}{\|u_n^+\|}\right\} \subseteq L^2(\Omega)$$

is bounded. Thus, we may assume that it is weakly convergent in $L^2(\Omega)$. Using again hypothesis (H_8) and reasoning as in [14, Proposition 5], we may find $\xi \in L^\infty(\Omega)_+$ such that

$$\frac{f(u_n^+(\,\cdot\,))}{\|u_n^+\|} \xrightarrow{\mathrm{w}} \xi y \quad \text{in } L^2(\Omega) \quad \text{and} \quad \lambda_1 \le \xi(x) \le \lambda_2 \quad \text{a.e. in } \Omega. \tag{3.17}$$

In (3.16) we choose $\varphi = y_n - y \in W_0^{1,2}(\Omega)$, and pass to the limit to obtain that

$$\int_\Omega \nabla y_n \nabla (y_n - y) \to 0$$

(see also (3.13)) which clearly implies that $y_n \to y$ strongly in $W_0^{1,2}(\Omega)$. From (3.15) we have

$$y \ge 0 \quad \text{and} \quad \|y\| = 1. \tag{3.18}$$

Passing again to the limit in (3.16) (recall that the right hand side tends to zero), we obtain

$$\int_\Omega \nabla y \nabla \varphi - \int_\Omega \xi y \varphi = 0, \quad \text{for all } \varphi \in W_0^{1,2}(\Omega),$$

that means, y satisfies, in the weak sense

$$\begin{cases} -\Delta y = \xi(x) y & \text{in } \Omega, \\ y = 0 & \text{in } \partial\Omega. \end{cases}$$

Note that $\lambda_1 \le \xi(x) \le \lambda_2$ almost everywhere in Ω. If $\xi \not\equiv \lambda_1, \lambda_2$, then the monotonicity properties of the weighted eigenvalues (see Remark 2.1) yield

$$\widehat{\lambda}_1(\xi) < \widehat{\lambda}_1(\lambda_1) = 1, \qquad \widehat{\lambda}_2(\xi) > \widehat{\lambda}_2(\lambda_2) = 1$$

which implies $y \equiv 0$, a contradiction.

If $\xi \equiv \lambda_2$ then, y would be nodal against (3.18). Thus, $\xi \equiv \lambda_1$. So, y turns out to be a λ_1-eigenfunction and hence, $y(x) > 0$ in Ω. Consequently,

$$u_n^+(x) = \|u_n^+\| y_n(x) \to +\infty, \quad \text{a.e. in } \Omega. \tag{3.19}$$

From assumption (H_7) we obtain that

$$\lim_{t \to +\infty} \big\{ t[f(t) + \lambda g_\lambda(x,t)] - 2[F(t) + \lambda G_\lambda(x,t)] \big\} = +\infty, \tag{3.20}$$

uniformly for almost all $x \in \Omega$.

From condition (jj) it follows that $\mathcal{F}_\lambda'(u_n)u_n \to 0$ and since

$$\int_\Omega f(u_n)u_n = \int_\Omega f(u_n^+)u_n^+, \qquad \int_\Omega F(u_n) = \int_\Omega F(u_n^+),$$

$$\int_\Omega g_\lambda(x,u_n)u_n = \int_\Omega g_\lambda(x,u_n^+)u_n^+ - \int_\Omega g_\lambda(x,-u_n^-)u_n^-,$$

$$\int_\Omega G_\lambda(x, u_n) = \int_\Omega G_\lambda(x, u_n^+) + \int_\Omega G_\lambda(x, -u_n^-),$$

we get that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\text{const.} &\geq |\mathcal{F}_\lambda(u_n) - \mathcal{F}'_\lambda(u_n)u_n| \\
&= \bigg| \int_\Omega \{[f(u_n)u_n + \lambda g_\lambda(x, u_n)u_n] - 2[F(u_n) + \lambda G_\lambda(x, u_n)]\} \bigg| \\
&= \bigg| \underbrace{\int_\Omega \big\{[f(u_n^+)u_n^+ + \lambda g_\lambda(x, u_n^+)u_n^+] - 2[F(u_n^+) + \lambda G_\lambda(x, u_n^+)]\big\}}_{J_n^\lambda} \\
&\qquad - \lambda \int_\Omega [g_\lambda(x, -u_n^-)u_n^- + 2G_\lambda(x, -u_n^-)] \bigg| \\
&\geq J_n^\lambda - \text{const.}\|u_n^-\|
\end{aligned}$$

(see (3.10), (3.11)). Combining (3.19) and (3.20) with Fatou's lemma, we deduce that $J_n^\lambda \to +\infty$, as $n \to \infty$. Recalling (3.13), i.e. $\|u_n^-\| \to 0$, from the above computation we reach a contradiction. Thus, $\{u_n^+\}$ is bounded, which together with (3.13) implies the boundedness of $\{u_n\}$ in $W_0^{1,2}(\Omega)$. In a standard way we conclude that $\{u_n\}$ admits a strongly convergent subsequence.

To proceed, define the homotopy $h_t^\lambda\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ for any $t \in [0, 1]$ by

$$h_t^\lambda(u) = (1 - t)\mathcal{F}_\lambda(u) + t\mathcal{G}(u),$$

where $\mathcal{G}$ is defined after Lemma 3.3.

Step 2. The assumptions of Lemma 3.3 *are satisfied.*

It is clear that $(t, u) \mapsto h_t^\lambda(u)$ belongs to $C^1([0, 1] \times W_0^{1,2}(\Omega))$, it is bounded and that conditions (a)–(c) of Lemma 3.3 are fulfilled.

We shall prove condition (d) arguing by contradiction. Some of the arguments will be close to those in Step 1. For completeness we give the details.

Suppose on the contrary that there exist sequences $\{t_n\} \subset [0, 1]$, $\{u_n\} \subset W_0^{1,2}(\Omega)$ such that

$$t_n \to t, \qquad h_{t_n}^\lambda(u_n) \to -\infty, \qquad (1 + \|u_n\|)\|(h_{t_n}^\lambda)'(u_n)\| \to 0. \tag{3.21}$$

We claim that $t = 0$. Due to (3.21) and by passing to subsequences, we may assume that $\|u_n\| \to +\infty$. Thus, for each $\varphi \in W_0^{1,2}(\Omega)$, we have

$$|(1 - t_n)\mathcal{F}'_\lambda(u_n)\varphi + t_n\mathcal{G}'(u_n)\varphi| \leq \frac{\varepsilon_n\|\varphi\|}{1 + \|u_n\|},$$

that is

$$(3.22)\qquad \bigg|\int_\Omega \nabla u_n\nabla\varphi-(1-t_n)\int_\Omega f(u_n)\varphi \\ -(1-t_n)\lambda\int_\Omega g_\lambda(x,u_n)\varphi-t_n\mu\int_\Omega u_n\varphi\bigg|\le\frac{\varepsilon_n\|\varphi\|}{1+\|u_n\|}.$$

Put $y_n=u_n/\|u_n\|$, $n\in\mathbb{N}$. Then, $\|y_n\|=1$ for all $n\in\mathbb{N}$, so it is bounded in $W_0^{1,2}(\Omega)$, and admits a subsequence which we still denote by y_n such that

$$y_n\xrightarrow{\mathrm{w}}y\quad\text{in } W_0^{1,2}(\Omega),\qquad y_n\to y\quad\text{in } L^2(\Omega),\qquad y_n\to y\quad\text{pointwisely in }\Omega.$$

Dividing by $\|u_n\|$ both members of the previous inequality and recalling (3.10), we obtain

$$(3.23)\qquad \bigg|\int_\Omega\nabla y_n\nabla\varphi-(1-t_n)\int_\Omega\frac{f(u_n)}{\|u_n\|}\,\varphi-t_n\mu\int_\Omega y_n\varphi\bigg| \\ \le\bigg(\frac{\varepsilon_n}{1+\|u_n\|}+\text{const.}\bigg)\frac{\|\varphi\|}{\|u_n\|},$$

for all $n\in\mathbb{N}$, $\varphi\in W_0^{1,2}(\Omega)$. Since the sequence

$$\bigg\{\frac{f(u_n(\,\cdot\,))}{\|u_n\|}\bigg\}\subseteq L^2(\Omega)$$

is bounded, there exists $\xi\in L^\infty(\Omega)$ such that

$$(3.24)\qquad \frac{f(u_n(\,\cdot\,))}{\|u_n\|}\xrightarrow{\mathrm{w}}\xi y\quad\text{in } L^2(\Omega)\quad\text{and}\quad \lambda_1\le\xi(x)\le\lambda_2,\quad\text{a.e. in }\Omega.$$

(We have used again hypotheses (H_4), (H_8).)

In (3.23) we choose $\varphi=y_n-y\in W_0^{1,2}(\Omega)$ and pass to the limit to deduce that

$$\int_\Omega\nabla y_n\nabla(y_n-y)\to0$$

which implies that $y_n\to y$ strongly in $W_0^{1,2}(\Omega)$, so $\|y\|=1$. Passing to the limit again in (3.23) we get

$$\int_\Omega\nabla y\nabla\varphi-(1-t)\int_\Omega\xi y\varphi-t\mu\int_\Omega y\varphi=0,\quad\text{for all }\varphi\in W_0^{1,2}(\Omega),$$

which means that y satisfies, in the weak sense, the problem

$$\begin{cases}-\Delta y=(1-t)\xi(x)y+t\mu y & \text{in }\Omega,\\ y=0 & \text{in }\partial\Omega.\end{cases}$$

Put $\xi_t(x)=(1-t)\xi(x)+t\mu$. Suppose that $t\in(0,1]$. Then, since $\xi(x)\in[\lambda_1,\lambda_2]$ and $\mu\in\,]\lambda_1,\lambda_2[$, their convex combination $\xi_t(x)\in\,]\lambda_1,\lambda_2[$ for almost all $x\in\Omega$. Now the monotonicity properties of the weighted eigenvalues (see Remark 2.1) yield $y\equiv0$ which is a contradiction as $\|y\|=1$. Thus, $t=0$.

Next we choose in (3.22) $\varphi = -u_n^-$, and noticing that $f(u_n)u_n^- = 0$ almost everywhere in Ω, we get

$$\begin{aligned}\left(1-\frac{t_n\mu}{\lambda_1}\right)\|u_n^-\|^2 &\le \|u_n^-\|^2 - t_n\mu\|u_n^-\|_2^2 \\ &\le \|u_n^-\|^2 + (1-t_n)\lambda\int_\Omega g_\lambda(x,u_n)u_n^- - t_n\mu\|u_n^-\|_2^2 \le \frac{\varepsilon_n\|u_n^-\|}{1+\|u_n\|} \le \varepsilon_n\|u_n^-\|,\end{aligned}$$

for $n \in \mathbb{N}$. As $t_n \to 0$, from the above inequalities, we obtain that $\|u_n^-\| \to 0$. Since $\|u_n\| \to +\infty$, it must be $\|u_n^+\| \to +\infty$.

Now, for all $\varphi \in W_0^{1,2}(\Omega)$, $n \in \mathbb{N}$, we have

$$\begin{aligned}\bigg|\int_\Omega \nabla u_n^+\nabla\varphi - \int_\Omega \nabla u_n^-\nabla\varphi - (1-t_n)\int_\Omega f(u_n^+)\varphi \\ - \lambda(1-t_n)\int_\Omega g_\lambda(x,u_n)\varphi - t_n\mu\int_\Omega u_n^+\varphi + t_n\mu\int_\Omega u_n^-\varphi\bigg| \le \frac{\varepsilon_n\|\varphi\|}{1+\|u_n\|}\end{aligned}$$

which implies that

$$\begin{aligned}\bigg|\int_\Omega \nabla u_n^+\nabla\varphi - (1-t_n)\int_\Omega f(u_n^+)\varphi - t_n\mu\int_\Omega u_n^+\varphi\bigg| \\ \le \left(\frac{\varepsilon_n}{1+\|u_n\|} + \|u_n^-\| + c_7 + t_n\mu c_8\|u_n^-\|\right)\|\varphi\|,\end{aligned} \tag{3.25}$$

where c_7, c_8 are positive constants (we have also used (3.10) and Hölder's inequality).

Put $\widehat{y}_n = u_n^+/\|u_n^+\|$, for $n \in \mathbb{N}$. Then, $\widehat{y}_n \ge 0$ and $\|\widehat{y}_n\| = 1$, for all $n \in \mathbb{N}$. By passing to subsequences we may assume that

$$\widehat{y}_n \xrightarrow{\text{w}} \widehat{y} \quad \text{in } W_0^{1,2}(\Omega), \qquad \widehat{y}_n \to \widehat{y} \quad \text{in } L^2(\Omega), \qquad \widehat{y}_n \to \widehat{y} \quad \text{pointwisely in } \Omega.$$

Dividing by $\|u_n^+\|$ both members of (3.25), we get

$$\bigg|\int_\Omega \nabla\widehat{y}_n\nabla\varphi - (1-t_n)\int_\Omega \frac{f(u_n^+)}{\|u_n^+\|}\,\varphi - t_n\mu\int_\Omega \widehat{y}_n\varphi\bigg| \le \delta_n\|\varphi\| \tag{3.26}$$

for some $\delta_n \to 0$. As above (exploiting (H_4), (H_8) once more), we may find $\widehat{\xi} \in L^\infty(\Omega)_+$ such that

$$\frac{f(u_n^+(\,\cdot\,))}{\|u_n^+\|} \xrightarrow{\text{w}} \widehat{\xi}\widehat{y} \quad \text{in } L^2(\Omega) \quad \text{and} \quad \lambda_1 \le \widehat{\xi}(x) \le \lambda_2 \quad \text{a.e. in } \Omega.$$

Since $\widehat{y}_n \to \widehat{y}$ strongly in $W_0^{1,2}(\Omega)$ (acting with $\varphi = \widehat{y}_n - \widehat{y}$ in (3.26)) we deduce that $\widehat{y} \ne 0$. Passing to the limit in (3.26) and recalling that $t_n \to 0$, one has that

$$\begin{cases} -\Delta\widehat{y} = \widehat{\xi}\widehat{y} & \text{in } \Omega, \\ \widehat{y} = 0 & \text{in } \partial\Omega. \end{cases}$$

Hence, $\widehat{y}$ is an eigenfunction of the above weighted problem. It is clear that the following situations cannot occur:

- $\widehat{\xi} \not\equiv \lambda_1$, $\widehat{\xi} \not\equiv \lambda_2$ as the monotonicity properties of the weighted eigenvalues (see Remark 2.1) would imply that $\widehat{y} = 0$;
- $\widehat{\xi} \equiv \lambda_2$ as $\widehat{y}$ would be nodal.

Thus, $\widehat{\xi} \equiv \lambda_1$ and $\widehat{y}(x) > 0$, for almost all $x \in \Omega$. So, $u_n^+(x) = ||u_n^+||\widehat{y}_n(x) \to +\infty$, for almost all $x \in \Omega$.

To proceed, we observe that for all $n \in \mathbb{N}$,

$$\begin{aligned} h^\lambda_{t_n}(u_n) &= (1-t_n)\mathcal{F}_\lambda(u_n) + t_n\mathcal{G}(u_n) \\ &= \frac{1}{2}\|u_n\|^2 - (1-t_n)\int_\Omega [F(u_n) + \lambda G_\lambda(x,u_n)] - \frac{t_n\mu}{2}\|u_n\|_2^2 \\ (h^\lambda_{t_n})'(u_n)u_n &= \|u_n\|^2 - (1-t_n)\int_\Omega [f(u_n) + \lambda g_\lambda(x,u_n)]u_n - t_n\mu\|u_n\|_2^2. \end{aligned}$$

Hence, for all $n \in \mathbb{N}$,

$$\begin{aligned} &2h^\lambda_{t_n}(u_n) - (h^\lambda_{t_n})'(u_n)u_n \\ &= (1-t_n)\int_\Omega \{[f(u_n) + \lambda g_\lambda(x,u_n)]u_n - 2[F(u_n) + \lambda G_\lambda(x,u_n)]\} \\ &= (1-t_n)\underbrace{\int_\Omega \big\{[f(u_n^+) + \lambda g_\lambda(x,u_n^+)]u_n^+ - 2[F(u_n^+) + \lambda G_\lambda(x,u_n^+)]\big\}}_{J_n^\lambda} \\ &\quad - (1-t_n)\lambda\int_\Omega [g_\lambda(x,-u_n^-)u_n^- + 2G_\lambda(x,-u_n^-)] \\ &\geq (1-t_n)J_n^\lambda - \text{const.}\|u_n^-\| \end{aligned}$$

(see (3.10), (3.11)). By (3.20) and from Fatou's lemma, we infer that $J_n^\lambda \to +\infty$ as $n \to \infty$ and since $\|u_n^-\| \to 0$, the right hand side of the above inequality tends to $+\infty$. Instead, the left hand side of the above inequality tends to $-\infty$ as it follows from (3.21). The above contradiction shows that all the assumptions of Lemma 3.3 are fulfilled. We conclude that

$$C_k(\mathcal{F}_\lambda, \infty) = C_k(\mathcal{G}, \infty), \quad \text{for all } k \geq 0.$$

In particular, $C_1(\mathcal{F}_\lambda, \infty) \neq 0$, thus, there exists a critical point v_λ of $\mathcal{F}_\lambda$ such that $C_1(\mathcal{F}_\lambda, v_\lambda) \neq 0$. Being v_λ a critical point of $\mathcal{F}_\lambda$, we have that $v_\lambda \geq \underline{u}_\lambda$. Also, following the proof of Lemma A.5 of [10], one can prove that $v_\lambda \in L^\infty(\Omega)$. Combining these outcomes, v_λ turns out to be a solution of $(\mathcal{P}_\lambda)$. The proof is concluded. □

The proofs of our Theorems 1.2 and 1.4 are the combination of Proposition 3.4 with Proposition 3.1 and Proposition 3.2, respectively.

PROOF OF THEOREM 1.2. From Proposition 3.1, we know that there exist $\underline{u} \in \operatorname{int} C_+$ with $0 < \underline{u} < \xi_0$ and $u \in \operatorname{int}_{C^1}[\underline{u}, \xi_0]$ which is a global minimizer of $\mathcal{E}$, where

$$\mathcal{E}(z) = \frac{1}{2}\,||z||^2 - \int_\Omega H(x,z)\,dx, \quad z \in W_0^{1,2}(\Omega),$$

and the function $H(\,\cdot\,,\,\cdot\,)$ is defined in Step 3 of the proof of Proposition 3.1.

Moreover, consider the functional $\mathcal{F}\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{F}(z) = \frac{1}{2}\,\|z\|^2 - \int_\Omega F(z)\,dx - \int_\Omega G(x,z)\,dx,$$

where the function $G(\,\cdot\,,\,\cdot\,)$ coincides with $G_\lambda(\,\cdot\,,\,\cdot\,)$ defined in Proposition 3.4 in the special case where $\lambda = 1$ and $\underline{u}_\lambda = \underline{u}$.

The definitions of H and G imply that for $t \in [\underline{u}(x), \xi_0]$, we have

$$H(x,t) = F(t) + G(x,t) + \int_0^{\underline{u}(x)} [f(\underline{u}(x)) - f(s)]\,ds$$

and hence, $\mathcal{F}(z) = \mathcal{E}(z)+$const. for all $z \in [\underline{u}, \xi_0]$. Thus, u is a C^1-local minimizer of $\mathcal{F}$. From [9, Theorem 1.1] (see also [6, Proposition 2.3]), u turns out to be a $W_0^{1,2}(\Omega)$-local minimizer of $\mathcal{F}$. If u is not isolated, then problem $(\mathcal{P})$ admits infinitely many solutions. If u is isolated then $C_1(\mathcal{F}, u) = 0$. Proposition 3.4 applied with $\lambda = 1$, ensures the existence of a weak solution v of $(\mathcal{P})$ such that $C_1(\mathcal{F}, v) \neq 0$, which clearly says that $v \neq u$. □

Proof of Theorem 1.4. Note first that since $f \geq 0$ (see hypothesis (H_5)), each weak solution of $(\mathcal{P}_\lambda)$, $\lambda > 0$, lies in $\operatorname{int} C_+$. To check this, one has to employ Lemma A.7 and Theorem B.1 of [9] in conjunction with the Strong Comparison Principle. The proof follows now as above from the combination of Proposition 3.2 with Proposition 3.4, working with the functionals $\mathcal{F}_\lambda, \mathcal{E}_\lambda$ and with the pair $\underline{u}_\lambda, \overline{u}_\lambda$ of sub and supersolutions. □

If we replace (H_8) with the double resonance hypothesis (H_9) at a nonprinciple spectral interval $[\lambda_k, \lambda_{k+1}]$ for some $k \geq 2$, we obtain in a similar, easier way the same conclusions:

Proof of Theorems 1.6 and 1.7. The existence of the first solution is a consequence of Propositions 3.1 and 3.2. In order to produce the second solution it is enough to apply the classical Mountain Pass Theorem instead of Morse Theory. Note that under (H_9), the functional $\mathcal{F}_\lambda$ is anti-coercive, i.e. $\mathcal{F}_\lambda(t\varphi_1) \to -\infty$, as $t \to +\infty$, for all $\lambda > 0$. □

4. Existence of multiple solutions: proofs of Theorems 1.8–1.10

In this section we will study multiplicity results for our class of singular problems when f has a suitable oscillatory behaviour.

Lemma 4.1. *Let $f\colon [0,+\infty[\,\to \mathbb{R}$ be a continuous function. For $\lambda > 0$ assume that there exist $0 < a < b$ such that*

$$f(t) + \lambda t^{-\gamma} \le 0, \quad \text{for every } t \in [a,b].$$

Define $h_\lambda\colon]0,+\infty[\,\to \mathbb{R}$ by

$$h_\lambda(t) = \begin{cases} f(t) + \lambda t^{-\gamma} & \text{if } \ 0 < t < a, \\ f(a) + \lambda a^{-\gamma} & \text{if } \ t \ge a, \end{cases}$$

and set

$$H_\lambda(t) = \int_0^{t^+} h_\lambda(s)\, ds, \quad t \in \mathbb{R}.$$

Then, the functional $\mathcal{E}_\lambda\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\,\|u\|^2 - \int_\Omega H_\lambda(u(x))\, dx$$

has a global minimizer $u_\lambda \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that $\|u_\lambda\|_\infty \le a$. Moreover, u_λ turns out to be a weak solution of problem $(\mathcal{P}_\lambda)$.

Proof. Let $F\colon [0,+\infty[\, \to \mathbb{R}$ be the function $F(s) = \int_0^{s^+} f(t)\, dt$. Since $0 < \gamma < 1$, H_λ is well defined and continuous on $\mathbb{R}$. In particular,

$$H_\lambda(t) = \begin{cases} 0 & \text{if } \ t \le 0, \\ F(t) + \dfrac{\lambda}{1-\gamma}\, t^{1-\gamma} & \text{if } 0 < t < a, \\ H_\lambda(a) + h_\lambda(a)(t-a) & \text{if } t \ge a. \end{cases}$$

Moreover, $H_\lambda(t^+) = H_\lambda(t)$, for all $t \in \mathbb{R}$ and $(H_\lambda)'(t) = h_\lambda(t)$, for all $t > 0$.

The functional $\mathcal{E}_\lambda$ is well defined on $W_0^{1,2}(\Omega)$, sequentially weakly lower semicontinuous and coercive. Thus, it has a global minimizer u_λ. In what follows and for sake of simplicity, we fix $\lambda > 0$ and we put

$$h \equiv h_\lambda, \quad H \equiv H_\lambda, \quad \mathcal{E} \equiv \mathcal{E}_\lambda, \quad u \equiv u_\lambda.$$

We can assume that $u \le a$. Indeed, if

$$v = \begin{cases} u & \text{if } 0 < u < a, \\ a & \text{if } u \ge a, \end{cases}$$

then $v \in W_0^{1,2}(\Omega)$ and in view of $h(a) \le 0$ we have the following inequality:

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_{\{u \le a\}} H(u) - \int_{\{u > a\}} H(a) - h(a)\int_{\{u>a\}} (u-a) \ge \mathcal{E}(v).$$

Let us prove now that u is a weak solution of problem $(\mathcal{P}_\lambda)$. We claim that $u \ge 0$. Note that by definition of h we get that

$$|h(t)| \le \max_{[0,a]} |f| + \lambda \max\{t^{-\gamma}, a^{-\gamma}\}, \quad \text{for all } t > 0. \tag{4.1}$$

For $0 < t < 1$, one has $(u + tu^-)^+ = u^+$, thus $H(u + tu^-) = H((u + tu^-)^+) = H(u^+) = H(u)$, so

$$\begin{aligned} 0 \le\, &\mathcal{E}(u + tu^-) - \mathcal{E}(u) \\ =\, &\frac{1}{2}\, t^2 \|u^-\|^2 - t\|u^-\|^2 - \int_\Omega (H(u + tu^-) - H(u)) = \left(\frac{t}{2} - 1\right) t\|u^-\|^2 \le 0. \end{aligned}$$

From the above computation, it follows that $u^- = 0$, so $u \ge 0$ almost everywhere in Ω.

Assume that there exists a set of positive measure A such that $u = 0$ in A. Let $\varphi\colon \Omega \to \mathbb{R}$ be a bounded function in $W_0^{1,2}(\Omega)$, positive in Ω. For $t > 0$ such that $t\|\varphi\|_\infty \le a/2$, one has $(u + t\varphi)^{1-\gamma} > u^{1-\gamma}$ almost everywhere in Ω and

$$\begin{aligned} &\int_\Omega \frac{H(u + t\varphi) - H(u)}{t} \\ &\qquad = \underbrace{\int_{\{u \le a/2\}} \frac{H(u + t\varphi) - H(u)}{t}}_{I_1^t} + \underbrace{\int_{\{u > a/2\}} \frac{H(u + t\varphi) - H(u)}{t}}_{I_2^t}. \end{aligned}$$

Thus,

$$\begin{aligned} I_1^t =& \int_{\{u \le a/2\}} \frac{F(u + t\varphi) - F(u)}{t} \\ &+ \frac{\lambda}{(1-\gamma)t^\gamma} \int_A \varphi^{1-\gamma} + \frac{\lambda}{(1-\gamma)} \int_{\Omega \setminus A} \frac{(u + t\varphi)^{1-\gamma} - u^{1-\gamma}}{t} \\ >& \int_{\{u \le a/2\}} \frac{F(u + t\varphi) - F(u)}{t} + \frac{\lambda}{(1-\gamma)t^\gamma} \int_A \varphi^{1-\gamma} \\ =& \int_0^1 \int_{\{u \le a/2\}} f(u(x) + st\varphi(x))\varphi(x)\, dx\, ds + \frac{\lambda}{(1-\gamma)t^\gamma} \int_A \varphi^{1-\gamma} \to +\infty, \end{aligned}$$

as $t \to 0^+$. (Note that the first double integral tends to $\int_{\{u \le a/2\}} f(u)\varphi$, thanks to Dominated Convergence Theorem.) In addition, for $t > 0$ small enough as above, we have

$$\begin{aligned} I_2^t &= \int_{\{u > a/2\}} \frac{H(u + t\varphi) - H(u)}{t} \\ &= \int_0^1 \int_{\{u > a/2\}} h(u(x) + st\varphi(x))\varphi(x)\, dx\, ds \to \int_{\{u > a/2\}} h(u)\varphi, \end{aligned}$$

as $t \to 0^+$. Indeed, (4.1) implies that if $u(x) > a/2$, then

$$|h(u(x) + st\varphi(x))\varphi(x)| \le \Big[\max_{[0,a]} |f| + (a/2)^{-\gamma}\Big] \cdot \|\varphi\|_\infty, \quad \text{for all } s, t > 0,$$

so, the thesis follows from the Dominated Convergence Theorem.

Putting together the above computations, we infer that

$$0 \leq \frac{\mathcal{E}(u+t\varphi)-\mathcal{E}(u)}{t} = \frac{1}{2}\,t\|\varphi\|^2 + \int_\Omega \nabla u \nabla \varphi - I_1^t - I_2^t \to -\infty,$$

as $t \to 0^+$. The contradiction ensures that $u > 0$.

Let us prove now that

$$u^{-\gamma}\varphi \in L^1(\Omega), \quad \text{for all } \varphi \in W_0^{1,2}(\Omega), \tag{4.2}$$

and

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega f(u)\varphi - \lambda \int_\Omega u^{-\gamma}\varphi \geq 0, \quad \text{for all } \varphi \in W_0^{1,2}(\Omega),\ \varphi \geq 0. \tag{4.3}$$

Choose $\varphi \in W_0^{1,2}(\Omega)$, $\varphi \geq 0$, φ bounded. Fix a decreasing sequence $\{t_n\} \subseteq \,]0,1]$ with $\lim\limits_n t_n = 0$. We can assume that for all $n \in \mathbb{N}$, $t_n\|\varphi\|_\infty \leq a/2$. As above we can write

$$\int_\Omega \frac{H(u+t_n\varphi)-H(u)}{t_n} = \underbrace{\int_{\{u\leq a/2\}} \frac{H(u+t_n\varphi)-H(u)}{t_n}}_{I_1^n} + \underbrace{\int_{\{u> a/2\}} \frac{H(u+t_n\varphi)-H(u)}{t_n}}_{I_2^n}.$$

One has

$$I_1^n = \int_{\{u\leq a/2\}} \frac{F(u+t_n\varphi)-F(u)}{t_n} + \frac{\lambda}{(1-\gamma)} \int_{\{u\leq a/2\}} \frac{(u(x)+t_n\varphi(x))^{1-\gamma}-u(x)^{1-\gamma}}{t_n}.$$

The functions

$$\widetilde{h}_n(x) = \frac{(u(x)+t_n\varphi(x))^{1-\gamma}-u(x)^{1-\gamma}}{t_n}$$

are measurable, nonnegative and $\lim\limits_n \widetilde{h}_n(x) = (1-\gamma)u(x)^{-\gamma}\varphi(x)$ for almost all $x \in \Omega$. From Fatou's lemma, we deduce that

$$(1-\gamma)\int_{\{u\leq a/2\}} u^{-\gamma}\varphi \leq \liminf_n \int_{\{u\leq a/2\}} h_n,$$

which implies at once

$$\liminf_n I_1^n \geq \int_{\{u\leq a/2\}} f(u)\varphi + \lambda\int_{\{u\leq a/2\}} u^{-\gamma}\varphi$$

and

$$I_2^n = \int_{\{u> a/2\}} \frac{H(u+t_n\varphi)-H(u)}{t_n} \to \int_{\{u>a/2\}} h(u)\varphi = \int_{\{a/2<u\leq a\}} (f(u)+\lambda u^{-\gamma})\varphi,$$

as $n \to +\infty$. (Recall also that $u \leq a$.) Thus,

$$\liminf_n (I_1^n + I_2^n) \geq \int_\Omega (f(u) + \lambda u^{-\gamma})\varphi.$$

Passing to the $\liminf_n$ in the inequality

$$0 \leq \frac{\mathcal{E}(u + t_n\varphi) - \mathcal{E}(u)}{t_n} = \frac{1}{2}\, t_n \|\varphi\|^2 + \int_\Omega \nabla u \nabla \varphi - I_1^n - I_2^n,$$

we obtain at once condition (4.2) (it is enough to prove the integrability for a nonnegative test function) and

$$\int_\Omega (f(u) + \lambda u^{-\gamma})\varphi \leq \int_\Omega \nabla u \nabla \varphi$$

which is claim (4.3) with φ bounded.

Choose now $\varphi \in W_0^{1,2}(\Omega), \varphi \geq 0$ and let $\{\varphi_n\}$ be a sequence in $C_0^1(\overline{\Omega})$ of nonnegative functions converging to φ in $W_0^{1,2}(\Omega)$. We have

$$\lambda \int_\Omega u^{-\gamma}\varphi_n \leq \int_\Omega \nabla u \nabla \varphi_n - \int_\Omega f(u)\varphi_n$$

and by Fatou's lemma, we get the desired inequality with $\varphi \geq 0$.

In order to prove

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega f(u)\varphi - \lambda \int_\Omega u^{-\gamma}\varphi \geq 0, \quad \text{for all } \varphi \in W_0^{1,2}(\Omega),$$

we proceed as follows. Notice that the function $\widetilde{\xi}(t) = \mathcal{E}((1+t)u)$ has a local minimum at zero and

$$0 = \widetilde{\xi}'(0) = \lim_{t \to 0} \frac{\mathcal{E}((1+t)u) - \mathcal{E}(u)}{t} = \|u\|^2 - \lim_{t \to 0} \frac{H((1+t)u) - H(u)}{t}.$$

For $t > 0$ small enough we have

$$\begin{aligned}
&\int_\Omega \frac{H((1+t)u) - H(u)}{t} \\
&= \int_{\{u \leq a/2\}} \frac{F((1+t)u) - F(u)}{t} + \frac{\lambda}{1-\gamma} \int_{\{u \leq a/2\}} \frac{(1+t)^{1-\gamma} - 1}{t}\, u^{1-\gamma} \\
&\quad + \int_{\{u > a/2\}} \frac{H((1+t)u) - H(u)}{t} \\
&\to \int_{\{u \leq a/2\}} f(u)u + \lambda \int_{\{u \leq a/2\}} u^{1-\gamma} + \int_{\{u > a/2\}} h(u)u \\
&= \int_\Omega (f(u)u + \lambda u^{1-\gamma}), \quad \text{as } t \to 0^+.
\end{aligned}$$

So, combining the above outcomes we obtain

$$\int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^{1-\gamma} + \int_\Omega f(u)u. \tag{4.4}$$

Let $\varphi \in W_0^{1,2}(\Omega)$ be fixed and $\varepsilon \in]0,1[$ such that $\varepsilon < a/(2\|u\|_\infty)$. Plug into (4.3) the test function $v = (u+\varepsilon\varphi)^+$. Hence, by using (4.4) we have

$$\begin{aligned}
0 \leq &\int_{\{u+\varepsilon\varphi\geq 0\}} \nabla u \nabla (u+\varepsilon\varphi) \\
&- \lambda \int_{\{u+\varepsilon\varphi\geq 0\}} u^{-\gamma}(u+\varepsilon\varphi) - \int_{\{u+\varepsilon\varphi\geq 0\}} f(u)(u+\varepsilon\varphi) \\
= &\int_\Omega |\nabla u|^2 + \varepsilon \int_\Omega \nabla u \nabla \varphi \\
&- \lambda \int_\Omega u^{1-\gamma} - \varepsilon\lambda \int_\Omega u^{-\gamma}\varphi - \int_\Omega f(u)u - \varepsilon \int_\Omega f(u)\varphi \\
&- \int_{\{u+\varepsilon\varphi< 0\}} |\nabla u|^2 - \varepsilon \int_{\{u+\varepsilon\varphi< 0\}} \nabla u \nabla \varphi \\
&+ \lambda \int_{\{u+\varepsilon\varphi< 0\}} u^{-\gamma}(u+\varepsilon\varphi) + \int_{\{u+\varepsilon\varphi< 0\}} f(u)(u+\varepsilon\varphi) \\
\leq &\varepsilon \bigg[\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega u^{-\gamma}\varphi - \int_\Omega f(u)\varphi \bigg] \\
&+ \int_{\{u+\varepsilon\varphi< 0,\, f(u)<0\}} f(u)(u+\varepsilon\varphi) - \varepsilon \int_{\{u+\varepsilon\varphi< 0\}} \nabla u \nabla \varphi,
\end{aligned}$$

and thus,

$$\begin{aligned}
(4.5) \quad 0 \leq \varepsilon \bigg[&\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega u^{-\gamma}\varphi - \int_\Omega f(u)\varphi \\
&+ \int_{\{u+\varepsilon\varphi< 0,\, f(u)<0\}} f(u)\varphi - \int_{\{u+\varepsilon\varphi< 0\}} \nabla u \nabla \varphi \bigg].
\end{aligned}$$

Notice that as $\varepsilon \to 0$, the measure of the set $\{u+\varepsilon\varphi < 0\} \to 0$, so

$$\int_{\{u+\varepsilon\varphi< 0,\, f(u)<0\}} f(u)\varphi \to 0, \qquad \int_{\{u+\varepsilon\varphi< 0\}} \nabla u \nabla \varphi \to 0.$$

Hence, dividing by ε and passing to the limit as $\varepsilon \to 0$ in (4.5), we get that

$$\int_\Omega \nabla u \nabla \varphi - \lambda \int_\Omega u^{s-1}\varphi - \int_\Omega f(u)\varphi \geq 0.$$

From the arbitrariness of φ, it follows that u is a weak solution of $(\mathcal{P}_\lambda)$. $\square$

Remark 4.2. If $\lambda = 0$, h can be defined in zero and the above conclusion holds with u nonnegative weak solution of $(\mathcal{P}_0)$.

Proof of Theorem 1.8. The proof of this result closely follows the idea of Theorem 1.2 of [13]. For completeness we give the details.

From (H_{11}) there exist $M_0 < 0$ and $\delta > 0$ such that

$$\frac{F(t)}{t^2} > M_0, \quad \text{for every } 0 < t < \delta.$$

Fix $x_0 \in \Omega$ and $0 < r < R$ such that $\overline{B(x_0, R)} \subset \Omega$. Choose $M_1 > 0$ large enough such that

$$\frac{1}{2}\,\omega_N \frac{(R^N - r^N)}{(R-r)^2} - M_1\omega_N r^N - M_0\omega_N(R^N - r^N) < 0,$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

Hypothesis (H_{11}) also enables us to choose a sequence of positive numbers $\{\xi_n\}$ such that

$$\xi_n \to 0^+, \quad \frac{F(\xi_n)}{\xi_n^2} > M_1, \quad \text{for every } n \in \mathbb{N}.$$

By virtue of hypothesis (H_{10}) and by continuity, we can construct three sequences of positive numbers $\{a_n\}$, $\{b_n\}$ and $\{\lambda_n\}$ such that $a_n \to 0^+$, $b_n \to 0^+$, $\lambda_n \downarrow 0^+$, $a_n < b_n < a_{n-1}$, $\xi_n \le a_n < \delta$ for all n and

$$f(t) + \lambda t^{-\gamma} \le 0, \quad \text{for every } t \in [a_n, b_n],\ \lambda \in [0, \lambda_n],\ n \in \mathbb{N}.$$

In particular, we deduce that

$$f(t) \le 0 \quad \text{for every } t \in [a_n, b_n],\ n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ and for each $\lambda \in [0, \lambda_n]$, define $h_{n,\lambda}\colon\,]0, +\infty[\, \to \mathbb{R}$ by

$$h_{n,\lambda}(t) = \begin{cases} f(t) + \lambda t^{-\gamma} & \text{if } 0 < t < a_n, \\ f(a_n) + \lambda a_n^{-\gamma} & \text{if } t \ge a_n, \end{cases}$$

and

$$H_{n,\lambda}(t) = \int_0^{t^+} h_{n,\lambda}(s)\, ds, \quad t \in \mathbb{R}.$$

Denote by $\mathcal{E}_{n,\lambda}\colon W_0^{1,2}(\Omega) \to \mathbb{R}$ the functionals defined by

$$\mathcal{E}_{n,\lambda}(u) = \frac{1}{2}\,\|u\|^2 - \int_\Omega H_{n,\lambda}(u(x))\, dx$$

and notice that, if $\|u\|_\infty \le a_n$,

$$\mathcal{E}_{n,\lambda}(u) = \mathcal{E}_{n,0}(u) - \frac{\lambda}{1-\gamma}\int_\Omega u^{1-\gamma}.$$

From Lemma 4.1 we deduce that for each $n \in \mathbb{N}$ and for each $\lambda \in [0, \lambda_n]$, there exists a global minimizer of $\mathcal{E}_{n,\lambda}$, denoted by $u_{n,\lambda}$, such that $\|u_{n,\lambda}\|_\infty \le a_n$, which is also a weak solution of $(\mathcal{P}_\lambda)$. Notice that since $a_{n+1} < a_n$, we have that for $\lambda < \lambda_{n+1}, \mathcal{E}_{n,\lambda}(u_{n,\lambda}) \le \mathcal{E}_{n,\lambda}(u_{n+1,\lambda}) = \mathcal{E}_{n+1,\lambda}(u_{n+1,\lambda})$.

Applying again Lemma 4.1 for $\lambda = 0$ and Remark 4.2, we deduce also the existence of a sequence $\{u_{n,0}\}$ of nonnegative weak solutions of the following problem:

$$(\mathcal{P}_0) \qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

such that, for each $n \in \mathbb{N}$, $u_{n,0}$ is a global minimizer of the functional $\mathcal{E}_{n,0}$ with $\|u_{n,0}\|_\infty \le a_n$.

We prove now that, up to a subsequence, $\{u_{n,0}\}$ has pairwise distinct terms. Define on Ω the continuous functions w_n, $n \in \mathbb{N}$, by

$$w_n(x) = \begin{cases} \xi_n & \text{if } \ x \in B(x_0, r), \\ \xi_n \dfrac{R - |x - x_0|}{R - r} & \text{if } x \in B(x_0, R) \setminus B(x_0, r), \\ 0 & \text{if } \ x \in \Omega \setminus B(x_0, R). \end{cases}$$

Then, $w_n \in W_0^{1,2}(\Omega)$, $0 \le w_n \le \xi_n \le a_n < \delta$, $\|w_n\|^2 = \omega_N (R^N - r^N)\xi_n^2/(R-r)^2$. Thus,

$$\begin{aligned} \mathcal{E}_{n,0}(w_n) &= \frac{1}{2}\,\omega_N \frac{(R^N - r^N)}{(R-r)^2}\,\xi_n^2 - \int_\Omega F(w_n) \\ &= \frac{1}{2}\,\omega_N \frac{(R^N - r^N)}{(R-r)^2}\,\xi_n^2 - \int_{B(x_0,r)} F(\xi_n) - \int_{B(x_0,R)\setminus B(x_0,r)} F(w_n) \\ &\le \left[\frac{1}{2}\,\omega_N \frac{(R^N - r^N)}{(R-r)^2} - M_1 \omega_N r^N - M_0 \omega_N (R^N - r^N)\right]\xi_n^2 < 0 \end{aligned}$$

by the above choice of M_1. Thus, $\mathcal{E}_{n,0}(u_{n,0}) \le \mathcal{E}_{n,0}(w_n) < 0$ for every $n \in \mathbb{N}$. Moreover, from the inequalities

$$0 > \mathcal{E}_{n,0}(u_{n,0}) \ge -a_n \max_{[0,a_1]} |f||\Omega|,$$

we deduce that

$$\lim_n \mathcal{E}_{n,0}(u_{n,0}) = \lim_n \mathcal{E}_{n,0}(w_n) = 0.$$

From above we conclude that there exists a subsequence which we still denote by $\{u_{n,0}\}$ of pairwise distinct solutions of $(\mathcal{P}_0)$.

Choose now, as in [13], an increasing sequence $\{\theta_n\}$ of negative numbers tending to zero, such that $\theta_n < \mathcal{E}_{n,0}(u_{n,0}) < \theta_{n+1}$, $n \in \mathbb{N}$. We notice that

$$\mathcal{E}_{n,\lambda}(u_{n,\lambda}) \le \mathcal{E}_{n,\lambda}(u_{n,0}) < \mathcal{E}_{n,0}(u_{n,0}) < \theta_{n+1},$$

and

$$\mathcal{E}_{n,\lambda}(u_{n,\lambda}) = \mathcal{E}_{n,0}(u_{n,\lambda}) - \frac{\lambda}{1-\gamma} \int_\Omega (u_{n,\lambda})^{1-\gamma} \ge \mathcal{E}_{n,0}(u_{n,0}) - \frac{\lambda}{1-\gamma}\,|\Omega| a_n^{1-\gamma},$$

for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and since $\mathcal{E}_{n,0}(u_{n,0}) > \theta_n$, we can choose $\lambda < \min\{\lambda_n, \widetilde{\lambda}_n\}$ where

$$\widetilde{\lambda}_n = (1-\gamma)\,\frac{\mathcal{E}_n(u_{n,0}) - \theta_n}{a_n^{1-\gamma}|\Omega|}$$

to get

$$\theta_n < \mathcal{E}_{n,\lambda}(u_{n,\lambda}) < \theta_{n+1}. \tag{4.6}$$

Fix $k \in \mathbb{N}$ and set $\lambda_k^\star = \min\{\lambda_1, \dots, \lambda_k, \widetilde{\lambda}_1, \dots, \widetilde{\lambda}_k\}$. If $\lambda \le \lambda_k^\star$, then the functions $u_{1,\lambda}, \dots, u_{k,\lambda}$ are weak solutions of problem $(\mathcal{P}_\lambda)$. They are distinct. Indeed, if $u_{i,\lambda} = u_{j,\lambda}$ for some $i < j$, then $\mathcal{E}_{i,\lambda}(u_{i,\lambda}) = \mathcal{E}_{i,\lambda}(u_{j,\lambda}) = \mathcal{E}_{j,\lambda}(u_{j,\lambda})$, against (4.6). □

In the next result we prove the existence of a sequence of solutions avoiding the parameter as the singular term itself gives a small contribution at infinity. However we need to strengthen the sign condition (H_{10}).

Proof of Theorem 1.10. From (H_{13}) there exist $M_0 < 0$ and $\delta > 0$ such that

$$\frac{F(t)}{t^2} > M_0 \quad \text{for every} \ \ t > \delta.$$

Fix $x_0 \in \Omega$ and $0 < r < R$ such that $\overline{B(x_0, R)} \subset \Omega$, and choose M_1 and a sequence $\{\xi_n\} \subset \mathbb{R}^+$ with $\xi_n \to +\infty$ such that

$$\frac{1}{2}\,\omega_N\,\frac{(R^N - r^N)}{(R-r)^2} - M_1\omega_N r^N - M_0\omega_N(R^N - r^N) < 0$$

and

$$\frac{F(\xi_n)}{\xi_n^2} > M_1 \quad \text{for every } n \in \mathbb{N}$$

(we have used again (H_{13})). Eventually passing to a subsequence we can suppose that $\delta < \xi_n \le t_n$ for every $n \in \mathbb{N}$ and

$$f(t_n) + t_n^{-\gamma} < 0 \quad \text{for every } n \in \mathbb{N}$$

(see hypothesis (H_{12})). By continuity we can construct two sequences of positive numbers $\{a_n\}$ and $\{b_n\}$ such that $a_n \to +\infty$, $b_n \to +\infty$, $a_n < b_n < a_{n+1}$, $\xi_n \le a_n$ and

$$f(t) + t^{-\gamma} \le 0 \quad \text{for every } t \in [a_n, b_n],\ n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ define $h_n \colon]0, +\infty[\to \mathbb{R}$ by

$$h_n(t) = \begin{cases} f(t) + t^{-\gamma} & \text{if } 0 < t < a_n, \\ f(a_n) + a_n^{-\gamma} & \text{if } t \ge a_n, \end{cases}$$

and

$$H_n(t) = \int_0^{t^+} h_n(s)\,ds.$$

Then, according to Lemma 4.1, the functional $\mathcal{E}_n \colon W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{E}_n(u) = \frac{1}{2}\,\|u\|^2 - \int_\Omega H_n(u(x))\,dx$$

has a global minimizer u_n such that $\|u_n\|_\infty \le a_n$. Also, u_n turns out to be a weak solution of $(\mathcal{P})$.

Let us prove that $\lim_n \mathcal{E}_n(u_n) = -\infty$. Observe that the sequence $\{\mathcal{E}_n(u_n)\}$ is decreasing. Indeed, for every $n \in \mathbb{N}$, since $\|u_n\|_\infty \le a_n < a_{n+1}$,

$$\mathcal{E}_{n+1}(u_{n+1}) \le \mathcal{E}_{n+1}(u_n) = \mathcal{E}_n(u_n).$$

As before, define on Ω the continuous functions w_n, $n \in \mathbb{N}$ by

$$w_n(x) = \begin{cases} \xi_n & \text{if } x \in B(x_0, r), \\ \xi_n \dfrac{R - |x - x_0|}{R - r} & \text{if } x \in B(x_0, R) \setminus B(x_0, r) = D, \\ 0 & \text{if } x \in \Omega \setminus B(x_0, R). \end{cases}$$

Then $w_n \in W_0^{1,2}(\Omega)$, $0 \le w_n \le \xi_n \le a_n$ and $\|w_n\|^2 = \omega_N(R^N - r^N)\xi_n^2/(R-r)^2$ for all $n \in \mathbb{N}$. Moreover, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{E}_n(w_n) &= \frac{1}{2}\,\|w_n\|^2 - \int_\Omega H_n(w_n)\,dx \\ &= \frac{1}{2}\,\omega_N\,\frac{(R^N - r^N)}{(R-r)^2}\,\xi_n^2 - \int_\Omega F(w_n) - \frac{1}{-\gamma+1}\int_\Omega w_n^{-\gamma+1} \\ &< \frac{1}{2}\,\omega_N\,\frac{(R^N - r^N)}{(R-r)^2}\,\xi_n^2 - \int_{B(x_0,r)} F(\xi_n) \\ &\quad - \int_{D\cap\{w_n>\delta\}} F(w_n) - \int_{D\cap\{w_n\le\delta\}} F(w_n) \\ &\le \left[\frac{1}{2}\,\omega_N\,\frac{(R^N - r^N)}{(R-r)^2} - M_1\omega_N r^N - M_0\omega_N(R^N - r^N)\right]\xi_n^2 \\ &\quad + \omega_N(R^N - r^N)\max_{[0,\delta]}|F|. \end{aligned}$$

By the choice of M_1, $\lim_n \mathcal{E}_n(w_n) = -\infty$, which immediately implies $\lim_n \mathcal{E}_n(u_n) = -\infty$. In particular, by passing eventually to a subsequence, we may assume that u_n, $n \in \mathbb{N}$, are pairwisely distinct.

Finally, suppose that $\{\|u_n\|_\infty\}$ is bounded, i.e. there exists a constant M_2 such that $\|u_n\|_\infty \le M_2$ for all $n \in \mathbb{N}$. Fix $\overline{n}$ such that $a_{\overline{n}} > M_2$. Then, for every $n \ge \overline{n}$, we have $u_n < a_{\overline{n}} \le a_n$, so $H_{\overline{n}}(u_n(\,\cdot\,)) = H_n(u_n(\,\cdot\,))$ and hence,

$$\mathcal{E}_n(u_n) = \mathcal{E}_{\overline{n}}(u_n) \ge \mathcal{E}_{\overline{n}}(u_{\overline{n}}),$$

which is in contradiction with the previous limit. It follows that $\{\|u_n\|_\infty\}$ is unbounded so, we may extract a subsequence which tends to $+\infty$, as $n \to \infty$.□

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