

**EXISTENCE AND CONCENTRATION
OF GROUND STATE SIGN-CHANGING SOLUTIONS
FOR KIRCHHOFF TYPE EQUATIONS
WITH STEEP POTENTIAL WELL AND NONLINEARITY**

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ABSTRACT. We study the following class of elliptic equations:

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + \lambda V(x)u = f(u), \quad x \in \mathbb{R}^3,$$

where $\lambda, a, b > 0$, $V \in C(\mathbb{R}^3, \mathbb{R})$ and $V^{-1}(0)$ has nonempty interior. First, we obtain one ground state sign-changing solution $u_{b,\lambda}$ applying the non-Nehari manifold method. We show that the energy of $u_{b,\lambda}$ is strictly larger than twice that of the ground state solutions of Nehari-type. Next we establish the convergence property of $u_{b,\lambda}$ as $b \searrow 0$. Finally, the concentration of $u_{b,\lambda}$ is explored on the set $V^{-1}(0)$ as $\lambda \rightarrow \infty$.

1. Introduction and preliminaries

In this paper, we are concerned with the following elliptic equations:

$$(1.1) \quad -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + \lambda V(x)u = f(u), \quad x \in \mathbb{R}^3,$$

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where $\lambda > 0$, $V \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$ and $V^{-1}(0)$ has nonempty interior, f is a continuous function, $a, b > 0$.

If $\lambda \equiv 0$, $f(u)$ is replaced by $f(x, u)$ and \mathbb{R}^3 is replaced by a bounded domain $\Omega \subset \mathbb{R}^3$ in (1.1), problem (1.1) reduces to the following nonlocal Kirchhoff type problem:

$$-\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u), \quad x \in \Omega.$$

This problem is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = f(x, u),$$

which was proposed by Kirchhoff in [17] as a model for the equation of elastic strings

$$(1.2) \quad \rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = f(x, u).$$

The Kirchhoff's model (1.2), which is an extension of the classical D'Alembert's wave equation, takes into account the changes in length of the string produced by transverse vibrations. Note that L, h, E, ρ, P_0 denote the length of the string, the area of the cross section, the Young modulus of the material, the mass density and the initial tension, respectively.

If $\lambda = 1$, then (1.1) reduces to the following Kirchhoff problem:

$$(1.3) \quad -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^3.$$

For study of (1.3) with variational methods we refer to [4], [6]–[8], [10], [12]–[14], [16], [19]–[22], [24]–[26], [28], [33], [34], [38], [39], [41], [42], [46], [47], [49]. Especially, Nie [25] proved the existence and multiplicity of nontrivial solutions when $N = 1, 2, 3$ under the following potential conditions:

- (V₁) $V(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq 0$ on \mathbb{R}^3 . Moreover, there exists a constant $L > 0$ such that the set $\mathcal{V}_L := \{x \in \mathbb{R}^3 : V(x) \leq L\}$ is nonempty and $\text{meas} \{x \in \mathbb{R}^3 : V(x) \leq L\} < +\infty$, where meas denotes the Lebesgue measure in \mathbb{R}^3 .

In order to obtain the concentration of solutions, the following additional assumption was posed on V in some papers (see [32], [43], [9]):

- (V₂) $\Omega = \text{int } V^{-1}(0)$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

It is worth mentioning that the above listed papers always assumed the potential V is positive so that we can get compact embedding. In order to solve this problem, in [42], [18], the author used the following condition to overcome the compactness of Sobolev embedding.

(V'₁) $V(x) \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $V(x) \geq 0$ on \mathbb{R}^3 . Moreover, for any $M > 0$, the set $\mathcal{V}_M := \{x \in \mathbb{R}^3 : V(x) \leq M\}$ is nonempty and $\text{meas}\{x \in \mathbb{R}^3 : V(x) \leq M\} < +\infty$, where meas denotes the Lebesgue measure in \mathbb{R}^3 .

Obviously, condition (V₁) is much weaker than condition (V'₁). But in this paper, we use (V'₁) to prove the existence and concentration of ground state sign-changing solutions. In order to study the concentration phenomenon of solutions, we need to add condition (V₂), which plays an important role in proving the concentration phenomenon. Besides, we are also interested in the case that the nonlinearity is a more general mixed nonlinearity involving a combination of superlinear and sublinear terms. Note that V satisfying conditions (V₁)–(V₂) is called steep potential well. Various elliptic equations with steep potential well are studied in [15], [32], [51], [44]. Especially, very recently, Zhang et al. [50] proved the existence of nontrivial solutions and the concentration phenomenon of solutions for Schrödinger–Poisson systems. Afterwards, Gao et al. [11] established the existence of nontrivial solutions and the concentration phenomenon of solutions for the fractional Schrödinger equation. To the best of our knowledge only [32], [43], [9] investigated the Kirchhoff-type problem. In particular, in [43], the authors considered problem (1.1) with steep well potential, and studied the existence of nontrivial solutions and the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \rightarrow \infty$ with the following assumptions on f :

- (f₁) $f \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $|f(x, u)| \leq c(1 + |u|^{q-1})$ for some $4 < q < 6$;
- (f₂) $f(x, u) = o(|u|)$ as $|u| \rightarrow 0$ uniformly for $x \in \mathbb{R}^3$;
- (f₃) there exists $\theta > 4$ such that $0 < \theta F(x, u) \leq uf(x, u)$ for every $x \in \mathbb{R}^3$ and $u \neq 0$, where $F(x, u) = \int_0^u f(x, t) dt$;
- (f₄) $f(x, u)/|u|^3$ is strictly increasing for $u > 0$;
- (f₅) $f(x, u) \equiv 0$ for all $u \leq 0$.

In [43], the authors established the following theorem.

THEOREM 1.1 ([43]). *Assume conditions (V₁)–(V₂) and (f₁)–(f₅) hold, then there exist two positive constants Λ_0 such that for every $\lambda > \Lambda_0$, problem (1.1) has at least one positive solution u_λ . Furthermore, $u_\lambda \rightarrow u$ in $H^1(\mathbb{R}^3)$ as $\lambda \rightarrow \infty$, where $u \in H_0^1(\Omega)$ is a positive solution of*

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Motivated by this result, in the present paper, we study the existence of ground sign-changing solutions and investigate the concentration phenomenon of steep well potential solutions on the set $V^{-1}(0)$ as $\lambda \rightarrow \infty$ under the following assumptions on f :

- (F₁) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(t) = o(t)$ as $t \rightarrow 0$;

(F₂) there exist constants $c_0 > 0$ and $p \in (4, 6)$ such that

$$|f(t)| \leq c_0(1 + |t|^{p-1}), \quad \text{for all } t \in \mathbb{R};$$

(F₃) $\lim_{|t| \rightarrow \infty} F(t)/t^4 = \infty$;

(F₄) there exists $\theta_0 \in (0, 1)$ such that for any $t > 0$ and $\tau \neq 0$

$$\left[\frac{f(\tau)}{\tau^3} - \frac{f(t\tau)}{(t\tau)^3} \right] \operatorname{sgn}(1-t) + \theta_0 V(x) \frac{|1-t^2|}{(t\tau)^2} \geq 0.$$

REMARK 1.2. (F₄) is much weaker than the following condition:

(Ne) $f(t)/|t|^3$ is increasing on $\mathbb{R} \setminus \{0\}$.

In [45], (Ne) was used to prove the existence of least energy nodal solutions for (1.3) and show that the sign-changing solution has an energy strictly larger than the least energy and less than twice the least energy. Moreover, (Ne) is much weaker than (f₄). Hence, our results are stronger and supplement the results obtained in [48], [43], [45].

Now, the working space E is given by

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \right\}$$

with the norm equipped with the inner product and the norm

$$(u, v) = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + V(x)uv \right) dx$$

and

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \right)^{1/2}, \quad \text{for all } u, v \in E.$$

Here, $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ for simplicity is a Hilbert space with the inner product

$$(u, v)_{D^{1,2}} = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v dx$$

and the corresponding norm

$$\|u\|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2},$$

see [40, p. 8]. It can be proved that E is a Hilbert space under condition (V'₁) and there is a continuous embedding $E \hookrightarrow H^1(\mathbb{R}^3)$. As the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$ is continuous for each $r \in [2, 6]$, for these r there exists $\gamma_r > 0$ such that

$$(1.4) \quad \|u\|_r \leq \gamma_r \|u\|, \quad u \in E,$$

where $\|\cdot\|_r$ denotes the usual $L^r(\mathbb{R}^3)$ norm. Moreover, under condition (V'₁), according to [2, Remark 3.5], the embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ is compact for each $r \in [2, 6)$.

For convenience, for each $\lambda > 0$, we also define an equivalent norm on E

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} [a|\nabla u|^2 + \lambda V(x)u^2] dx \right)^{1/2}, \quad u \in E,$$

and the corresponding inner product

$$(u, v)_\lambda = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + \lambda V(x)uv) dx, \quad u, v \in E.$$

It is clear that

$$(1.5) \quad \|u\| \leq \frac{1}{\min\{a^{1/2}, \lambda^{1/2}\}} \|u\|_\lambda = \frac{1}{a_\lambda^{1/2}} \|u\|_\lambda, \quad u \in E,$$

where $1/a_\lambda^{1/2} = 1/\min\{a^{1/2}, \lambda^{1/2}\}$, specially, $a_\lambda^{1/2} = a^{1/2}$ is independent of $\lambda \in [a, \infty)$, where $a > 0$. Thus, it follows from (1.4) that for each $\lambda > 0$,

$$(1.6) \quad \|u\|_r \leq \frac{\gamma_r}{a_\lambda^{1/2}} \|u\|_\lambda, \quad u \in E.$$

Define the energy functional

$$(1.7) \quad \mathcal{J}_{b,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda V(x)u^2) dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \int_{\mathbb{R}^3} F(u) dx,$$

where $F(u) = \int_0^u f(s) ds$. The functional $\mathcal{J}_{b,\lambda}$ is well defined for every $u \in E$ and $\mathcal{J}_{b,\lambda} \in C^1(E, \mathbb{R})$. Moreover, for any $u, \varphi \in E$, we have

$$(1.8) \quad \begin{aligned} \langle \mathcal{J}'_{b,\lambda}(u), \varphi \rangle &= \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla \varphi + \lambda V(x)u\varphi) dx \\ &\quad + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx - \int_{\mathbb{R}^3} f(u)\varphi dx. \end{aligned}$$

Clearly, the critical points of $\mathcal{J}_{b,\lambda}(u)$ are weak solutions of (1.1). Furthermore, if $u \in E$ is a solution of (1.1) and $u^\pm \neq 0$, then u is a sign-changing solution of (1.1), where

$$u^+(x) := \max\{u(x), 0\} \quad \text{and} \quad u^-(x) := \min\{u(x), 0\}.$$

If $b = 0$, then (1.1) is reduced to the following equation:

$$(1.9) \quad -a\Delta u + \lambda V(x)u = f(u)$$

where $u \in H^1(\mathbb{R}^3)$. Problem (1.9) possesses a least energy sign-changing solution when \mathbb{R}^3 is replaced by Ω if

(BWW) $f(t)/|t|$ is increasing on $\mathbb{R}^3 \setminus \{0\}$,

this was proved by Bartsch, Weth and Willem.

A variety of ways are used to get the sign-changing solutions, e.g., by constructing invariant sets and descending flow (see [1]), adopting the Ekeland's variational principle and the implicit function theorem (see [27]), applying variational methods together with the Brouwer degree theory (see [3]), and using

diagonal principle with the non-Nehari manifold method (see [5], [35]–[37], [48]). The following decomposition plays an important role in seeking for sign-changing solutions to (1.9), for any $u \in E$,

$$\begin{aligned} \mathcal{J}'_{0,\lambda}(u) &= \mathcal{J}'_{0,\lambda}(u^+) + \mathcal{J}'_{0,\lambda}(u^-), \\ \langle \mathcal{J}'_{0,\lambda}(u), u^+ \rangle &= \langle \mathcal{J}'_{0,\lambda}(u^+), u^+ \rangle, \quad \langle \mathcal{J}'_{0,\lambda}(u), u^- \rangle = \langle \mathcal{J}'_{0,\lambda}(u^-), u^- \rangle, \end{aligned}$$

where $\mathcal{J}_{0,\lambda}: E \rightarrow \mathbb{R}$ is the energy functional of (1.9) given by

$$\mathcal{J}_{0,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda V(x)|u|^2) dx - \int_{\mathbb{R}^3} F(u) dx$$

and

$$\langle \mathcal{J}'_{0,\lambda}(u), \varphi \rangle = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla \varphi + \lambda V(x)u\varphi) dx - \int_{\mathbb{R}^3} f(u)\varphi dx.$$

Moreover, for the functional $\mathcal{J}_{b,\lambda}$, we have

$$(1.10) \quad \mathcal{J}_{b,\lambda}(u) = \mathcal{J}_{b,\lambda}(u^+) + \mathcal{J}_{b,\lambda}(u^-) + \frac{b}{2} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2,$$

$$(1.11) \quad \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle = \langle \mathcal{J}'_{b,\lambda}(u^+), u^+ \rangle + b\|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2,$$

$$(1.12) \quad \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle = \langle \mathcal{J}'_{b,\lambda}(u^-), u^- \rangle + b\|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2.$$

We will consider the following minimization problems:

$$m_{b,\lambda} := \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) \quad \text{and} \quad m_{0,\lambda} := \inf_{u \in \mathcal{M}_{0,\lambda}} \mathcal{J}_{0,\lambda}(u),$$

where

$$\mathcal{M}_{b,\lambda} := \{u \in E : u^\pm \neq 0, \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle = \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle = 0\},$$

$$\mathcal{M}_{0,\lambda} := \{u \in E : u^\pm \neq 0, \langle \mathcal{J}'_{0,\lambda}(u), u^+ \rangle = \langle \mathcal{J}'_{0,\lambda}(u), u^- \rangle = 0\},$$

whose minimizers correspond to the sign-changing solutions for problems (1.1) and (1.9), respectively.

The following Nehari manifolds will be used to seek for the ground state solutions of Nehari type for (1.1) and (1.9) as minimizers of the corresponding energy functionals $\mathcal{J}_{b,\lambda}$ and $\mathcal{J}_{0,\lambda}$:

$$\mathcal{N}_{b,\lambda} := \{u \in E : u \neq 0, \langle \mathcal{J}'_{b,\lambda}(u), u \rangle = 0\},$$

$$\mathcal{N}_{0,\lambda} := \{u \in E : u \neq 0, \langle \mathcal{J}'_{0,\lambda}(u), u \rangle = 0\}$$

with

$$c_{b,\lambda} := \inf_{u \in \mathcal{N}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) \quad \text{and} \quad c_{0,\lambda} := \inf_{u \in \mathcal{N}_{0,\lambda}} \mathcal{J}_{0,\lambda}(u).$$

Now, we state our main results on the existence and concentration of ground state sign-changing solutions.

THEOREM 1.3. *Suppose (V_1) , (V_2) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then problem (1.1) has a sign-changing solution $u_{b,\lambda} \in \mathcal{M}_{b,\lambda}$ such that $\mathcal{J}_{b,\lambda}(u_{b,\lambda}) = \inf_{\mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda} > 0$, which has precisely two nodal domains.*

THEOREM 1.4. *Suppose (V'_1) , (V_2) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then problem (1.1) has a solution $\bar{u}_\lambda \in \mathcal{N}_{b,\lambda}$ such that $\mathcal{J}_{b,\lambda}(\bar{u}_\lambda) = \inf_{\mathcal{N}_{b,\lambda}} \mathcal{J}_{b,\lambda}$, moreover, $m_{b,\lambda} > 2c_{b,\lambda}$.*

THEOREM 1.5. *Suppose (V'_1) , (V_2) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then problem (1.9) has a sign-changing solution $v_{0,\lambda} \in \mathcal{M}_{0,\lambda}$ such that $\mathcal{J}_{0,\lambda}(v_{0,\lambda}) = \inf_{\mathcal{M}_{0,\lambda}} \mathcal{J}_{0,\lambda} > 0$, which has precisely two nodal domains. Furthermore, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \rightarrow \infty$, there exists a subsequence which we label in the same way, such that $u_{b_n,\lambda} \rightarrow u_{0,\lambda}$ in E where $u_{0,\lambda} \in \mathcal{M}_{0,\lambda}$ is a sign-changing solution of (1.9) with $\mathcal{J}_{0,\lambda}(u_{0,\lambda}) = \inf_{\mathcal{M}_{0,\lambda}} \mathcal{J}_{0,\lambda} > 0$.*

THEOREM 1.6. *Suppose (V'_1) , (V_2) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. For any sequence $\{\lambda_n\} \subset (\max\{a, \theta_0\}, \infty)$ with $\lambda_n \rightarrow \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $u_n := u_{b,\lambda_n} \rightarrow u_0 := u_{b,0}$ in E , where $u_0 \in H_0^1(\Omega)$ and u_0 is a ground state sign-changing solution of the limit system*

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

REMARK 1.7. In this paper, our results on the existence and concentration of ground state sign-changing solutions for Kirchhoff type equation are new, especially on the concentration. Compared with [38], our results are supplement.

REMARK 1.8. When $l = 0$ (see Remark 1.3 in [18]), our method on proving the existence of sign-changing solutions are different from [18]. Moreover, we discuss the ground state of sign-changing solutions. But in [18], the authors only studied the existence of sign-changing solutions.

This paper is organized as follows. In Section 2, we present some lemmas, which are crucial in establishing our results. Section 3 is devoted to the proof of Theorem 1.3. Furthermore, we complete the proofs of Theorems 1.4–1.6 in Sections 4–6, respectively.

Throughout this paper, positive constants possibly different in different places, are denoted by C .

2. Some lemmas

In this section, we present some useful lemmas and corollaries.

LEMMA 2.1. *Suppose that (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then*

$$(2.1) \quad \begin{aligned} \mathcal{J}_{b,\lambda}(u) &\geq \mathcal{J}_{b,\lambda}(su^+ + tu^-) + \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle \\ &\quad + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle + \frac{(1-\theta_0/\lambda)(1-s^2)^2}{4} \|u^+\|_\lambda^2 \\ &\quad + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(s^2-t^2)^2}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2, \end{aligned}$$

for all $u \in E$ and $s, t \geq 0$.

PROOF. By (F₄), for any $x \in \mathbb{R}^3$, $s, t \geq 0$, $\tau \in \mathbb{R} \setminus \{0\}$, one has

$$(2.2) \quad \begin{aligned} &\left[\frac{1-t^4}{4} \tau f(\tau) + F(t\tau) - F(\tau) \right] + \frac{\theta_0 V(x)}{4} (1-t^2)^2 \tau^2 \\ &= \int_t^1 \left\{ \left[\frac{f(\tau)}{\tau^3} - \frac{f(s\tau)}{(s\tau)^3} \right] + \theta_0 V(x) \frac{(1-s^2)}{(s\tau)^2} \right\} s^3 \tau^4 ds \geq 0. \end{aligned}$$

Hence, from (1.7), (1.8), (1.11), (1.12) and (2.2), for any $s, t \geq 0$, we have

$$\begin{aligned} &\mathcal{J}_{b,\lambda}(u) - \mathcal{J}_{b,\lambda}(su^+ + tu^-) \\ &= \frac{1}{2} (\|u\|_\lambda^2 - \|su^+ + tu^-\|_\lambda^2) + \frac{b}{4} (\|\nabla u\|_2^4 - \|s\nabla u^+ + t\nabla u^-\|_2^4) \\ &\quad + \int_{\mathbb{R}^3} [F(su^+ + tu^-) - F(u)] dx \\ &= \frac{1-s^4}{4} (\|u^+\|_\lambda^2 + b\|\nabla u^+\|_2^4) + \frac{1-t^4}{4} (\|u^-\|_\lambda^2 + b\|\nabla u^-\|_2^4) \\ &\quad + \frac{(1-s^2)^2}{4} \|u^+\|_\lambda^2 + \frac{(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(1-s^2t^2)}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 \\ &\quad + \int_{\mathbb{R}^3} [F(su^+) + F(tu^-) - F(u^+) - F(u^-)] dx \\ &= \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle + \frac{(1-s^2)^2}{4} \|u^+\|_\lambda^2 \\ &\quad + \frac{(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(s^2-t^2)^2}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 \\ &\quad + \int_{\mathbb{R}^3} \left[\frac{1-s^4}{4} f(u^+)u^+ + F(su^+) - F(u^+) \right] dx \\ &\quad + \int_{\mathbb{R}^3} \left[\frac{1-t^4}{4} f(u^-)u^- + F(tu^-) - F(u^-) \right] dx \\ &\geq \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle + \frac{(1-\theta_0/\lambda)(1-s^2)^2}{4} \|u^+\|_\lambda^2 \\ &\quad + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(s^2-t^2)^2}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 \\ &\quad + \int_{\mathbb{R}^3} \left\{ \left[\frac{1-s^4}{4} f(u^+)u^+ + F(su^+) - F(u^+) \right] + \frac{\theta_0 V(x)}{4} (1-s^2)^2 |u^+|^2 \right\} dx \end{aligned}$$

$$\begin{aligned}
 & + \int_{\mathbb{R}^3} \left\{ \left[\frac{1-t^4}{4} f(u^-)u^- + F(tu^-) - F(u^-) \right] + \frac{\theta_0 V(x)}{4} (1-t^2)^2 |u^-|^2 \right\} dx \\
 \geq & \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^+ \rangle + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u^- \rangle + \frac{(1-\theta_0/\lambda)(1-s^2)^2}{4} \|u^+\|_\lambda^2 \\
 & + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(s^2-t^2)^2}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2,
 \end{aligned}$$

which implies that (2.1) holds. \square

COROLLARY 2.2. *Suppose (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. If $u = u^+ + u^- \in \mathcal{M}_{b,\lambda}$, then*

$$\begin{aligned}
 \mathcal{J}_{b,\lambda}(u) \geq & \mathcal{J}_{b,\lambda}(su^+ + tu^-) + \frac{(1-\theta_0/\lambda)(1-s^2)^2}{4} \|u^+\|_\lambda^2 \\
 & + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u^-\|_\lambda^2 + \frac{b(s^2-t^2)^2}{4} \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2,
 \end{aligned}$$

for all $s, t \geq 0$.

COROLLARY 2.3. *Suppose (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. If $u = u^+ + u^- \in \mathcal{M}_{b,\lambda}$, then*

$$\mathcal{J}_{b,\lambda}(u^+ + u^-) = \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-).$$

LEMMA 2.4. *Suppose (V'₁) and (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. If $u \in E$ with $u^\pm \neq 0$, then there exists a unique pair (s_u, t_u) of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_{b,\lambda}$.*

PROOF. We will first prove the existence of (s_u, t_u) . Set

$$\begin{aligned}
 (2.3) \quad g_1(s, t) = & s^2 \|u^+\|_\lambda^2 + bs^4 \|\nabla u^+\|_2^4 \\
 & + bs^2 t^2 \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 - \int_{\mathbb{R}^3} f(su^+) su^+ dx,
 \end{aligned}$$

$$\begin{aligned}
 (2.4) \quad g_2(s, t) = & t^2 \|u^-\|_\lambda^2 + bt^4 \|\nabla u^-\|_2^4 \\
 & + bs^2 t^2 \|\nabla u^+\|_2^2 \|\nabla u^-\|_2^2 - \int_{\mathbb{R}^3} f(tu^-) tu^- dx.
 \end{aligned}$$

It follows from (F₁) and (F₃) that $g_1(s, s) > 0$, $g_2(s, s) > 0$ for $s > 0$ small and $g_1(t, t) < 0$ and $g_2(t, t) < 0$ for t large. Thus, there exist $0 < a_1 < a_2$ such that

$$(2.5) \quad g_1(a_1, a_1) > 0, \quad g_2(a_1, a_1) > 0, \quad g_1(a_2, a_2) < 0, \quad g_2(a_2, a_2) < 0.$$

By (2.3)–(2.5), we have

$$g_1(a_1, t) > 0, \quad g_1(a_2, t) < 0 \quad \text{for all } t \in [a_1, a_2],$$

and

$$g_2(s, a_1) > 0, \quad g_2(s, a_2) < 0 \quad \text{for all } s \in [a_1, a_2].$$

By Miranda's Theorem [23], there exists a pair (s_u, t_u) with $a_1 < s_u, t_u < a_2$ such that $g_1(s_u, t_u) = g_2(s_u, t_u) = 0$. Hence $s_u u^+ + t_u u^- \in \mathcal{M}_{b,\lambda}$.

Next, we prove the uniqueness. Let (s_1, t_1) and (s_2, t_2) be such that $s_i u^+ + t_i u^- \in \mathcal{M}_{b,\lambda}$, where $i = 1, 2$. In view of Corollary 2.2, we have

$$\begin{aligned} \mathcal{J}_{b,\lambda}(s_1 u^+ + t_1 u^-) &\geq \mathcal{J}_{b,\lambda}(s_2 u^+ + t_2 u^-) \\ &+ \frac{(1 - \theta_0/\lambda)(s_1^2 - s_2^2)^2}{4s_1^2} \|u^+\|_\lambda^2 + \frac{(1 - \theta_0/\lambda)(t_1^2 - t_2^2)^2}{4t_1^2} \|u^-\|_\lambda^2 \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_{b,\lambda}(s_2 u^+ + t_2 u^-) &\geq \mathcal{J}_{b,\lambda}(s_1 u^+ + t_1 u^-) \\ &+ \frac{(1 - \theta_0/\lambda)(s_1^2 - s_2^2)^2}{4s_2^2} \|u^+\|_\lambda^2 + \frac{(1 - \theta_0/\lambda)(t_1^2 - t_2^2)^2}{4t_2^2} \|u^-\|_\lambda^2, \end{aligned}$$

which implies that $(s_1, t_1) = (s_2, t_2)$. \square

LEMMA 2.5. *Suppose (V'_1) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then*

$$\inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) = m_{b,\lambda} = \inf_{u \in E, u^\pm \neq 0} \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-).$$

PROOF. By Corollary 2.3, we obtain

$$\begin{aligned} (2.6) \quad \inf_{u \in E, u^\pm \neq 0} \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-) &\leq \inf_{u \in \mathcal{M}_{b,\lambda}} \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-) \\ &= \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) = m_{b,\lambda}. \end{aligned}$$

Moreover, for any $u \in E$ with $u^\pm \neq 0$, it follows from Lemma 2.4 that

$$\max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-) \geq \mathcal{J}_{b,\lambda}(su^+ + tu^-) \geq \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) = m_{b,\lambda},$$

which implies that

$$(2.7) \quad \inf_{u \in E, u^\pm \neq 0} \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su^+ + tu^-) \geq \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) = m_{b,\lambda}.$$

Hence, it follows from (2.6) and (2.7) that conclusion holds. \square

LEMMA 2.6. *Suppose (F_4) is satisfied. Then, for any $\tau \in \mathbb{R}$,*

$$(2.8) \quad \frac{1}{4} \tau f(\tau) - F(\tau) + \frac{\theta_0 V(x)}{4} \tau^2 \geq 0.$$

PROOF. Taking $t = 0$ in (2.2), we can get the conclusion. This completes the proof. \square

LEMMA 2.7. *Suppose (V'_1) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then $m_{b,\lambda} > 0$ can be achieved.*

PROOF. Let $\{u_n\} \subset \mathcal{M}_{b,\lambda}$ be such that $\mathcal{J}_{b,\lambda}(u_n) \rightarrow m_{b,\lambda}$. According to (1.7), (1.8) and (2.8), for large $n \in \mathbb{N}$, one has

$$(2.9) \quad \begin{aligned} 1 + m_{b,\lambda} &\geq \mathcal{J}_{b,\lambda}(u_n) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_n), u_n \rangle \\ &= \frac{1 - \theta_0/\lambda}{4} \|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \left\{ \left[\frac{1}{4} f(u_n)u_n - F(u_n) \right] + \frac{\theta_0 V(x)}{4} u_n^2 \right\} dx \\ &\geq \frac{1 - \theta_0/\lambda}{4} \|u_n\|_\lambda^2. \end{aligned}$$

This shows that $\{u_n\}$ is bounded in E due to $0 < \theta_0 < 1$ and $\lambda > \theta_0$, and then, there exists a $u_{b,\lambda} \in E$ such that $u_n^\pm \rightharpoonup u_{b,\lambda}^\pm$ in E . Since $\langle \mathcal{J}'_{b,\lambda}(u), u \rangle = 0$, for all $u \in \mathcal{M}_{b,\lambda}$, then by (F₁)–(F₃) and the Sobolev embedding theorem, for any $\varepsilon > 0$, we have

$$\begin{aligned} \|u\|_\lambda^2 &\leq \int_{\mathbb{R}^3} (a|\nabla u|^2 + \lambda V(x)u^2) dx + b \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 = \int_{\mathbb{R}^3} f(u)u dx \\ &\leq \varepsilon \int_{\mathbb{R}^3} |u|^2 dx + C_\varepsilon \int_{\mathbb{R}^3} |u|^p dx \leq \varepsilon \|u\|^2 + C_\varepsilon \|u\|^p \leq \varepsilon C \|u\|_\lambda^2 + C \|u\|_\lambda^p, \end{aligned}$$

where C_ε is a positive constant. We can choose $\varepsilon = 1/(2C)$, so there exists a constant $\alpha > 0$ such that $\|u\|_\lambda^2 \geq \alpha$. Moreover, by (V₁'), (F₁)–(F₃), (1.8) and [40, A.2], one can conclude that

$$(2.10) \quad \begin{aligned} 0 < \alpha &\leq \|u_n^\pm\|_\lambda^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} |\nabla u_n^\pm|^2 dx \\ &= \int_{\mathbb{R}^3} f(u_n^\pm)u_n^\pm dx = \int_{\mathbb{R}^3} f(u_{b,\lambda}^\pm)u_{b,\lambda}^\pm dx + o(1), \end{aligned}$$

which yields that $u_{b,\lambda}^\pm \neq 0$. Furthermore, by (2.10), the weak semicontinuity of norm and Fatou's Lemma, we get

$$(2.11) \quad \begin{aligned} \|u_{b,\lambda}^\pm\|_\lambda^2 + b \int_{\mathbb{R}^3} |\nabla u_{b,\lambda}^\pm|^2 dx &\int_{\mathbb{R}^3} |\nabla u_{b,\lambda}^\pm|^2 dx \\ &\leq \liminf_{n \rightarrow \infty} \left[\|u_n^\pm\|_\lambda^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\mathbb{R}^3} |\nabla u_n^\pm|^2 dx \right] = \int_{\mathbb{R}^3} f(u_{b,\lambda}^\pm)u_{b,\lambda}^\pm dx. \end{aligned}$$

This shows that

$$(2.12) \quad \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda}^\pm \rangle \leq 0.$$

Thus, by (1.7), (1.8), (2.1), (2.8), (2.12), the weak semicontinuity of norm, Fatou's Lemma and Lemma 2.5, we obtain

$$\begin{aligned} m_{b,\lambda} &= \lim_{n \rightarrow \infty} \left[\mathcal{J}_{b,\lambda}(u_n) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_n), u_n \rangle \right] \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{1}{4} \|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(u_n)u_n - F(u_n) \right] dx \right\} \\ &\geq \frac{1}{4} \liminf_{n \rightarrow \infty} \left[\int_{\mathbb{R}^3} a|\nabla u_n|^2 dx + \left(1 - \frac{\theta_0}{\lambda} \right) \int_{\mathbb{R}^3} \lambda V(x)u_n^2 dx \right] \end{aligned}$$

$$\begin{aligned}
& + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left\{ \left[\frac{1}{4} f(u_n) u_n - F(u_n) \right] + \frac{\theta_0}{4} V(x) u_n^2 \right\} dx \\
& \geq \frac{1}{4} \left[\int_{\mathbb{R}^3} |\nabla u_{b,\lambda}|^2 dx + \left(1 - \frac{\theta_0}{\lambda} \right) \int_{\mathbb{R}^3} \lambda V(x) u_{b,\lambda}^2 dx \right] \\
& \quad + \int_{\mathbb{R}^3} \left\{ \left[\frac{1}{4} f(u_{b,\lambda}) u_{b,\lambda} - F(u_{b,\lambda}) \right] + \frac{\theta_0}{4} V(x) |u_{b,\lambda}|^2 \right\} dx \\
& = \frac{1}{4} \|u_{b,\lambda}\|_\lambda^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(u_{b,\lambda}) u_{b,\lambda} - F(u_{b,\lambda}) \right] dx \\
& = \mathcal{J}_{b,\lambda}(u_{b,\lambda}) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda} \rangle \\
& \geq \sup_{s,t \geq 0} \left[\mathcal{J}_{b,\lambda}(s u_{b,\lambda}^+ + t u_{b,\lambda}^-) + \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda}^+ \rangle \right. \\
& \quad \left. + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda}^- \rangle \right] - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda} \rangle \\
& \geq \sup_{s,t \geq 0} \mathcal{J}_{b,\lambda}(s u_{b,\lambda}^+ + t u_{b,\lambda}^-) \geq m_{b,\lambda},
\end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} |\nabla u_{b,\lambda}|^2 dx$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(x) |u_n|^2 dx = \int_{\mathbb{R}^3} V(x) |u_{b,\lambda}|^2 dx.$$

Hence, $u_n \rightarrow u_{b,\lambda}$ in E , then $\mathcal{J}_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda}$ and $u_{b,\lambda} \in \mathcal{M}_{b,\lambda}$. \square

LEMMA 2.8. *Let (V'_1) and (F_1) – (F_4) be satisfied and $\lambda > \max\{a, \theta_0\}$. If $u_0 \in \mathcal{M}_{b,\lambda}$ and $\mathcal{J}_{b,\lambda}(u_0) = m_{b,\lambda}$, then u_0 is a critical point of $\mathcal{J}_{b,\lambda}$.*

PROOF. Let $u_0 = u_0^+ + u_0^- \in \mathcal{M}_{b,\lambda}$, $\mathcal{J}_{b,\lambda}(u_0) = m_{b,\lambda}$ and $\mathcal{J}'_{b,\lambda}(u_0) \neq 0$. Then there exist $\delta > 0$ and $\sigma > 0$ such that

$$u \in E, \|u - u_0\|_\lambda \leq 3\delta \Rightarrow \|\mathcal{J}_{b,\lambda}(u)\| \geq \sigma.$$

By Corollary 2.2, one has

$$\begin{aligned}
(2.13) \quad & \mathcal{J}_{b,\lambda}(s u_0^+ + t u_0^-) \leq \mathcal{J}_{b,\lambda}(u_0) \\
& - \frac{(1 - \theta_0/\lambda)(1 - s^2)^2}{4} \|u_0^+\|_\lambda^2 - \frac{(1 - \theta_0/\lambda)(1 - t^2)^2}{4} \|u_0^-\|_\lambda^2 \\
& = m_{b,\lambda} - \frac{(1 - \theta_0/\lambda)(1 - s^2)^2}{4} \|u_0^+\|_\lambda^2 - \frac{(1 - \theta_0/\lambda)(1 - t^2)^2}{4} \|u_0^-\|_\lambda^2.
\end{aligned}$$

Let $D = (0.5, 1.5) \times (0.5, 1.5)$. It follows from (2.13) that

$$\kappa := \max_{(s,t) \in \partial D} \mathcal{J}_{b,\lambda}(s u_0^+ + t u_0^-) < m_{b,\lambda}.$$

For $\varepsilon := \min\{(m_{b,\lambda} - \kappa)/3, 1, \sigma\delta/8\}$, $S := B(u_0, \delta)$, [40, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0, 1] \times E, E)$ such that

- (i) $\eta(1, u) = u$ if $u \notin \mathcal{J}_{b,\lambda}^{-1}([m_{b,\lambda} - 2\varepsilon, m_{b,\lambda} + 2\varepsilon]) \cap S_{2\delta}$;
- (ii) $\eta(1, \mathcal{J}_{b,\lambda}^{m_{b,\lambda} + \varepsilon} \cap B(u_0, \delta)) \subset \mathcal{J}_{b,\lambda}^{m_{b,\lambda} - \varepsilon}$;
- (iii) $\mathcal{J}_{b,\lambda}(\eta(1, u)) \leq \mathcal{J}_{b,\lambda}(u)$, for all $u \in E$.

By Corollary 2.3, $\mathcal{J}_{b,\lambda}(su_0^+ + tu_0^-) \leq \mathcal{J}_{b,\lambda}(u_0) = m_{b,\lambda}$ for $s, t \geq 0$, then it follows from (ii) that

$$(2.14) \quad \begin{aligned} \mathcal{J}_{b,\lambda}(\eta(1, su_0^+ + tu_0^-)) &\leq m_{b,\lambda} - \varepsilon, \quad \text{for all } s, t \geq 0, \\ |s - 1|^2 + |t - 1|^2 &< \delta^2 / \|u_0\|_\lambda^2. \end{aligned}$$

On the other hand, by (iii) and (2.13), for any $s, t \geq 0$, $|s - 1|^2 + |t - 1|^2 \geq \delta^2 / \|u_0\|_\lambda^2$, one has

$$(2.15) \quad \begin{aligned} \mathcal{J}_{b,\lambda}(\eta(1, su_0^+ + tu_0^-)) &\leq \mathcal{J}_{b,\lambda}(su_0^+ + tu_0^-) \\ &\leq m_{b,\lambda} - \frac{(1 - \theta_0/\lambda)(1 - s^2)^2}{4} \|u_0^+\|_\lambda^2 - \frac{(1 - \theta_0/\lambda)(1 - t^2)^2}{4} \|u_0^-\|_\lambda^2 \\ &\leq m_{b,\lambda} - \frac{(1 - \theta_0/\lambda)\delta^2}{8\|u_0\|_\lambda^2} \min\{\|u_0^+\|_\lambda^2, \|u_0^-\|_\lambda^2\}. \end{aligned}$$

Combining (2.14) with (2.15), we get $\max_{(s,t) \in \bar{D}} \mathcal{J}_{b,\lambda}(\eta(1, su_0^+ + tu_0^-)) < m_{b,\lambda}$. Moreover, $g(s, t) := su_0^+ + tu_0^-$. By an argument similar as [30, 31], we can get $\eta(1, g(D)) \cap \mathcal{M}_{b,\lambda} \neq \emptyset$. Since $m_{b,\lambda} := \inf_{u \in \mathcal{M}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u)$, this is a contradiction. \square

3. Sign-changing solutions

PROOF OF THEOREM 1.3. In view of Lemmas 2.7 and 2.8, there exists $u_{b,\lambda} \in \mathcal{M}_{b,\lambda}$ such that $\mathcal{J}_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda}$ and $\mathcal{J}'_{b,\lambda}(u_{b,\lambda}) = 0$. Thus $u_{b,\lambda}$ is a sign-changing solution of (1.1). Next, we prove that $u_{b,\lambda}$ has exactly two nodal domains. Let $u_{b,\lambda} = u_{1,\lambda} + u_{2,\lambda} + u_{3,\lambda}$, where

$$\begin{aligned} u_{1,\lambda} &\geq 0, \quad u_{2,\lambda} \leq 0, \quad \Omega_1 \cap \Omega_2 = \emptyset, \\ u_{1,\lambda}|_{\Omega_2 \cup \Omega_3} &= u_{2,\lambda}|_{\Omega_1 \cap \Omega_3} = u_{3,\lambda}|_{\Omega_1 \cap \Omega_2} = 0, \\ \Omega_1 &:= \{x \in \mathbb{R}^3 : u_{1,\lambda}(x) > 0\}, \quad \Omega_2 := \{x \in \mathbb{R}^3 : u_{2,\lambda}(x) < 0\}, \\ \Omega_3 &:= \mathbb{R}^3 \setminus (\Omega_1 \cup \Omega_2), \end{aligned}$$

and Ω_1, Ω_2 are connected open subsets of \mathbb{R}^3 .

Setting $v_\lambda = u_{1,\lambda} + u_{2,\lambda}$, we see that $v_\lambda^+ = u_{1,\lambda}$ and $v_\lambda^- = u_{2,\lambda}$, i.e. $v_\lambda^\pm \neq 0$. By (1.7), (1.8), (2.1) and (2.8), we have

$$\begin{aligned} m_{b,\lambda} &= \mathcal{J}_{b,\lambda}(u_{b,\lambda}) = \mathcal{J}_{b,\lambda}(u_{b,\lambda}) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_{b,\lambda}), u_{b,\lambda} \rangle \\ &= \mathcal{J}_{b,\lambda}(v_\lambda) + \mathcal{J}_{b,\lambda}(u_{3,\lambda}) + \frac{b}{2} \|\nabla v_\lambda\|_2^2 \|\nabla u_{3,\lambda}\|_2^2 \\ &\quad - \frac{1}{4} [\langle \mathcal{J}'_{b,\lambda}(v_\lambda), v_\lambda \rangle + \langle \mathcal{J}'_{b,\lambda}(u_{3,\lambda}), u_{3,\lambda} \rangle + 2b \|\nabla v_\lambda\|_2^2 \|\nabla u_{3,\lambda}\|_2^2] \end{aligned}$$

$$\begin{aligned}
&\geq \sup_{s,t \geq 0} \left[\mathcal{J}_{b,\lambda}(sv_\lambda^+ + tv_\lambda^-) + \frac{1-s^4}{4} \langle \mathcal{J}'_{b,\lambda}(v_\lambda), v_\lambda^+ \rangle + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(v_\lambda), v_\lambda^- \rangle \right] \\
&\quad - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(v_\lambda), v_\lambda \rangle + \mathcal{J}_{b,\lambda}(u_{3,\lambda}) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_{3,\lambda}), u_{3,\lambda} \rangle \\
&\geq \sup_{s,t \geq 0} \left[\mathcal{J}_{b,\lambda}(sv_\lambda^+ + tv_\lambda^-) + \frac{bs^4}{4} \|\nabla v_\lambda^+\|_2^2 \|\nabla u_{3,\lambda}\|_2^2 + \frac{bt^4}{4} \|\nabla v_\lambda^-\|_2^2 \|\nabla u_{3,\lambda}\|_2^2 \right] \\
&\quad + \frac{1}{4} \|u_{3,\lambda}\|_\lambda^2 + \int_{\mathbb{R}^3} \left[\frac{1}{4} f(u_{3,\lambda}) u_{3,\lambda} - F(u_{3,\lambda}) \right] dx,
\end{aligned}$$

which implies that

$$m_{b,\lambda} \geq \sup_{s,t \geq 0} \mathcal{J}_{b,\lambda}(sv_\lambda^+ + tv_\lambda^-) + \frac{(1-\theta_0/\lambda)}{4} \|u_{3,\lambda}\|_\lambda^2 \geq m_{b,\lambda} + \frac{(1-\theta_0/\lambda)}{4} \|u_{3,\lambda}\|_\lambda^2.$$

Thus $u_{3,\lambda} = 0$ due to $\theta_0 \in (0, 1)$ and $\lambda > \theta_0$. Therefore, $u_{b,\lambda}$ has exactly two nodal domains. \square

4. Nehari type of ground state solutions

In this section, we will use non-Nehari manifold method to seek ground state solutions of Nehari type for (1.1). Before stating our results, we want to give the following lemmas and corollaries, which can be proved in the same as Section 2.

LEMMA 4.1. *Suppose (F₁)–(F₄) are satisfied. Then*

$$\mathcal{J}_{b,\lambda}(u) \geq \mathcal{J}_{b,\lambda}(tu) + \frac{1-t^4}{4} \langle \mathcal{J}'_{b,\lambda}(u), u \rangle + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u\|_\lambda^2,$$

for all $u \in E$, $t \geq 0$.

COROLLARY 4.2. *Suppose (F₁)–(F₄) are satisfied. Then*

$$\mathcal{J}_{b,\lambda}(u) \geq \mathcal{J}_{b,\lambda}(tu) + \frac{(1-\theta_0/\lambda)(1-t^2)^2}{4} \|u\|_\lambda^2, \quad \text{for all } t \geq 0,$$

for any $u \in \mathcal{N}_{b,\lambda}$.

COROLLARY 4.3. *Suppose (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then, for any $u \in \mathcal{N}_{b,\lambda}$,*

$$\mathcal{J}_{b,\lambda}(u) = \max_{t \geq 0} \mathcal{J}_{b,\lambda}(tu).$$

LEMMA 4.4. *Suppose (V'₁) and (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then, for any $u \in E \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_{b,\lambda}$.*

LEMMA 4.5. *Suppose (V'₁) and (F₁)–(F₄) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then*

$$\inf_{u \in \mathcal{N}_{b,\lambda}} \mathcal{J}_{b,\lambda}(u) = c_{b,\lambda} = \inf_{u \in E, u \neq 0} \max_{t \geq 0} \mathcal{J}_{b,\lambda}(tu).$$

Similarly to $m_{b,\lambda} > 0$, we can also prove $c_{b,\lambda} > 0$. Then we can get the following lemma.

LEMMA 4.6. *Suppose (V'_1) and (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then there exist a constant $c_\lambda^* \in (0, c_{b,\lambda}]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$(4.1) \quad \mathcal{J}_{b,\lambda}(u_n) \rightarrow c_\lambda^*, \quad \|\mathcal{J}'_{b,\lambda}(u_n)\|_\lambda(1 + \|u_n\|_\lambda) \rightarrow 0.$$

PROOF. Since (F_1) , (F_2) and (1.8) hold, there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$u \in E, \quad \|u\|_\lambda = \delta_0 \Rightarrow \mathcal{J}_{b,\lambda}(u) \geq \rho_0.$$

Choose $v_k \in \mathcal{N}_{b,\lambda}$ such that

$$(4.2) \quad m_{b,\lambda} \leq \mathcal{J}_{b,\lambda}(v_k) < m_{b,\lambda} + \frac{1}{k}, \quad k \in \mathbb{N}.$$

Since $\mathcal{J}_{b,\lambda}(tv_k) < 0$ for large $t > 0$, then according to [22] and the Mountain Pass Lemma, we can derive that there exists a sequence $\{u_{k,n}\}_{n \in \mathbb{N}} \subset E$ satisfying

$$(4.3) \quad \mathcal{J}_{b,\lambda}(u_{k,n}) \rightarrow c_k, \quad \|\mathcal{J}'_{b,\lambda}(u_{k,n})\|_\lambda(1 + \|u_{k,n}\|_\lambda) \rightarrow 0, \quad k \in \mathbb{N},$$

where $c_k \in [\rho_0, \sup_{t \geq 0} \mathcal{J}_{b,\lambda}(tv_k)]$. By virtue of Corollary 4.2, one has

$$\mathcal{J}_{b,\lambda}(v_k) \geq \mathcal{J}_{b,\lambda}(tv_k), \quad \text{for all } t \geq 0,$$

which implies $\mathcal{J}_{b,\lambda}(v_k) = \sup_{t \geq 0} \mathcal{J}_{b,\lambda}(tv_k)$. Hence, by (4.2) and (4.3), we have

$$\mathcal{J}_{b,\lambda}(u_{k,n}) < c_{b,\lambda} + \frac{1}{k}, \quad \|\mathcal{J}'_{b,\lambda}(u_{k,n})\|_\lambda(1 + \|u_{k,n}\|_\lambda) \rightarrow 0, \quad k \in \mathbb{N}.$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\mathcal{J}_{b,\lambda}(u_{k,n_k}) < c_{b,\lambda} + \frac{1}{k}, \quad \|\mathcal{J}'_{b,\lambda}(u_{k,n_k})\|_\lambda(1 + \|u_{k,n_k}\|_\lambda) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Let $u_k = u_{k,n_k}$, where $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\mathcal{J}_{b,\lambda}(u_n) \rightarrow c_\lambda^* \in [\rho_0, c_{b,\lambda}], \quad \|\mathcal{J}'_{b,\lambda}(u_n)\|_\lambda(1 + \|u_n\|_\lambda) \rightarrow 0. \quad \square$$

PROOF OF THEOREM 1.4. By Lemma 4.6, we can deduce that there exists a sequence $\{u_n\} \subset E$ satisfying (4.1) such that

$$(4.4) \quad \mathcal{J}_{b,\lambda}(u_n) \rightarrow c_\lambda^*, \quad \langle \mathcal{J}'_{b,\lambda}(u_n), u_n \rangle \rightarrow 0.$$

From (1.7), (1.8), (2.8) and (4.4), one has for large $n \in \mathbb{N}$

$$1 + c_\lambda^* \geq \mathcal{J}_{b,\lambda}(u_n) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_n), u_n \rangle \geq \frac{1 - \theta_0/\lambda}{4} \|u_n\|_\lambda^2.$$

This implies that $\{u_n\}$ is bounded in E . By a standard argument, we can prove that there exists $u_{0,\lambda} \in E \setminus \{0\}$ such that $\mathcal{J}'_{b,\lambda}(u_{0,\lambda}) = 0$. This shows that $u_{0,\lambda} \in \mathcal{N}_{b,\lambda}$ is a nontrivial solution of (1.1) and $\mathcal{J}_{b,\lambda}(u_{0,\lambda}) \geq c_{b,\lambda}$. On the other

hand, by using (1.7), (1.8), (2.8), the weak semicontinuity of norm and Fatou's Lemma, we have

$$\begin{aligned}
c_{b,\lambda} &\geq c_\lambda^* = \lim_{n \rightarrow \infty} \left(\mathcal{J}_{b,\lambda}(u_n) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_n), u_n \rangle \right) \\
&= \lim_{n \rightarrow \infty} \left[\frac{1}{4} \|u_n\|_\lambda^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u_n)u_n - F(u_n) \right) \right] \\
&\geq \frac{1}{4} \liminf_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} a |\nabla u_n|^2 dx + (1 - \theta_0/\lambda) \int_{\mathbb{R}^3} \lambda V(x) u_n^2 dx \right) \\
&\quad + \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left\{ \left(\frac{1}{4} f(u_n)u_n - F(u_n) \right) + \frac{\theta_0 V(x)}{4} u_n^2 \right\} dx \\
&\geq \frac{1}{4} \left(\int_{\mathbb{R}^3} a |\nabla u_{0,\lambda}|^2 dx + (1 - \theta_0/\lambda) \int_{\mathbb{R}^3} \lambda V(x) u_{0,\lambda}^2 dx \right) \\
&\quad + \int_{\mathbb{R}^3} \left\{ \left(\frac{1}{4} f(u_{0,\lambda})u_{0,\lambda} - F(u_{0,\lambda}) \right) + \frac{\theta_0 V(x)}{4} u_{0,\lambda}^2 \right\} dx \\
&= \frac{1}{4} \|u_{0,\lambda}\|_\lambda^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u_{0,\lambda})u_{0,\lambda} - F(u_{0,\lambda}) \right) dx \\
&= \mathcal{J}_{b,\lambda}(u_{0,\lambda}) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda}(u_{0,\lambda}), u_{0,\lambda} \rangle = \mathcal{J}_{b,\lambda}(u_{0,\lambda}).
\end{aligned}$$

Hence, we have $\mathcal{J}_{b,\lambda}(u_{0,\lambda}) \leq c_\lambda^*$ and so $\mathcal{J}_{b,\lambda}(u_{0,\lambda}) = c_{b,\lambda} = \inf_{\mathcal{N}_{b,\lambda}} \mathcal{J}_{b,\lambda} > 0$.

By virtues of Theorem 1.1, there exists $u_{b,\lambda} \in \mathcal{M}_{b,\lambda}$ such that $\mathcal{J}_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda}$. Thus, by (1.7), Corollary 2.3 and Lemma 4.5, one has

$$\begin{aligned}
m_{b,\lambda} &= \mathcal{J}_{b,\lambda}(u_{b,\lambda}) = \sup_{s,t \geq 0} \mathcal{J}_{b,\lambda}(su_{b,\lambda}^+ + tu_{b,\lambda}^-) \\
&= \sup_{s,t \geq 0} \left[\mathcal{J}(su_{b,\lambda}^+) + \mathcal{J}(tu_{b,\lambda}^-) + \frac{bs^2t^2}{2} \|\nabla u_{b,\lambda}^+\|_2^2 \|\nabla u_{b,\lambda}^-\|_2^2 \right] \\
&> \sup_{s \geq 0} \mathcal{J}(su_{b,\lambda}^+) + \sup_{t \geq 0} \mathcal{J}(tu_{b,\lambda}^-) \geq 2c_{b,\lambda}.
\end{aligned}$$

This completes the proof. \square

5. The convergence property

In this section, we will give the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. In Section 2, $b = 0$ is allowed in the argument. Therefore, under the assumptions of Theorem 1.3, there exists $v_0 \in \mathcal{M}_{0,\lambda}$ such that

$$\mathcal{J}'_{0,\lambda}(v_0) = 0 \quad \text{and} \quad \mathcal{J}_{0,\lambda}(v_0) = m_{0,\lambda} = \inf_{u \in \mathcal{M}_{0,\lambda}} \mathcal{J}_{0,\lambda}(u),$$

that is, (1.4) has at least energy sign-changing solution, which changes sign only once.

For any $b > 0$, let $u_{b,\lambda} \in \mathcal{M}_{b,\lambda}$ be a sign-changing solution of (1.1) obtained in Theorem 1.3, which changes sign only once and satisfies $\mathcal{J}_{b,\lambda}(u_{b,\lambda}) = m_{b,\lambda}$.

Choose $\omega_0 \in C_0^\infty(\mathbb{R}^3)$ such that $\omega_0^\pm \neq 0$. From (F₁)–(F₃), there exist $\beta_1 > 0$ and $\beta_2 \geq \max\{\|\nabla\omega_0^+\|_2^4, \|\nabla\omega_0^-\|_2^4\}$ such that

$$\int_{\mathbb{R}^3} F(s\omega_0^+) dx \geq \beta_2|s|^4 - \beta_1, \quad \int_{\mathbb{R}^3} F(t\omega_0^-) dx \geq \beta_2|t|^4 - \beta_1,$$

for all $s, t \in \mathbb{R}$. For any $b \in [0, 1]$, it follows from (1.7) and Lemma 2.5 that

$$\begin{aligned} \mathcal{J}_{b,\lambda}(u_{b,\lambda}) &= m_b \leq \max_{s,t \geq 0} \mathcal{J}_{b,\lambda}(s\omega_0^+ + t\omega_0^-) \\ &= \max_{s,t \geq 0} \left\{ \frac{s^2}{2} \|\omega_0^+\|_\lambda^2 + \frac{bs^4}{4} \|\nabla\omega_0^+\|_2^4 - \int_{\mathbb{R}^3} F(s\omega_0^+) dx \right. \\ &\quad \left. + \frac{t^2}{2} \|\omega_0^-\|_\lambda^2 + \frac{bt^4}{4} \|\nabla\omega_0^-\|_2^4 - \int_{\mathbb{R}^3} F(t\omega_0^-) dx + \frac{bs^2t^2}{2} \|\nabla\omega_0^+\|_2^2 \|\nabla\omega_0^-\|_2^2 \right\} \\ &\leq \max_{s,t \geq 0} \left\{ \frac{s^2}{2} \|\omega_0^+\|_\lambda^2 + \frac{bs^4}{2} \|\nabla\omega_0^+\|_2^4 + 2\beta_1 - \beta_2s^4 + \frac{t^2}{2} \|\omega_0^-\|_\lambda^2 \right. \\ &\quad \left. + \frac{bt^4}{4} \|\nabla\omega_0^-\|_2^4 - \beta_2t^4 + \frac{bs^2t^2}{2} \|\nabla\omega_0^+\|_2^2 \|\nabla\omega_0^-\|_2^2 \right\} \\ &\leq \max_{s \geq 0} \left[\frac{s^2}{2} \|\omega_0^+\|_\lambda^2 - \frac{s^4}{2} \|\nabla\omega_0^+\|_2^4 \right] \\ &\quad + \max_{t \geq 0} \left[\frac{t^2}{2} \|\omega_0^-\|_\lambda^2 - \frac{t^4}{2} \|\nabla\omega_0^-\|_2^4 \right] + 2\beta_1 := \Lambda_0 > 0. \end{aligned}$$

By (1.7), (1.8) and (2.8), we get

$$\Lambda_0 + 1 \geq \mathcal{J}_{b_n,\lambda}(u_{b_n,\lambda}) - \frac{1}{4} \langle \mathcal{J}'_{b_n}(u_{b_n,\lambda}), u_{b_n,\lambda} \rangle \geq \frac{(1 - \theta_0/\lambda)}{4} \|u_{b_n,\lambda}\|_\lambda^2,$$

which implies that $\{u_{b_n,\lambda}\}$ is bounded in E due to $0 < \theta_0 < 1$ and $\lambda > \theta_0$. Hence there exists a subsequence of $\{b_n\}$, still denoted by $\{b_n\}$ and $u_{0,\lambda} \in E$ such that $u_{b_n,\lambda} \rightharpoonup u_{0,\lambda}$ in E . Similarly to Lemma 2.6, we conclude that $u_{b_n,\lambda}^\pm \rightarrow u_{0,\lambda}^\pm \neq 0$ in E . Note that

$$\begin{aligned} \langle \mathcal{J}'_{0,\lambda}(u_{0,\lambda}), \varphi \rangle &= \int_{\mathbb{R}^3} (\nabla u_{0,\lambda} \cdot \nabla \varphi + V(x)u_{0,\lambda}\varphi) dx - \int_{\mathbb{R}^3} f(u_{0,\lambda})\varphi dx \\ &= \lim_{n \rightarrow \infty} \left[\int_{\mathbb{R}^3} (\nabla u_{b_n,\lambda} \cdot \nabla \varphi + V(x)u_{b_n,\lambda}\varphi) dx - \int_{\mathbb{R}^3} f(u_{b_n,\lambda})\varphi dx \right] \\ &= \lim_{n \rightarrow \infty} \langle \mathcal{J}'_{b_n,\lambda}(u_{b_n,\lambda}), \varphi \rangle = 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. This shows that $\mathcal{J}'_{0,\lambda}(u_{0,\lambda}) = 0$, and then $u_{0,\lambda} \in \mathcal{M}_{0,\lambda}$ and $\mathcal{J}_{0,\lambda}(u_{0,\lambda}) \geq m_{0,\lambda}$.

Next, we prove that $\mathcal{J}_{0,\lambda}(u_{0,\lambda}) = m_{0,\lambda}$. Let $b_n \in [0, 1]$. Then it follows from (F₃) that there exists $K_0 > 0$ such that

$$\begin{aligned}
(5.1) \quad & \mathcal{J}_{b_n,\lambda}(sv_0^+ + tv_0^-) \\
&= \frac{s^2}{2} \|v_0^+\|_\lambda^2 + \frac{b_n s^4}{4} \|\nabla v_0^+\|_2^4 - \int_{\mathbb{R}^3} F(x, sv_0^+) dx + \frac{t^2}{2} \|v_0^-\|_\lambda^2 \\
&\quad + \frac{b_n t^4}{4} \|\nabla v_0^-\|_2^4 - \int_{\mathbb{R}^3} F(tv_0^-) dx + \frac{b_n s^2 t^2}{2} \|\nabla v_0^+\|_2^2 \|\nabla v_0^-\|_2^2 \\
&\leq \frac{s^2}{2} \|v_0^+\|_\lambda^2 + \frac{b_n s^4}{4} \|\nabla v_0^+\|_2^4 - \int_{\mathbb{R}^3} F(sv_0^+) dx \\
&\quad + \frac{t^2}{2} \|v_0^-\|_\lambda^2 + \frac{b_n t^4}{4} \|\nabla v_0^-\|_2^4 - \int_{\mathbb{R}^3} F(tv_0^-) dx < 0,
\end{aligned}$$

for all $s + t \geq K_0$. In view of Lemma 2.4, there exists (s_n, t_n) such that $s_n v_0^+ + t_n v_0^- \in \mathcal{M}_{b_n,\lambda}$, which, together with (5.1), implies $0 < s_n, t_n < K_0$. Hence, from (1.7), (1.8) and (2.1), we have

$$\begin{aligned}
m_{0,\lambda} = \mathcal{J}_{0,\lambda}(v_0) &= \mathcal{J}_{b_n,\lambda}(v_0) - \frac{b_n}{4} \|\nabla v_0\|_2^4 \\
&\geq \mathcal{J}_{b_n,\lambda}(s_n v_0^+ + t_n v_0^-) + \frac{1 - s_n^4}{4} \langle \mathcal{J}'_{b_n,\lambda}(v_0), v_0^+ \rangle \\
&\quad + \frac{1 - t_n^4}{4} \langle \mathcal{J}'_{b_n,\lambda}(v_0), v_0^- \rangle - \frac{b_n}{4} \|\nabla v_0\|_2^4 \\
&\geq m_{b_n,\lambda} - \frac{1 + K_0^4}{4} |\langle \mathcal{J}'_{b_n,\lambda}(v_0), v_0^+ \rangle| - \frac{1 + K_0^4}{4} |\langle \mathcal{J}'_{b_n,\lambda}(v_0), v_0^- \rangle| - \frac{b_n}{4} \|\nabla v_0\|_2^4 \\
&= m_{b_n,\lambda} - \frac{(1 + K_0^4)b_n}{4} \|\nabla v_0\|_2^2 \|\nabla v_0^+\|_2^2 - \frac{(1 + K_0^4)b_n}{4} \|\nabla v_0\|_2^2 \|\nabla v_0^-\|_2^2 - \frac{b_n}{4} \|\nabla v_0\|_2^4,
\end{aligned}$$

which implies that

$$(5.2) \quad \limsup_{n \rightarrow \infty} m_{b_n,\lambda} \leq m_{0,\lambda}.$$

By (1.7) and (5.2), one has

$$m_{0,\lambda} \leq \mathcal{J}_{0,\lambda}(u_{0,\lambda}) = \limsup_{n \rightarrow \infty} \mathcal{J}_{b_n,\lambda}(u_{b_n,\lambda}) = \limsup_{n \rightarrow \infty} m_{b_n,\lambda} \leq m_{0,\lambda}.$$

This shows that $\mathcal{J}_{0,\lambda}(u_{0,\lambda}) = m_{0,\lambda}$. \square

6. Concentration of ground state sign-changing solutions

In this section, we will give the proof of Theorem 1.6. Now, let us define a manifold

$$\mathcal{M}_{b,0} = \{u \in H_0^1(\Omega) : u^\pm \neq 0, \langle \mathcal{J}'_{b,0}(u), u^+ \rangle = \langle \mathcal{J}'_{b,0}(u), u^- \rangle = 0\}$$

and

$$m_{b,0} = \inf_{\mathcal{M}_{b,0}} \mathcal{J}_{b,0}.$$

LEMMA 6.1. *Suppose (V'_1) , (F_1) – (F_4) are satisfied and $\lambda > \max\{a, \theta_0\}$. Then $m_{b,\lambda} \leq m_{b,0}$.*

PROOF. Let any fixed $\eta \in \mathcal{M}_{b,0}$, then $\zeta := \eta\chi_\Omega \in E$, where

$$\chi_\Omega = \begin{cases} 1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \Omega. \end{cases}$$

It follows that $\zeta \in \mathcal{M}_{b,\lambda}$ for all $\lambda > 0$, and

$$\begin{aligned} & \mathcal{J}_{b,\lambda}(s\zeta^+ + t\zeta^-) \\ &= \frac{1}{2} \|s\zeta^+ + t\zeta^-\|_\lambda^2 + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla(s\zeta^+ + t\zeta^-)|^2 dx \right)^2 - \int_{\mathbb{R}^3} F(s\zeta^+ + t\zeta^-) dx \\ &= \frac{1}{2} s^2 \|\zeta^+\|_\lambda^2 + \frac{b}{4} s^4 \left(\int_\Omega |\nabla\zeta^+|^2 dx \right)^2 \\ & \quad - \int_\Omega F(s\zeta^+) dx + \frac{bs^2t^2}{2} \int_\Omega |\nabla\zeta^+|^2 dx \int_\Omega |\nabla\zeta^-|^2 dx \\ & \quad + \frac{1}{2} t^2 \|\zeta^-\|_\lambda^2 + \frac{b}{4} t^4 \left(\int_\Omega |\nabla\zeta^-|^2 dx \right)^2 - \int_\Omega F(t\zeta^-) dx = \mathcal{J}_{b,0}(s\zeta^+ + t\zeta^-) \end{aligned}$$

for all $s, t \geq 0$. Thus $m_{b,\lambda} \leq \mathcal{J}_{b,\lambda}(\zeta) = \mathcal{J}_{b,0}(\zeta)$. According to the arbitrariness of $\zeta \in \mathcal{M}_{b,0}$, we have that $m_{b,\lambda} \leq m_{b,0}$, where $m_{b,0}$ is independent of $\lambda \in (\theta_0, \infty)$. \square

PROOF OF THEOREM 1.6 (concentration). By the existence of ground state sign-changing solutions to (1.1), for any sequence $\{\lambda_n\} \subset (\max\{a, \theta_0\}, \infty)$ with $\lambda_n \rightarrow \infty$, there exists a critical point sequence $\{u_{b,\lambda_n}\}$ denoted by $u_n := u_{b,\lambda_n}$ of $\mathcal{J}_{b,\lambda_n}$ satisfying $\mathcal{J}_{b,\lambda_n}(u_n) = m_{b,\lambda_n}$ and $\mathcal{J}'_{b,\lambda_n}(u_n) = 0$ with $u_n^\pm \neq 0$, where u_n is the corresponding ground state sign-changing solution. We have that

$$\begin{aligned} m_{b,0} &\geq \mathcal{J}_{b,\lambda_n}(u_n) - \frac{1}{4} \langle \mathcal{J}'_{b,\lambda_n}(u_n), u_n \rangle \\ &= \frac{1 - \theta_0/\lambda_n}{4} \|u_n\|_{\lambda_n}^2 + \int_{\mathbb{R}^3} \left\{ \left[\frac{1}{4} f(u_n)u_n - F(u_n) \right] + \frac{\theta_0 V(x)}{4} u_n^2 \right\} dx \\ &\geq \frac{1 - \theta_0/\lambda_n}{4} \|u_n\|_{\lambda_n}^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded uniformly, that is,

$$(6.1) \quad \sup_{n \geq 1} \|u_n\|_{\lambda_n}^2 \leq \frac{4}{1 - \theta_0/\lambda_n} m_{b,0}.$$

It follows that $\{u_n\}$ is bounded in E due to $0 < \theta_0 < 1$ and $\lambda_n > \theta_0$. Therefore, up to a subsequence, there is $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in E . By the compactness of Sobolev embedding $E \hookrightarrow L^r(\mathbb{R}^3)$ for $r \in [2, 6)$, we get that $u_n \rightarrow u_0$ in $L^r(\mathbb{R}^3)$ for all $r \in [2, 6)$. Up to a subsequence, we may assume that $u_n(x) \rightarrow u_0(x)$

almost everywhere on $x \in \mathbb{R}^3$. Since $V(x) \geq 0$, it follows from Fatou's Lemma and (6.1) that

$$\int_{\mathbb{R}^3} V u_0^2 dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} V u_n^2 dx \leq \liminf_{n \rightarrow \infty} \frac{\|u_n\|_{\lambda_n}^2}{\lambda_n} = 0.$$

By condition (V_1') , we deduce that $u_0(x) = 0$, almost everywhere in $\mathbb{R}^3 \setminus V^{-1}(0)$ and $u_0 \in H_0^1(\Omega)$. It follows from $\mathcal{J}'_{b,\lambda_n}(u_n) = 0$ that

$$\int_{\Omega} a \nabla u_n \cdot \nabla \psi dx + b \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \int_{\Omega} \nabla u_n \cdot \nabla \psi dx - \int_{\mathbb{R}^3} f(x, u_n) \psi dx = 0,$$

for $\psi \in H_0^1(\Omega)$. In order to prove that u_0 is a ground state sign-changing solution of the limit system, it is sufficient to show $u_0^\pm \neq 0$ and

$$(6.2) \quad \int_{\Omega} a \nabla u_0 \cdot \nabla \psi dx + b \int_{\mathbb{R}^3} |\nabla u_0|^2 dx \int_{\Omega} \nabla u_0 \cdot \nabla \psi dx - \int_{\mathbb{R}^3} f(x, u_0) \psi dx = 0,$$

for $\psi \in H_0^1(\Omega)$. First of all, we prove (6.2). Going if necessary to a subsequence, we may assume that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = A^2$$

exists. It follows that

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 dx \leq A^2.$$

Applying Lemma A.2 in [40] and $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, we can get

$$(6.3) \quad \int_{\Omega} a \nabla u_0 \cdot \nabla \psi dx + b A^2 \int_{\Omega} \nabla u_0 \cdot \nabla \psi dx - \int_{\mathbb{R}^3} f(x, u_0) \psi dx = 0,$$

for $\psi \in H_0^1(\Omega)$. From (6.2) and (6.3), it is sufficient to prove that

$$A^2 = \int_{\mathbb{R}^3} |\nabla u_0|^2 dx = \int_{\Omega} |\nabla u_0|^2 dx.$$

Since $\langle \mathcal{J}'_{b,\lambda_n}(u_n), u_n \rangle = 0$, then

$$(6.4) \quad \int_{\mathbb{R}^3} (a |\nabla u_n|^2 + \lambda_n V(x) u_n^2) dx + b \left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 - \int_{\mathbb{R}^3} f(x, u_n) u_n dx = 0.$$

By (6.4) and letting $n \rightarrow \infty$, we know that

$$(6.5) \quad a A^2 + b A^4 + \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 dx - \int_{\mathbb{R}^3} f(x, u_0) u_0 dx = 0.$$

Take $\psi = u_0$ in (6.3), we can get

$$(6.6) \quad a \int_{\mathbb{R}^3} |\nabla u_0|^2 dx + b A^2 \int_{\mathbb{R}^3} |\nabla u_0|^2 dx - \int_{\mathbb{R}^3} f(x, u_0) u_0 dx = 0.$$

It follows from (6.5) and (6.6) that

$$\int_{\mathbb{R}^3} |\nabla u_0|^2 dx = A^2 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \lambda_n V(x) u_n^2 dx = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(x) u_n^2 dx = 0.$$

Thus we obtain that $\|u_n\|^2 \rightarrow \|u_0\|^2$ and $u_n \rightarrow u_0$ in E and also in $H^1(\mathbb{R}^3)$.

Finally, we prove that $u_0^\pm \neq 0$. Since $u_n \in \mathcal{M}_{b, \lambda_n}$, it follows from the proof of Lemma 2.7 that $0 < \alpha \leq \|u_n\|_{\lambda_n}^2$, where α is independent of n since $\lambda_n \in (\max\{a, \theta_0\}, \infty)$ for n large enough. Moreover, by $u_n \rightarrow u_0$ in E , we have that $u_n^\pm \rightharpoonup u_0^\pm$. Thus, this implies that

$$0 < \alpha \leq \int_{\mathbb{R}^3} f(x, u_0^\pm) u_0^\pm dx$$

and thus $u_0^\pm \neq 0$. This completes the proof. \square

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