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EXISTENCE OF THREE NONTRIVIAL SOLUTIONS FOR A CLASS OF FOURTH-ORDER ELLIPTIC EQUATIONS

Chun Li — Ravi P. Agarwal — Zeng-Qi Ou

Abstract. The existence of three nontrivial solutions is established for a class of fourth-order elliptic equations. Our technical approach is based on Linking Theorem and (∇)-Theorem.

1. Introduction and main results

We consider the fourth-order elliptic equation

$$\begin{cases} \Delta^2 u + c\Delta u = \mu u + f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 4)$ is a bounded smooth domain, $c \in \mathbb{R}$ and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$. Δ is the Laplace operator and Δ^2 is the biharmonic operator.

Let $0 < \lambda_1 < \ldots < \lambda_k < \ldots$ be the distinct eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. The eigenvalue problem

$$\begin{cases} \Delta^2 u + c\Delta u = \mu u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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has distinct eigenvalues $\mu_i = \lambda_i(\lambda_i - c)$, $i = 1, 2, \ldots$ We will always assume $c < \lambda_1$. The Hilbert space $E = H^2(\Omega) \cap H_0^1(\Omega)$ is equipped with the inner product

$$\langle u, v\rangle = \int_\Omega (\Delta u \Delta v - c\nabla u \nabla v)\, dx.$$

So, the norm on E is given by

$$\|u\| = \bigg(\int_\Omega (|\Delta u|^2 - c|\nabla u|^2)\, dx\bigg)^{1/2}.$$

A weak solution of problem (1.1) is $u \in E$ such that

$$\int_\Omega (\Delta u \Delta v - c\nabla u \nabla v - \mu uv)\, dx - \int_\Omega f(x, u)v\, dx = 0 \tag{1.3}$$

for all $v \in E$.

If $i > 1$, we denote

$$H_{i-1} = \bigoplus_{j \le i-1} \ker(\Delta^2 + c\Delta - \mu_j), \qquad H_i^0 = \ker(\Delta^2 + c\Delta - \mu_i)$$

and $H_i^\perp$ is the orthogonal complement of H_i in E, moreover, $\dim H_i < +\infty$. Let $P\colon E \to H_i^0$ and $Q\colon E \to H_{i-1} \oplus H_i^\perp$ be the orthogonal projections. For any $i \ge 1$, we have the following inequalities

$$\begin{aligned} &\|u\|^2 \le \mu_i \|u\|_{L^2}^2 && \text{for all } u \in H_i, \\ &\|u\|^2 \ge \mu_{i+1} \|u\|_{L^2}^2 && \text{for all } u \in H_i^\perp. \end{aligned} \tag{1.4}$$

For convenience, we introduce some notations. Let

$$S_i^+(\rho) = \{u \in H_i^\perp : \|u\| = \rho\}, \qquad B_i^+(\rho) = \{u \in H_i^\perp : \|u\| \le \rho\},$$
$$T_{i-1,i}(R) = \{u \in H_{i-1} : \|u\| \le R\} \cup \{u \in H_i : \|u\| = R\}.$$

In recent years, fourth-order problems have been studied by many authors (see [2], [3], [10]–[12], [23]–[25]). In [2], Lazer and McKenna pointed out that the problem (1.1) furnishes a model to study travelling waves in suspension bridges if $f(x, u) = b((u + 1)^+ - 1)$, where $u^+ = \max\{u, 0\}$ and $b \in \mathbb{R}$. Since then, more general nonlinear fourth-order elliptic boundary value problems have been studied. In [10, 11], Micheletti and Pistoia proved that the problem

$$\begin{cases} \Delta^2 u + c\Delta u = f(x, u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \Omega, \end{cases} \tag{1.5}$$

admits two or three solutions by the variational method. In [24], Zhang proved the existence of weak solutions for the problem (1.5) when $f(x, u)$ is sublinear at ∞. In [25], Zhang and Li showed that the problem (1.5) has at least two nontrivial solutions by means of Morse theory and local linking.

Recently, the (∇)-Theorem (see [7]) of Marino and Saccon is widely used to study the multiplicity of solutions for differential equations and variational inequalities (see [4]–[9], [12]–[19], [21]–[23]). In [4], Magrone, Mugnai and Servadei used the Linking Theorem and (∇)-Theorem to overcome the lack of the Palais–Smale condition, and proved the existence of three distinct nontrivial solutions for a class of semilinear elliptic variational inequalities involving a superlinear nonlinearity. In [21], [22], Wang proved that some nonlinear Schrödinger equations have at least three nontrivial solutions by means of the Linking Theorem and (∇)-Theorem. In [15], Mugnai studied the Dirichlet problem

$$\begin{cases} -\Delta u - \lambda u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a bounded smooth subset of $\mathbb{R}^N$, $N \geq 3$, and a Carathéodory function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is superlinear and subcritical in the usual senses. By using the Linking Theorem and (∇)-Theorem, Mugnai proved the existence of three nontrivial solutions for the problem (1.6).

In this paper, we shall study the multiplicity of nontrivial solutions for the problem (1.1). The technical approach is based on the Linking Theorem and (∇)-Theorem. Our main result is the following.

THEOREM 1.1. *Assume that the nonlinearity* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$. *Let* $F(x,t) = \int_0^t f(x,s)\,ds$, *suppose that the following conditions hold:*

- (f_1) *There are constants* $k_0 > 0$ *and* $1 < s < (N+4)/(N-4)$ *such that*
$$|f(x,t)| \leq k_0(1+|t|^s) \quad \textit{for all } (x,t) \in \Omega \times \mathbb{R}.$$
- (f_2) $f(x,t) = o(|t|)$ *as* $t \to 0$ *uniformly in* $x \in \Omega$.
- (f_3) *There exist constants* $\beta > \max\{N(s-1)/4, s\}$, $k_1 > 0$ *and* $L > 0$ *such that*
$$\begin{aligned} &f(x,t)t - 2F(x,t) > 0 && \textit{for all } (x,t) \in \Omega \times \mathbb{R} \setminus \{0\}, \\ &f(x,t)t - 2F(x,t) \geq k_1|t|^\beta && \textit{for all } x \in \Omega,\ |t| \geq L. \end{aligned}$$
- (F_1) $F(x,t)/t^2 \to +\infty$ *as* $|t| \to \infty$ *uniformly in* $x \in \Omega$.
- (F_2) $F(x,t) \geq 0$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.

Then, for all $i \geq 2$, *there is* $\delta_i > 0$ *such that for* $\mu \in (\mu_i - \delta_i, \mu_i)$, *the problem* (1.1) *has at least three nontrivial solutions.*

By the Sobolev theorem, E is continuously embedded in $L^\theta(\Omega)$ for $1 \leq \theta \leq 2N/(N-4)$. If $1 \leq \theta < 2N/(N-4)$, the embedding is compact. There exists a positive constant K such that

$$\|u\|_{L^\theta} \leq K\|u\| \quad \text{for all } u \in E, \tag{1.7}$$

for $\theta = 1, 2, s+1, 2N/(N-4) = 2^{**}$, where $\|\cdot\|_{L^\theta}$ denotes the norm of $L^\theta(\Omega)$. Define a functional

$$\varphi_\mu(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \quad \text{for all } u \in E. \tag{1.8}$$

It is easy to see that $\varphi_\mu \in C^1(E,\mathbb{R})$ under conditions of Theorem 1.1 and it is well known that a critical point of the functional φ_μ in E corresponds to a weak solution of the problem (1.1).

2. Proof of the main results

Here, we recall two variational theorems for the reader's convenience. The first one is a classical Linking Theorem, and the second one, called (∇)-Theorem, is a recent result in variational theory of mixed type, with assumptions on the values of the functional on some suitable sets and on the value of its gradient.

Theorem 2.1 (Classical Linking Theorem, [1], [20]). *Let X be a real Banach space with $X = X_1 \oplus X_2$, with $\dim X_1 < \infty$. Suppose $I \in C^1(X,\mathbb{R})$ satisfies the* (PS)*-condition, and*

(a) *there are constants ρ and $\alpha > 0$ such that $I|_{\partial B_\rho \cap X_2} \geq \alpha$, and*
(b) *there is an $e \in \partial B_1 \cap X_2$ and $R > \rho$ such that if $Q = (\overline{B}_R \cap X_1) \oplus \{re : 0 < r < R\}$, then $I|_{\partial Q} \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q}(I(h(u))),$$

where $\Gamma = \{h \in C(\overline{Q}, X) \mid h = \mathrm{id}$ on $\partial Q\}$.

Definition 2.2. Let X be a Hilbert space, $I\colon X \to \mathbb{R}$ be a C^1 functional, M a closed subspace of X, $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$. We say that the condition $(\nabla)(I, M, a, b)$ holds if there is $\gamma > 0$ such that

$$\inf\{\|P_M \nabla I(u)\| : a \leq I(u) \leq b,\ \mathrm{dist}(u, M) \leq \gamma\} > 0,$$

where $P_M\colon X \to M$ is the orthogonal projection of X onto M.

Theorem 2.3 (∇-Theorem, [7]). *Let X be a Hilbert space and X_i, $i = 1, 2, 3$, three subspaces of X such that $X = X_1 \oplus X_2 \oplus X_3$, and $\dim X_i < \infty$ for $i = 1, 2$. Denote by P_i the orthogonal projection of X onto X_i. Let $I\colon X \to \mathbb{R}$ be a $C^{1,1}$ functional and $\rho, \rho', \rho'', \rho_1$ be constants such that $\rho_1 > 0$, $0 \leq \rho' < \rho < \rho''$. Define*

$$\Gamma = \{u \in X_1 \oplus X_2 : \rho' \leq \|P_2 u\| \leq \rho'',\ \|P_1 u\| \leq \rho_1\} \quad \text{and} \quad T = \partial_{X_1\oplus X_2}\Gamma,$$
$$S_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| = \rho\} \quad \text{and} \quad B_{23}(\rho) = \{u \in X_2 \oplus X_3 : \|u\| \leq \rho\}.$$

Assume that $a' = \sup I(T) < \inf I(S_{23}(\rho)) = a''$. Let a and b be such that $a' < a < a''$ and $b > \sup I(\Gamma)$. Assume that $(\nabla)(I, X_1 \oplus X_3, a, b)$ holds and that

$(\mathrm{PS})_c$ *holds at any* c *in* $[a,b]$. *Then* I *has at least two critical points in* $I^{-1}([a,b])$. *Moreover, if* $\inf I(B_{23}(\rho)) > -\infty$, $a_1 < \inf I(B_{23}(\rho))$ *and* $(\mathrm{PS})_c$ *holds at any* c *in* $[a_1,b]$, *then* I *has another critical level in* $[a_1,a']$.

LEMMA 2.4. *Assume that conditions* (f_1)–(f_3) *hold. Then, for any* δ *with* $\min\{\mu_{i+1}-\mu_i,\mu_i-\mu_{i-1}\} > \delta > 0$, *for some* $\varepsilon_0 > 0$ *such that* $\mu \in [\mu_i-\delta,\mu_i+\delta]$, *the unique critical point* u *of* φ_μ *constrained on* $H_{i-1}\oplus H_i^\perp$ *such that* $\varphi_\mu(u) \in [-\varepsilon_0,\varepsilon_0]$, *is the trivial one.*

PROOF. Assume by contradiction that there exist $\delta>0$, $\{\overline{\mu}_n\}\subset[\mu_i-\delta,\mu_i+\delta]$, and $\{u_n\}\subset H_{i-1}\oplus H_i^\perp\backslash\{0\}$ such that, for all $v\in H_{i-1}\oplus H_i^\perp$,

$$\begin{aligned}\varphi_{\overline{\mu}_n}(u_n) &= \frac{1}{2}\int_\Omega(|\Delta u_n|^2 - c|\nabla u_n|^2 - \overline{\mu}_n u_n^2)\,dx \\ &\quad - \int_\Omega F(x,u_n)\,dx \to 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\langle\varphi'_{\overline{\mu}_n}(u_n),v\rangle &= \int_\Omega(\Delta u_n\Delta v - c\nabla u_n\nabla v - \overline{\mu}_n u_n v)\,dx \\ &\quad - \int_\Omega f(x,u_n)v\,dx = 0.\end{aligned} \tag{2.2}$$

Here, going if necessary to a subsequence, we can assume that $\overline{\mu}_n\to\mu\in[\mu_i-\delta,\mu_i+\delta]$ as $n\to\infty$. From (f_3), there exists a positive constant k_2 such that

$$f(x,t)t - 2F(x,t) \ge k_1|t|^\beta - k_2 \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}. \tag{2.3}$$

Let $v=u_n$ in (2.2), by (2.3), one has

$$\begin{aligned}2\varphi_{\overline{\mu}_n}(u_n) - \langle\varphi'_{\overline{\mu}_n}(u_n),u_n\rangle &= \int_\Omega(f(x,u_n)u_n - 2F(x,u_n))\,dx \\ &\ge k_1\int_\Omega|u_n|^\beta\,dx - k_2|\Omega|.\end{aligned} \tag{2.4}$$

We deduce from (2.1), (2.2) and (2.4) that there exists a positive constant L_1 such that

$$\int_\Omega|u_n|^\beta\,dx < L_1 \quad \text{for all } n. \tag{2.5}$$

Choose $v_n\in H_{i-1}$ and $w_n\in H_i^\perp$ so that $u_n=v_n+w_n$, for all n. And let $v=w_n-v_n$ in (2.2), we get

$$\begin{aligned}&\int_\Omega(|\Delta w_n|^2 - c|\nabla w_n|^2)\,dx - \int_\Omega(|\Delta v_n|^2 - c|\nabla v_n|^2)\,dx \\ &\qquad - \overline{\mu}_n\int_\Omega(w_n^2 - v_n^2)\,dx = \int_\Omega f(x,u_n)(w_n-v_n)\,dx.\end{aligned} \tag{2.6}$$

As for the first line in (2.6), by (1.4), one has

$$
\begin{aligned}
(2.7)\qquad &\int_\Omega (|\Delta w_n|^2 - c|\nabla w_n|^2)\,dx - \overline{\mu}_n \int_\Omega w_n^2\,dx \\
&\qquad - \int_\Omega (|\Delta v_n|^2 - c|\nabla v_n|^2)\,dx + \overline{\mu}_n \int_\Omega v_n^2\,dx \\
&= \int_\Omega (|\Delta w_n|^2 - c|\nabla w_n|^2)\,dx - \frac{\overline{\mu}_n}{\mu_{i+1}}\mu_{i+1} \int_\Omega w_n^2\,dx \\
&\qquad - \int_\Omega (|\Delta v_n|^2 - c|\nabla v_n|^2)\,dx + \frac{\overline{\mu}_n}{\mu_{i-1}}\mu_{i-1} \int_\Omega v_n^2\,dx \\
&\geq \bigg(1 - \frac{\overline{\mu}_n}{\mu_{i+1}}\bigg)\|w_n\|^2 + \bigg(\frac{\overline{\mu}_n}{\mu_{i-1}} - 1\bigg)\|v_n\|^2.
\end{aligned}
$$

It follows from (2.6) and (2.7) that there exists $k_3 > 0$, independent of n, such that

$$
(2.8)\qquad k_3\|u_n\|^2 \leq \int_\Omega f(x,u_n)(w_n - v_n)\,dx \quad \text{for all } n \in \mathbb{N}.
$$

We have

$$
\beta > \frac{N}{4}(s-1) \quad \text{and} \quad \frac{N}{4}(s-1) < \frac{2N}{N+4}s < \frac{2N}{N-4}.
$$

First, we consider the case $N(s-1)/4 < \beta < 2Ns/(N+4)$. Put

$$
\alpha = \frac{2sN - (N+4)\beta}{2N - (N-4)\beta},
$$

one has $0 < \alpha < 1$. Let $p = \beta/(s-\alpha) > 1$, we can obtain from Hölder's inequality and (1.7) that

$$
\begin{aligned}
(2.9)\qquad &\int_\Omega |u_n|^s|w_n - v_n|\,dx = \int_\Omega |u_n|^{s-\alpha}|u_n|^\alpha|w_n - v_n|\,dx \\
&= \int_\Omega |u_n|^{\beta/p}|u_n|^\alpha|w_n - v_n|\,dx \\
&\leq \bigg(\int_\Omega (|u_n|^{\beta/p})^p\,dx\bigg)^{1/p}\bigg(\int_\Omega (|u_n|^\alpha|w_n - v_n|)^q\,dx\bigg)^{1/q} \\
&\leq L_1^{1/p}\bigg(\int_\Omega (|u_n|^{q\alpha})^{2^{**}/(q\alpha)}\,dx\bigg)^{\alpha/2^{**}}\bigg(\int_\Omega (|w_n - v_n|^q)^{2^{**}/q}\,dx\bigg)^{1/2^{**}} \\
&\leq L_1^{1/p}\|u_n\|_{L^{2^{**}}}^\alpha\|w_n - v_n\|_{L^{2^{**}}} \leq L_1^{1/p}K^{\alpha+1}\|u_n\|^\alpha\|w_n - v_n\|
\end{aligned}
$$

for all n, where $q = p/(p-1) = 2^{**}/(\alpha+1)$. By ($f_1$), (2.5), (2.8) and (2.9), for all n, one has

$$
\begin{aligned}
(2.10)\qquad k_3\|u_n\|^2 &\leq \bigg|\int_\Omega f(x,u_n)(w_n - v_n)\,dx\bigg| \leq \int_\Omega |f(x,u_n)||w_n - v_n|\,dx \\
&\leq k_0\int_\Omega (|u_n|^s|w_n - v_n| + |w_n - v_n|)\,dx
\end{aligned}
$$

$$\leq k_0 L_1^{1/p} K^{\alpha+1} \|u_n\|^{\alpha} \|w_n - v_n\| + k_0 K \|w_n - v_n\|$$
$$\leq 2k_0 L_1^{1/p} K^{\alpha+1} \|u_n\|^{\alpha+1} + 2k_0 K \|u_n\|.$$

On the other hand, if β satisfies

$$\frac{2N}{N+4} s \leq \beta < \frac{2N}{N-4},$$

then $1 \leq \beta/(\beta - s) \leq 2N/(N-4)$. So, we get

$$\|u\|_{L^{\beta/(\beta-s)}} \leq K\|u\|, \quad u \in E. \tag{2.11}$$

By (f_1) and (2.11), we obtain

$$\begin{aligned} k_3\|u_n\|^2 &\leq \left| \int_\Omega f(x,u_n)(w_n - v_n)\,dx \right| \\ &\leq k_0 \int_\Omega (|u_n|^s |w_n - v_n| + |w_n - v_n|)\,dx \\ &\leq k_0 \bigg(\int_\Omega (|u_n|^s)^{\beta/s}\,dx \bigg)^{s/\beta} \cdot \bigg(\int_\Omega |w_n - v_n|^{\beta/(\beta-s)}\,dx \bigg)^{(\beta-s)/\beta} \\ &\quad + k_0 K \|w_n - v_n\| \\ &\leq k_0 K \|u_n\|_{L^\beta}^s \cdot \|w_n - v_n\| + k_0 K \|w_n - v_n\| \\ &\leq 2k_0 K \|u_n\|_{L^\beta}^s \cdot \|u_n\| + 2k_0 K \|u_n\| \end{aligned} \tag{2.12}$$

for all n. Note that $\beta \geq 2Ns/(N+4)$ and $N > 4$ imply that $\beta > s$. So, it follows from (2.5), (2.10) and (2.12) that $\{u_n\}$ is bounded in E. There is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ and $u \in E$ such that $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^\theta(\Omega)$ for all $\theta \in (1, 2^{**})$ and $u_n(x) \rightharpoonup u(x)$ for almost every $x \in \Omega$. Let $v = u_n$ in (2.2), by (2.1) and Fatou's lemma, we get

$$\begin{aligned} 0 = \lim_{n\to\infty} 2\varphi_{\overline{\mu}_n}(u_n) - \langle \varphi'_{\overline{\mu}_n}(u_n), u_n \rangle &= \lim_{n\to\infty} \int_\Omega (f(x,u_n)u_n - 2F(x,u_n))\,dx \\ \geq \int_\Omega \liminf_{n\to\infty} (f(x,u_n)u_n - 2F(x,u_n))\,dx &\geq \int_\Omega (f(x,u)u - 2F(x,u))\,dx. \end{aligned}$$

Then, (f_3) and the above expression imply that $u = 0$. Since $\|u_n\|^2 = \|v_n\|^2 + \|w_n\|^2$, from (1.7), (2.8) and Hölder's inequality we have

$$\begin{aligned} k_3\|u_n\|^2 &\leq \bigg(\int_\Omega |f(x,u_n)|^{(1+s)/s}\,dx \bigg)^{s/(s+1)} \bigg(\int_\Omega |w_n - v_n|^{1+s}\,dx \bigg)^{1/(1+s)} \\ &\leq 2K\|u_n\| \bigg(\int_\Omega |f(x,u_n)|^{(1+s)/s}\,dx \bigg)^{s/(s+1)}, \end{aligned}$$

that is

$$k_3\|u_n\| \leq 2K \bigg(\int_\Omega |f(x,u_n)|^{(1+s)/s}\,dx \bigg)^{s/(s+1)}. \tag{2.13}$$

If $u_n \to 0$, then (f$_2$) and (2.13) would show

$$1 \leq \lim_{n\to\infty} \frac{K}{k_3\|u_n\|}\bigg(\int_\Omega |f(x,u_n)|^{(1+s)s}\,dx\bigg)^{s/(s+1)} = 0,$$

which is a contradiction. So, there should exist $\alpha_1 > 0$ such that $\|u_n\| \geq \alpha_1$ for all n. Then, we can obtain from (f$_2$) and (2.13) that

$$k_3\alpha_1 \leq \lim_{n\to\infty} K\bigg(\int_\Omega |f(x,u_n)|^{(1+s)/s}dx\bigg)^{s/(s+1)} = 0.$$

This is a contradiction. □

Lemma 2.5. *Assume that conditions* (f$_1$) *and* (f$_3$) *hold,* $\mu \in (\mu_{i-1}, \mu_{i+1})$ *and* $\{u_n\} \subset E$ *is such that* $\{\varphi_\mu(u_n)\}$ *is bounded,* $Pu_n \to 0$ *and* $Q\varphi'_\mu(u_n) \to 0$*. Then* $\{u_n\}$ *is bounded in* E*.*

Proof. By contradiction, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $u_n = Pu_n + Qu_n$, by (f$_1$) and Hölder's inequality, there exists a constant $k_4 > 0$ such that

$$\begin{aligned}\bigg|\int_\Omega f(x,u_n)Pu_n\,dx\bigg| &\leq \int_\Omega |f(x,u_n)||Pu_n|\,dx \\ &\leq k_0\int_\Omega (|Pu_n| + |u_n|^s|Pu_n)|)\,dx \leq k_0\|Pu_n\|_\infty\bigg(1+\int_\Omega |u_n|^s dx\bigg) \\ &\leq k_0\|Pu_n\|_\infty\bigg(1+|\Omega|^{(\beta-s)/\beta}\bigg(\int_\Omega (|u_n|^s)^{\beta/s}\,dx\bigg)^{s/\beta}\bigg) \\ &\leq k_4\|Pu_n\|_\infty(1+\|u_n\|^s_{L^\beta}).\end{aligned}$$

So, we can obtain from (2.4) and the above expression that

$$\begin{aligned}2\varphi_\mu(u_n) - \langle Q\varphi'_\mu(u_n), u_n\rangle &= 2\varphi_\mu(u_n) - \langle \varphi'_\mu(u_n), u_n\rangle + \langle P\varphi'_\mu(u_n), u_n\rangle \\ = &\int_\Omega (f(x,u_n)u_n - 2F(x,u_n))\,dx \\ &+ \int_\Omega (|\Delta Pu_n|^2 - c|\nabla Pu_n|^2 - \mu|Pu_n|^2)\,dx - \int_\Omega f(x,u_n)Pu_n\,dx \\ \geq &\, k_1\int_\Omega |u_n|^\beta dx - k_2|\Omega| + \|Pu_n\|^2 - \mu\|Pu_n\|^2_{L^2} - k_4\|Pu_n\|_\infty(1+\|u_n\|^s_{L^\beta}).\end{aligned}$$

Since $1 < s < \beta$, $\dim(H_i^0) < \infty$ and $\|Pu_n\|_\infty \to 0$ as $n \to \infty$, the above inequality implies that

$$\frac{\|u_n\|^s_{L^\beta}}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{2.14}$$

Let $Qu_n = v_n + w_n$, $v_n \in H_{i-1}$, $w_n \in H_i^\perp$. On one hand, from (2.9), for

$$\frac{N}{4}(s-1) < \beta < \frac{2N}{N+4}s \quad \text{and} \quad \alpha = \frac{2sN-(N+4)\beta}{2N-(N-4)\beta},$$

we can obtain by Hölder's inequality and (1.7) that

$$\int_\Omega |u_n|^s|v_n|\,dx \le K^{\alpha+1}\|u_n\|^\alpha\|v_n\|\bigg(\int_\Omega |u_n|^\beta\,dx\bigg)^{1/p} \tag{2.15}$$

and

$$\int_\Omega |u_n|^s|w_n|\,dx \le K^{\alpha+1}\|u_n\|^\alpha\|w_n\|\bigg(\int_\Omega |u_n|^\beta\,dx\bigg)^{1/p} \tag{2.16}$$

for all n, where $p = \beta/(s-\alpha) > 1$ and $q = p/(p-1) = 2^{**}/(\alpha+1)$. So, we can get from ($\mathrm{f}_1$), (1.4), (1.7) and (2.15) that

$$\begin{aligned}
\langle Q\varphi'_\mu(u_n), -v_n\rangle &= -\|v_n\|^2 + \mu\int_\Omega v_n^2 dx + \int_\Omega f(x,u_n)v_n\,dx \\
&\ge \bigg(-1+\frac{\mu}{\mu_{i-1}}\bigg)\|v_n\|^2 + \int_\Omega f(x,u_n)v_n\,dx \\
&\ge \frac{\mu-\mu_{i-1}}{\mu_{i-1}}\|v_n\|^2 - k_0\int_\Omega (|v_n| + |u_n|^s|v_n|)\,dx \\
&\ge \frac{\mu-\mu_{i-1}}{\mu_{i-1}}\|v_n\|^2 - k_0K^{\alpha+1}\|u_n\|^\alpha\|v_n\|\bigg(\int_\Omega |u_n|^\beta\,dx\bigg)^{1/p} - k_0\|v_n\|_{L^1} \\
&\ge \frac{\mu-\mu_{i-1}}{\mu_{i-1}}\|v_n\|^2 - k_0K\|v_n\|(1+K^\alpha\|u_n\|_{L^\beta}^{\beta/p}\|u_n\|^\alpha)
\end{aligned}$$

for all n. So, from (2.14), one has

$$\frac{\|v_n\|}{\|u_n\|} \to 0 \quad \text{as } n\to\infty. \tag{2.17}$$

From (f_1), (1.4), (1.7) and (2.16), we have

$$\begin{aligned}
\langle Q\varphi'_\mu(u_n), w_n\rangle &= \|w_n\|^2 - \mu\int_\Omega w_n^2\,dx - \int_\Omega f(x,u_n)w_n\,dx \\
&\ge \bigg(1-\frac{\mu}{\mu_{i+1}}\bigg)\|w_n\|^2 - \int_\Omega f(x,u_n)w_n\,dx \\
&\ge \frac{\mu_{i+1}-\mu}{\mu_{i+1}}\|w_n\|^2 - k_0\int_\Omega (|w_n| + |u_n|^s|w_n|)\,dx \\
&\ge \frac{\mu_{i+1}-\mu}{\mu_{i+1}}\|w_n\|^2 - k_0K^{\alpha+1}\|u_n\|^\alpha\|w_n\|\bigg(\int_\Omega |u_n|^\beta\,dx\bigg)^{1/p} - k_0\|w_n\|_{L^1} \\
&\ge \frac{\mu_{i+1}-\mu}{\mu_{i+1}}\|w_n\|^2 - k_0K\|w_n\|(1+K^\alpha\|u_n\|_{L^\beta}^{\beta/p}\|u_n\|^\alpha)
\end{aligned}$$

for all n. By (2.14), we get

$$\frac{\|w_n\|}{\|u_n\|} \to 0 \quad \text{as } n\to\infty. \tag{2.18}$$

On the other hand, if

$$\frac{2N}{N+4}\,s \le \beta < \frac{2N}{N-4},$$

we can get

$$\int_\Omega |u_n|^s |v_n|\,dx \le K\|u_n\|_{L^\beta}^s \|v_n\| \tag{2.19}$$

and

$$\int_\Omega |u_n|^s |w_n|\,dx \le K\|u_n\|_{L^\beta}^s \|w_n\| \tag{2.20}$$

for all n. It follows from (f$_1$), (1.4), (1.7) and (2.19) that

$$\langle Q\varphi'_\mu(u_n), -v_n\rangle \ge \frac{\mu-\mu_{i-1}}{\mu_{i-1}}\|v_n\|^2 - k_0K\|v_n\|(1+K^\alpha\|u_n\|_{L^\beta}^s)$$

for all n. Combining (2.14) and the above expression, we get

$$\frac{\|v_n\|}{\|u_n\|} \to 0 \quad \text{as } n\to\infty. \tag{2.21}$$

We conclude from (f$_1$), (1.4), (1.7) and (2.20) that

$$\langle Q\varphi'_\mu(u_n), w_n\rangle \ge \frac{\mu_{i+1}-\mu}{\mu_{i+1}}\|w_n\|^2 - k_0K\|w_n\|(1+\|u_n\|_{L^\beta}^s)$$

for all n. The above expression and (2.14) imply that

$$\frac{\|w_n\|}{\|u_n\|} \to 0 \quad \text{as } n\to\infty. \tag{2.22}$$

It is easy to see that

$$\frac{\|Pu_n\|}{\|u_n\|} \to 0 \quad \text{as } n\to\infty. \tag{2.23}$$

Then, we can get from (2.17), (2.18), (2.21)–(2.23) that

$$1=\frac{\|u_n\|}{\|u_n\|} \le \frac{\|v_n\|+\|Pu_n\|+\|w_n\|}{\|u_n\|} \to 0$$

as $n\to\infty$, which is a contradiction. Hence, $\{u_n\}$ must be bounded. □

Lemma 2.6. *Assume that conditions* (f$_1$)–(f$_3$) *hold. Then, for any δ with* $\min\{\mu_{i+1}-\mu_i, \mu_i-\mu_{i-1}\} > \delta > 0$ *and for some $\varepsilon > 0$, $\mu \in [\mu_i-\delta, \mu_i+\delta]$, and for any $a, b \in (0,\varepsilon)$ with $a < b$, the condition $(\nabla)(\varphi_\mu, H_{i-1}\oplus H_i^\perp, a, b)$ holds.*

Proof. Assume by contradiction that there exists $\delta > 0$ such that for all $\varepsilon_0 > 0$, there exist $\mu\in[\mu_i-\delta,\mu_i+\delta]$ and $a,b\in(0,\varepsilon_0)$ such that $a<b$ and the condition $(\nabla)(\varphi_\mu, H_{i-1}\oplus H_i^\perp, a, b)$ does not hold.

Take $\varepsilon_0 > 0$ as given by Lemma 2.4. There exists $\{u_n\}$ in E such that $d(u_n, H_{i-1}\oplus H_i^\perp)\to 0$, $\varphi_\mu(u_n)\in(a,b)$ and $Q\varphi'_\mu(u_n)\to 0$. So, from Lemma 2.5, $\{u_n\}$ is bounded. We can assume that $u_n \rightharpoonup u$.

Taking into account that $Q\varphi'_\mu(u_n) = u_n - Pu_n + (\Delta^2+c\Delta)^{-1}(\mu u_n + f(x,u_n))$, where $f(x,u_n)\to f(x,u)$ in $L^{1+1/s}(\Omega)$ by (f$_1$) and $(\Delta^2+c\Delta)^{-1}\colon E\to E$ is a compact operator. We have $u_n\to u$ and $u=0$ is a critical point of φ_μ

constrained on $H_{i-1} \oplus H_i^{\perp}$ by Lemma 2.4. But $0 < a \le \varphi_\mu(u_n)$ for all n and the continuity of φ_μ imply a contradiction. □

Lemma 2.7. *Assume that* (f_1), (f_2) *and* (F_2) *hold. Then, for any* $\mu \in (\mu_{i-1}, \mu_i)$*, there are* R, ρ *with* $R > \rho > 0$ *such that*

$$\sup \varphi_\mu(T_{i-1,i}(R)) < \inf \varphi_\mu(S_{i-1}^{+}(\rho)).$$

Proof. For any $u \in H_{i-1}$ and $\mu \in (\mu_{i-1}, \mu_i)$, by ($\mathrm{F}_2$), one has

$$\begin{aligned}\varphi_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \\ &\le \frac{\mu_{i-1}-\mu}{2\mu_{i-1}}\|u\|^2 \le 0.\end{aligned} \tag{2.24}$$

By (F_1), there exists $L_2 > 0$ such that

$$F(x,t) \ge (\mu_i - \mu)t^2 - L_2 \tag{2.25}$$

for all $(x,t) \in \Omega \times \mathbb{R}$. So, from (1.4) and (2.25), we have

$$\begin{aligned}\varphi_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \\ &\le \frac{\mu_i-\mu}{2\mu_i}\|u\|^2 - (\mu_i-\mu)\|u\|_{L^2}^2 + L_2|\Omega| \le \frac{\mu-\mu_i}{2\mu_i}\|u\|^2 + L_2|\Omega|\end{aligned}$$

for any $u \in H_i$. We obtain from the above expression that

$$\varphi_\mu(u) \to -\infty \quad \text{as } \|u\| \to \infty. \tag{2.26}$$

Moreover, by (f_1) and (f_2), for $\varepsilon = (\mu_i - \mu)/2 > 0$, there exists $L_3(\varepsilon) > 0$ such that

$$F(x,t) \le \frac{\varepsilon}{2}|t|^2 + L_3|t|^{s+1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{2.27}$$

For any $u \in H_{i-1}^{\perp}$, $\mu \in (\mu_{i-1}, \mu_i)$, by (1.7) and (2.27), we have

$$\begin{aligned}\varphi_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \\ &\ge \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \frac{\varepsilon}{2}\int_\Omega u^2\,dx - L_3\int_\Omega |u|^{s+1}\,dx \\ &\ge \frac{\mu_i-\mu-\varepsilon}{2\mu_i}\|u\|^2 - L_3\|u\|_{L^{s+1}}^{s+1} \ge \frac{\mu_i-\mu}{4\mu_i}\|u\|^2 - L_3K^{s+1}\|u\|^{s+1}.\end{aligned} \tag{2.28}$$

Since $s+1 > 2$, from (2.24), (2.26) and (2.28), there are $R > 0$ and $\rho > 0$ such that

$$\sup I_\lambda(T_{i-1,i}(R)) < \inf I_\lambda(S_{i-1}^{+}(\rho)).$$ □

Lemma 2.8. *Assume that condition* (F_2) *holds, then for any* $\varepsilon > 0$ *and for any* $R_1 > 0$*, there is* $\delta_i' > 0$ *such that for any* $\mu \in (\mu_i - \delta_i', \mu_i)$*, one has*

$$\sup \varphi_\mu(B_i(R_1)) < \varepsilon,$$

where $B_i(R_1) = \{u \in H_i : \|u\| \le R_1\}$.

Proof. By (F_2) and $\mu < \mu_i$, for any $u \in H_i$, we have

$$\varphi_\mu(u) = \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \le \frac{\mu_i - \mu}{2\mu_i}\,\|u\|^2.$$

Hence, it is easy to see that the conclusion holds. □

From Lemmas 2.7 and 2.8, we can choose $a \in (0, \inf \varphi_\mu(S_{i-1}^+(\rho)))$ and $b > \sup \varphi_\mu(B_i(R))$ such that $0 < a < b < \varepsilon_0$. Then the condition $(\nabla)(\varphi_\mu, H_{i-1} \oplus H_i^\perp, a, b)$ from Lemma 2.6 holds.

Proof of Theorem 1.1. The argument will be divided into two steps.

Step 1. Two critical points are obtained. First of all, from Lemmas 2.6–2.8, the condition $(\nabla)(\varphi_\mu, H_{i-1} \oplus H_i^\perp, a, b)$ holds. According to Theorem 2.3, if we can prove that the (PS) condition holds, there are two critical points u_1, u_2 such that $\varphi_\mu(u_i) \in [a,b]$, $i = 1,2$. From the proof of Lemma 2.5, if for any (PS) sequence $\{u_n\}$ of φ_μ, one has $\|Pu_n\|/\|u_n\| \to 0$ as $n \to \infty$, then $\{u_n\}$ is bounded. Moreover, by (f_1), a standard argument implies that the (PS) condition holds. By (f_3), there exist $k_5 > 0$ and $k_6 > 0$ such that

$$f(x,t)t - 2F(x,t) \ge k_5|t| - k_6 \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

which implies that

$$\begin{aligned}
&2\varphi_\mu(u_n) - \langle \varphi_\mu'(u_n), u_n\rangle \\
&\quad = \int_\Omega (f(x,u_n)u_n - 2F(x,u_n))\,dx \ge \int_\Omega (k_5|u_n| - k_6)\,dx \\
&\quad \ge \int_\Omega (k_5|Pu_n| - k_5|v_n| - k_5|w_n| - k_6)\,dx \ge k_7\|Pu_n\| - k_8(\|v_n\| + \|w_n\| + 1),
\end{aligned}$$

where k_7, k_8 are two positive constants. Combining (2.17), (2.18), (2.21), (2.22) with the above expression, we have $\|Pu_n\|/\|u_n\| \to 0$ as $n \to \infty$.

Step 2. The third critical point is obtained. By Theorem 2.1 it suffices to prove that there are $\delta_i'' > 0$ and $R_2 > \rho_1 > 0$ such that for any $\mu \in (\mu_i - \delta_i'', \mu_i)$,

$$\inf \varphi_\mu(S_i^+(\rho_1)) > \sup \varphi_\mu(T_{i,i+1}(R_2)). \tag{2.29}$$

Hence, there is a critical point u of φ_μ such that $\varphi_\mu(u) > \inf \varphi_\mu(S_i^+(\rho_1))$. For $u \in H_i^\perp$, by (2.27), one has

$$\begin{aligned}
\varphi_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \int_\Omega F(x,u)\,dx \\
&\ge \frac{1}{2}\int_\Omega (|\Delta u|^2 - c|\nabla u|^2 - \mu u^2)\,dx - \frac{\varepsilon}{2}\int_\Omega u^2\,dx - L_3\int_\Omega |u|^{s+1}\,dx \\
&\ge \frac{\mu_{i+1} - \mu - \varepsilon}{2\mu_{i+1}}\|u\|^2 - L_3\|u\|_{L^{s+1}}^{s+1} \ge \frac{\mu_{i+1} - \mu}{4\mu_{i+1}}\|u\|^2 - L_3K^{s+1}\|u\|^{s+1}.
\end{aligned}$$

Since $s+1>2$, there are $\rho_1>0$ and $\alpha>0$ such that

$$\inf \varphi_\mu(S_i^+(\rho_1)) \geq \alpha. \tag{2.30}$$

Moreover, for any $u\in H_i$, by (F$_2$), one has

$$\varphi_\mu(u)=\frac{1}{2}\int_\Omega (|\Delta u|^2-c|\nabla u|^2-\mu u^2)\,dx-\int_\Omega F(x,u)\,dx \leq \frac{\mu_i-\mu}{2\mu_i}\|u\|^2.$$

There are $\delta_i''>0$ and $R_2>0$ such that $\mu\in(\mu_i-\delta_i'',\mu_i)$. From the above expression we have $\varphi_\mu(u)<\alpha$ for all $\|u\|\leq R_2$. It follows from (F$_1$) that there exists $L_4>0$ such that

$$F(x,t)\geq(\mu_{i+1}-\mu)t^2-L_4 \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}. \tag{2.31}$$

For any $u\in H_{i+1}$ and $\mu\in(\mu_i-\delta_i'',\mu_i)$, by (2.31), we get

$$\begin{aligned}\varphi_\mu(u)&=\frac{1}{2}\int_\Omega (|\Delta u|^2-c|\nabla u|^2-\mu u^2)\,dx-\int_\Omega F(x,u)\,dx\\ &\leq \frac{\mu_{i+1}-\mu}{2\mu_{i+1}}\|u\|^2-(\mu_{i+1}-\mu)\|u\|_{L^2}^2+L_4|\Omega|\leq -\frac{\mu_{i+1}-\mu}{2\mu_{i+1}}\|u\|^2+L_4|\Omega|.\end{aligned}$$

Hence, we have

$$\varphi_\mu(u)\to-\infty \quad \text{as } \|u\|\to\infty. \tag{2.32}$$

Therefore, from (2.30) and (2.32), the conclusion (2.29) holds. By Theorem 2.1, there is a critical point u of φ_μ such that $\varphi_\mu(u)>\alpha$. Finally, we take $\delta_i=\min\{\delta_i',\delta_i''\}$ where δ_i' is given in Lemma 2.8 and Theorem 1.1 is proved. □

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Chun Li
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P.R. CHINA
E-mail address: Lch1999@swu.edu.cn

Ravi P. Agarwal
Department of Mathematics
Texas A&M University
Kingsville, TX 78363, USA
and
Distinguished University Professor of Mathematics
Florida Institute of Technology
150 West University Boulevard
Melbourne, FL 32901, USA

Zeng-Qi Ou
School of Mathematics and Statistics
Southwest University
Chongqing 400715, P.R. CHINA