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EXISTENCE OF A WEAK SOLUTION FOR THE FRACTIONAL p-LAPLACIAN EQUATIONS WITH DISCONTINUOUS NONLINEARITIES VIA THE BERKOVITS–TIENARI DEGREE THEORY

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Abstract. We are concerned with the following nonlinear elliptic equations of the fractional p-Laplace type:

$$\begin{cases} (-\Delta)_p^s u \in \lambda[\underline{f}(x,u(x)),\overline{f}(x,u(x))] & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)_p^s$ is the fractional p-Laplacian operator, λ is a parameter, $0 < s < 1 < p < +\infty$, $sp < N$, and the measurable functions $\underline{f}$, $\overline{f}$ are induced by a possibly discontinuous at the second variable function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$. By using the Berkovits–Tienari degree theory for upper semicontinuous set-valued operators of type (S_+) in reflexive Banach spaces, we show that our problem with the discontinuous nonlinearity f possesses at least one nontrivial weak solution. In addition, we show the existence of two nontrivial weak solutions in which one has negative energy and another has positive energy.

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1. Introduction

In the present paper, we consider the existence of nontrivial weak solutions to a boundary value problem of the form:

$$\text{(B)} \qquad \begin{cases} (-\Delta)_p^s u \in \lambda[\underline{f}(x,u(x)),\overline{f}(x,u(x))] & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where the fractional p-Laplacian operator $(-\Delta)_p^s$ is defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N.$$

Here, Ω is a smooth bounded domain in $\mathbb{R}^N$, λ is a parameter, $0 < s < 1 < p < +\infty$, $sp < N$, $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$ and the measurable functions $\underline{f}$, $\overline{f}$ are induced by a possibly discontinuous at the second variable function $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$. The study on boundary value problems of some partial differential equations with discontinuous nonlinearities has been investigated principally by Chang [15], [16]. Afterwards, variational problems for elliptic equation with this nonlinearity have been widely studied in recent years; see for example [2], [5], [4], [7], [10], [11], [21], [30] and the references therein.

In the last years a great attention has been drawn to the study of fractional and nonlocal problems of elliptic type in view of applications of mathematical models to certain phenomena in social sciences, fractional quantum mechanics, materials science, continuum mechanics, phase transition phenomena, image process, game theory and Levy processes, etc.; see, e.g. [8], [9], [14], [19], [22], [25], [26] and the references therein. In the local case, formally $s = 1$, the problem with $p > 1$ which involves the p-Laplace operator $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ has a variational nature and its solutions can be constructed as critical points of the associated Euler–Lagrange functional. A natural question is whether or not these topological and variational methods may be adapted to equation (B) and to its generalization in order to extend the classical results known for the p-Laplacian to a non-local setting. In this direction, the existence of a weak solution for problem (B) is established by utilizing the topological method such as a degree theory. As compared with the local case, the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on the whole Ω, but actually on the entire space $\mathbb{R}^N$. Hence this operator has more complicated nonlinearities than the p-Laplacian equation, so more complicated analysis has to be carefully carried out when we investigate the variational framework for problem (B) and some basic properties of the considered functional spaces. In this respect, many researchers have extensively studied the fractional Laplacian problems in various way; see [6], [12], [20], [23], [28], [33], [37] and the references therein. Concerning the existence of nontrivial solutions to nonlinear elliptic problems involving the fractional p-Laplacian with continuous nonlinearities, most of results have been obtained by a critical point theory which is initially introduced by A. Ambrosetti and P. Rabinowitz in [3] and is one of the main tools for finding solutions to elliptic equations of variational type; see for example [6], [17], [20], [23], [24], [29], [32], [33], [37]. Especially, Iannizzotto et al. [20] have investigated the existence and multiplicity results for the fractional p-Laplacian problems:

$$\begin{cases} (-\Delta)_p^s u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where f satisfies a Carathéodory condition; see [29] for $p = 2$.

Recently, the authors of [21] attempted another approach to attack the existence of a weak solution for elliptic problems with discontinuous nonlinearity. Namely, they used the Berkovits–Tienari degree theory in [7] for upper semicontinuous set-valued operators of class (S_+) which is a generalization of the single-valued version due to Browder [13]. In this respect, we first observe the existence of at least one weak solution for our problem by applying the degree theory of this type. Even if our first main theorem is motivated by the works [7], [21], a functional corresponding to the nonlinear term is slightly different from that in [7], [21] because we establish the existence of a weak solution after showing the existence of critical points for an energy functional associated with problem (B). Hence we analyze some properties for this functional and also give the fact that solutions for our problem like the local case can be constructed as critical points of the associated Euler–Lagrange functional. However, the argument that uses this type degree theory gives no further information on the existence of at least two solutions. To overcome this difficulty, we utilize the existence of a nontrivial global minimizer for the energy functional to find another solution. Roughly speaking, the second aim of this paper is to obtain the existence of at least two nontrivial weak solutions for problem (B) without employing the mountain pass theorem which plays an important role in obtaining the existence and multiplicity results of solutions. As far as we are aware, there were no such existence results for fractional p-Laplacian problems in this situation although the principal part $(-\Delta)_p^s$ is replaced by the p-Laplacian.

This paper is structured as follows. First we recall briefly the Berkovits–Tienari degree theory for weakly upper semicontinuous set-valued operators of class (S_+) and some basic properties for locally Lipschitz continuous functional in reflexive Banach spaces. Next, using the fact that every critical point of the energy functional associated with (B) is a weak solution for our problem, we obtain the existence of at least one or two nontrivial weak solutions for problem (B).

2. Preliminaries and main result

Let X be a real Banach space and X^* be its dual space with dual pairing $\langle \cdot, \cdot \rangle$. Given a nonempty subset U of X, let $\overline{U}$ and ∂U denote the closure and the boundary of U in X, respectively. Let $B_r(x)$ denote the open ball in X of radius $r > 0$ centered at x. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Definition 2.1. Let U be an open set in X and let Y be another real Banach space. Suppose that $\mathcal{I}\colon \overline{U} \to 2^Y$ is a set-valued operator. Then

(a) $\mathcal{I}$ is an *upper semicontinuous* if the set $\mathcal{I}^{-1}(A) = \{u \in \overline{U} \mid \mathcal{I}(u) \cap A \neq \emptyset\}$ is closed for all closed sets A in Y.
(b) $\mathcal{I}$ is a *weakly upper semicontinuous* if $\mathcal{I}^{-1}(A)$ is closed for all weakly closed sets A in Y.
(c) $\mathcal{I}$ is a *compact* if it is upper semicontinuous and the image of any bounded set is relatively compact.
(d) $\mathcal{I}$ is a *bounded* if $\mathcal{I}$ maps bounded sets into bounded sets.

Definition 2.2. Let U be an open set in X and let Y be another real Banach space. Suppose that $\mathcal{I}\colon \overline{U} \to 2^{X^*} \setminus \emptyset$ is a set-valued operator. Then

(a) $\mathcal{I}$ is of type (S_+) if for any sequence $\{u_n\}$ in $\overline{U}$ and for any sequence $\{w_n\}$ in X^* with $w_n \in \mathcal{I}(u_n)$ such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and
$$\limsup_{n\to\infty} \langle w_n, u_n - u \rangle \leq 0,$$
we have $u_n \to u$ in X as $n \to \infty$.
(b) $\mathcal{I}$ is *pseudomonotone* if for any sequences $\{u_n\}$ in $\overline{U}$, with $u_n \rightharpoonup u$ in X as $n \to \infty$, and $\{w_n\}$ in X^*, with $w_n \in \mathcal{I}(u_n)$, and such that
$$\limsup_{n\to\infty} \langle w_n, u_n - u \rangle \leq 0,$$
we have $\lim_{n\to\infty} \langle w_n, u_n - u \rangle = 0$; and if $u \in \overline{U}$ and $w_j \rightharpoonup w$ in X^* for some subsequence $\{w_j\}$ of $\{w_n\}$, then $w \in \mathcal{I}(u)$.

The following assertion means that operators of type (S_+) are invariant under quasimonotone perturbations.

Lemma 2.3 ([21]). *Let U be an open set in a real reflexive Banach space X. Suppose that $\mathcal{I}\colon \overline{U} \to 2^{X^*}$ is a set-valued operator with nonempty values and $S\colon \overline{U} \to X^*$ is a single-valued operator. If $\mathcal{I}$ is quasimonotone and S is of type* (S_+)*, then the sum $\mathcal{I} + S$ is of type* (S_+).

Remark 2.4. It is known that in this case the duality operator $J\colon X \to X^*$ is injective, bounded, continuous, and of type (S_+), and such that $\langle Ju, u \rangle = \|u\|_X^2$ and $\|Ju\|_{X^*} = \|u\|_X$ for $u \in X$; for instance, see [35].

As a crucial tool for obtaining our result, we present the Berkovits–Tienari degree theory for weakly upper semicontinuous set-valued operators of type (S_+) in reflexive Banach spaces.

LEMMA 2.5 ([4]). *Let G be any bounded open subset of X and let $\mathcal{I}\colon \overline{G} \to 2^{X^*}$ be of type (S_+), locally bounded, and weakly upper semicontinuous such that $\mathcal{I}(u)$ is nonempty, closed, and convex for each $u \in \overline{G}$. If $w \notin \mathcal{I}(\partial G)$, then an integer $d(\mathcal{I}, G, w)$ is defined, called the degree of $\mathcal{I}$ on G over w, and the degree has the following properties*:

(a) (Existence) *If $d(\mathcal{I}, G, w) \neq 0$, then $w \in \mathcal{I}(G)$.*

(b) (Homotopy Invariance) *Suppose that $\mathcal{H}\colon [0,1] \times \overline{G} \to 2^{X^*}$ is of type (S_+), locally bounded, and weakly upper semicontinuous such that $\mathcal{H}(t,u)$ is nonempty, closed, and convex for each $(t,u) \in [0,1] \times \overline{G}$. If $c\colon [0,1] \to X^*$ is a continuous path in X^* such that $c(t) \notin \mathcal{H}(t,u)$ for all $(t,u) \in [0,1] \times \partial G$, then $d(\mathcal{H}(t,\,\cdot\,), G, c(t))$ is constant for all $t \in [0,1]$.*

(c) (Normalization) *Let $J\colon X \to X^*$ be the duality operator. If $w \in J(G)$, then we have $d(J, G, w) = 1$. In particular, if $0 \in G$, then $d(\varepsilon J, G, 0) = 1$ for every $\varepsilon > 0$.*

Here, a homotopy $\mathcal{H}\colon [0,1] \times \overline{G} \to 2^{X^*}$ is of type (S_+) in the sense that for any sequence $\{(t_n, u_n)\}$ in $[0,1] \times \overline{G}$ and for any sequence $\{w_n\}$ in X^* with $w_n \in \mathcal{H}(t_n, u_n)$ such that $t_n \to t$, $u_n \rightharpoonup u$ in X as $n \to \infty$, and

$$\limsup_{n\to\infty} \langle w_n, u_n - u \rangle \leq 0,$$

we have $u_n \to u$ in X as $n \to \infty$.

Next we recall some definitions and properties of locally Lipschitz continuous functional. For a real Banach space $(X, \|\cdot\|_X)$, we say that a functional $\mathcal{I}\colon X \to \mathbb{R}$ is locally Lipschitz when, for every $u \in X$, there is a neighbourhood U of u and a constant $\mathcal{C} \geq 0$ such that

$$|\mathcal{I}(v) - \mathcal{I}(w)| \leq \mathcal{C}\|v - w\|_X \quad \text{for all } v, w \in U. \tag{2.1}$$

Let $u, \varphi \in X$. The symbol $\mathcal{I}^\circ(u;\varphi)$ indicates the generalized directional derivative of h at a point u along the direction φ, namely

$$\mathcal{I}^\circ(u;\varphi) := \limsup_{w\to u,\, t\to 0^+} \frac{\mathcal{I}(w + t\varphi) - \mathcal{I}(w)}{t}.$$

The generalized gradient of the function $\mathcal{I}$ at u, denoted by $\partial\mathcal{I}(u)$, is the set

$$\partial\mathcal{I}(u) := \{u^* \in X^* : \langle u^*, \varphi \rangle \leq \mathcal{I}^\circ(u;\varphi) \text{ for all } \varphi \in X\}.$$

A functional $\mathcal{I}\colon X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\nu \in X^*$ (denoted by $\mathcal{I}'(u)$) such that

$$\lim_{t\to 0^+} \frac{h(u + t\varphi) - h(u)}{t} = \mathcal{I}'(u)(\varphi)$$

for all $\varphi \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \mapsto \mathcal{I}'(u)$ is a continuous map from X to its dual X^*. We recall that if h is continuously Gâteaux differentiable then it is locally Lipschitz and one has $\mathcal{I}^\circ(u;\varphi) = \mathcal{I}'(u)(\varphi)$ for all $u, \varphi \in X$. If $\mathcal{I} \colon X \to \mathbb{R}$ is a locally Lipschitz functional and $x \in X$, then we say that x is a critical point of h if it satisfies the inequality

$$\mathcal{I}^\circ(x;y) \geq 0$$

for all $y \in X$ or, equivalently, $0 \in \partial\mathcal{I}(x)$.

We give some properties of the locally Lipschitz functional which will be used later.

Proposition 2.6 ([18]). *Let $\mathcal{I} \colon X \to \mathbb{R}$ be locally Lipschitz functional. Then*

(a) $(-\mathcal{I})^\circ(u;z) = \mathcal{I}^\circ(u;-z)$ *for all* $u, z \in X$.

(b) $\mathcal{I}^\circ(u;z) = \max\{\langle u^*, z\rangle_X : u^* \in \partial\mathcal{I}(u)\} \leq \mathcal{C}\|z\|$ *with* $\mathcal{C}$ *as in* (2.1) *for all* $u, z \in X$.

(c) *Let* $j \colon X \to \mathbb{R}$ *be a continuously differentiable function. Then* $(\mathcal{I} + j)^\circ(u;z) = \mathcal{I}^\circ(u;z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$.

(d) (Lebourg's mean value theorem) *Let u and φ be two points in X. Then, there is a point w in the open segment between u and φ, and a $u^*_\omega \in \partial\mathcal{I}(\omega)$ such that*

$$\mathcal{I}(u) - \mathcal{I}(\varphi) = \langle u^*_\omega, u - \varphi\rangle_X.$$

(e) *Let Y be a Banach space and $j \colon Y \to X$ a continuously differentiable function. Then $\mathcal{I} \circ j$ is locally Lipschitz and*

$$\partial(\mathcal{I} \circ j)(u) \subseteq \partial\mathcal{I}(j(y)) \circ j'(y) \quad \textit{for all } y \in Y.$$

(f) *If $\mathcal{I}_1, \mathcal{I}_2 \colon X \to \mathbb{R}$ are locally Lipschitz, then*

$$\partial(\mathcal{I}_1 + \mathcal{I}_2)(u) \subseteq \partial\mathcal{I}_1(u) + \partial\mathcal{I}_2(u).$$

(g) *$\partial\mathcal{I}(u)$ is convex and weakly* compact and the set-valued mapping $\partial\mathcal{I} \colon X \to 2^{X^*}$ is weakly* u.s.c.*

(h) $\partial(\lambda\mathcal{I})(u) = \lambda\partial\mathcal{I}(u)$ *for any* $\lambda \in \mathbb{R}$.

Lemma 2.7 ([34]). *Let $\mathcal{I} \colon X \to \mathbb{R}$ be a locally Lipschitz functional with compact gradient. Then, $\mathcal{I}$ is sequentially weakly continuous, i.e., if $\{u_n\}$ is a sequence in X such that $u_n \rightharpoonup u$ in X as $n \to \infty$, then $\mathcal{I}(u_n) \to \mathcal{I}(u)$ in X as $n \to \infty$.*

Henceforth we study the boundary value problem (B) with discontinuous nonlinearity which involves the fractional p-Laplacian. Let $s \in (0,1)$ and $p \in (1,+\infty)$. To do this, we list definitions and some basic properties for the fractional Sobolev spaces which will be used in obtaining our main result.

We define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as follows:

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \big(\|u\|^p_{L^p(\mathbb{R}^N)} + |u|^p_{W^{s,p}(\mathbb{R}^N)} \big)^{1/p},$$

where

$$\|u\|^p_{L^p(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |u|^p \, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy.$$

Let $s \in (0,1)$ and $1 < p < +\infty$. Then $W^{s,p}(\mathbb{R}^N)$ is a separable and reflexive Banach space. Also, for any $s \in (0,1)$ and $1 < p < +\infty$, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, that is $W_0^{s,p}(\mathbb{R}^N) = W^{s,p}(\mathbb{R}^N)$ (see e.g. [1], [27]).

We consider problem (B) in the closed linear subspace defined by

$$X_s^p(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega \right\},$$

which can be equivalently renormed by setting $\| \cdot \|_{X_s^p(\Omega)} = | \cdot |_{W^{s,p}(\mathbb{R}^N)}$ (see [27, Theorem 7.1]). Then it is readily seen that $X_s^p(\Omega)$ is a uniformly convex Banach space; see Lemma 2.4 in [33]. Thanks to the following lemma, it is possible to obtain the estimate of a positive constant denoted by $\mathcal{S}_p$ which is crucial to obtain the existence of at least one nontrivial weak solution for our problem (B).

LEMMA 2.8. ([28]) *Let Ω be a bounded open set in $\mathbb{R}^N$, $0 < s < 1 < p < +\infty$ and $sp < N$. Then*

$$\|u\|^p_{L^p(\Omega)} \le \mathcal{S}_p \, |u|^p_{W^{s,p}(\mathbb{R}^N)}$$

for any $u \in \widetilde{W}^{s,p}(\mathbb{R}^N)$, where

$$\mathcal{S}_p = \frac{sp \, |\Omega|^{sp/N}}{2\omega_N^{sp/N+1}}.$$

Here ω_N denotes the volume of the N-dimensional unit ball and we denote by $\widetilde{W}^{s,p}(\mathbb{R}^N)$ the space of all $u \in X_s^p(\Omega)$ such that $\widetilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\widetilde{u}$ is the extension by zero of u.

LEMMA 2.9 ([27]). *Let $\Omega \subset \mathbb{R}^N$ a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $p \in (1, +\infty)$. Then we have the following continuous embeddings:*

$$X_s^p(\Omega) \hookrightarrow L^q(\Omega) \qquad \text{for all } q \in [1, p_s^*], \qquad \text{if } sp \le N,$$
$$X_s^p(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) \quad \text{for all } \lambda < s - N/p \quad \text{if } sp > N,$$

where p_s^* is the fractional critical Sobolev exponent, that is

$$p_s^* := \begin{cases} \dfrac{Np}{N - sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \ge N. \end{cases}$$

In particular, the space $X_s^p(\Omega)$ *is compactly embedded in* $L^q(\Omega)$ *for any* $q \in [1, p_s^*)$ *if* $sp \le N$.

Let us define a functional $\Phi_{s,p}\colon X_s^p(\Omega) \to \mathbb{R}$ by

$$\Phi_{s,p}(u) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy.$$

Then the functional $\Phi_{s,p}$ is well defined on $X_s^p(\Omega)$, $\Phi_{s,p} \in C^1(X_s^p(\Omega), \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'_{s,p}(u), \varphi\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy,$$

for any $\varphi \in X_s^p(\Omega)$ where $\langle \cdot\,, \cdot\rangle$ denotes the pairing of $X_s^p(\Omega)$ and its dual $(X_s^p(\Omega))^*$; see [28].

Lemma 2.10 ([28]). *Let* $0 < s < 1 < p < +\infty$ *and* $sp < N$. *Then the functional* $\Phi'_{s,p}\colon X_s^p(\Omega) \to (X_s^p(\Omega))^*$ *is of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* $X_s^p(\Omega)$ *as* $n \to \infty$ *and* $\limsup_{n\to\infty} \langle \Phi'_{s,p}(u_n) - \Phi'_{s,p}(u), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* $X_s^p(\Omega)$ *as* $n \to \infty$.

Corollary 2.11. *Let* $0 < s < 1 < p < +\infty$ *and* $sp < N$. *Then the functional* $\Phi'_{s,p}$ *is strictly monotone, coercive and hemicontinuous on* $X_s^p(\Omega)$. *Furthermore, the operator* $\Phi'_{s,p}$ *is a bounded homeomorphism onto* $(X_s^p(\Omega))^*$.

Proof. It is obvious that the operator $\Phi'_{s,p}$ is strictly monotone, coercive and hemicontinuous on X. By the Browder–Minty theorem, the inverse operator $(\Phi'_{s,p})^{-1}$ exists; see Theorem 26.A in [36]. If we apply Lemma 2.10, then the proof of continuity of the inverse operator $(\Phi'_{s,p})^{-1}$ is analogous to that in the case of the p-Laplacian and is omitted. □

We assume that

(F1) f is measurable with respect to each variable separately.
(F2) There exist nonnegative functions $\rho \in L^{p'}(\Omega)$ and $\sigma \in L^\infty(\Omega)$ such that

$$|f(x,t)| \le \rho(x) + \sigma(x)\,|t|^{p-1} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

Moreover, we denote by $\mathfrak{LB}$ the family of all locally bounded functions $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(f_1) $f(\cdot\,, z)$ is measurable for every $z \in \mathbb{R}$.
(f_2) there exists a set $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 0$ such that the set

$$\mathfrak{D}_f := \bigcup_{x \in \Omega\setminus\Omega_0} \{z \in \mathbb{R} : f(x, \cdot\,) \text{ is discontinuous at } z\}$$

has measure zero.

(f_3) For $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ and $(x,t)\in\Omega\times\mathbb{R}$, the functions

$$\underline{f}(x,z) := \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|\xi-z|<\delta} f(x,\xi) \quad \text{and} \quad \overline{f}(x,z) := \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|\xi-z|<\delta} f(x,\xi)$$

are superpositionally measurable, that is, $\underline{f}(\,\cdot\,,u(\,\cdot\,))$ and $\overline{f}(\,\cdot\,,u(\,\cdot\,))$ are measurable on Ω for any measurable function $u\colon\Omega\to\mathbb{R}$.

Clearly, if $f\in\mathfrak{LB}$ then f satisfies (F1). For fixed $x\in\Omega$, as the function of t, the function F is defined by

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi \quad \text{for } (x,t)\in\Omega\times\mathbb{R}.$$

The generalized gradients of the function t exist, that is,

$$\partial F(x,t) := \partial_t F(x,t) = \partial_z F^\circ(x,t;\theta),$$

where

$$F^\circ(x,t;z) = \limsup_{h\to 0,\,\eta\downarrow 0} \frac{f(x,t+h+\eta z)-f(x,t+h)}{h},$$

and ∂_z is the subdifferential in z. Define the functional $\Psi\colon X_s^p(\Omega)\to\mathbb{R}$ by

$$\Psi(u) = \int_\Omega F(x,u)\,dx.$$

Next we define a functional $I_\lambda\colon X_s^p(\Omega)\to\mathbb{R}$ by

$$I_\lambda(u) = \Phi_{s,p}(u) - \lambda\Psi(u).$$

Proposition 2.12 ([16]). *If $f\in\mathfrak{LB}$ satisfies* (F2)*, then we have*

$$\partial\Psi(u) \subseteq \partial F(x,u) = [\underline{f}(x,u(x)),\overline{f}(x,u(x))] \quad \textit{for almost all } x\in\Omega.$$

Definition 2.13. Let $0<s<1<p<+\infty$. We say that $u\in X_s^p(\Omega)$ is a weak solution of problem (B) if there exists a function $w\in\partial F(x,u)$ for almost all $x\in\Omega$ such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}\,dx\,dy = \lambda\int_\Omega w(x)\varphi(x)\,dx$$

for all $\varphi\in X_s^p(\Omega)$.

This equation in Definition 2.13 corresponds to the following operator problem:

$$0\in\Phi'_{s,p}(u) - \lambda\partial F(x,u) \quad \text{for a.a. } x\in\Omega.$$

In view of Proposition 2.12, we deduce that $0\in(\Phi'_{s,p}-\partial(\lambda\Psi))(u)$ implies $0\in\Phi'_{s,p}(u)-\lambda\partial F(x,u)$ for almost all $x\in\Omega$.

Lemma 2.14. *Let $0<s<1<p<+\infty$ and $sp<N$. If $f\in\mathfrak{LB}$ satisfies* (F2)*, then $\Psi\colon X_s^p(\Omega)\to\mathbb{R}$ is a locally Lipschitz functional with compact gradient.*

Proof. We firstly prove that Ψ is locally Lipschitz functional. Let $u,\varphi \in X_s^p(\Omega)$. Apply Lemma 2.9 and the Hölder inequality to obtain

$$\begin{aligned}
&|\Psi(u)-\Psi(\varphi)| \le \int_\Omega |F(x,u)-F(x,\varphi)|\,dx\\
&\le \int_\Omega (\rho(x)+\sigma(x)\,|u(x)|^{p-1}+\rho(x)+\sigma(x)\,|\varphi(x)|^{p-1})\,|u(x)-\varphi(x)|\,dx\\
&\le 2\|\rho\|_{L^{p'}(\Omega)}\|u-\varphi\|_{L^p(\Omega)}+\|\sigma\|_{L^\infty(\Omega)}(\|u\|^{p-1}_{L^p(\Omega)}+\|\varphi\|^{p-1}_{L^p(\Omega)})\|u-\varphi\|_{L^p(\Omega)}\\
&\le 2C\|\rho\|_{L^{p'}(\Omega)}\|u-\varphi\|_{X_s^p(\Omega)}+C\|\sigma\|_{L^\infty(\Omega)}(\|u\|^{p-1}_{X_s^p(\Omega)}+\|\varphi\|^{p-1}_{X_s^p(\Omega)})\|u-\varphi\|_{X_s^p(\Omega)}
\end{aligned}$$

for a positive constant C. From the above computation, it is obvious that Ψ is a locally Lipschitz functional.

Now, we prove that $\partial\Psi(u)$ is compact. Apply the Lebourg mean value theorem and Proposition 2.6 (b), for every $\varphi \in X_s^p(\Omega)$ we choose an element u in $X_s^p(\Omega)$ and $u^* \in \partial\Psi(u)$ such that

$$\langle u^*,\varphi\rangle \le \Psi^\circ(u;\varphi) \tag{2.2}$$

and $\Psi^\circ(u;\,\cdot\,)\colon L^p(\Omega)\to\mathbb{R}$ is a subadditive function. Furthermore, $u^*\in(X_s^p(\Omega))^*$ is also continuous with respect to the topology induced on $X_s^p(\Omega)$ by the norm $\|\cdot\|_{L^p(\Omega)}$. In fact, if we set a Lipschitz constant $\mathcal{C}>0$ for Ψ in a neighbourhood of u, then it follows from Proposition 2.6 (b) that for all $z\in X_s^p(\Omega)$ we obtain

$$\langle u^*,z\rangle \le \mathcal{C}\|z\|_{L^p(\Omega)}, \quad \langle u^*,-z\rangle \le \mathcal{C}\|-z\|_{L^p(\Omega)},$$

and thus $\langle u^*,z\rangle \le \mathcal{C}\|z\|_{L^p(\Omega)}$. Hence, from the Hahn–Banach theorem, u^* can be extended to an element of the dual $L^p(\Omega)$ (complying with (2.2)) for every $\varphi\in L^p(\Omega)$, this means that u^* can be regarded as an element of $L^{p'}(\Omega)$ and write for every $\varphi\in L^p(\Omega)$

$$\langle u^*,\varphi\rangle = \int_\Omega u^*(x)\varphi(x)\,dx. \tag{2.3}$$

Let $\{u_n\}$ be a sequence in $X_s^p(\Omega)$ such that $\|u_n\|_{X_s^p(\Omega)}\le M$ for all $n\in\mathbb{N}$ and for some positive constant M, and take $u^*_{F_n}\in\partial\Psi(u_n)$ for all $n\in\mathbb{N}$. From (F2) and (2.3) we have

$$\begin{aligned}
\langle u^*_{F_n},\varphi\rangle &= \int_\Omega u^*_{F_n}\varphi(x)\,dx \le \int_\Omega |u^*_{F_n}||\varphi(x)|\,dx\\
&\le \int_\Omega (\rho(x)+\sigma(x)\,|u(x)|^{p-1})|\varphi(x)|\,dx\\
&\le \|\rho\|_{L^{p'}(\Omega)}\|\varphi\|_{L^p(\Omega)}+\|\sigma\|_{L^\infty(\Omega)}\|u\|^{p-1}_{L^p(\Omega)}\|\varphi\|_{L^p(\Omega)}\\
&\le (\mathcal{S}_p^{1/p}+\mathcal{S}_p)(\|\rho\|_{L^{p'}(\Omega)}+\|\sigma\|_{L^\infty(\Omega)}M^{p-1})\|\varphi\|_{X_s^p(\Omega)}
\end{aligned}$$

for all $n\in\mathbb{N}$ and for all $u\in X_s^p(\Omega)$. Hence

$$\|u^*_{F_n}\|_{(X_s^p(\Omega))^*} \le 2\mathcal{S}_p(\|\rho\|_{L^{p'}(\Omega)}+\|\sigma\|_{L^\infty(\Omega)}M^{p-1}),$$

namely, the sequence $\{u^*_{F_n}\}$ is bounded. So, passing to a subsequence, we have that the sequence $\{u^*_{F_n}\}$ weakly converges to u^*_F in $(X^p_s(\Omega))^*$ as $n \to \infty$. We will prove that $\{u^*_{F_n}\}$ has strong convergence in $(X^p_s(\Omega))^*$. Suppose to the contrary that there exists some $\varepsilon_0 > 0$ such that

$$\|u^*_{F_n} - u^*_F\|_{(X^p_s(\Omega))^*} > \varepsilon_0$$

for all $n \in \mathbb{N}$. Then there is $\varphi_n \in B_1(0)$ such that

$$\langle u^*_{F_n} - u^*_F, \varphi_n\rangle > \varepsilon_0. \tag{2.4}$$

Noting that $\{\varphi_n\}$ is a bounded sequence and passing to a subsequence, one has

$$\varphi_n \rightharpoonup \varphi \quad \text{in } X^p_s(\Omega), \qquad \|\varphi_n - \varphi\|_{L^p(\Omega)} \to 0 \quad \text{as } n \to \infty$$

by Lemma 2.9. So, for n large enough, we have

$$\big|\langle u^*_{F_n} - u^*_F, \varphi\rangle\big| < \frac{\varepsilon_0}{3}, \qquad |\langle u^*_F, \varphi_n - \varphi\rangle| < \frac{\varepsilon_0}{3},$$

$$\|\varphi_n - \varphi\|_{L^p(\Omega)} < \frac{\varepsilon_0}{3(\|\rho\|_{L^{p'}(\Omega)} + \|\sigma\|_{L^\infty(\Omega)}\mathcal{S}_p^{(p-1)/p}M^{p-1})}.$$

Then

$$\begin{aligned}
\langle u^*_{F_n} - u^*_F, \varphi_n\rangle &= \langle u^*_{F_n} - u^*_F, \varphi\rangle + \langle u^*_{F_n}, \varphi_n - \varphi\rangle - \langle u^*_F, \varphi_n - \varphi\rangle \\
&< \frac{2\varepsilon_0}{3} + \int_\Omega |u^*_{F_n}||\varphi_n(x) - \varphi(x)|\,dx \\
&\le \frac{2\varepsilon_0}{3} + \int_\Omega (\rho(x) + \sigma(x)\,|u_n(x)|^{p-1})|\varphi_n(x) - \varphi(x)|\,dx \\
&\le \frac{2\varepsilon_0}{3} + \|\rho\|_{L^{p'}(\Omega)}\|\varphi_n - \varphi\|_{L^p(\Omega)} + \|\sigma\|_{L^\infty(\Omega)}\|u_n\|^{p-1}_{L^p(\Omega)}\|\varphi_n - \varphi\|_{L^p(\Omega)} \\
&\le \frac{2\varepsilon_0}{3} + (\|\rho\|_{L^{p'}(\Omega)} + \|\sigma\|_{L^\infty(\Omega)}\mathcal{S}_p^{(p-1)/p}M^{p-1})\|\varphi_n - \varphi\|_{L^p(\Omega)} < \varepsilon_0,
\end{aligned}$$

which contradicts (2.4). □

Now we show that every critical point of the functional I_λ is a weak solution for our problem. The basic idea of the proof comes from [10]; see also [5].

LEMMA 2.15. *Let $0 < s < 1 < p < +\infty$ and $sp < N$. Assume that $f \in \mathfrak{LB}$ satisfies* (F2). *Then the critical points of the functional I_λ are weak solutions for problem* (B).

PROOF. If $u_0 \in X^p_s(\Omega)$ is a critical point of I_λ, then one has

$$(\Phi_{s,p} - \lambda\Psi)^\circ(u_0;\varphi) \ge 0 \quad \text{for all } \varphi \in X^p_s(\Omega).$$

Since $\Phi_{s,p} \in C^1(X^p_s(\Omega),\mathbb{R})$, we have, by Proposition 2.6,

$$\begin{aligned}
0 \le (\Phi_{s,p} - \lambda\Psi)^\circ(u_0;\varphi) &\le \Phi^\circ_{s,p}(u_0;\varphi) + (-\lambda\Psi)^\circ(u_0;\varphi) \\
&= \langle \Phi'_{s,p}(u_0), \varphi\rangle + (-\lambda\Psi)^\circ(u_0;\varphi),
\end{aligned}$$

whence

$$-\langle \Phi'_{s,p}(u_0), \varphi \rangle \le (-\lambda\Psi)^\circ(u_0; \varphi) \quad \text{for all } \varphi \in X_s^p(\Omega).$$

This means $-\Phi'_{s,p}(u_0) \in \partial(-\lambda\Psi)(u_0)$, namely

$$\Phi'_{s,p}(u_0) \in \partial(\lambda\Psi)(u_0). \tag{2.5}$$

Since $X_s^p(\Omega)$ is compactly embedded and dense in $L^p(\Omega)$, from Theorem 2.2 in [16] one has

$$\partial(-\Psi)(u_0) \subseteq \partial(-\Psi_{|L^p(\Omega)})(u_0).$$

From (F2) and because $-\lambda f \in \mathfrak{LB}$, we deduce that $\overline{f}$, $\underline{f}$, $-\lambda\overline{f}$, and $-\lambda\underline{f}$ satisfy all the assumptions in Theorem 2.1 of [16]. Thus

$$\partial(\Psi_{|L^p(\Omega)})(u_0) \subseteq [\underline{f}(\,\cdot\,, u_0(\,\cdot\,)), \overline{f}(\,\cdot\,, u_0(\,\cdot\,))]_{p'}$$

where

$$[\underline{f}(\,\cdot\,, u_0(\,\cdot\,)), \overline{f}(\,\cdot\,, u_0(\,\cdot\,))]_{p'} = \{\omega \in L^{p'}(\Omega) : \omega(x) \in [\underline{f}(x, u_0(x)), \overline{f}(x, u_0(x))] \text{ a.e. in } \Omega\}.$$

Using (2.5),

$$\Phi'_{s,p}(u_0) \in \lambda[\underline{f}(\,\cdot\,, u_0(\,\cdot\,)), \overline{f}(\,\cdot\,, u_0(\,\cdot\,))]_{p'}$$

and then we have

$$(-\Delta)_p^s u_0 \in \lambda[\underline{f}(\,\cdot\,, u_0(\,\cdot\,)), \overline{f}(\,\cdot\,, u_0(\,\cdot\,))]_{p'}.$$

Thus, the Radon–Nikodym theorem implies that there exists a unique

$$\omega_0(\,\cdot\,) \in \lambda[\underline{f}(\,\cdot\,, u_0(\,\cdot\,)), \overline{f}(\,\cdot\,, u_0(\,\cdot\,))]_{p'}$$

such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p-2}(u_0(x) - u_0(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+ps}}\,dx\,dy = \int_\Omega \omega_0(x)\varphi(x)\,dx$$

for each $\varphi \in X_s^p(\Omega)$. This means that u_0 is a weak solution of the problem

$$\begin{cases} (-\Delta)_p^s u = \omega_0(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

This completes the proof. □

Now, we are ready to state the main result of this paper. We investigate the solvability of nonlinear elliptic equations involving the fractional p-Laplacian, by using the Berkovits–Tienari degree theory for set-valued operators of type (S_+).

THEOREM 2.16. *Let $0 < s < 1 < p < +\infty$ and $sp < N$. Assume that $f \in \mathfrak{LB}$ satisfies* (F2). *If $\lambda \mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)} < 1$, then problem* (B) *admits at least one nontrivial weak solution in $X_s^p(\Omega)$.*

PROOF. Note by Corollary 2.11, Lemmas 2.7, 2.10 and 2.14 that the bounded continuous operator $\Phi'_{s,p}$ is of type (S_+) and the sequentially weakly continuous operator Ψ is quasimonotone. Hence the sum $\Phi'_{s,p} - \partial(\lambda\Psi)$ is bounded, upper semicontinuous, and of type (S_+) or pseudomonotone. For each $u \in X_s^p(\Omega)$, we can write

$$\langle w, u \rangle = \int_\Omega wu\, dx$$

for some $w \in \partial\Psi(u)$. Taking into account the Hölder inequality, one has

$$\begin{aligned} -\int_\Omega wu\, dx &\geq -\int_\Omega (\rho(x) + \sigma(x)\,|u(x)|^{p-1}) u(x)\, dx \\ &\geq -\|\sigma\|_{L^\infty(\Omega)} \int_\Omega |u(x)|^p\, dx - \left(\int_\Omega |\rho(x)|^{p'}\, dx\right)^{1/p'} \left(\int_\Omega |u(x)|^p\, dx\right)^{1/p} \\ &\geq -\mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)} \|u\|^p_{X_s^p(\Omega)} - \mathcal{S}_p^{1/p} \|\rho\|_{L^{p'}(\Omega)} \|u\|_{X_s^p(\Omega)} \end{aligned}$$

and hence

$$\begin{aligned} \langle \Phi'_{s,p}(u) - \lambda w, u \rangle &= \|u\|^p_{X_s^p(\Omega)} - \lambda \int_\Omega wu\, dx \\ &\geq \big(1 - \lambda \mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)}\big) \|u\|^p_{X_s^p(\Omega)} - \lambda \mathcal{S}_p^{1/p} \|\rho\|_{L^{p'}(\Omega)} \|u\|_{X_s^p(\Omega)}. \end{aligned}$$

This implies, owing to $\lambda \mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)} < 1$ and $p > 1$, that there exists a positive constant R_0 such that

$$\langle \Phi'_{s,p}(u) - \lambda w, u \rangle > 0 \quad \text{for all } u \in X_s^p(\Omega) \text{ with } \|u\|_{X_s^p(\Omega)} \geq R_0 \text{ and } w \in \partial\Psi(u).$$

Observe by Remark 2.4 that the duality operator $J\colon X_s^p(\Omega) \to (X_s^p(\Omega))^*$ is injective, bounded, continuous, and of type (S_+), and such that $\langle Ju, u \rangle = \|u\|^2_{X_s^p(\Omega)}$ and $\|Ju\|_{(X_s^p(\Omega))^*} = \|u\|_{X_s^p(\Omega)}$ for $u \in X_s^p(\Omega)$. Let $\varepsilon > 0$ be arbitrary but fixed. We consider a homotopy $\mathcal{H}\colon [0,1] \times \overline{B_{R_0}(0)} \to 2^{(X_s^p(\Omega))^*}$ defined by

$$\mathcal{H}(t,u) := (1-t)(\Phi'_{s,p} - \partial(\lambda\Psi))(u) + \varepsilon Ju \quad \text{for } (t,u) \in [0,1] \times \overline{B_{R_0}(0)}.$$

Then the operators $(\Phi'_{s,p} - \partial(\lambda\Psi)) + \varepsilon J$ and J are of type (S_+), the affine homotopy $\mathcal{H}$ is also of type (S_+). Moreover, we have $0 \notin H(t,u)$ for all $(t,u) \in [0,1] \times \partial B_{R_0}(0)$. The homotopy invariance and normalization properties of the degree d in Lemma 2.5 imply that

$$d((\Phi'_{s,p} - \partial(\lambda\Psi)) + \varepsilon J, B_{R_0}(0), 0) = d(\varepsilon J, B_{R_0}(0), 0) = 1.$$

Put $\varepsilon = 1/n$ for each $n \in \mathbb{N}$. The existence property of the degree yields that there exist points $u_n \in B_{R_0}(0)$ and $w_n \in (\Phi'_{s,p} - \partial(\lambda\Psi))(u_n)$ such that

$$w_n + \frac{1}{n} Ju_n = 0$$

for each $n \in \mathbb{N}$. Passing to a subsequence, if necessary, we may suppose that $u_n \rightharpoonup u$ in $X_s^p(\Omega)$ for some $u \in X_s^p(\Omega)$ as $n \to \infty$. Then it follows from the boundedness of (Ju_n) that $w_n \to 0$ in $(X_s^p(\Omega))^*$ as $n \to \infty$ and hence

$$\lim_{n\to\infty} \langle w_n, u_n - u\rangle = 0.$$

Note that the weak limit u belongs to the closed convex hull of the open ball $B_{R_0}(0)$ and so $u \in \overline{B_{R_0}(0)} \subset X_s^p(\Omega)$. Since $\Phi'_{s,p} - \partial(\lambda\Psi)$ is pseudomonotone, we assert that the inclusion $0 \in (\Phi'_{s,p} - \partial(\lambda\Psi))(u)$ has a solution in $X_s^p(\Omega)$ and so u is a critical point of the functional $\Phi_{s,p} - \lambda\Psi$. In view of Lemma 2.15, the conclusion holds. □

Theorem 2.17. *Let $0 < s < 1 < p < +\infty$ and $sp < N$. Assume that $f \in \mathfrak{LB}$ satisfies*

(F2′) *There exist nonnegative functions $\rho \in L^{\gamma'}(\Omega)$ and $\sigma \in L^\infty(\Omega)$ such that*

$$|f(x,t)| \le \rho(x) + \sigma(x)\,|t|^{\gamma-1}$$

for all $(x,t) \in \Omega \times \mathbb{R}$ where $1 < \gamma < p$.

Then problem (B) *admits at least one nontrivial weak solution in $X_s^p(\Omega)$ for all $\lambda > 0$.*

Proof. If we assume the condition (F2′) instead of (F2) and repeat the analogous arguments as in the proofs of Lemmas 2.14 and 2.15, then it follows from Lemma 2.9 that the same assertions hold. Let us denote by $\mathcal{S}_\gamma$ the embedding's constant of $X_s^p(\Omega) \hookrightarrow L^\gamma(\Omega)$. Invoking the Hölder inequality, one has

$$\begin{aligned}\langle \Phi'_{s,p}(u) - \lambda w, u\rangle &= \|u\|^p_{X_s^p(\Omega)} - \lambda \int_\Omega wu\,dx \\ &\ge \|u\|^p_{X_s^p(\Omega)} - \lambda\mathcal{S}_\gamma\|\sigma\|_{L^\infty(\Omega)}\|u\|^\gamma_{X_s^p(\Omega)} - \lambda\mathcal{S}_\gamma^{1/\gamma}\|\rho\|_{L^{\gamma'}(\Omega)}\|u\|_{X_s^p(\Omega)}\end{aligned}$$

for some $w \in \partial\Psi(u)$. Since $p > \gamma > 1$, there is a constant $R_1 > 0$ such that

$$\langle \Phi'_{s,p}(u) - \lambda w, u\rangle > 0 \quad \text{for all } u \in X_s^p(\Omega) \text{ with } \|u\|_{X_s^p(\Omega)} \ge R_1 \text{ and } w \in \partial\Psi(u).$$

Therefore it follows upon proceeding the same way as in the proof of Theorem 2.16 that the conclusion holds. □

In the rest of this paper, we consider the existence of two distinct nontrivial weak solutions for problem (B). To do this, we assume that f satisfies the following condition:

(F3) There exists $\delta > 0$ such that $f(x,t) \ge s(x)t^{\gamma_0-1}$ for almost all $x \in \Omega$ and $0 < t \le \delta$, where $s \ge 0, s \in C(\Omega,\mathbb{R})$ and $1 < \gamma_0 < p$.

Theorem 2.18. *Let $0 < s < 1 < p < +\infty$ and $sp < N$. Assume that $f \in \mathfrak{LB}$ satisfies* (F2)–(F3)*. Then there exists a positive constant λ^* such that problem* (B) *admits two nontrivial weak solutions in $X_s^p(\Omega)$ in which one has negative energy and another has positive energy for any $\lambda \in (0,\lambda^*)$.*

Proof. We first claim that there exists $\phi \in X_s^p(\Omega)$ such that $\phi \geq 0$, $\phi \neq 0$ and $I_\lambda(\eta\phi) < 0$ for $\eta > 0$ small enough. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ be such that $\phi(x) \equiv 1$ for all $x \in B_R(x_0)$; $0 \leq \phi(x) \leq 1$ and $(|\phi(x) - \phi(y)|^p)/|x-y|^{N+ps} \leq 1/R$ for all $x, y \in \mathbb{R}^N$, where $B_R(x_0) := \{x \in \Omega : |x - x_0| \leq R\}$. Then for any $\eta \in (0,1)$ it follows from (F3) that

$$\begin{aligned}
I_\lambda(\eta\phi) &= \Phi_{s,p}(\eta\phi) - \lambda\Psi(\eta\phi) \\
&= \frac{1}{p}\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\eta\phi(x) - \eta\phi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \lambda\int_{B_{2R}(x_0)} F(x,\eta\phi)\,dx \\
&\leq \frac{\eta^p}{p}\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \lambda\int_{B_{2R}(x_0)} \frac{\eta^{\gamma_0}}{\gamma_0}\,|s(x)|\,|\phi|^{\gamma_0}\,dx \\
&\leq \frac{\eta^p}{p}\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{\lambda\eta^{\gamma_0}}{\gamma_0}\int_{B_{2R}(x_0)} |s(x)|\,dx.
\end{aligned}$$

Choose a positive constant δ such that

$$0 < \delta < \min\left\{1, \frac{\dfrac{\lambda p}{\gamma_0}\displaystyle\int_{B_{2R}(x_0)} |s(x)|\,dx}{\displaystyle\int_{B_{2R}(x_0)}\int_{B_{2R}(x_0)} \frac{|\phi(x) - \phi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}\right\},$$

then $\eta < \delta^{1/(p-\gamma_0)}$ implies that $I_\lambda(\eta\phi) < 0$, as claimed. Also, if we define the quantity

$$\lambda^* = \min\bigl\{\varrho^{p-1}\bigl(\mathcal{S}_p\|\sigma\|_{L^\infty(\Omega)}\varrho^{p-1} + p\mathcal{S}_p^{1/p}\|\rho\|_{L^{p'}(\Omega)}\bigr)^{-1}, \bigl(\mathcal{S}_p\|\sigma\|_{L^\infty(\Omega)}\bigr)^{-1}\bigr\},$$

then, for any $u \in X_s^p(\Omega)$ and $\lambda \in (0, \lambda^*)$, it follows from the assumption (F2) that

$$\begin{aligned}
I_\lambda(u) &= \Phi_{s,p}(u) - \lambda\Psi(u) \\
&= \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \lambda\int_\Omega F(x,u)\,dx \\
&\geq \frac{1}{p}\left(\|u\|^p_{X_s^p(\Omega)} - \lambda\|\sigma\|_{L^\infty(\Omega)}\int_\Omega |u|^p\,dx\right) \\
&\quad - \left(\int_\Omega |\rho(x)|^{p'}\,dx\right)^{1/p'}\left(\int_\Omega |u|^p\,dx\right)^{1/p} \\
&\geq \frac{1}{p}\,(1 - \lambda\mathcal{S}_p\|\sigma\|_{L^\infty(\Omega)})\|u\|^p_{X_s^p(\Omega)} - \lambda\mathcal{S}_p^{1/p}\|\rho\|_{L^{p'}(\Omega)}\|u\|_{X_s^p(\Omega)}.
\end{aligned} \tag{2.6}$$

Since $\lambda\mathcal{S}_p\|\sigma\|_{L^\infty(\Omega)} < 1$ and $p > 1$, we conclude that

$$I_\lambda(u) \to +\infty \quad \text{as } \|u\|_{X_s^p(\Omega)} \to +\infty \text{ for all } u \in X_s^p(\Omega) \text{ and } \lambda \in (0,\lambda^*).$$

This means that I_λ is coercive for all $\lambda \in (0, \lambda^*)$. By the coercivity of the functional I_λ, we get that there exists a global minimizer $u_1 \in X_s^p(\Omega)$ of I_λ

(Theorem 1.2 in [31]). This together with the above claim yields that

$$I_\lambda(u_1) = \inf_{u \in X_s^p(\Omega)\setminus\{0\}} I_\lambda(u) < 0.$$

Hence we deduce that u_1 is a nontrivial global minimizer of the functional I_λ in $X_s^p(\Omega)$ for any $\lambda \in (0, \lambda^*)$.

Finally, we establish that u_2 is another weak solution with positive energy. As in Theorem 2.16, we deduce that the functional I_λ has a nontrivial critical point u. Denote it by $u = u_2$ with $\|u_2\|_{X_s^p(\Omega)} = \varrho > 0$. By the inequality (2.6), we yield

$$\begin{aligned} I_\lambda(u_2) &\geq \frac{1}{p}\,(1 - \lambda \mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)})\|u_2\|^p_{X_s^p(\Omega)} - \lambda \mathcal{S}_p^{1/p}\|\rho\|_{L^{p'}(\Omega)}\|u_2\|_{X_s^p(\Omega)} \\ &= \frac{1}{p}\,(1 - \lambda \mathcal{S}_p \|\sigma\|_{L^\infty(\Omega)})\varrho^p - \lambda \mathcal{S}_p^{1/p}\|\rho\|_{L^{p'}(\Omega)}\varrho \end{aligned}$$

and therefore we conclude that $I_\lambda(u_2) > 0$ for any $\lambda \in (0, \lambda^*)$. □

Remark 2.19. Under the same conditions given in Theorem 2.17, it is clear that

$$I_\lambda(u) \to +\infty \quad \text{as } \|u\|_{X_s^p(\Omega)} \to +\infty$$

for all $\lambda \in \mathbb{R}$, being $\gamma < p$. Furthermore, let us assume the condition (F3) with $1 < \gamma_0 < \gamma$ instead of the relation $1 < \gamma_0 < p$ and denote $\lambda^* > 0$ by

$$\lambda^* = \varrho^{p-1}\big(\mathcal{S}_\gamma \|\sigma\|_{L^\infty(\Omega)}\varrho^{\gamma-1} + p\mathcal{S}_\gamma^{1/\gamma}\|\rho\|_{L^{\gamma'}(\Omega)}\big)^{-1}.$$

Then obvious modification of the proof of Theorem 2.18 yield that (B) admits two nontrivial weak solutions in $X_s^p(\Omega)$ in which one has negative energy and another has positive energy for any $\lambda \in (0, \lambda^*)$.

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