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GENERAL AND OPTIMAL DECAY FOR A VISCOELASTIC EQUATION WITH BOUNDARY FEEDBACK

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Abstract. We establish a general decay rate for a viscoelastic problem with a nonlinear boundary feedback and a relaxation function satisfying $g'(t) \leq -\xi(t)g^p(t)$, $t \geq 0$, $1 \leq p < 3/2$. This work generalizes and improves earlier results in the literature. In particular those of [5], [11] and [17].

1. Introduction

In this work, we investigate the following viscoelastic wave equation with boundary feedback:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds = 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-s)\,\dfrac{\partial u}{\partial \nu}(s)\,ds + h(u_t) = 0 & \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, with $\operatorname{meas}(\Gamma_0) > 0$, ν is the unit outward normal to $\partial\Omega$, and g, h are specific functions.

In the absence of the viscoelastic term ($g = 0$), problem (1.1) has been investigated by many authors and several stability results were established. We refer the reader to the work of Lasiecka and Tataru [8], Alabau-Boussouira [1], Cavalcanti et al. [3] and the references therein.

In the presence of viscoelastic term ($g \neq 0$), Cavalcanti et al. [2] treated, in a bounded domain, a quasilinear equation of the form

$$
\begin{cases}
|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} \\
\qquad + \displaystyle\int_0^t g(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = 0 & \text{in } \Omega \times (0,+\infty), \\
u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t \geq 0, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{for } x \in \Omega,
\end{cases}
\tag{1.2}
$$

with $\rho > 0$, and established a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This latter result was extended to a situation, where a nonlinear source term is competing with the strong mechanism damping and the one induced by the viscosity, by Messaoudi and Tatar [18]. Furthermore, Messaoudi and Tatar [19], [20] established, for $\gamma = 0$, exponential and polynomial decay results in the absence, as well as in the presence, of a source term. Also, Messaoudi [11] studied the following problem:

$$
\begin{cases}
u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds + |u|^\gamma u = 0 & \text{in } \Omega \times (0,+\infty), \\
u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \Omega,
\end{cases}
$$

for relaxation functions satisfying, for a positive constant ξ,

$$
g'(t) \leq -\xi g^p(t), \qquad t \geq 0,\ 1 \leq p < \frac{3}{2}. \tag{1.3}
$$

He showed that the energy decays exponentially for $p = 1$ and polynomially for $p > 1$. In 2008, Messaoudi [12], [13] generalized the decay rates allowing an extended class of relaxation functions and gave general decay rates from which the exponential and the polynomial decay rates are only special cases however, the optimality in the polynomial decay case was not obtained. Precisely, he considered relaxation functions that satisfy

$$
g'(t) \leq -\xi(t)g(t), \quad t \geq 0, \tag{1.4}
$$

where $\xi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function and showed that the rate of the decay of the energy is the same as the rate of decay of g, which is not necessarily of exponential or polynomial decay type. Mustafa and Messaoudi [22] established an explicit and general decay rate for relaxation function satisfying

$$g'(t) \leq -H(g(t)), \tag{1.5}$$

where $H \in \mathcal{C}^1(\mathbb{R}_+)$, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex $\mathcal{C}^2$ function near the origin. Lasiecka and Wang [9] improved the results of [22] by extending the range of optimality in the case of polynomial decay rate. Moreover, the authors obtained this result in a more general semi-linear abstract viscoelastic problem. In [4], Cavalcanti et al. considered (1.2), with $\gamma = 0$, and a relaxation function satisfying (1.5) and in addition, they required $\liminf_{x\to 0^+} \{x^2H''(x) - xH'(x) + H(x)\} \geq 0$ and $y^{1-\alpha_0} \in L^1(1, +\infty)$, for some $\alpha_0 \in [0, 1)$, where $y(t)$ is the solution of the problem

$$y'(t) + H(y(t)) = 0, \quad y(0) = g(0) > 0.$$

They characterized the decay of the energy by the solution of a corresponding ODE as in [8]. Recently, Messaoudi and Al-Khulaifi [15] treated (1.2) with $\gamma = 0$ and a relaxation function satisfying

$$g'(t) \leq -\xi(t)g^p(t), \quad \text{for all } t \geq 0,\ 1 \leq p < \frac{3}{2}.$$

They obtained a more general stability result from which the results of [12], [13] are only special cases. Moreover, the optimal decay rate for the polynomial case is achieved without any extra work and conditions as in [7] and [8]. Messaoudi [14] investigated the problem

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)\,ds + a|u_t|^{m-2}u_t = 0 & \text{in } \Omega \times (0, +\infty), \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{for } x \in \Omega, \end{cases} \tag{1.6}$$

where $m > 1$ and $a > 0$ are constants, with relaxation functions satisfying (1.4). He established general decay rates for $1 < m < 2$ and $m \geq 2$, but still, these results are laking optimality for relaxation functions decaying polynomially. For stabilization by means of boundary feedback, Cavalcanti et al. [6] studied (1.1) and proved a global existence result for weak and strong solutions. Moreover, they gave some uniform decay rate results under some restrictive assumptions on both the kernel g and the damping function h. These restrictions had been relaxed by Cavalcanti et al. [5] and further they established a uniform stability depending on the behavior of h near the origin and on the behavior of g at infinity. Messaoudi and Mustafa [17] exploited some properties of convex functions [1] and the multiplier method to extend and improve these results, by considering relaxation functions g satisfying (1.4). In addition they gave an explicit and general decay rate result from which the polynomial and the exponential decay rates are only special cases. In [23], Wu proved an existence result for the following problem:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds = a|u|^{p-1}u & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-s)\,\frac{\partial u}{\partial \nu}(s)\,ds + h(u_t) = b|u|^{k-1}u & \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{for } x \in \Omega, \end{cases}$$

where $a > 0$, $b > 0$, $p > 1$, $k > 1$ and improved the result of [17] by establishing an explicit and general decay rate and further he proved some blow-up results. Recenlty, Wu [24] obtained the same stability result for the problem

$$\begin{cases} u_{tt} - M(t)\Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds = |u|^{p-1}u & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ M(t)\dfrac{\partial u}{\partial \nu} - \displaystyle\int_0^t g(t-s)\,\frac{\partial u}{\partial \nu}(s)\,ds + h(u_t) = |u|^{k-1}u & \text{on } \Gamma_1 \times (0,+\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{for } x \in \Omega, \end{cases}$$

where

$$M(t) = a + b\|\nabla u\|_2^2 + \sigma \int_\Omega \nabla u \cdot \nabla u_t\,dx, \quad a > 0,\ b > 0 \text{ and } \sigma > 0.$$

Our goal is to establish a more general decay rate from which the exponential decay and the polynomial decay are only special cases. Moreover, the optimal polynomial decay is easily and directly obtained without restrictive conditions (see Example 4.1). In fact, our decay formulae extend and improve the results in [5], [11]–[13] and [17]. We also simplify significantly the conditions of [10] and obtain a sharper general decay result.

2. Preliminaries

In this section we present some material needed in the proof of our result and state, without proof, the global existence result of [6]. Throughout this paper, C denotes a generic positive constant. We impose the following assumptions on g and h.

(H1) $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing differentiable function such that

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(s)\,ds = l > 0.$$

(H2) There exists a nonincreasing differentiable function $\xi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\xi(0) > 0$, and satisfying

$$g'(t) \le -\xi(t) g^p(t), \quad \text{for all } t \ge 0,\ 1 \le p < \frac{3}{2}.$$

(H3) $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a nondecreasing continuous function such that there exist positive constants c_1, c_2, ε and a strictly increasing function $H \in C^1([0,+\infty))$, with $H(0) = 0$, and H is linear or strictly convex $\mathcal{C}^2$ function on $(0,\varepsilon]$ such that

$$\begin{aligned} c_1|s| \le |h(s)| \le c_2|s|, &\quad \text{for all } |s| \ge \varepsilon, \\ s^2 + h^2(s) \le H^{-1}(sh(s)), &\quad \text{for all } |s| \le \varepsilon. \end{aligned}$$

Remark 2.1. Hypothesis (H3) implies that $sh(s) > 0$, for all $s \neq 0$.

Remark 2.2. It is worth to mention that condition (H3), with $\varepsilon = 1$, was introduced for the first time by Lasiecka and Tataru [8]. They also showed that the monotonicity and continuity of the function h guarantee the existence of the function H with the properties stated in (H3).

Remark 2.3. In condition (H2), we restrict the interval of p to be $(0, 3/2]$ where we obtained the optimal decay for the polynomial case. We point out that Lasiecka et al. [7] and Cavalcanti et al. [4] used iteration calculation to extend this interval to $(0, 2]$ in order to attain the optimal polynomial decay. However, our objective is to derive decay formulas so one can easily obtain the exponential and polynomial decays as special cases. See Example 4.2.

We start with the following crucial lemma which will be used in the proof of our result.

Lemma 2.4. *Assume that g satisfies* (H1) *and* (H2). *Then*

$$\int_0^{+\infty} \xi(t) g^{1-\sigma}(t)\,dt < +\infty, \quad \textit{for all } \sigma < 2 - p.$$

Proof. Recalling (H2), we easily see that

$$\xi(t) g^{1-\sigma}(t) = \xi(t) g^{1-\sigma}(t) g^p(t) g^{-p}(t) \le -g'(t) g^{1-\sigma-p}(t).$$

Integration then gives

$$\int_0^{+\infty} \xi(t) g^{1-\sigma}(t)\,dt \le -\int_0^{+\infty} g'(t) g^{1-\sigma-p}(t)\,dt = -\left.\frac{g^{2-p-\sigma}(t)}{2-p-\sigma}\right]_0^{+\infty} < +\infty,$$

since $\sigma < 2 - p$. $\square$

Let $V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$. For completeness, we state the global existence result of [6].

Theorem 2.5. *Let* $(u_0, u_1) \in V \times L^2(\Omega)$ *be given. Assume that* (H1)–(H3) *are satisfied. Then problem* (1.1) *has a unique global* (*weak*) *solution*

$$u \in \mathcal{C}(\mathbb{R}_+; V) \cap \mathcal{C}^1(\mathbb{R}_+; L^2(\Omega)).$$

Moreover, if $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V$ *and satisfies the compatibility condition*

$$\frac{\partial u_0}{\partial \nu} + h(u_1) = 0 \quad \text{on } \Gamma_1$$

then the solution satisfies

$$u \in L^\infty(\mathbb{R}_+; H^2(\Omega) \cap V) \cap W^{1,\infty}(\mathbb{R}_+; V) \cap W^{2,\infty}(\mathbb{R}_+; L^2(\Omega)).$$

We introduce the "modified" energy functional

$$E(t) := \frac{1}{2}\bigg(1 - \int_0^t g(s)\,ds\bigg)\|\nabla u(t)\|_2^2 + \frac{1}{2}\,\|u_t\|_2^2 + \frac{1}{2}\,(g \circ \nabla u)(t),$$

where, for any $v \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\Omega))$, we set

$$(g \circ v)(t) = \int_0^t g(t-s)\|v(t) - v(s)\|_2^2\,ds.$$

A direct differentiation, using (1.1), and some manipulation as in [5] leads to

$$E'(t) = \frac{1}{2}\,(g' \circ \nabla u)(t) - \frac{1}{2}\,g(t)\|\nabla u(t)\|_2^2 - \int_{\Gamma_1} u_t h(u_t)\,d\Gamma \le 0. \tag{2.1}$$

The next lemma and corollary are essential for the proof of our main result.

Lemma 2.6 ([11]). *Assume that* g *satisfies* (H1) *and* (H2) *and let* u *be the solution of* (1.1). *Then, for* $0 < \sigma < 1$, *we have*

$$(g \circ \nabla u)(t) \le C\bigg[\bigg(\int_0^t g^{1-\sigma}(t)\,dt\bigg)E(0)\bigg]^{(p-1)/(p-1+\sigma)}(g^p \circ \nabla u)^{\sigma/(p-1+\sigma)}(t).$$

By taking $\sigma = 1/2$, we get

$$(g \circ \nabla u)(t) \le C\bigg[\int_0^t g^{1/2}(s)\,ds\bigg]^{(2p-2)/(2p-1)}(g^p \circ \nabla u)^{1/(2p-1)}(t). \tag{2.2}$$

Corollary 2.7. *Assume that* g *satisfies* (H1) *and* (H2) *and* u *is the solution of* (1.1). *Then*

$$\xi(t)(g \circ \nabla u)(t) \le C[-E'(t)]^{1/(2p-1)}.$$

Proof. Multiply both sides of (2.2) by $\xi(t)$ and recall Lemma 2.4 and (2.1) to get

$$\begin{aligned}\xi(t)(g\circ\nabla u)(t) &\le C\xi^{(2p-2)/(2p-1)}(t)\\ &\quad\cdot\left[\int_0^t g^{1/2}(s)\,ds\right]^{(2p-2)/(2p-1)}\xi^{1/(2p-1)}(t)(g^p\circ\nabla u)^{1/(2p-1)}(t)\\ &\le C\left[\int_0^t \xi(s)g^{1/2}(s)\,ds\right]^{(2p-2)/(2p-1)}(\xi g^p\circ\nabla u)^{1/(2p-1)}(t)\\ &\le C\left[\int_0^{+\infty} \xi(s)g^{1/2}(s)\,ds\right]^{(2p-2)/(2p-1)}(-g'\circ\nabla u)^{1/(2p-1)}(t)\\ &\le C[-E'(t)]^{1/(2p-1)}. \end{aligned}$$

□

We also recall the well-known Jensen inequality which will be of essential use in obtaining our result. If G is a concave function on $[a,b]$ ($-G$ is convex), $f\colon\Omega\to[a,b]$ and K are integrable functions on Ω, with $K(x)\ge 0$, and $\int_\Omega K(x)\,dx = k>0$, then Jensen's inequality states that

$$\frac{1}{k}\int_\Omega G[f(x)]K(x)\,dx \le G\left[\frac{1}{k}\int_\Omega f(x)K(x)\,dx\right].$$

For the special case $G(y)=y^{1/p}$, $y\ge 0$, $p>1$, we have

$$\frac{1}{k}\int_\Omega [f(x)]^{1/p}K(x)\,dx \le \left[\frac{1}{k}\int_\Omega f(x)K(x)\,dx\right]^{1/p}. \tag{2.3}$$

3. Decay of solutions

In this section we state and prove the main result of our work. For this purpose, we adopt, without proofs, the following two results from [17] and [21].

Lemma 3.1. [17, (3.7)] *There exist positive constants N_1, N_2, m, t_0 such that the functional*

$$F(t) := N_1E(t) + \Psi(t) + N_2\chi(t)$$

is equivalent to E and satisfies

$$F'(t) \le -mE(t) + C(g\circ\nabla u)(t) + C\int_{\Gamma_1} h^2(u_t)\,d\Gamma, \quad \textit{for all } t\ge t_0, \tag{3.1}$$

where

$$\Psi(t) := \int_\Omega uu_t\,dx \quad \textit{and} \quad \chi(t) := -\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx.$$

Lemma 3.2. [21, (3.8)–(3.10)] *Under assumptions* (H1)–(H3)*, the solution satisfies the estimate*

$$\int_{\Gamma_1} (u_t^2 + h^2(u_t))\,d\Gamma \le -CE'(t), \quad \textit{for all } t\ge t_0, \tag{3.2}$$

if H is linear; and

$$\int_{\Gamma_1} (u_t^2 + h^2(u_t))\, d\Gamma \le CH^{-1}(\lambda(t)) - CE'(t), \quad \text{for all } t \ge t_0, \tag{3.3}$$

if H is nonlinear, where

$$\lambda(t) := \frac{1}{|\Gamma_{12}|} \int_{\Gamma_{12}} u_t h(u_t)\, d\Gamma \quad \text{and} \quad \Gamma_{12} = \{x \in \Gamma_1 : |u_t| \le \varepsilon_1\}.$$

Theorem 3.3. *Let $(u_0, u_1) \in V \times L^2(\Omega)$ be given. Assume that* (H1)–(H3) *are satisfied. Then there exist strictly positive constants $k_1, \ldots, k_4$ such that the solution of* (1.1) *satisfies, for all $t \ge t_0$,*

$$E(t) \le k_1 H_1^{-1}\bigg(k_2 \int_{t_0}^{t} \xi(s)\, ds\bigg), \qquad p = 1, \tag{3.4}$$

$$E(t) \le k_3 H_1^{-1}\bigg(k_4 \int_{t_0}^{t} \xi^{2p-1}(s)\, ds\bigg), \quad 1 < p < \frac{3}{2}. \tag{3.5}$$

Moreover, if

$$\int_0^{+\infty} H_1^{-1}\bigg(k_4 \int_{t_0}^{t} \xi^{2p-1}(s)\, ds\bigg)\, dt < +\infty, \quad 1 < p < \frac{3}{2}, \tag{3.6}$$

then

$$E(t) \le k_3 (\widehat{H}_1)^{-1}\bigg(k_4 \int_{t_0}^{t} \xi^{p}(s)\, ds\bigg), \quad \text{for all } t \ge t_0,\ p > 1. \tag{3.7}$$

where

$$H_1(t) = \int_t^1 \frac{1}{s^{2p-1} H'(\varepsilon_0 s)}\, ds \quad \text{and} \quad \widehat{H}_1(t) = \int_t^1 \frac{1}{s^{p} H'(\varepsilon_0 s)}\, ds.$$

Remark 3.4. It is clear that (3.5) and (3.6) yield

$$\int_{t_0}^{+\infty} E(t)\, dt < +\infty.$$

Proof. First, we add the positive term $\int_{\Gamma_1} u_t^2\, d\Gamma$ to the right hand side of (3.1) to get

$$F'(t) \le -mE(t) + C(g \circ \nabla u)(t) + C \int_{\Gamma_1} (u_t^2 + h^2(u_t))\, d\Gamma, \quad \text{for all } t \ge t_0. \tag{3.8}$$

Multiplying (3.8) by $\xi(t)$ gives

$$\xi(t)F'(t) \le -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) + C\xi(t) \int_{\Gamma_1} (u_t^2 + h^2(u_t))\, d\Gamma, \tag{3.9}$$

for all $t \ge t_0$.

When $p = 1$, we refer the reader to Messaoudi and Mustafa [17]. So we only consider the case $p > 1$.

Case of H is linear. To establish (3.5), we consider (3.9) and use (3.2) and the fact that ξ is bounded to get

$$\xi(t)F'(t) \le -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) - CE'(t), \quad \text{for all } t \ge t_0.$$

Let $L(t) := \xi(t)F(t) + CE(t)$ then clearly $L \sim E$ and, we have

$$L'(t) \le -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t), \quad \text{for all } t \ge t_0. \tag{3.10}$$

Use of Corollary 2.7 in (3.10) gives

$$L'(t) \le -m\xi(t)E(t) + C[-E'(t)]^{1/(2p-1)}, \quad \text{for all } t \ge t_0.$$

Multiplication of the last inequality by $\xi^\alpha E^\alpha(t)$, where $\alpha = 2p-2$, leads to

$$\xi^\alpha E^\alpha(t) L'(t) \le -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C(\xi E)^\alpha(t)[-E'(t)]^{1/(\alpha+1)}, \quad \text{for all } t \ge t_0.$$

Use of Young's inequality, with $q = \alpha+1$ and $q' = (\alpha+1)/\alpha$, yields

$$\begin{aligned} \xi^\alpha E^\alpha(t) L'(t) &\le -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C[\varepsilon \xi^{\alpha+1}(t)E^{\alpha+1}(t) - C_\varepsilon E'(t)] \\ &= -(m - \varepsilon C)\xi^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t), \end{aligned} \tag{3.11}$$

for all $\varepsilon > 0$ and all $t \ge t_0$. We then choose $\varepsilon < m/C$, and recall that $\xi' \le 0$ and $E' \le 0$, to get

$$(\xi^\alpha E^\alpha L)'(t) \le \xi^\alpha(t)E^\alpha(t)L'(t) \le -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t) - CE'(t),$$

for all $t \ge t_0$, which implies

$$(\xi^\alpha E^\alpha L + CE)'(t) \le -c_1\xi^{\alpha+1}(t)E^{\alpha+1}(t).$$

Let $W = \xi^\alpha E^\alpha L + CE \sim E$. Then

$$W'(t) \le -C\xi^{\alpha+1}(t)W^{\alpha+1}(t) = -C\xi^{2p-1}(t)W^{2p-1}(t), \quad \text{for all } t \ge t_0. \tag{3.12}$$

Integrating over (t_0, t) and using the fact that $W \sim E$, we obtain

$$E(t) \le C\left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)\,ds + 1}\right]^{1/(2p-2)} \tag{3.13}$$

for all $t \ge t_0$. Since, in this case, $H(s) = cs$ we have $H_1(t) = C(t^{2-2p}-1)/(2p-2)$ and

$$E(t) \le C\left[\frac{1}{\int_{t_0}^t \xi^{2p-1}(s)\,ds + 1}\right]^{1/(2p-2)} = C_1 H_1^{-1}\left(C_2 \int_{t_0}^t \xi^{2p-1}(s)\,ds\right)$$

for all $t \ge t_0$. To establish (3.7), we consider (3.10) and recall Remark 3.4. So, we have

$$\begin{aligned} L'(t) &\le -m\xi(t)E(t) + C\xi(t)(g \circ \nabla u)(t) \\ &= -m\xi(t)E(t) + C\,\frac{\eta(t)}{\eta(t)} \int_0^t [\xi^p(s)g^p(s)]^{1/p} \|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds, \end{aligned} \tag{3.14}$$

where

$$
\begin{aligned}
(3.15)\quad \eta(t) &= \int_0^t \|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds \le C\int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\,ds \\
&\le C\int_0^t [E(t)+E(t-s)]\,ds \le 2C\int_0^t E(t-s)\,ds \\
&= 2C\int_0^t E(s)\,ds < 2C\int_0^{+\infty} E(s)\,ds < +\infty.
\end{aligned}
$$

Applying Jensen's inequality (2.3) for the second term of the right hand side of (3.14), with $G(y) = y^{1/p}$, $y > 0$, $f(s) = \xi^p(s)g^p(s)$, $K(s) = \|\nabla u(t) - \nabla u(t-s)\|_2^2$, we get

$$
\begin{aligned}
(3.16)\quad L'(t) &\le -m\xi(t)E(t) \\
&\quad + C\eta(t)\bigg[\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds\bigg]^{1/p}
\end{aligned}
$$

where we assume that $\eta(t) > 0$, otherwise we get $\|\nabla u(t) - \nabla u(t-s)\| = 0$ and hence from (3.8) and (3.2) we have $E(t) \le Ce^{-mt}$. Therefore, we obtain

$$
\begin{aligned}
(3.17)\quad L'(t) \le &- m\xi(t)E(t) \\
&+ C\eta^{(p-1)/p}(t)\bigg[\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds\bigg]^{1/p} \\
\le &- m\xi(t)E(t) + C(-g'\circ\nabla u)^{1/p}(t) \le -m\xi(t)E(t) + C(-E'(t))^{1/p}.
\end{aligned}
$$

Multiplying by $\xi^\alpha(t)E^\alpha(t)$, for $\alpha = p-1$, and repeating the same computations as in above, we arrive at

$$
(3.18)\qquad E(t) \le C\bigg[\frac{1}{\int_{t_0}^t \xi^p(s)\,ds + 1}\bigg]^{1/(p-1)}, \quad \text{for all } t \ge t_0.
$$

Since, in this case, $H(s) = \sqrt{s}h_0(s) = cs$ we have $\widehat{H}_1(t) = C(t^{p-1}-1)/(p-1)$ and

$$
E(t) \le C\bigg[\frac{1}{\int_{t_0}^t \xi^p(s)\,ds + 1}\bigg]^{1/(p-1)} = C_1\widehat{H}_1^{-1}\bigg(C_2\int_{t_0}^t \xi^p(s)\,ds\bigg), \quad \text{for all } t \ge t_0.
$$

Case of H is nonlinear. Again we consider (3.9) and use (3.3) to get

$$
(3.19)\qquad L_2'(t) \le -m\xi(t)E(t) + C\xi(t)(g\circ\nabla u)(t) + C\xi(t)H^{-1}(\lambda(t)),
$$

for all $t \ge t_0$, where $L_2 = \xi F + CE$ which is clearly equivalent to E.

From Corollary 2.7, we obtain

$$
(3.20)\qquad L_2'(t) \le -m\xi(t)E(t) + C[-E'(t)]^{1/(2p-1)} + C\xi(t)H^{-1}(\lambda(t)),
$$

for all $t \geq t_0$. Multiplying (3.20) by $\xi^{\alpha}(t)E^{\alpha}(t)$, where $\alpha = 2p-2$ and repeating the calculations as in (3.11)–(3.12), we arrive at

$$W_1' \leq -m\xi^{\alpha+1}(t)E^{\alpha+1}(t) + C\xi^{\alpha+1}(t)E^{\alpha}(t)H^{-1}(\lambda(t)), \quad \text{for all } t \geq t_0, \tag{3.21}$$

where $W_1 = \xi^{\alpha}E^{\alpha}L_2 + CE$ and is also equivalent to E. For $\varepsilon_0 < r^2$ and $c_0 > 0$, let

$$F_1(t) := H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)W_1(t) + c_0 E(t).$$

Clearly F_1 satisfies, for some positive constants α_1, α_2,

$$\alpha_1 F_1 \leq E(t) \leq \alpha_2 F_1 \tag{3.22}$$

and

$$\begin{aligned} F_1'(t) = {} & \varepsilon_0 \frac{E'(t)}{E(0)} H''\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)W_1(t) + H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)W_1'(t) + c_0 E'(t) \\ \leq {} & - m\xi^{\alpha+1}(t)E^{\alpha+1}(t)\, H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & + C\xi^{\alpha+1}(t)E^{\alpha}(t)\, H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)H^{-1}(\lambda(t)) + c_0 E'(t), \end{aligned} \tag{3.23}$$

for all $t \geq t_0$. Let $H^*(s) := \sup_{\tau \in (0,r^2]} \{s\tau - H(\tau)\}$ for $s \in (0, H'(r^2)]$ denote the dual function of H. From (H3) we conclude that H' is increasing and defines a bijection from $(0, r^2]$ to $(0, H'(r^2)]$ and then for any $s \in (0, H'(r^2)]$, the function $\tau \mapsto s\tau - H(\tau)$ reaches its maximum on $(0, r^2]$ at the unique point $(H'(s))^{-1}$. Hence

$$H^*(s) = s(H')^{-1}(s) - H((H')^{-1}(s)), \quad \text{for all } s \in (0, H'(r^2)]$$

and $H^*(s)$ satisfies the general Young inequality:

$$AB \leq H^*(A) + H(B), \quad \text{for all } A \in (0, H'(r^2)],\ B \in (0, r^2]. \tag{3.24}$$

We apply (3.24) on the second term on the right hand side of (3.23) with $A = H'(\varepsilon_0 E(t)/E(0))$ and $B = H^{-1}(\lambda(t))$ and use (2.1) and the fact that $H^*(s) \leq s(H')^{-1}(s)$ to arrive at

$$\begin{aligned} F_1'(t) \leq {} & - m\xi^{\alpha+1}(t)E^{\alpha+1}(t)\, H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) + \varepsilon_0 C\xi^{\alpha+1}(t)\, \frac{E^{\alpha+1}(t)}{E(0)}\, H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & + C\xi^{\alpha+1}(t)E^{\alpha}(t)\lambda(t) + c_0 E'(t) \\ \leq {} & - (mE^{\alpha+1}(0) - \varepsilon_0 C)\xi^{\alpha+1}(t)\left(\frac{E(t)}{E(0)}\right)^{\alpha+1} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) \\ & + (c_0 - C\xi^{\alpha+1}(0)E^{\alpha}(0))E'(t), \end{aligned}$$

for all $t \geq t_0$. With a suitable choice of ε_0 and c_0, we obtain

$$F_1'(t) \leq -C\xi^{\alpha+1}(t)\left(\frac{E(t)}{E(0)}\right)^{\alpha+1} H'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right) = -C\xi^{2p-1}(t) H_2\left(\frac{E(t)}{E(0)}\right), \tag{3.25}$$

for all $t \geq t_0$, where $H_2(\tau) := \tau^{\alpha+1}H'(\varepsilon_0\tau) = \tau^{2p-1}H'(\varepsilon_0\tau)$.

From the properties of H and keeping in mind that $p > 1$, we find that

$$H_2'(\tau) = (2p-1)\tau^{2p-2}H'(\varepsilon_0\tau) + \varepsilon_0\tau^{2p-1}H''(\varepsilon_0\tau) > 0,$$

for all $\tau \in (0,1]$. Therefore the functional R defined by

$$R(t) := \frac{\alpha_1 F_1(t)}{E(0)}$$

is equivalent to E and, in addition, taking in account (3.22) and (3.25), we obtain

$$R'(t) \leq -C\xi^{2p-1}(t)H_2(R(t)), \quad \text{for all } t \geq t_0.$$

Thus

$$-\int_{t_0}^{t} \frac{R'(s)}{H_2(R(s))}\, ds \geq C\int_{t_0}^{t} \xi^{2p-1}(s)\, ds, \quad \text{for all } t \geq t_0,$$

and with substitution $y = R(t)$ on the left hand side, we get

$$\int_{R(t)}^{1} \frac{1}{H_2(y)}\, dy \geq \int_{R(t)}^{R(t_0)} \frac{1}{H_2(y)}\, dy \geq C\int_{t_0}^{t} \xi^{2p-1}(s)\, ds, \quad \text{for all } t \geq t_0. \tag{3.26}$$

Since $H_2(\tau) > 0$ for all $\tau \in (0,1]$, the function H_1 defined as

$$H_1(\tau) := \int_{\tau}^{1} \frac{1}{H_2(s)}\, ds$$

is strictly decreasing on $(0,1]$, thus, using $R \sim E$, (3.26) becomes

$$E(t) \leq C_1 H_1^{-1}\left(C_2\int_{t_0}^{t} \xi^{2p-1}(s)\, ds\right), \quad \text{for all } t \geq t_0. \tag{3.27}$$

To establish (3.7) we consider (3.19) and repeat all the steps of (3.14)–(3.17) to reach

$$L_2'(t) \leq -m\xi(t)E(t) + C(-E'(t))^{1/p} + C\xi(t)H^{-1}(\lambda(t)), \quad \text{for all } t \geq t_0.$$

Multiplication of the last inequality by $\xi^{\alpha}(t)E^{\alpha}(t)$ where $\alpha = p-1$ and repeating, again, the same procedure as in (3.21)–(3.26) we arrive at

$$E(t) \leq C_3\widehat{H}_1^{-1}\left(C_4\int_{t_0}^{t} \xi^{p}(s)\, ds\right), \quad \text{for all } t \geq t_0.$$

where

$$\widehat{H}_1(\tau) := \int_{\tau}^{1} \frac{1}{\widehat{H}_2(s)}\, ds \quad \text{and} \quad \widehat{H}_2(s) = s^p H'(\varepsilon_0 s).$$

This completes the proof of our main result. □

4. Examples

The following examples illustrate our result and show the optimal decay rate in the polynomial case.

Example 4.1. Let $g(t) = a/(1+t)^\nu$, $\nu > 2$, where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)\,dt < 1$ and assume that H is linear. We have

$$g'(t) = -\frac{a\nu}{(1+t)^{\nu+1}} = -b\biggl(\frac{a}{(1+t)^\nu}\biggr)^{(\nu+1)/\nu} = -bg^p(t),$$

where $p = (\nu+1)/\nu < 3/2$, $b > 0$. Therefore (3.6), with $\xi(t) = b$ and $H_1^{-1}(t) = (1/(Ct+1))^{1/(2p-2)}$, yields

$$\int_0^{+\infty} \left(\frac{1}{Cb^{2p-1}t+1}\right)^{1/(2p-2)} dt < +\infty$$

and hence by (3.7) we get

$$E(t) \le \frac{C}{(1+t)^{1/(p-1)}} = \frac{C}{(1+t)^\nu},$$

which is the optimal decay rate.

Example 4.2. Let $g(t) = ae^{-(1+t)^\nu}$, $0 < \nu \le 1$, where $0 < a < 1$ is chosen so that $\int_0^{+\infty} g(t)\,dt < 1$ and assume that H is linear. Then

$$g'(t) = -a\nu(1+t)^{\nu-1}e^{-(1+t)^\nu} = -\xi(t)g(t)$$

where $\xi(t) = \nu(1+t)^{\nu-1}$ which is a decreasing function and $\xi(0) > 0$. Therefore we can use (3.4) to deduce

$$E(t) \le Ce^{-\lambda(1+t)^\nu}.$$

Remark 4.3. Note that our result and that of [4] agree in giving the optimal decay for the polynomial case in a certain range ($1 < p < 3/2$). However, we obtain our result directly, without solving any extra ODE. In addition, we do not see how their result can be applied in a direct way to Example 4.2.

Example 4.4. If $h_0(s) = s^q$ where $q > 1$ then $H(s) = s^{(q+1)/2}$ is a strictly convex C^2 function on $(0,\infty)$. Therefore Therorem 3.3 is applicable and, with $H_1^{-1}(t) = (Ct+1)^{-2/(q+4p-5)}$, we obtain

$$E(t) \le k_1\biggl(k_2\int_{t_0}^t \xi(s)\,ds\biggr)^{-2/(q-1)} \qquad \text{if } p = 1,$$

$$E(t) \le k_3\biggl(k_4\int_{t_0}^t \xi^{2p-1}(s)\,ds\biggr)^{-2/(q+4p-5)} \qquad \text{if } 1 < p < \frac{3}{2}.$$

If (3.6) is satisfied, i.e.

$$\int_0^{+\infty} (Ct\xi^{2p-1}(t)+1)^{-2/(q+4p-5)}\,dt < +\infty,$$

then we have the improved decay rate

$$E(t) \leq k_3 \bigg(k_4 \int_{t_0}^{t} \xi^p(s)\, ds \bigg)^{-2/(q+4p-5)} \quad \text{if } 1 < p < \frac{3}{2}.$$

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