Topological **M**ethods in **N**onlinear **A**nalysis
Volume 51, No. 2, 2018, 389–411
DOI: 10.12775/TMNA.2017.065

EXISTENCE OF SOLUTIONS FOR A CLASS OF DEGENERATE ELLIPTIC EQUATIONS IN $P(X)$-SOBOLEV SPACES

Benali Aharrouch — Mohamed Boukhrij — Jaouad Bennouna

Abstract. We study the Dirichlet problem for degenerate elliptic equations of the form

$$-\operatorname{div} a(x,u,\nabla u)+H(x,u,\nabla u)=f \quad \text{in } \Omega,$$

where $a(x,u,\nabla u)$ is allowed to degenerate with respect to the unknown u, and $H(x,u,\nabla u)$ is a nonlinear term without sign condition. Under suitable conditions on a and H, we prove the existence of bounded and unbounded solution for a datum $f\in L^m$, with $1\le m\le\infty$.

1. Introduction

Let Ω be a bounded subset of $\mathbb{R}^N$, $N\ge 2$. In [10], the authors have studied the quasi-linear elliptic problem

$$A(u)+H(x,u,\nabla u)=f \quad \text{in } \Omega,$$

where $Au=-\operatorname{div}((a(x,u)\nabla u))$ is a Leray–Lions operator from $H_0^1(\Omega)$, the Carathéodory function H satisfies the growth conditions and no sign condition is posed (i.e. $H(x,s,\xi)s\ge 0$), the data f belongs to $L^m(\Omega)$. They showed the existence of weak solutions if $m>N/2$, and existence of entropy solutions if

2010 *Mathematics Subject Classification.* Primary: 47A15; Secondary: 46A32, 47D20.
Key words and phrases. Weak and entropy solutions; degenerate elliptic equations; Sobolev spaces with variable exponent; Stampacchia methods.

$2N/(N+2) \leq m < N/2$. These results were extended by Porretta and Segura de León in 2006, [17], to the model case

$$\begin{cases} -\operatorname{div}(\alpha(u)|\nabla u|^{p-2}\nabla u) = \beta(u)|\nabla u|^p + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $1 < p < \infty$. Also under the additional hypothesis

$$\lim_{s\to\infty} \frac{\beta(s)}{\alpha(s)} = 0$$

they obtained an L^∞-estimate. Aharrouch and al. [1] proved the existence results in the setting of Orlicz space for the unilateral problem associated to the equation

$$A(u) + H(x,u,\nabla u) = f \quad \text{in } \Omega,$$

where $Au = -\operatorname{div}((a(x,u)\nabla u))$ is a Leray–Lions operator and no sign condition is posed on H, and $f \in L^1(\Omega)$.

To deal with this kind of problems, it is natural to work under the framework of Sobolev spaces with variable exponents. The study of differential equations with variable exponents has been a very active field in recent years, with applications in electro-rheological fluids and image processing, and so on. We refer the readers to [15] and references therein.

In [3] Azroul, Hjaij and Touzani proved the existence of entropy solutions for the following problem:

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + H(x,u,\nabla u) = f - \operatorname{div}(\phi) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^1(\Omega)$ and $\phi \in C^0(\mathbb{R},\mathbb{R}^N)$.

Our purpose is to study the existence of a solution for the following degenerate problem:

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) + H(x,u,\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the setting of the Sobolev space with variable exponent $W_0^{1,p(\,\cdot\,)}(\Omega)$, where Ω be a bounded subset of $\mathbb{R}^N$, $N \geq 2$, a and H are a Carathéodory functions. We assume that there exists a continuous function $\alpha\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\alpha(0) = 0$, such that $a(x,s,\xi)\xi \geq \alpha(s)|\xi|^{p(x)}$ for $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

There exist two main difficulties in dealing with this problem, which are related to the facts that the main operator is degenerate for the subset $\{x \in \Omega : u(x) = 0\}$ and we cannot use the classical method of Stampacchia to prove L^∞-estimates for the solution. To overcome these difficulties, we shall employ a test function method with respect to the boundary of α, and then following the ideas of [6], we make a partition of $\overline{\Omega}$ into a finite number of balls B_i (such that for all continuous functions $f < g$ on Ω, we have $\sup(f) < \inf(g)$ on B_i), and assure conditions of [19, Lemma 4] are verified.

This paper is organized as follows: in Section 2 we recall some preliminaries and useful lemmas. In Section 3, we first prove an estimation for solutions in $L^\infty(\Omega)$, then we prove the existence of the weak solution when $f \in L^m(\Omega)$, $m > N/p(\cdot)$ and $m \geq p'(\cdot)$. In the last section, we prove the existence of the entropy solution when $f \in L^1(\Omega)$.

2. Preliminaries

In this section we define Lebesgue and Sobolev spaces with variable exponent and recall some of their properties. Let Ω be an open bounded set in $\mathbb{R}^N$, $N \geq 2$. The function $p(\cdot)$ satisfies the log-Hölder continuity on Ω if

$$|p(y) - p(x)| \leq \frac{C}{|\log|y-x||}, \quad \text{for all } x, y \in \overline{\Omega} \text{ such that } |y-x| \leq \frac{1}{2}, \tag{2.1}$$

with C being a positive constant.

We denote $\mathcal{C}_+(\overline{\Omega}) = \{p\colon \overline{\Omega} \to \mathbb{R}$ is a log-Hölder continuous function such that $p(x) > 1$ for any $x \in \overline{\Omega}\}$. For every $p \in \mathcal{C}_+(\overline{\Omega})$ we put

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

The variable exponent Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \left\{u : u \text{ is a measurable real-valued function}, \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\}.$$

We can introduce the norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many aspects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p_- < p_+ < \infty$ and continuous functions are dense in $L^{p(\cdot)}(\Omega)$ if $p_+ < \infty$ (see Kováčik and Rákosník [18]).

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $1/p(\cdot) + 1/p'(\cdot) = 1$ (see [13], [14]). For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the generalized Hölder inequality

$$\left|\int_\Omega uv\right| \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|u\|_{p(x)}\|v\|_{p'(x)},$$

holds true.

Proposition 2.1 (see [12], [21]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx, \quad \textit{for all } u \in L^{p(\cdot)}(\Omega),$$

then the following assertions hold:

(a) $\|u\|_{p(\,\cdot\,)} < 1$ (*resp.* $= 1, > 1$) *if and only if* $\rho(u) < 1$ (*resp.* $= 1, > 1$),

(b) *if* $\|u\|_{p(\,\cdot\,)} > 1$ *then* $\|u\|_{p(\,\cdot\,)}^{p_-} \le \rho(u) \le \|u\|_{p(\,\cdot\,)}^{p_+}$,
if $\|u\|_{p(\,\cdot\,)} < 1$ *then* $\|u\|_{p(\,\cdot\,)}^{p_+} \le \rho(u) \le \|u\|_{p(\,\cdot\,)}^{p_-}$,

(c) $\|u\|_{p(\,\cdot\,)} \to 0$ *if and only if* $\rho(u) \to 0$,
$\|u\|_{p(\,\cdot\,)} \to \infty$ *if and only if* $\rho(u) \to \infty$.

We define the variable Sobolev space by

$$W^{1.p(\,\cdot\,)}(\Omega) = \{u \in L^{p(\,\cdot\,)}(\Omega) : |\nabla u| \in L^{p(\,\cdot\,)}(\Omega)\},$$

normed by

$$\|u\|_{1.p(\,\cdot\,)} = \|u\|_{p(\,\cdot\,)} + \|\nabla u\|_{p(\,\cdot\,)} \quad \text{for all } u \in W^{1.p(\,\cdot\,)}(\Omega). \tag{2.2}$$

We denote by $W_0^{1.p(\,\cdot\,)}(\Omega)$ *the closure of* $\mathcal{C}_0^\infty(\Omega)$ *in* $W^{1.p(\,\cdot\,)}(\Omega)$ *and*

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N, \\ \infty & \text{for } p(x) \ge N. \end{cases}$$

Proposition 2.2 (see [12]).

(a) *Assuming* $p_- > 1$, *the spaces* $W^{1.p(\,\cdot\,)}(\Omega)$ *and* $W_0^{1.p(\,\cdot\,)}(\Omega)$ *are separable and reflexive Banach spaces.*

(b) *If* $q \in \mathcal{C}_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then*

$$W_0^{1.p(\,\cdot\,)}(\Omega) \hookrightarrow L^{p(\,\cdot\,)}(\Omega) \tag{2.3}$$

is compact and continuous (*for more details see* [11, Theorem 8.4.2]).

(c) (The Poincaré inequality) *There exists a constant* $C > 0$ *such that*

$$\|u\|_{p(\,\cdot\,)} \le C\|\nabla u\|_{p(\,\cdot\,)} \quad \text{for all } u \in W_0^{1.p(\,\cdot\,)}(\Omega).$$

(d) (The Sobolev inequality) *There exists a constant* $C > 0$ *such that*

$$\|u\|_{p^*(\,\cdot\,)} \le C\|\nabla u\|_{p(\,\cdot\,)} \quad \text{for all } u \in W_0^{1.p(\,\cdot\,)}(\Omega).$$

Remark 2.3. By (c) of Proposition 2.2, we know that $\|\nabla u\|_{p(\,\cdot\,)}$ and $\|u\|_{1.p(\,\cdot\,)}$ are equivalent norms on $W_0^{1.p(\,\cdot\,)}(\Omega)$.

Some technical lemmas.

Lemma 2.4 ([4]). *Let* $q \in \mathcal{C}_+(\overline{\Omega})$, $g \in L^{q(\,\cdot\,)}(\Omega)$ *and* $(g_n)_n \in L^{q(\,\cdot\,)}(\Omega)$ *with* $\|g_n\|_{q(\,\cdot\,)} \le C$. *If* $g_n(x) \to g(x)$ *almost everywhere in* Ω, *then* $g_n(x) \to g(x)$ *in* $L^{q(\,\cdot\,)}(\Omega)$, *where* C *is a positive constant.*

LEMMA 2.5 ([4]). *Let $F\colon \mathbb{R} \to \mathbb{R}$ be a uniformly Lipshitz function, with $F(0) = 0$ and $p \in \mathcal{C}_+(\overline{\Omega})$. If $u \in W_0^{1.p(\,\cdot\,)}(\Omega)$, then $F(u) \in W_0^{1.p(\,\cdot\,)}(\Omega)$. Moreover, if the set of discontinuity points of F'' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\,\dfrac{\partial u}{\partial x_i} & \text{for a.e. } x \notin \Omega \setminus u(x) \in D, \\ 0 & \text{for a.e. } x \in \Omega \setminus u(x) \in D. \end{cases}$$

LEMMA 2.6 ([12]). *Let $u \in W_0^{1.p(\,\cdot\,)}(\Omega)$, then $T_k(u) \in W_0^{1.p(\,\cdot\,)}(\Omega)$, with $k > 0$. Moreover, $T_k(u) \to u \in W_0^{1.p(\,\cdot\,)}(\Omega)$ when $k \to \infty$.*

LEMMA 2.7 ([3]). *Let $(u_n)_n$ be a sequence in $W_0^{1.p(\,\cdot\,)}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1.p(\,\cdot\,)}(\Omega)$, then $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1.p(\,\cdot\,)}(\Omega)$.*

LEMMA 2.8 ([4]). *Assume that* (3.1)–(3.3) *hold and there exists $\lambda > 0$ such that $\alpha(\,\cdot\,) \geq \lambda$. Let $(u_n)_n$ be a sequence in $W_0^{1.p(\,\cdot\,)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1.p(\,\cdot\,)}(\Omega)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u) \to 0.$$

Then $u_n \to u$ in $W_0^{1.p(\,\cdot\,)}(\Omega)$.

3. Basic assumptions, notations and definitions

First, we suppose that the functional $-\operatorname{div}(a(x, u, \nabla u))$ is a Leray–Lions operator defined on $W_0^{1.p(\,\cdot\,)}(\Omega)$ into $W^{-1.p'(\,\cdot\,)}(\Omega)$, where $a\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, satisfies the following assumptions:

$$|a(x, s, \xi)| \leq L(x) + |s|^{p(x)-1} + |\xi|^{p(x)-1}, \tag{3.1}$$

$$a(x, s, \xi)\xi \geq \alpha(s)|\xi|^{p(x)}, \tag{3.2}$$

$$[a(x, s, \xi) - a(x, s, \xi')][\xi - \xi'] > 0 \quad \text{for } \xi \neq \xi', \tag{3.3}$$

for almost every $x \in \Omega$, for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $L(\,\cdot\,)$ is a positive function of $L^{p'(\,\cdot\,)}(\Omega)$. Moreover, $H(x, s, \xi)$ is a Carathéodory function satisfying

$$|H(x, s, \xi)| \leq \beta(s)|\xi|^{p(x)}, \tag{3.4}$$

where $\alpha, \beta\colon \mathbb{R} \mapsto \mathbb{R}^+$ are continuous functions, with $\alpha > 0$,

$$\frac{\beta}{\alpha} \in L^1(\mathbb{R}), \qquad \alpha^{1/(p^\pm - 1)} \notin L^1([0, \infty[) \cup L^1([-\infty, 0[),$$
$$f \in L^m(\Omega). \tag{3.5}$$

We define

$$\gamma(s) = \int_0^s \frac{\beta(\tau)}{\alpha(\tau)}\, d\tau, \tag{3.6}$$

$$A(s) = \int_0^s \alpha(\tau)^{1/(p^+ - 1)}\, d\tau \quad \text{if } \alpha \text{ is unbounded}, \tag{3.7}$$

$$A(s) = \int_0^s \alpha(\tau)^{1/(p^- -1)}\, d\tau \quad \text{if } \alpha \text{ is bounded,} \tag{3.8}$$

4. Existence of bounded solutions

Definition 4.1. For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\,\cdot\,)$ can be defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \le k, \\ k \cdot \operatorname{sign}(s) & \text{if } |s| > k, \end{cases}$$

and we define $G_k(s) = s - T_k(s)$.

Definition 4.2. A measurable function $u \in W_0^{1,p(\,\cdot\,)}(\Omega)$, is a weak solution of (1.1), if $a(x,u,\nabla u) \in L^{p'(\,\cdot\,)}(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$ and

$$\int_\Omega a(x,u,\nabla u)\nabla\varphi\, dx + \int_\Omega H(x,u,\nabla u)\varphi\, dx = \int_\Omega f\varphi\, dx \tag{4.1}$$

for all $\varphi \in W_0^{1,p(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.3. *Assume* (3.1)–(3.5) *hold with $m \ge p'(\,\cdot\,)$ and $m > N/p(\,\cdot\,)$, if u is a weak solution of* (1.1) *such that $A(u)$ may be taken as a test function, then $\|u\|_\infty \le C$, where $C > 0$ only depends on m, on the norm of f in $L^m(\Omega)$ and on the parameters of* (1.1).

Proof. *Case 1.* The function α is unbounded, i.e. $\lim_{s\to+\infty} \alpha(s) = +\infty$. Then there exists $\alpha_0 > 0$ such that $\alpha(s) > 1$ for all $s \ge \alpha_0$ since $\lim_{s\to+\infty} \alpha(s) = +\infty$. For every $k > A(\alpha_0)$, with

$$A(s) = \int_0^s \alpha(\tau)^{1/(p^+ -1)}\, d\tau,$$

taking $v = e^{\gamma(u)}(G_k(A(u)))^+$ as an admissible test function in (1.1), we have

$$\begin{aligned}\int_{\{A(u)\ge k\}} & a(x,u,\nabla u)\nabla u \alpha(u)^{1/(p^+-1)} e^{\gamma(u)}\, dx \\ &+ \int_{\{A(u)\ge k\}} a(x,u,\nabla u)\nabla u \,\frac{\beta(u)}{\alpha(u)}\, e^{\gamma(u)} G_k(A(u))\, dx \\ &+ \int_{\{A(u)\ge k\}} H(x,u,\nabla u) e^{\gamma(u)} G_k(A(u))\, dx \\ = & \int_{\{A(u)\ge k\}} f e^{\gamma(u)} G_k(A(u))\, dx.\end{aligned}$$

On the other hand, by (3.2) and (3.4), we have

$$\begin{aligned}\int_{\{A(u)\ge k\}} & a(x,u,\nabla u)\, \nabla u\, \frac{\beta(u)}{\alpha(u)}\, e^{\gamma(u)} G_k(A(u))\, dx \\ &\ge \int_{\{A(u)\ge k\}} \beta(u)|\nabla u|^{p(x)} e^{\gamma(u)} G_k(A(u))\, dx,\end{aligned}$$

and

$$\int_{\{A(u)\geq k\}} H(x,u,\nabla u)e^{\gamma(u)}G_k(A(u))\,dx \geq -\int_{\{A(u)\geq k\}} \beta(u)|\nabla u|^{p(x)}e^{\gamma(u)}G_k(A(u))\,dx.$$

So we conclude that

$$\int_{\{A(u)\geq k\}} a(x,u,\nabla u)\nabla u\alpha(u)^{1/(p^+-1)}e^{\gamma(u)}\,dx \leq \int_{\{A(u)\geq k\}} fe^{\gamma(u)}G_k(A(u))\,dx.$$

By assumption (3.2) we have

$$\begin{aligned}\int_{\{A(u)\geq k\}} & a(x,u,\nabla u)\nabla u\alpha(u)^{1/(p^+-1)}e^{\gamma(u)}\,dx \\ &\geq C\int_{\{A(u)\geq k\}} |\nabla u|^{p(x)}\alpha(u)^{p^+/(p^+-1)}\,dx \\ &\geq C\int_{\{A(u)\geq k\}} |\nabla u|^{p(x)}\alpha(u)^{p(x)/(p^+-1)}\,dx \\ &\geq C\int_{\{A(u)\geq k\}} |\nabla A(u)|^{p(x)}\,dx,\end{aligned}$$

and consequently

$$\int_{\{A(u)\geq k\}} |\nabla A(u)|^{p(x)}\,dx \leq C'\int_{\{A(u)\geq k\}} |f||G_k(A(u))|\,dx,$$

which give

$$\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx \leq C'\int_\Omega |f||G_k(A(u))|\,dx.$$

Case 2. The function α is bounded, i.e. there exists a constant $M>0$ such that $\alpha(s)\leq M$ for every $s\in[0,+\infty[$.

Taking $v=e^{\gamma(u)}(G_k(A(u)))^+$ as an admissible test function in (1.1), with

$$A(s)=\int_0^s \alpha(\tau)^{1/(p^--1)}\,d\tau,$$

we have

$$\begin{aligned}\int_{\{A(u)\geq k\}} & a(x,u,\nabla u)\nabla u\alpha(u)^{1/(p^--1)}e^{\gamma(u)}\,dx \\ &+\int_{\{A(u)\geq k\}} a(x,u,\nabla u)\nabla u\,\frac{\beta(u)}{\alpha(u)}e^{\gamma(u)}G_k(A(u))\,dx \\ &+\int_{\{A(u)\geq k\}} H(x,u,\nabla u)e^{\gamma(u)}G_k(A(u))\,dx \\ =&\int_{\{A(u)\geq k\}} fe^{\gamma(u)}G_k(A(u))\,dx.\end{aligned}$$

The reasoning as above gives

$$\int_{\{A(u)\geq k\}} a(x,u,\nabla u)\nabla u\alpha(u)^{1/(p^{-}-1)}e^{\gamma(u)}\,dx \leq \int_{\{A(u)\geq k\}} fe^{\gamma(u)}G_k(A(u))\,dx.$$

By assumption (3.2) we have

$$\begin{aligned}\int_{\{A(u)\geq k\}} &a(x,u,\nabla u)\nabla u\alpha(u)^{1/(p^{-}-1)}e^{\gamma(u)}\,dx\\ &\geq C_1\int_{\{A(u)\geq k\}} |\nabla u|^{p(x)}\alpha(u)^{p^{-}/(p^{-}-1)}\,dx\\ &\geq C_1'\int_{\{A(u)\geq k\}} |\nabla u|^{p(x)}\alpha(u)^{p(x)/(p^{-}-1)}\,dx\\ &\geq C_1'\int_{\{A(u)\geq k\}} |\nabla A(u)|^{p(x)}\,dx,\end{aligned}$$

and consequently,

$$\int_{\{A(u)\geq k\}} |\nabla A(u)|^{p(x)}\,dx \leq C_1''\int_{\{A(u)\geq k\}} |f||G_k(A(u))|\,dx. \tag{4.2}$$

Now let $e^{\gamma(u)}(G_k(A(u)))^{-}$ be an admissible test function in (1.1), and reasoning as above we get

$$\int_{\{A(u)< k\}} |\nabla A(u)|^{p(x)}dx \leq C_2\int_{\{A(u)< k\}} |f||G_k(A(u))|dx, \tag{4.3}$$

(4.2) and (4.3) give

$$\int_\Omega |\nabla G_kA(u)|^{p(x)}\,dx \leq C_3\int_\Omega |f||G_k(A(u))|\,dx. \tag{4.4}$$

By the Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}\int_\Omega |\nabla G_kA(u)|^{p(x)}\,dx &\leq c_3\|f\chi_{A_k}\|_{p_*'(\,\cdot\,)}.\|G_k(A(u))\|_{p_*(\,\cdot\,)}\\ &\leq c_3\|f\chi_{A_k}\|_{p_*'(\,\cdot\,)}.\|\nabla G_k(A(u))\|_{p(\,\cdot\,)}\\ &\leq c_3\|f\chi_{A_k}\|_{p_*'(\,\cdot\,)}\bigg(\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx\bigg)^{1/\gamma_1},\end{aligned}$$

with

$$\gamma_1 = \begin{cases} p^{-} & \text{if } \|\nabla G_k(A(u))\|_{p(\,\cdot\,)} \geq 1,\\ p^{+} & \text{if } \|\nabla G_k(A(u))\|_{p(\,\cdot\,)} < 1,\end{cases}$$

and $A_k = \{x\in\Omega, |A(u)| > k\}$. The Young and Hölder inequalities give

$$c''\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx \leq c_1'\|f\chi_{A_k}\|_{p_*'(\,\cdot\,)}^{\gamma_1'} + c_2'\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx,$$

and

$$c'''\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx \le c_1'\|f\chi_{A_k}\|_{p_*'(\cdot)}^{\gamma_1'} \le c_1'\bigg(\int_{A_k}|f|^{p_*'(x)}\,dx\bigg)^{\gamma_1'/\gamma_2} \tag{4.5}$$
$$\le c_1'\||f|^{p_*'}\|_{s(\cdot)/p_*'(\cdot)}^{\gamma_1'/\gamma_2}.\|\chi_{A_k}\|_{s(\cdot)/(s(\cdot)-p_*'(\cdot))}^{\gamma_1'/\gamma_2}$$
$$\le c_3'(\Phi(k))^{\gamma_1'/(\gamma_2.\gamma_5)} \le c_3'(\Phi(k))^{\gamma_1'/(\gamma_2.\gamma_5)},$$

with $c''' = c'' - c_2' > 0$, $\Phi(k) = \operatorname{mes}(A_k)$ and

$$\gamma_2 = \begin{cases} (p_*')^- & \text{if } \|f\chi_{A_k}\|_{p_*'(\cdot)} \ge 1,\\ (p_*')^+ & \text{if } \|f\chi_{A_k}\|_{p_*'(\cdot)} < 1,\end{cases}$$

$$\gamma_5 = \begin{cases} \bigg(\dfrac{s(x)}{s(x)-p_*'(x)}\bigg)^- & \text{if } \|\chi_{A_k}\|_{s(\cdot)/(s(\cdot)-p_*'(\cdot))} \ge 1,\\ \bigg(\dfrac{s(x)}{s(x)-p_*'(x)}\bigg)^+ & \text{if } \|\chi_{A_k}\|_{s(\cdot)/(s(\cdot)-p_*'(\cdot))} < 1.\end{cases}$$

By Sobolev embedding, we have

$$\int_\Omega |\nabla G_k(A(u))|^{p(x)}\,dx \ge c_4\bigg(\int_\Omega |G_k(A(u))|^{p_*(x)}\,dx\bigg)^{\gamma_4/\gamma_3}, \tag{4.6}$$

where

$$\gamma_3 = \begin{cases} (p_*)^- & \text{if } \|G_k(A(u))\|_{p_*(\cdot)} \ge 1,\\ (p_*)^+ & \text{if } \|G_k(A(u))\|_{p_*(\cdot)} < 1,\end{cases}$$

$$\gamma_4 = \begin{cases} p^- & \text{if } \|\nabla G_k(A(u))\|_{p(\cdot)} \ge 1,\\ p^+ & \text{if } \|\nabla G_k(A(u))\|_{p(\cdot)} < 1.\end{cases}$$

So, by (4.5) and (4.6), we get

$$\int_\Omega |G_k(A(u))|^{p_*(x)}\,dx \le c_4'(\Phi(k))^{\gamma_1'.\gamma_3/(\gamma_2.\gamma_5.\gamma_4)}. \tag{4.7}$$

Choose h such that $h-k>1$ and in $A_h = \{x\in\Omega : |A(u)| > h\}$ we have $h-k < G_k(u)$. Hence, in view of (4.7), we obtain

$$\Phi(h) \le \frac{C}{(h-k)^{(p_*)^-}}\,(\Phi(k))^{\gamma_1'.\gamma_3/(\gamma_2.\gamma_5.\gamma_4)}.$$

First, let p^+ be a constant satisfying $p^+ < \min_{x\in\overline{\Omega}}(1+1/N)p(x)$ which implies that $p^+ < \min_{x\in\overline{\Omega}}(Np(x)/(N-p(x)))$, then $\gamma_3/\gamma_4 > 1$ and $\gamma_1'/\gamma_2 > 1$. By a suitable choice of $s(\cdot)$, we have $\beta = \gamma_1'.\gamma_3/(\gamma_2.\gamma_5.\gamma_4) > 1$. Now, we use the result of Stampacchia [19]; then there exists a constant C, such that $\|u\|_\infty \le C$.

Now, let $p \in \mathcal{C}_+(\overline{\Omega})$ be such that

$$p(x) < \frac{Np(x)}{N-p(x)} \quad\text{and}\quad p(x) < \bigg(1+\frac{1}{N}\bigg)p(x).$$

By the continuity of $p(\,\cdot\,)$ on $\overline{\Omega}$, there exist two constants $\delta_1, \delta_2 > 0$ such that

$$\max_{y\in B(x,\delta_1)\cap\Omega} p(y) < \min_{y\in B(x,\delta_1)\cap\Omega} \frac{Np(y)}{N-p(y)}, \tag{4.8}$$

$$\max_{y\in B(x,\delta_2)\cap\Omega} p(y) < \inf_{y\in B(x,\delta_2)\cap\Omega} \bigg(1+\frac{1}{N}\bigg)p(y) \tag{4.9}$$

for all $x \in \overline{\Omega}$. So, recalling that $\overline{\Omega}$ is compact, we can cover it with a finite number of balls $(B_j)_{j=1,\ldots,k}$. Moreover, there exists a constant $\lambda > 0$ such that

$$\min(\delta_1,\delta_2) > |\Omega_i| > \lambda, \quad \Omega_i = B_i \cap \Omega, \quad \text{for all } i = 1,\ldots,k.$$

We denote by $(p_j)^+$ and $(p_{*j})^+$ the local maxima of p and $p_* = Np/(N-p)$ on $\overline{\Omega_j}$ (respectively, $(p_j)^-$ and $(p_{*j})^-$, the local minima of p and p_* on $\overline{\Omega_j}$). By (4.7) and the fact that $(p_{*i})^- \le p_* = Np/(N-p)$ on Ω_i, we have

$$\int_{\Omega_i} |G_k(A(u))|^{(p_{*i})^-}\,dx \le c'_4(\Phi_i(k))^{(\gamma_1^i)'\cdot\gamma_3^i/(\gamma_2^i\cdot\gamma_5^i\cdot\gamma_4^i)}, \quad i = 1,\ldots,k, \tag{4.10}$$

with $\Phi_i(k) = \operatorname{mes}(\{x \in \Omega_i : |A(u)| > k\})$ and γ_j^i are the restrictions of γ_j on Ω_i.

Choose h such that $h-k > 1$, and in $A_h^i = \{x \in \Omega_i : |A(u)| > h\}$ we have $h-k < G_k(u)$. Hence, in view of (4.10), we obtain

$$\Phi(h) \le \frac{C}{(h-k)^{(p_{*i})^-}}\,(\Phi(k))^{(\gamma_1^i)'\cdot\gamma_3^i/(\gamma_2^i\cdot\gamma_5^i\cdot\gamma_4^i)}, \quad i = 1,\ldots,k.$$

It follows from (4.8)–(4.9) that $\gamma_3^j/\gamma_4^j > 1$ and $(\gamma_1^j)'/\gamma_2^j > 1$ for all $x \in \overline{\Omega}$ and $j = 1,\ldots,k$, which give $\gamma_3^j(\gamma_1^j)'/(\gamma_4^j\gamma_2^j) > 1$ and, by a suitable choice of $s(\,\cdot\,)$, we have $(\gamma_1^i)'.\gamma_3^i/(\gamma_2^i.\gamma_5^i.\gamma_4^i) > 1$, for all $x \in \overline{\Omega}$ and $i = 1,\ldots,k$. By Lemma 4 of [19] we get $\|u\|_\infty \le C$. □

Theorem 4.4. *Under assumptions* (3.1)–(3.5)*, there exists a weak solution of* (1.1) *in the sense of Definition* 4.1.

Proof. We obtain the solution u by approximation. Consider the following sequence of problems:

$$\begin{cases} -\operatorname{div} a_n(x,u_n,\nabla u_n) + H_n(x,u_n,\nabla u_n) = f & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.11}$$

where

$$a_n(x,s,\xi) = a(x,T_n(s),\xi),$$
$$H_n(x,s,\xi) = \min\,[T_n(\beta_n(s)|\xi|^{p(x)}), \quad \max\,(-T_n(\beta_n(s)|\xi|^{p(x)}), H_n(x,s,\xi))],$$
$$\alpha_n(s) = \alpha(T_n(s)), \quad \beta_n(s) = \alpha_n(s)\,\frac{\beta(s)}{\alpha(s)}.$$

Note that α is continuous, so there exists $\lambda \geq 0$ such that $\alpha_n(s) \leq \lambda$ and since $\alpha_n \notin L^1([0,\infty[) \cup L^1(]-\infty.0])$, we obtain

$$(a(x,s,\xi))\xi \geq \alpha_n(s)|\xi|^{p(x)} \geq \lambda|\xi|^{p(x)}. \tag{4.12}$$

On the other hand, the function H_n is bounded and

$$|H_n(x,s,\xi)| \leq T_n(\beta_n(s)|\xi|^{p(x)-1}) \leq \beta_n(s)|\xi|^{p(x)-1}. \tag{4.13}$$

We also observe that $\beta_n/\alpha_n = \beta/\alpha \in L^1(\mathbb{R})$ and

$$\beta_n \leq \max\{\alpha(s) : |s| \leq n\}\,\frac{\beta}{\alpha}, \quad \text{so that} \quad \beta_n \in L^1(\mathbb{R}).$$

Consider

$$A_n(s) = \int_0^s \alpha_n(\tau)^{1/(p^+-1)}\,d\tau.$$

Applying the classical result by Lions [14], for each $n \in \mathbb{N}$, there exists a weak solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$, which is an admissible test function in the weak sense (4.11). By Theorem 4.3, we have $u_n \in L^\infty(\Omega)$, and so $A_n(u_n) \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ for all $n \in \mathbb{N}$.

Estimates for the sequences $\{u_n\}$. We have $u_n \in L^\infty(\Omega)$, so let $v = e^{\gamma(u_n)}u_n^+$ be a test function in (4.11),

$$\begin{aligned}\int_{u_n\geq 0} a(x,u_n,\nabla u_n)\nabla u_n e^{\gamma(u_n)}\,dx \\ + \int_{u_n\geq 0} a(x,u_n,\nabla u_n)u_n\,\frac{\beta(u_n)}{\alpha(u_n)}\,e^{\gamma(u_n)}\nabla u_n\,dx \\ + \int_{u_n\geq 0} H(x,u_n,\nabla u_n)u_n e^{\gamma(u_n)}\,dx = \int_{u_n\geq 0} f u_n e^{\gamma(u_n)}\,dx.\end{aligned} \tag{4.14}$$

On the other hand, by (3.2) and (3.4) we have

$$\int_{u_n\geq 0} a(x,u_n,\nabla u_n)u_n\,\frac{\beta(u_n)}{\alpha(u_n)}\,e^{\gamma(u_n)}\nabla u_n\,dx \geq \int_{u_n\geq 0} \beta(u_n)|u_n||\nabla u_n|^{p(x)}e^{\gamma(u_n)}\,dx,$$

and

$$\int_{u_n\geq 0} H(x,u_n,\nabla u_n)u_n e^{\gamma(u_n)}\,dx \geq -\int_{u_n\geq 0} \beta(u_n)|u_n||\nabla u_n|^{p(x)}e^{\gamma(u_n)}\,dx.$$

So (4.14) becomes

$$\int_{u_n\geq 0} a(x,u_n,\nabla u_n)\nabla u_n e^{\gamma(u_n)}\,dx \leq \int_{u_n\geq 0} |f||u_n|e^{\gamma(u_n)}\,dx,$$

or γ is bounded, so for some $C_6 > 0$, we have

$$\int_{u_n\geq 0} |\nabla u_n|^{p(x)}\,dx \leq C_6 \int_{u_n\geq 0} f^+|u_n|\,dx \leq C_6\|u_n\|_{L^\infty(\Omega)} \int_\Omega |f|\,dx. \tag{4.15}$$

Now, let $v = -e^{-\gamma(u_n)}u_n^-$ be a test function in (4.11), by the same way as before we get, for some $C_7 > 0$,

$$\int_{u_n\le 0} |\nabla u_n|^{p(x)}\,dx \le C_7\|u_n\|_{L^\infty(\Omega)}\int_\Omega |f|\,dx. \tag{4.16}$$

This estimate proves that $(u_n)_n$ is bounded in $W_0^{1,p(\,\cdot\,)}(\Omega)$. Hence, up to subsequences, $(u_n)_n$ converges weakly; moreover, Rellich–Kondrachov's theorem implies that we may also assume that converges almost everywhere in Ω. Let u be that limit; then $u \in W_0^{1,p(\,\cdot\,)}(\Omega)\cap L^\infty(\Omega)$

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p(\,\cdot\,)}(\Omega), \tag{4.17}$$

$$u_n \to u \quad \text{strongly in } L^{p(\,\cdot\,)}(\Omega) \text{ and a.e. in } \Omega. \tag{4.18}$$

Strong convergence of $\{u_n\}$. Let $v = e^{\gamma(u_n)}(u_n-u)^+$ be a test function in (4.11), then we have

$$\begin{aligned}\int_{u_n\ge u} & a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)e^{\gamma(u_n)}\,dx \\ &+\int_{u_n\ge u} H(x,u_n,\nabla u_n)(u_n-u)e^{\gamma(u_n)}\,dx \\ &+\int_{u_n\ge u} a(x,u_n,\nabla u_n)(u_n-u)\,\frac{\beta(u_n)}{\alpha(u_n)}\,e^{\gamma(u_n)}\nabla u_n\,dx \\ &=\int_{u_n\ge u} f(u_n-u)e^{\gamma(u_n)}\,dx.\end{aligned} \tag{4.19}$$

In view of (3.2) and (3.4); we conclude that

$$\begin{aligned}\int_{u_n\ge u} & H(x,u_n,\nabla u_n)(u_n-u)e^{\gamma(u_n)}\,dx \\ &+\int_{u_n\ge u} a(x,u_n,\nabla u_n)(u_n-u)\,\frac{\beta(u_n)}{\alpha(u_n)}\,e^{\gamma(u_n)}\nabla u_n\,dx \ge 0.\end{aligned}$$

Consequently, we have

$$\begin{aligned}\int_{u_n\ge u} & a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)e^{\gamma(u_n)}\,dx \\ &\le \int_{u_n\ge u}|f|(u_n-u)e^{\gamma(u_n)}\,dx \le C_8\int_{u_n\ge u}|f|(u_n-u)\,dx.\end{aligned} \tag{4.20}$$

Now let $v = -e^{-\gamma(u_n)}(u_n-u)^-$ be a test function in (4.11), we obtain

$$\begin{aligned}\int_{u_n\le u} & a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)e^{-\gamma(u_n)}\,dx \\ &-\int_{u_n\le u} H(x,u_n,\nabla u_n)(u_n-u)^-e^{-\gamma(u_n)}\,dx\end{aligned} \tag{4.21}$$

$$+\int_{u_n\le u} a(x,u_n,\nabla u_n)(u_n-u)^-\,\frac{\beta(u_n)}{\alpha(u_n)}\,e^{-\gamma(u_n)}\nabla u_n\,dx$$
$$=-\int_{u_n\le u} f(u_n-u)^- e^{-\gamma(u_n)}\,dx.$$

By the same way as above we, get

$$\int_{u_n\le u} a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)e^{-\gamma(u_n)}\,dx \le \int_{u_n\le u}|f|(u_n-u)^- e^{-\gamma(u_n)}\,dx \le C_9\int_{u_n\le u}|f||u_n-u|\,dx. \tag{4.22}$$

Adding up (4.20) and (4.22), we conclude that there exists $C_{10}>0$ such that

$$\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)\le C_{10}\|f\|_{p'(x)}\|u_n-u\|_{p(x)}. \tag{4.23}$$

On the other hand we have

$$\begin{aligned}\int_\Omega &(a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))(\nabla u_n-\nabla u)\,dx\\ &=\int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)-\int_\Omega a(x,u_n,\nabla u)(\nabla u_n-\nabla u)\,dx\\ &\le C_{10}\|f\|_{p'(x)}\|u_n-u\|_{p(x)}-\int_\Omega a(x,u_n,\nabla u)(\nabla u_n-\nabla u).\end{aligned} \tag{4.24}$$

Then, by letting n tend to infinity in the right-hand side of (4.24), we conclude that

$$\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla u))(\nabla u_n-\nabla u)\,dx\to 0. \tag{4.25}$$

In view of Lemma 2.8, we deduce that

$$u_n\to u\quad\text{in } W_0^{1,p(\cdot)}(\Omega),\ \text{a.e. in }\Omega. \tag{4.26}$$

The equi-integrability of $(H(x,u_n,\nabla u_n))_n$. Since, by (3.4) and (4.26), we already know that $H(x,u_n,\nabla u_n)\to H(x,u,\nabla u)$ almost everywhere in Ω, it is enough to see the equi-integrability of this sequence and then apply Vitali's convergence theorem. Observe that $\beta_n(u_n)=\beta(u_n)$ for n big enough, so that the sequence $(\beta_n(u_n))_n$ is bounded, there is $C_{11}>0$ such that

$$|H(x,u_n,\nabla u_n)|\le\beta_n(u_n)|\nabla u_n|^{p(x)}\le C_{11}|\nabla u_n|^{p(x)}.$$

Finally, the equi-integrability of $(|\nabla u_n|^{p(x)})_n$, which follows from (4.26), implies that of $H(x,u_n,\nabla u_n)$, so we have

$$H(x,u_n,\nabla u_n)\to H(x,u,\nabla u)\quad\text{in } L^1(\Omega). \tag{4.27}$$

By the condition (3.1), we have

$$a(x,u_n,\nabla u_n)\to a(x,u,\nabla u)\quad\text{in } L^{p(\cdot)}(\Omega). \tag{4.28}$$

Let $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, then

$$\int_\Omega a(x,u_n,\nabla u_n)\varphi\,dx + \int_\Omega H(x,u_n,\nabla u_n)\varphi\,dx = \int_\Omega f\varphi\,dx, \tag{4.29}$$

it follows from (4.27) and (4.28) that we may pass to the limit in (4.29) obtaining that u is a weak solution of (1.1). □

5. Existence of unbounded solutions

Definition 5.1. We will say that a function $u \in W_0^{1,p(\cdot)}(\Omega)$ is an entropy solution of (1.1) if $H(x,u,\nabla u) \in L^1(\Omega)$ and

$$\int_\Omega a(x,u,\nabla u)\nabla T_k[u-\varphi]\,dx + \int_\Omega H(x,u,\nabla u)T_k[u-\varphi]\,dx \le \int_\Omega fT_k[u-\varphi]\,dx \tag{5.1}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Theorem 5.2. *Assume that* (3.1)–(3.8) *hold and* $f \in L^1(\Omega)$*, then problem* (1.1) *has at least one entropy solution.*

Proof. *Approximate problem.* Let $(f_n)_{u\in\mathbb{N}}$ be a sequence of smooth functions such $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \le |f|$, we consider the following problem:

$$\begin{cases} -\operatorname{div} a(x,u_n,\nabla u_n) + H(x,u_n,\nabla u_n) = f_n & \text{in } \Omega, \\ u_n \in W_0^{1,p(\cdot)}(\Omega). \end{cases} \tag{5.2}$$

By Theorem 4.3 we have the existence of a weak solution to problem (5.2).

A priori estimates of $(T_k(A(u_n)))_n$. Let $(T_k(A(u_n)))^+e^{\gamma(u_n)}$ be a test function in (5.2), we have

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(A(u_n))e^{\gamma(u_n)}\,dx \\ &\quad + \int_\Omega H(x,u_n,\nabla u_n)(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx \\ &\quad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\,\frac{\beta(u_n)}{\alpha(u_n)}\,(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx \\ &= \int_\Omega f_n(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx. \end{aligned} \tag{5.3}$$

By (3.2) and (3.4), we have

$$\int_\Omega H(x,u_n,\nabla u_n)(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\,\frac{\beta(u_n)}{\alpha(u_n)}\,(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx \ge 0.$$

It follows that

$$\int_{A(u_n)>0} a(x,u_n,\nabla u_n)\nabla T_k(A(u_n))e^{\gamma(u_n)}\,dx \le \int_\Omega |f_n|(T_k(A(u_n)))^+e^{\gamma(u_n)}\,dx,$$

so we get, using (3.7),

$$\int_{0<A(u_n)\le k} a(x,u_n,\nabla u_n)\nabla u_n\alpha^{1/(p^+-1)}(u_n)e^{\gamma(u_n)}\,dx \le \int_{0<A(u_n)} |f_n||(T_k(A(u_n)))|e^{\gamma(u_n)}\,dx.$$

By assumption (3.2), there exists $C_{12}>0$ such that

$$\int_{0<A(u_n)\le k} |\nabla A(u_n)|^{p(x)}\,dx \le C_{12}k\|f_n\|_{L^1}, \tag{5.4}$$

and, by (3.8), we obtain

$$\int_{0<A(u_n)\le k} a(x,u_n,\nabla u_n)\nabla u_n\alpha^{1/(p^--1)}(u_n)e^{\gamma(u_n)}\,dx \le \int_{0<A(u_n)} |f_n||(T_k(A(u_n)))|e^{\gamma(u_n)}\,dx.$$

By using the test function $-(T_k(A(u_n)))^-e^{-\gamma(u_n)}$, and reasoning as before, we get

$$\int_{-k<A(u_n)\le 0} |\nabla A(u_n)|^{p(x)}\,dx \le C_{13}k\|f_n\|_{L^1}. \tag{5.5}$$

Combining (5.4) and (5.5), we get

$$\int_\Omega |\nabla T_k(A(u_n))|^{p(x)}\,dx \le C_{14}k. \tag{5.6}$$

Therefore,

$$\|\nabla T_k(A(u_n))\|_{p(\,\cdot\,)}^{\theta_4} dx \le C_{14}k \tag{5.7}$$

with

$$\theta_4 = \begin{cases} p^+ & \text{if } \|\nabla T_k(A(u_n))\|_{p(\,\cdot\,)} < 1,\\ p^- & \text{if } \|\nabla T_k(A(u_n))\|_{p(\,\cdot\,)} \ge 1.\end{cases}$$

Let $k\ge 1$, we have

$$k\,\mathrm{mes}\{|A(u_n)|>k\} = \int_{|A(u_n)|>k} |T_k(A(u_n))|\,dx \le C_{14}k^{1/\theta_4},$$

which implies that

$$\mathrm{mes}\{|A(u_n)|>k\} \le C_{14}\,\frac{1}{k^{1-1/\theta_4}} \to 0 \quad \text{as } k\to\infty.$$

By the usual method we get that $T_k(A(u_n))$ is bounded in $W_0^{1,p(\,\cdot\,)}(\Omega)$, then there exists a subsequence still denoted $(T_k(A(u_n)))_{n\in\mathbb{N}}$ such that

$$T_k(A(u_n)) \rightharpoonup \eta_k \quad \text{weakly in } W_0^{1,p(\,\cdot\,)}(\Omega),$$

and, by the compact embedding, we have

$$T_k(A(u_n)) \to \eta_k \quad \text{strongly in } L^{p(\,\cdot\,)}(\Omega), \text{ and a.e. in } \Omega.$$

Consequently, we can assume that $(T_k(A(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus,

$$\operatorname{mes}\{|T_k(A(u_n)) - T_k(A(u_m))| > \delta\} \le \frac{\varepsilon}{2} \quad \text{for all } m,n \ge n_0(\delta,\varepsilon).$$

We conclude that for all $\delta,\varepsilon > 0$ there exists $n_0 = n_0(\delta,\varepsilon)$ such that $\operatorname{mes}\{|u_n - u_m| > \delta\} \le \varepsilon$ for all $\delta,\varepsilon > 0$.

It follows that $(T_k(A(u_n)))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then it converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{aligned} &T_k(A(u_n)) \rightharpoonup T_k(A(u)) \quad \text{weakly in } W_0^{1,p(\,\cdot\,)}(\Omega),\\ &T_k(A(u_n)) \to T_k(A(u)) \quad \text{strongly in } L^{p(\,\cdot\,)}(\Omega), \text{ and a.e. in } \Omega. \end{aligned}$$

Strong convergence of truncations. Let $w_n^+ e^{\gamma(u_n)}$ be a test function in problem (5.2), where $w_n = T_{2k}(Z_n)$ with $Z_n = (A(u_n) - T_h(A(u_n)) + T_k(A(u_n)) + T_k(A(u)))$, for $h > k > 0$. Taking $M = 4k + h$ we have

$$\begin{aligned} \int_{w_n>0} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx + \int_{w_n>0} H(x,u_n,\nabla u_n) w_n^+ e^{\gamma(u_n)}\,dx \\ + \int_{w_n>0} a(x,u_n,\nabla u_n)\nabla u_n \frac{\beta(u_n)}{\alpha(u_n)}\, w_n^+ e^{\gamma(u_n)}\,dx = \int_{w_n>0} f_n w_n^+ e^{\gamma(u_n)}\,dx, \end{aligned} \tag{5.8}$$

by (3.2) and (3.4), we have

$$\int_{w_n>0} H(x,u_n,\nabla u_n) w_n^+ e^{\gamma(u_n)}\,dx + \int_{w_n>0} a(x,u_n,\nabla u_n)\nabla u_n \frac{\beta(u_n)}{\alpha(u_n)}\, w_n^+ e^{\gamma(u_n)}\,dx \ge 0.$$

So, we get that

$$\int_{w_n>0} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx \le \int_{w_n>0} f_n w_n^+ e^{\gamma(u_n)}\,dx.$$

We have

$$\begin{aligned}
(5.9)\qquad &\int_{\{w_n>0\}} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx \\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx \\
&\quad + \int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx.
\end{aligned}$$

Concerning the first term in the right-hand side of (5.9); since $\nabla w_n = 0$ on $\{|A(u_n)| > M\}$, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a(x,u_n,\nabla u_n)\nabla w_n e^{\gamma(u_n)}\,dx \\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla z_n e^{\gamma(u_n)}\,dx \\
&\ge -\int_{\{w_n>0\}\cap\{|A(u_n)|>k\}} a(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla T_k(A(u)) e^{\gamma(u_n)}\,dx \\
&\ge -e^{\gamma(\infty)}\int_{\{|A(u_n)|>k\}} a(x,T_{\widehat{M}}(u_n),\nabla T_{\widehat{M}}(u_n))\nabla T_k(A(u))\,dx \ge -\varepsilon_0(n),
\end{aligned}$$

with $\widehat{M} = A^{-1}(M)$. For the second term in the right-hand side of (5.9); we have for $\widehat{k} = A^{-1}(k)$

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx \\
&\qquad\qquad \le e^{\gamma(\infty)}\int_{\{w_n>0\}} |f_n||w_n|\,dx + \varepsilon_0(n).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx \\
&= \int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} (a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u))) \\
&\qquad \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx \\
&\quad + \int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u)) \\
&\qquad \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx.
\end{aligned}$$

The second and third terms in the right-hand side tend to 0, as n tends to infinity. So, we have

$$
\begin{aligned}
(5.10)\qquad &\int_{\{w_n>0\}\cap\{|A(u_n)|\le k\}} (a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) \\
&\qquad - a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u)))(\nabla T_k(A(u_n)) - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx \\
&= \int_{\{|A(u_n)|\le k\}} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) \\
&\qquad\qquad - \nabla T_k(A(u)))e^{\gamma(u_n)}\,dx + \varepsilon_1(n)) \\
&\le e^{\gamma(\infty)} \int_{\{w_n>0\}} |f_n(x)||w_n|\,dx + \varepsilon_1(n) + \varepsilon_0(n) \le \varepsilon_2(n),
\end{aligned}
$$

as $f_n \to f$ strongly in $L^1(\Omega)$, and $w_n \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$.

Let $-(w_n)^- \exp(-\gamma(u_n))$ be a test function in problem (5.2), we obtain

$$
\begin{aligned}
&\int_{w_n<0} a(x,u_n,\nabla u_n)\nabla w_n \exp(-\gamma(u_n))\,dx \\
&\qquad + \int_{w_n<0} a(x,u_n,\nabla u_n)(w_n)^-\nabla u_n \frac{\beta(u_n)}{\alpha(u_n)} \exp(-\gamma(u_n))\,dx \\
&\qquad + \int_{w_n<0} H(x,u_n,\nabla u_n) w_n \exp(-\gamma(u_n))\,dx \\
&= \int_{w_n<0} f_n w_n \exp(-\gamma(u_n))\,dx,
\end{aligned}
$$

so we get

$$
\int_{w\le 0} a(x,u_n,\nabla u_n)\nabla w_n \exp(-\gamma(u_n))\,dx \le \int_{w\le 0} |f_n||w_n| \exp(-\gamma(u_n))\,dx.
$$

Reasoning as before, we get that

$$
\int_{\{w\le 0\}\cap\{|A(u_n)|>k\}} a(x,u_n,\nabla u_n)\nabla w_n \exp(-\gamma(u_n))dx \ge -\varepsilon_3(n),
$$

where $\varepsilon_3(n)$ tends to 0 as n tends to infinity.

$$
\begin{aligned}
&\int_{w\le 0} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_{\widehat{k}}(u_n) - \nabla T_{\widehat{k}}(u)) \exp(-\gamma(u_n))\,dx \\
&\qquad\qquad \le \int_{w\le 0} |f_n||w_n| \exp(-\gamma(u_n))\,dx + \varepsilon_3(n).
\end{aligned}
$$

Since w_n tends to 0 weakly * in $L^\infty(\Omega)$ and f_n converges strongly to f in $L^1(\Omega)$, we conclude that

$$
\int_{w\le 0} a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))(\nabla T_k(A(u_n)) - \nabla T_k(A(u))) \exp(-\gamma(u_n))\,dx \le \varepsilon_4(n).
$$

By adding the term to the last expression, we get

$$\begin{aligned}(5.11)\quad \int_{w\le 0} & [a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))] \\ & \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) \exp(-\gamma(u_n))\, dx \le \varepsilon_5(n).\end{aligned}$$

Combining (5.10) and (5.11), we get

$$\begin{aligned}\int_{\Omega} & [a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n)) - a(x,T_{\widehat{k}}(u_n),\nabla T_{\widehat{k}}(u_n))] \\ & \times (\nabla T_k(A(u_n)) - \nabla T_k(A(u))) \exp(-\gamma(u_n))\, dx \le \varepsilon_6(n),\end{aligned}$$

so, by Lemma 2.8, we conclude that $T_k(A(u_n)) \to T_k(A(u))$ in $W_0^{1,p(\,\cdot\,)}(\Omega)$, $\nabla A(u_n) \to \nabla A(u)$ almost everywhere in Ω.

By assumption (3.7), we have

$$\begin{aligned}|\nabla T_k(A(u_n))|^{p(x)} &= |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)|\le k} \\ &= \alpha^{p(x)/(p^+-1)}(u_n)\chi_{|A(u_n)|\le k}|\nabla u_n|^{p(x)}, \\ \alpha^{p(x)/(p^+-1)}(u_n)\chi_{|u_n|\le \widehat{k}}|\nabla u_n|^{p(x)} &= \alpha^{p(x)/(p^+-1)}(T_{\widehat{k}}(u_n))|\nabla T_{\widehat{k}}(u_n)|^{p(x)}, \\ |\nabla T_{\widehat{k}}(u_n)|^{p(x)} &= \frac{|\nabla T_k(A(u_n))|^{p(x)}}{\alpha^{p(x)/(p^+-1)}(T_{\widehat{k}}(u_n))},\end{aligned}$$

and, by (3.8), we have

$$\begin{aligned}|\nabla T_k(A(u_n))|^{p(x)} &= |\nabla A(u_n)|^{p(x)} \chi_{|A(u_n)|\le k} \\ &= \alpha^{p(x)/(p^--1)}(u_n)\chi_{|A(u_n)|\le k}|\nabla u_n|^{p(x)}, \\ \alpha^{p(x)/(p^--1)}(u_n)\chi_{|u_n|\le \widehat{k}}|\nabla u_n|^{p(x)} &= \alpha^{p(x)/(p^--1)}(T_{\widehat{k}}(u_n))|\nabla T_m(u_n)|^{p(x)}, \\ |\nabla T_{\widehat{k}}(u_n)|^{p(x)} &= \frac{|\nabla T_k(A(u_n))|^{p(x)}}{\alpha^{p(x)/(p^--1)}(T_{\widehat{k}}(u_n))}.\end{aligned}$$

Since α is continuous we have $\alpha(T_{\widehat{k}}(u_n)) \ge \min\limits_{[0,\widehat{k}]}(\alpha(s)) = \alpha_{\widehat{k}}$. Finally, we have

$$(5.12)\qquad |\nabla T_m(u_n)|^{p(x)} \le c|\nabla T_k(A(u_n))|^{p(x)}.$$

The equi-integrability of $H(x,u_n,\nabla u_n)$. In order to pass to the limit in the approximate problem, we shall show that

$$H(x,u_n,\nabla u_n) \to H(x,u_n,\nabla u_n) \quad \text{in } L^1(\Omega).$$

Let E be a set of Ω such that $\mathrm{mes}(E)=0$ and $l>0$. We have

$$\begin{aligned}\int_E |H(x,u_n,\nabla u_n)|\,dx &\le \int_E \beta(u_n)|\nabla u_n|^{p(x)}\,dx\\ &= \int_{E\cap|u_n|>l}\beta(u_n)|\nabla u_n|^{p(x)}\,dx+\int_{E\cap|u_n|\le l}\beta(u_n)|\nabla u_n|^{p(x)}\,dx\\ &= \int_{E\cap|u_n|>l}\beta(u_n)|\nabla u_n|^{p(x)}\,dx+\int_{E\cap|u_n|\le l}\beta(T_l(u_n))|\nabla T_l(u_n)|^{p(x)}\,dx\\ &\le \int_{E\cap|u_n|>l}\beta(u_n)|\nabla u_n|^{p(x)}\,dx+\max_{[0,l]}(\beta(s))\int_{E\cap|u_n|\le l}|\nabla T_l(u_n)|^{p(x)}\,dx.\end{aligned}$$

From (5.12), we deduce that the second term in the right-hand side of the last inequality equals to 0 as $\mathrm{mes}(E)=0$. We prove that

$$\int_{E\cap|u_n|>l}\beta(u_n)|\nabla T_l(u_n)|^{p(x)}\,dx\to 0.$$

Let $(T_1(u_n-T_l(u_n)))^+\exp(2\gamma(u_n))$ be a test function in problem (5.2). We have

$$\begin{aligned}&\int_{l<u_n\le l+1} a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_l(u_n))\exp(2\gamma(u_n))\,dx\\ &\qquad+\int_{l<u_n} 2a(x,u_n,\nabla u_n)\nabla u_n\,\frac{\beta(u_n)}{\alpha(u_n)}\,(T_1(u_n-T_l(u_n)))^+\exp(2\gamma(u_n))\,dx\\ &\qquad+\int_{l<u_n} H(x,u_n,\nabla u_n)T_1(u_n-T_l(u_n))^+\exp(2\gamma(u_n))\,dx\\ &\quad=\int_{l<u_n} f_n(T_1(u_n-T_l(u_n)))^+\exp(2\gamma(u_n))\,dx.\end{aligned}$$

By assumption (3.2) we have

$$\begin{aligned}(5.13)\qquad \int_{l<u_n}|\nabla u_n|^{p(x)}\beta(u_n)(T_1(u_n-T_l(u_n)))^+\exp(2\gamma(u_n))\,dx&\\ \le C_{15}\int_{l<u_n}|f|\,dx.&\end{aligned}$$

Let $-(T_1(u_n-T_l(u_n)))^-\exp(-2\gamma(u_n))$ be a test function as in problem (5.2). Reasoning as above, we get

$$\begin{aligned}(5.14)\qquad \int_{u_n<-l}|\nabla u_n|^{p(x)}\beta(u_n)(T_1(u_n-T_l(u_n)))^+\exp(2\gamma(u_n))\,dx&\\ \le C_{16}\int_{u_n<-l}|f|\,dx.&\end{aligned}$$

From (5.13) and (5.14) we conclude that

$$(5.15)\qquad \int_{l<|u_n|}|\nabla u_n|^{p(x)}\beta(u_n)\,dx\le C_{17}\int_{l<|u_n|}|f|\,dx.$$

Let l tend to infinity. We get

$$\int_{l<|u_n|} |\nabla u_n|^{p(x)} \beta(u_n)\, dx \to 0.$$

Finally, we get the equi-integrability of H.

Passage to the limit. Let $\phi \in W_0^{1,p(\,\cdot\,)}(\Omega) \cap L^\infty(\Omega)$. Taking $T_k(u_n - \phi)$ as a test function in the approximate problem, we get

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \phi)\, dx \\ + \int_\Omega H(x, u_n, \nabla u_n) T_k(u_n - \phi)\, dx + \int_\Omega f_n T_k(u_n - \phi)\, dx.$$

Choosing $M = k + \|\phi\|_\infty$, if $|u_n| > M$ then $|u_n - \phi| \geq |u_n - \|\phi\|_\infty| > k$. Therefore $\{|u_n - \phi|\} \subset \{|u_n| \leq u_n\}$. Now, we can write the first term in the right-hand side of the above relation as

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \phi)\, dx \\ &= \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla\phi)\chi_{|u_n-\phi|\leq k}\, dx \\ &= \int_\Omega (a(x, T_M(u_n), \nabla T_M(u_n)) - a(x, T_M(u_n), \nabla\phi)) \\ &\qquad \times (\nabla T_M(u_n) - \nabla\phi)\chi_{|u_n-\phi|\leq k}\, dx \\ &\quad + \int_\Omega a(x, T_M(u_n), \nabla\phi)(\nabla T_M(u_n) - \nabla\phi)\chi_{|u_n-\phi|\leq k}\, dx. \end{aligned}$$

According to Fatou's lemma we obtain

$$\begin{aligned} (5.16) \quad &\liminf \int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \phi)\, dx \\ &= \liminf \int_\Omega a(x, T_M(u_n), \nabla T_M(u_n))(\nabla T_M(u_n) - \nabla\phi)\chi_{|u_n-\phi|\leq k}\, dx \\ &\geq \int_\Omega (a(x, T_M(u), \nabla T_M(u)) - a(x, T_M(u), \nabla\phi))(\nabla T_M(u) - \nabla\phi)\chi_{\{|u-\phi|\}} \\ &\quad + \lim_{n\to\infty} \int_\Omega a(x, T_M(u_n), \nabla\phi)(\nabla T_M(u_n) - \nabla\phi)\chi_{|u_n-\phi|\leq k}\, dx. \end{aligned}$$

The second term in the right-hand side of (5.16) is equal to

$$\int_\Omega a(x, T_M(u), \nabla\phi)(\nabla T_M(u) - \nabla\phi)\chi_{|u-\phi|\leq k}\, dx.$$

Therefore, we get

$$\begin{aligned}\liminf \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx \\ \geq \int_\Omega a(x,T_M(u),\nabla T_M(u))(\nabla T_M(u)-\nabla\phi)\chi_{|u-\phi|\leq k}\,dx \\ = \int_\Omega a(x,u,\nabla u)\nabla(T_k(u)-\phi)\,dx.\end{aligned}$$

On the other hand, as $T_k(u_n-\phi) \rightharpoonup T_k(u-\phi)$ weakly* in $L^\infty(\Omega)$ and $f_n \to f$ in $L^1(\Omega)$, we deduce that

$$\int_\Omega f_n T_k(u_n-\phi)\,dx \to \int_\Omega f T_k(u-\phi)\,dx.$$

Hence, putting all the terms together, we complete the proof of Theorem 5.2. □

References

[1] L. Aharouch, A. Benkirane and M. Rhoudaf, *Existence results for some unilateral problems without sign condition with obstacle free in Orlicz spaces*, Nonlinear Anal. **68** (2008), 2362–380.

[2] L. Aharouch and J. Bennouna, *Existence and uniqueness of solutions of unilateral problems in Orlicz spaces*, Nonlinear Anal. **72** (2010), no. 9–10, 3553–3565.

[3] E. Azroul, H. Hjiaj and A. Touzani, *Existence and regularity of entropy solutions for strongly nonlinear $p(x)$-elliptic equations*, Electron. J. Differential Equations **68** (2013), 1–27.

[4] M.B. Benboubker, E. Azroul and A. Barbara, *Quasilinear elliptic problems with nonstandard growth*, Electron. J. Differential Equations **62** (2011), 1–16.

[5] M. Bendahmane and F. Mokhtari, *Nonlinear elliptic systems with variable exponents and measure data*, Moroccan J. Pure Appl. Anal. (MJPAA) **1** (2015), 108–125.

[6] M. Bendahmane and P. Wittbold, *Renormalized solutions for nonlinear elliptic equations with variable exponents and L1 data*, Nonlinear Anal. **70** (2009), 567–583.

[7] Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J.L. Vazquez, *An L_1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1995), 241–273.

[8] L. Boccardo, T. Gallouet and L.Orsina, *Existence and nonexistence of solutions for some nonlinear elliptic equations*, J. Anal. Math. **73**, 203–223 (1997).

[9] L. Boccardo, F. Murat and J.P. Puel, *Existence de solutions non bornes pour certaines equations quasi-lineaires*, Portugal. Math. **41** (1982), 507–534.

[10] L. Boccardo, S. Segura and C. Trombeti, *Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term*, J. Math. Pures Appl. **80** (2001), 919–940.

[11] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011.

[12] X.L. Fan and D. Zhao, *On the generalised Orlicz–Sobolev space $W^{k.p(\,\cdot\,)}(\Omega)$*, J. Gansu Educ. College **12** (1998), 1–6.

[13] O. Kováčik and J. Rákosník, *On spaces $L^{p(\,\cdot\,)}$ and $W^{1.p(\,\cdot\,)}$*, Czechoslovak Math. J. **41** (1991), no. 116, 592–618.

[14] J.-L. LIONS, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Gauthier-Villars, Paris, 1969.

[15] I. NYANQUINI AND S. OUARO, *Entropy solution for nonlinear elliptic problem involving variable exponent and Fourier type boundary condition*, Afr. Mat. **23** (2012), no. 2, 205–228.

[16] A. PORRETTA, *Nonlinear equations with natural growth terms and measure data*, Electron. J. Differ. Equ. Conf. **9** (2002), 183–202.

[17] A. PORRETTA AND S. SEGURA DE LEÓN, *Nonlinear elliptic equations having a gradient term with natural growth*, J. Math. Pures Appl. (9) **85** (2006), no. 3, 465–492.

[18] J.M. RAKOTOSON, *Réarrangement relatif dans les équations qausi-linéaires avec un second membre distribution: application à un théorème d'existence et de régularité*, J. Differential Equations **66** (1987), 391–419.

[19] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre á coefficients discontinus*, Ann. Inst. Fourier (Grenoble) **15** (1965), 189–258.

[20] D. ZHAO AND X.L. FAN, *On the Nemytskiĭ operators from $L^{p_1(\cdot)}$ to $L^{p_2(\cdot)}$*, J. Lanzhou Univ. **34** (1998), 1–5.

[21] D. ZHAO, W.J. QIANG AND X.L. FAN, *On generalized Orlicz spaces $L^p(\cdot)(\Omega)$*, J. Gansu Sci. **9** (1997), 1–7.

[22] W. ZOU AND F. LI, *Existence of solutions for degenerate quasilinear elliptic equations*, J. Nonlinear Anal. **73** (2010), 3069–3082.

Manuscript received February 15, 2017

accepted January 4, 2018

BENALI AHARROUCH, MOHAMED BOUKHRIJ AND JOUAD BENNOUNA
University of Fez
Faculty of Sciences Dhar El Mahraz
Laboratory LAMA
Department of Mathematics
B.P. 1796 Atlas Fez, MOROCCO

E-mail address: bnaliaharrouch@gmail.com
jbennouna@hotmail.com
mohamed.boukhrij@gmail.com