Topological Methods in Nonlinear Analysis
Volume 52, No. 2, 2018, 665–675
DOI: 10.12775/TMNA.2017.060

NONZERO POSITIVE SOLUTIONS OF A MULTI-PARAMETER ELLIPTIC SYSTEM WITH FUNCTIONAL BCS

Gennaro Infante

This paper is dedicated to the memory of the late Panagiotis K. Palamides, for his teachings and friendship

Abstract. We prove, by topological methods, new results on the existence of nonzero positive weak solutions for a class of multi-parameter second order elliptic systems subject to functional boundary conditions. The setting is fairly general and covers the case of multi-point, integral and nonlinear boundary conditions. We also present a non-existence result. We provide some examples to illustrate the applicability of our theoretical results.

1. Introduction

In this paper we discuss the solvability of the multi-parameter system of second order elliptic equations subject to functional boundary conditions (BCs)

$$\begin{cases} L_i u_i(x) = \lambda_i f_i(x, u(x)), & x \in \Omega, \quad i = 1, \ldots, n, \\ B_i u_i(x) = \eta_i h_i[u], & x \in \partial\Omega, \ \ i = 1, \ldots, n, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^m$ ($m \geq 2$) is a bounded domain with sufficiently regular boundary, L_i is a strongly uniformly elliptic operator, B_i is a first order boundary operator,

2010 *Mathematics Subject Classification.* Primary: 35J47; Secondary: 35B09, 35J57, 35J60, 47H10.

Key words and phrases. Positive solution; elliptic system; functional boundary condition; cone; fixed point index.

G. Infante was partially supported by G.N.A.M.P.A. – INdAM (Italy).

$u = (u_1, \dots, u_n)$, f_i is a continuous function, h_i is a suitable compact functional, λ_i, η_i are parameters.

A motivation for studying this kind of boundary value problems (BVPs) is that they often occur in physical applications. In order to illustrate this fact, take $n = 1$, $m = 2$ and consider the BVP

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & \|x\|_2 < 1, \\ u(x) = \eta u(0), & \|x\|_2 = 1, \end{cases} \tag{1.2}$$

where $\|\cdot\|_2$ is the Euclidean norm. The BVP (1.2) can be used as a model for the steady-states of the temperature of a heated disk of radius 1, where a controller located in the border of the disk adds or removes heat in manner proportional to the temperature registered by a sensor located in the center of the disk. In the context of ODEs, a good reference for this kind of thermostat problems is the recent paper [25].

The assumptions we make on the functionals h_i that occur in (1.1) are fairly weak and allow to cover, *for example*, the special cases of *multi-point* BCs of the form

$$h_i[u] = \sum_{k=1}^{n} \sum_{j=1}^{N} \widehat{\alpha}_{ijk} u_k(\omega_j), \tag{1.3}$$

where $\widehat{\alpha}_{ijk}$ are non-negative coefficients and $\omega_j \in \Omega$, or *integral* BCs of the type

$$h_i[u] = \sum_{k=1}^{n} \int_{\Omega} \widehat{\alpha}_{ik}(\omega) u_k(\omega)\, d\omega, \tag{1.4}$$

where $\widehat{\alpha}_{ik}$ are non-negative continuous functions on $\overline{\Omega}$. Note that the functionals h_i in (1.3) and (1.4) allow an interaction between the components of the solution.

There exists a wide literature on multi-point, integral and, more in general, nonlocal BCs. As far as we know multi-point BCs have been studied firstly by Picone [23] in the context of ODEs. For an introduction to nonlocal BCs, we refer the reader to the reviews [6], [17], [20], [24], [27] and the papers [13], [14], [22], [26].

Note that our approach is not restricted to *linear* functionals like (1.3) and (1.4), we may also deal with the case of *nonlinear* BCs. These type of BCs also make physical sense; for example the BVP (1.2) might be modified in order to take into account a nonlinear response of the controller, by having a nonlinear, nonlocal BC of the form

$$u(x) = \widehat{h}(u(0)), \quad \|x\|_2 = 1, \tag{1.5}$$

where $\widehat{h}$ is a continuous function. In the context of radial solutions of PDEs on annular domains, conditions similar to (1.5) have been investigated recently in [4], [7]–[10]. We stress that nonlinear BCs have been widely studied for different classes of differential equations, nonlinearities and domains, we refer the reader to [2]–[4], [11], [18], [19], [21], [29] and references therein; in particular, the method of upper and lower solutions has been employed for the system (1.1) in the case of *non-homogeneus* (not necessarily constant) BCs in [2] and in the case of nonlinear BCs (where $\lambda_i = \eta_i = 1$) in [18], [21].

We highlight that the existence of positive solutions of the system (1.1) with *homogeneous* BCs has been recently discussed in [15], [16] (in the *sublinear* case) and in [5] (under monotonicity assumptions on the nonlinearities). Our theory can be applied also in this case, by considering $h_i[u] \equiv 0$. We do not assume global restrictions on the growth nor we assume monotonicity of the nonlinearities, thus complementing the results in [5], [15], [16].

We prove, by means of classical fixed point index, the existence of one nontrivial weak solution of the system (1.1). We also prove, via an elementary argument, a non-existence result. We provide some examples in order to illustrate the applicability of our theoretical results.

2. Existence and non-existence results

In what follows, for every $\widehat{\mu} \in (0,1)$ we denote by $C^{\widehat{\mu}}(\overline{\Omega})$ the space of all $\widehat{\mu}$-Hölder continuous functions $g\colon \overline{\Omega} \to \mathbb{R}$ and, for every $k \in \mathbb{N}$, we denote by $C^{k+\widehat{\mu}}(\overline{\Omega})$ the space of all functions $g \in C^k(\overline{\Omega})$ such that all the partial derivatives of g of order k are $\widehat{\mu}$-Hölder continuous in $\overline{\Omega}$ (for more details see [2, Examples 1.13 and 1.14]). We make the following assumptions on the domain Ω and the operators L_i and B_i that occur in (1.1) (see [2, Section 4 of Chapter 1] and [15], [16]):

(1) $\Omega \subset \mathbb{R}^m$, $m \geq 2$, is a bounded domain such that its boundary $\partial\Omega$ is an $(m-1)$-dimensional $C^{2+\widehat{\mu}}$−manifold for some $\widehat{\mu} \in (0,1)$, such that Ω lies locally on one side of $\partial\Omega$ (see [28, Section 6.2] for more details).

(2) L_i is a the second order elliptic operator given by

$$L_i u(x) = -\sum_{j,l=1}^{m} a_{ijl}(x)\,\frac{\partial^2 u}{\partial x_j \partial x_l}(x) + \sum_{j=1}^{m} a_{ij}(x)\,\frac{\partial u}{\partial x_j}(x) + a_i(x)u(x),$$

for $x \in \Omega$, where $a_{ijl}, a_{ij}, a_i \in C^{\widehat{\mu}}(\overline{\Omega})$ for $j,l = 1,\dots,m$, $a_i(x) \geq 0$ on $\overline{\Omega}$, $a_{ijl}(x) = a_{ijl}(x)$ on $\overline{\Omega}$ for $j,l = 1,\dots,m$. Moreover L_i is strongly uniformly elliptic, that is, there exists $\overline{\mu}_{i0} > 0$ such that

$$\sum_{j,l=1}^{m} a_{ijl}(x)\xi_j\xi_l \geq \overline{\mu}_{i0}\|\xi\|^2, \quad \text{for } x \in \Omega \text{ and } \xi = (\xi_1,\xi_2,\dots,\xi_m) \in \mathbb{R}^m.$$

(3) B_i is a boundary operator given by

$$B_i u(x) = b_i(x)u(x) + \delta_i\,\frac{\partial u}{\partial \nu}(x), \quad \text{for } x \in \partial\Omega,$$

where ν is an outward pointing and nowhere tangent vector field on $\partial\Omega$ of class $C^{1+\widehat{\mu}}$ (not necessarily a unit vector field), $\partial u/\partial\nu$ is the directional derivative of u with respect to ν, $b_i\colon \partial\Omega \to \mathbb{R}$ is of class $C^{1+\widehat{\mu}}$ and moreover, one of the following conditions holds:

(a) $\delta_i = 0$ and $b_i(x) \equiv 1$ (Dirichlet boundary operator).

(b) $\delta_i = 1$, $b_i(x) \equiv 0$ and $a_i(x) \not\equiv 0$ (Neumann boundary operator).

(c) $\delta_i = 1$, $b_i(x) \geq 0$ and $b_i(x) \not\equiv 0$ (Regular oblique derivative boundary operator).

It is known (see [2, Section 4]) that, under the previous conditions, a strong maximum principle holds and, furthermore, given $g \in C^{\widehat{\mu}}(\overline{\Omega})$, the boundary value problem

$$\begin{cases} L_i u(x) = g(x), & x \in \Omega, \\ B_i u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

admits a unique classical solution $u \in C^{2+\widehat{\mu}}(\overline{\Omega})$.

In order to seek solutions of the system (1.1), we work in a suitable cone of positive functions. We recall that a *cone* P of a real Banach space X is a closed set with $P + P \subset P$, $\lambda P \subset P$ for all $\lambda \geq 0$ and $P \cap (-P) = \{0\}$. A cone P induces a partial ordering in X by means of the relation

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

The cone P is *normal* if there exists $d > 0$ such that for all $x, y \in X$ with $0 \leq x \leq y$, $\|x\| \leq d\|y\|$. Note that every (closed) cone P has the *Archimedean property*, that is, $nx \leq y$ for all $n \in \mathbb{N}$ and some $y \in X$ implies $x \leq 0$. In what follows, with abuse of notation, we will use the same symbol "$\geq$" for the different cones appearing in the paper.

Now consider the (normal) cone of non-negative functions $P = C(\overline{\Omega}, \mathbb{R}_+)$. Then the solution operator $K_i\colon C^{\widehat{\mu}}(\overline{\Omega}) \to C^{2+\widehat{\mu}}(\overline{\Omega})$ defined as $K_i g = u$ is linear, continuous and (due to the maximum principle) positive, that is $K_i(P) \subset P$. It is known that K can be extended uniquely to a continuous, linear and compact operator $K_i\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ (that we denote again by the same name). The following result (see [1, Lemma 5.3]) provides further positivity properties of the generalized solution operator.

Proposition 2.1. *Let $e_i = K_i 1 \in C(\overline{\Omega}, \mathbb{R}_+) \setminus \{0\}$. Then $K_i\colon C(\overline{\Omega}) \to C^1(\overline{\Omega}) \subset C(\overline{\Omega})$ is e-positive (and in particular positive), that is, for each $g \in C(\overline{\Omega}, \mathbb{R}_+) \setminus \{0\}$ there exist $\alpha_g > 0$ and $\beta_g > 0$ such that $\alpha_g e_i \leq K_i g \leq \beta_g e_i$.*

Denote by $r(K_i)$ the spectral radius of K_i. As a consequence of Proposition 2.1 and the Krein–Rutman theorem, it is known (for details see, for example, Lemma 3.3 of [16]) that $r(K_i) \in (0, +\infty)$ and there exists $\varphi_i \in P \setminus \{0\}$ such that

$$\varphi_i = \mu_i K_i \varphi_i, \tag{2.2}$$

where $\mu_i = 1/r(K_i)$.

We utilize the space $C(\overline{\Omega}, \mathbb{R}^n)$, endowed with the norm

$$\|u\| := \max_{i=1,\dots,n}\{\|u_i\|_\infty\}, \quad \text{where } \|z\|_\infty = \max_{x\in\overline{\Omega}} |z(x)|,$$

and consider (with abuse of notation) the cone $P = C(\overline{\Omega}, \mathbb{R}^n_+)$. Given a nonempty set $D \subset C(\overline{\Omega}, \mathbb{R}^n)$ we define

$$D_I = \{u \in D : u(x) \in I \text{ for all } x \in \overline{\Omega}\},$$

$I = \prod_{i=1}^{n} I_i \subset \mathbb{R}^n$, where each $I_i \subset \mathbb{R}$ is a closed nonempty interval.

Given a function $f_i\colon \overline{\Omega} \times I \to \mathbb{R}$, we define the Nemytskiĭ (or superposition) operator F_i in the following way:

$$F_i(u)(x) := f_i(x, u(x)), \quad \text{for } u \in C(\overline{\Omega}, I) \text{ and } x \in \overline{\Omega}.$$

We now fix $I = \prod_{i=1}^{n} [0, \rho_i]$ and rewrite the elliptic system (1.1) as a fixed point problem in the product space of continuous functions by considering the operators $T, \Gamma\colon C(\overline{\Omega}, I) \to C(\overline{\Omega}, \mathbb{R}^n)$ given by

$$T(u) := (\lambda_i K_i F_i(u))_{i=1,\dots,n}, \quad \Gamma(u) := (\eta_i \gamma_i h_i[u])_{i=1,\dots,n}, \tag{2.3}$$

where $\gamma_i \in C^{2+\widehat{\mu}}(\overline{\Omega})$ is the unique solution (non-negative, due to the maximum principle, see [2, Section 4 of Chapter 1]) of the BVP

$$\begin{cases} L_i u(x) = 0, & x \in \Omega, \\ B_i u(x) = 1, & x \in \partial\Omega. \end{cases}$$

Definition 2.2. We say that $u \in C(\overline{\Omega}, I)$ is a *weak solution* of the system (1.1) if and only if u is a fixed point of the operator $T + \Gamma$, that is,

$$u = Tu + \Gamma u = (\lambda_i K_i F_i(u) + \eta_i \gamma_i h_i[u])_{i=1,\dots,n};$$

if, furthermore, the components of u are non-negative with $u_j \not\equiv 0$ for some j we say that u is a *nonzero positive solution*.

In the following proposition we recall the main properties of the classical fixed point index for compact maps, for more details see [2], [12]. In what follows the closure and the boundary of subsets of a cone $\widehat{P}$ are understood to be relative to $\widehat{P}$.

Proposition 2.3. *Let X be a real Banach space and let $\widehat{P} \subset X$ be a cone. Let D be an open bounded set of X with $0 \in D \cap \widehat{P}$ and $\overline{D \cap \widehat{P}} \neq \widehat{P}$. Assume that $T\colon \overline{D \cap \widehat{P}} \to \widehat{P}$ is a compact operator such that $x \neq Tx$ for $x \in \partial(D \cap \widehat{P})$. Then the fixed point index $i_{\widehat{P}}(T, D \cap \widehat{P})$ has the following properties*:

(a) *If there exists $e \in \widehat{P} \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial(D \cap \widehat{P})$ and all $\lambda > 0$, then $i_{\widehat{P}}(T, D \cap \widehat{P}) = 0$.*

(b) *If* $Tx \neq \lambda x$ *for all* $x \in \partial(D \cap \widehat{P})$ *and all* $\lambda > 1$, *then* $i_{\widehat{P}}(T, D \cap \widehat{P}) = 1$.

(c) *Let* D^1 *be open bounded in* X *such that* $\overline{(D^1 \cap \widehat{P})} \subset (D \cap \widehat{P})$. *If* $i_{\widehat{P}}(T, D \cap \widehat{P}) = 1$ *and* $i_{\widehat{P}}(T, D^1 \cap \widehat{P}) = 0$, *then* T *has a fixed point in* $(D \cap \widehat{P}) \setminus \overline{(D^1 \cap \widehat{P})}$. *The same holds if* $i_{\widehat{P}}(T, D \cap \widehat{P}) = 0$ *and* $i_{\widehat{P}}(T, D^1 \cap \widehat{P}) = 1$.

With these ingredients we can now state a result regarding the existence of positive solutions for the system (1.1).

Theorem 2.4. *Let* $I = \prod_{i=1}^{n} [0, \rho_i]$ *and assume the following conditions hold*:

(a) *For every* $i = 1, \dots, n$, $f_i \in C(\overline{\Omega} \times I)$ *and* $f_i \geq 0$. *Set*

$$M_i := \max_{(x,u) \in \overline{\Omega} \times I} f_i(x, u).$$

(b) *There exist* $\delta \in (0, +\infty)$, $i_0 \in \{1, \dots, n\}$ *and* $\rho_0 \in \Big(0, \min_{i=1,\dots,n} \rho_i\Big)$ *such that*

$$f_{i_0}(x, u) \geq \delta u_{i_0}, \quad \text{for every } (x, u) \in \overline{\Omega} \times I_0,$$

where $I_0 := \prod_{i=1}^{n} [0, \rho_0]$.

(c) *For every* $i = 1, \dots, n$, $h_i \colon P_I \to [0, +\infty)$ *is continuous and*

$$H_i := \sup_{u \in P_I} h_i[u] < +\infty.$$

(d) *For every* $i = 1, \dots, n$, *the following two inequalities are satisfied*:

$$\frac{\mu_{i_0}}{\delta} \leq \lambda_{i_0} \quad \text{and} \quad \lambda_i M_i \|K_i(1)\|_\infty + \eta_i H_i \|\gamma_i\|_\infty \leq \rho_i. \tag{2.4}$$

Then the system (1.1) *has a nonzero positive weak solution* u *such that*

$$\rho_0 \leq \|u\| \quad \text{and} \quad \|u_i\|_\infty \leq \rho_i, \quad \text{for every } i = 1, \dots, n.$$

Proof. Take $P = C(\overline{\Omega}, \mathbb{R}^n_+)$. Due to the assumptions above the operator $T + \Gamma$ maps P_I into P and is compact (the compactness of T is well known and Γ is a finite rank operator). If $T + \Gamma$ has a fixed point either on ∂P_I or ∂P_{I_0} we are done.

Assume now that $T + \Gamma$ is fixed point free on $\partial P_I \cup \partial P_{I_0}$, we are going to prove that $T + \Gamma$ has a fixed point in $P_I \setminus (\partial P_I \cup P_{I_0})$. We firstly prove, by means of (a), (c) and (d), that

$$\sigma u \neq Tu + \Gamma u, \quad \text{for every } u \in \partial P_I \text{ and every } \sigma > 1.$$

If this does not hold, then there exist $u \in \partial P_I$ and $\sigma > 1$ such that $\sigma u = Tu + \Gamma u$. Note that $\|u_j\|_\infty = \rho_j$ for some j and $\|u_i\|_\infty \leq \rho_i$ for every i. Furthermore, for every $x \in \overline{\Omega}$, we obtain

$$\begin{aligned}\sigma u_j(x) &= \lambda_j K_j F_j(u)(x) + \eta_j h_j[u] \gamma_j(x) \leq \|\lambda_j K_j F_j(u) + \eta_j h_j[u] \gamma_j\|_\infty \\ &\leq \|\lambda_j K_j(M_j)\|_\infty + \|\eta_j H_j \gamma_j\|_\infty = \lambda_j M_j \|K_j(1)\|_\infty + \eta_j H_j \|\gamma_j\|_\infty \leq \rho_j.\end{aligned}$$

Taking the supremum over $\overline{\Omega}$ we obtain $\sigma\rho_j \le \rho_j$, a contradiction which yields

$$i_P(T+\Gamma, P_I \setminus \partial P_I) = 1.$$

We now consider $\varphi = (\varphi_1, \dots, \varphi_n)$ where φ_i is given by (2.2) and use (b) and (d) to show that

$$u \ne Tu + \Gamma u + \sigma\varphi, \quad \text{for every } u \in \partial P_{I_0} \text{ and every } \sigma > 0.$$

If not, there exist $u \in \partial P_{\rho_0}$ and $\sigma > 0$ such that $u = Tu + \Gamma u + \sigma\varphi$. Then we have $u \ge \sigma\varphi$ and, in particular, $u_{i_0} \ge \sigma\varphi_{i_0}$. For every $x \in \overline{\Omega}$ we have

$$\begin{aligned} u_{i_0}(x) &= (\lambda_{i_0} K_{i_0} F_{i_0} u)(x) + \eta_{i_0} h_{i_0}[u]\gamma_{i_0}(x) + \sigma\varphi_{i_0}(x) \\ &\ge (\lambda_{i_0} K_{i_0} \delta u_{i_0})(x) + \sigma\varphi_{i_0}(x) \ge (\lambda_{i_0} \delta K_{i_0}(\sigma\varphi_{i_0}))(x) + \sigma\varphi_{i_0}(x) \\ &= \frac{\sigma\lambda_{i_0}\delta}{\mu_{i_0}}\varphi_{i_0}(x) + \sigma\varphi_{i_0}(x) \ge 2\sigma\varphi_{i_0}(x). \end{aligned}$$

By iterating the process, for $x \in \overline{\Omega}$, we get $u_{i_0}(x) \ge n\sigma\varphi_{i_0}(x)$ for every $n \in \mathbb{N}$, a contradiction, since u is bounded. Thus we obtain $i_P(T+\Gamma, P_{I_0} \setminus \partial P_{I_0}) = 0$. Therefore we have

$$i_P(T+\Gamma, P_I \setminus (\partial P_I \cup P_{I_0})) = i_P(T+\Gamma, P_I \setminus \partial P_I) - i_P(T+\Gamma, P_{I_0} \setminus \partial P_{I_0}) = 1,$$

which proves the result. □

Remark 2.5. Note that, in the applications, sometimes it could be useful to replace the constants M_i and H_i with some majorants, say $\widehat{M}_i$ and $\widehat{H}_i$, at the cost of having to deal with the condition

$$\lambda_i \widehat{M}_i \|K_i(1)\|_\infty + \eta_i \widehat{H}_i \|\gamma_i\|_\infty \le \rho_i, \quad \text{for every } i = 1, \dots, n,$$

which is more stringent than the corresponding one occurring in (2.4).

We now illustrate the applicability of Theorem 2.4.

Example 2.6. Take $\Omega = \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$, and consider the system

$$\begin{cases} -\Delta u_1 = \lambda_1(|(u_1,u_2)|^{1/2} + \tan|(u_1,u_2)|) & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2(1-\sin(u_2))|(u_1,u_2)|^2 & \text{in } \Omega, \\ u_1 = \eta_1 h_1[u], \quad u_2 = \eta_2 h_2[u] & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

where $|(u_1,u_2)| = \max\{|u_1|,|u_2|\}$,

$$h_1[u] = (u_1(0))^2 + (u_2(0))^{1/2} \quad \text{and} \quad h_2[u] = (u_1(0))^{1/4} + \left(\int_\Omega u_2(\xi)\,d\xi\right)^2.$$

By direct calculation we obtain $K_1(1) = K_2(1) = (1 - x_1^2 - x_2^2)/4$, where $x = (x_1, x_2)$, and we may take $\gamma_1 = \gamma_2 \equiv 1$, this gives $\|K_i(1)\|_\infty = 1/4$ and $\|\gamma_i\|_\infty = 1$ for $i = 1, 2$.

Fix $\rho_1, \rho_2 = 15\pi/64$ and set

$$f_1(u_1,u_2) = |(u_1,u_2)|^{1/2} + \tan|(u_1,u_2)|,$$
$$f_2(u_1,u_2) = (1-\sin(u_2))|(u_1,u_2)|^2.$$

First of all, note that given $\delta > 0$, f_1 satisfies condition (b) in Theorem 2.4 for ρ_0 sufficiently small, due to the behaviour near the origin.

In the reminder of this example the numbers are rounded from above to the third decimal place unless exact.

We have $M_1 = f_1(15\pi/64, 15\pi/64) \approx 1.765$ and $M_2 = f_2(15\pi/64, 0) = (15\pi/64)^2 \approx 0.543$. Moreover, we can use the estimates $H_1 \le (15\pi/64)^2 + (15\pi/64)^{1/2} \approx 1.401$ and $H_2 \le (15\pi/64)^{1/4} + (15\pi^2/64)^2 \approx 6.278$.

By Theorem 2.4, the system (2.5) has a nonzero positive solution (u_1,u_2) such that $0 < \|(u_1,u_2)\| \le 15\pi/64$ for every $\lambda_1, \lambda_2, \eta_1, \eta_2 > 0$ with

$$1.765 \times \frac{\lambda_1}{4} + 1.401 \times \eta_1 \le \frac{15}{64}\pi \quad \text{and} \quad 0.543 \times \frac{\lambda_2}{4} + 6.278 \times \eta_2 \le \frac{15}{64}\pi.$$

We now prove, via an elementary argument, a non-existence result.

Theorem 2.7. *Let $I = \prod_{i=1}^{n}[0,\rho_i]$ and assume that for every $i = 1, \dots, n$ we have*:

(a) *$f_i \in C(\overline{\Omega} \times I)$ and there exist $\tau_i \in (0,+\infty)$ such that*

$$0 \le f_i(x,u) \le \tau_i u_i, \quad \textit{for every } (x,u) \in \overline{\Omega} \times I,$$

(b) *$h_i \colon P_I \to [0,+\infty)$ is continuous and there exist $\xi_i \in (0,+\infty)$ and*

$$h_i[u] \le \xi_i \|u\|, \quad \textit{for every } u \in P_I,$$

(c) *the following inequality holds*:

$$\lambda_i \tau_i \|K_i(1)\|_\infty + \eta_i \xi_i \|\gamma_i\|_\infty < 1. \tag{2.6}$$

Then the system (1.1) *has at most the zero solution in P_I.*

Proof. Assume, on the contrary, that there exists $u \in P_I$, $\|u\| = \sigma > 0$, such that $u = Tu + \Gamma u$. Then there exists j such that $\|u_j\|_\infty = \sigma$. For $x \in \overline{\Omega}$ we have

$$\begin{aligned} u_j(x) &= \lambda_j K_j F_j(u)(x) + \eta_j h_j[u]\gamma_j(x) \le \|\lambda_j K_j F_j(u) + \eta_j h_j[u]\gamma_j\|_\infty \\ &\le \|\lambda_j K_j(\tau_j \sigma)\|_\infty + \|\eta_j \xi_j \sigma \gamma_j\|_\infty = (\lambda_j \tau_j \|K_j(1)\|_\infty + \eta_j \xi_j \|\gamma_j\|_\infty)\sigma < \sigma. \end{aligned}$$

By taking the supremum over $\overline{\Omega}$, we obtain $\sigma < \sigma$, a contradiction. □

We conclude by illustrating in the next example the applicability of Theorem 2.7.

EXAMPLE 2.8. Take $\Omega = \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$ and consider the system

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1^2 \sin(u_2) & \text{in } \Omega, \\ -\Delta u_2 = \lambda_2 u_2^4 \cos(u_1) & \text{in } \Omega, \\ u_1 = \eta_1 h_1[u], \quad u_2 = \eta_2 h_2[u] & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $h_1[u] = u_1(0) + (u_2(0))^2$ and $h_2[u] = u_1(0) + (u_2(0))^3$. First of all note that the trivial solution satisfies the system (2.7). Let us fix $I = [0, \pi/4] \times [0, \pi/2]$ and note that for every $(x, u_1, u_2) \in \overline{\Omega} \times [0, \pi/4] \times [0, \pi/2]$ we have

$$0 \le u_1^2 \sin(u_2) \le \frac{\pi}{4}\, u_1, \qquad 0 \le u_2^4 \cos(u_1) \le \frac{\pi^3}{8}\, u_2.$$

Furthermore, for $u \in P_I$, we have

$$0 \le h_1[u] \le \left(\frac{\pi}{2} + 1\right)\|u\|, \qquad 0 \le h_2[u] \le \left(\frac{\pi^2}{4} + 1\right)\|u\|.$$

Thus, in this case, condition (2.6) reads

$$\frac{\pi}{4}\, \lambda_1 + \left(\frac{\pi}{2} + 1\right)\eta_1 < 1 \quad \text{and} \quad \frac{\pi^3}{8}\, \lambda_2 + \left(\frac{\pi^2}{4} + 1\right)\eta_2 < 1. \tag{2.8}$$

Therefore if (2.8) is satisfied, by Theorem 2.7 the system (2.7) admits only the trivial solution in P_I.

Acknowledgements. The author wishes to thank the Referees for their constructive comments.

REFERENCES

[1] H. AMANN, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Funct. Anal. **11** (1972), 346–384.

[2] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM. Rev. **18** (1976), 620–709.

[3] A. CABADA, *An overview of the lower and upper solutions method with nonlinear boundary value conditions*, Bound. Value Probl. (2011), Art. ID 893753, 18 pp.

[4] F. CIANCIARUSO, G. INFANTE AND P. PIETRAMALA, *Solutions of perturbed Hammerstein integral equations with applications*, Nonlinear Anal. Real World Appl. **33** (2017), 317–347.

[5] J.A. CID AND G. INFANTE, *A non-variational approach to the existence of nonzero positive solutions for elliptic systems*, J. Fixed Point Theory Appl. **19** (2017), 3151–3162.

[6] R. CONTI, *Recent trends in the theory of boundary value problems for ordinary differential equations*, Boll. Unione Mat. Ital. **22** (1967), 135–178.

[7] J.D.B. DE GODOI, O.H. MIYAGAKI AND R.S. RODRIGUES, *A class of nonlinear elliptic systems with Steklov–Neumann nonlinear boundary conditions*, Rocky Mountain J. Math. **46** (2016), 1519–1545.

[8] D.R. DUNNINGER AND H. WANG, *Multiplicity of positive solutions for a nonlinear differential equation with nonlinear boundary conditions*, Ann. Polon. Math. **69** (1998), 155–165.

[9] C.S. Goodrich, *Perturbed Hammerstein integral equations with sign-changing kernels and applications to nonlocal boundary value problems and elliptic PDEs*, J. Integral Equations Appl. **28** (2016), 509–549.

[10] C.S. Goodrich, *A new coercivity condition applied to semipositone integral equations with nonpositive, unbounded nonlinearities and applications to nonlocal BVPs*, J. Fixed Point Theory Appl. **19** (2017), 1905–1938.

[11] C.S. Goodrich, *The effect of a nonstandard cone on existence theorem applicability in nonlocal boundary value problems*, J. Fixed Point Theory Appl. **19** (2017), 2629–2646.

[12] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, 1988.

[13] G.L. Karakostas and P.Ch. Tsamatos, *Existence of multiple positive solutions for a nonlocal boundary value problem*, Topol. Methods Nonlinear Anal. **19** (2002), 109–121.

[14] G.L. Karakostas and P.Ch. Tsamatos, *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems*, Electron. J. Differential Equations **2002**, 17 pp.

[15] K.Q. Lan, *Nonzero positive solutions of systems of elliptic boundary value problems*, Proc. Amer. Math. Soc. **139** (2011), 4343–4349.

[16] K.Q. Lan, *Existence of nonzero positive solutions of systems of second order elliptic boundary value problems*, J. Appl. Anal. Comput. **1** (2011), 21–31.

[17] R. Ma, *A survey on nonlocal boundary value problems*, Appl. Math. E-Notes **7** (2007), 257–279.

[18] R. Ma, R. Chen and Y. Lu, *Method of lower and upper solutions for elliptic systems with nonlinear boundary condition and its applications*, J. Appl. Math. (2014), Art. ID 705298, 7 pp.

[19] J. Mawhin and K. Schmitt, *Upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **97** (1984), 199–207.

[20] S.K. Ntouyas, *Nonlocal initial and boundary value problems: a survey*, Handbook of Differential Equations: Ordinary Differential Equations, Vol. II, Elsevier B.V., Amsterdam, 2005, 461–557.

[21] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[22] C.V. Pao and Y.M. Wang, *Nonlinear fourth-order elliptic equations with nonlocal boundary conditions*, J. Math. Anal. Appl. **372** (2010), 351–365.

[23] M. Picone, *Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (1908), 1–95.

[24] A. Štikonas, *A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions*, Nonlinear Anal. Model. Control **19** (2014), 301–334.

[25] J.R.L. Webb, *Existence of positive solutions for a thermostat model*, Nonlinear Anal. Real World Appl. **13** (2012), 923–938.

[26] J.R.L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc. **74** (2006), 673–693.

[27] W.M. Whyburn, *Differential equations with general boundary conditions*, Bull. Amer. Math. Soc. **48** (1942), 692–704.

[28] E. Zeidler, *Nonlinear Functional Analysis and its Applications.* I. *Fixed-Point Theorems*, Springer, New York, 1986.

[29] Y. ZHANG AND M. WANG, *Bifurcation from trivial solution for elliptic systems with nonlinear boundary conditions*, Complex Var. Elliptic Equ. **60** (2015), 951–967.

Manuscript received July 11, 2017
accepted October 17, 2017

GENNARO INFANTE
Dipartimento di Matematica e Informatica
Università della Calabria
87036 Arcavacata di Rende, Cosenza, ITALY
E-mail address: gennaro.infante@unical.it