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DYNAMICS OF THE BBM EQUATION WITH A DISTRIBUTION FORCE IN LOW REGULARITY SPACES

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Abstract. The Benjamin–Bona–Mahony equation with a distribution force on torus is studied in low regularity spaces. The global well-posedness and the existence of a global attractor in $\dot{H}^{s,p}(\mathbb{T})$ are proved.

1. Introduction

There are a lot of studies devoted to the global attractor of dynamical systems generated by nonlinear partial differential equations. The dynamical system, due to the damped effect, is usually dissipative in some Banach space X, namely it has a bounded absorbing set in X. To prove the compact property of solution semigroup, one may try to control the nonlinear term by Sobolev embedding and the dissipative bound in X. This is the reason why some growth restrictions need to be posed on the nonlinear terms. Following this line, roughly speaking, the growth restrictions on nonlinear term can be relaxed if the phase space is more regular, see [2] for a discussion on this topic for reaction diffusion equations. In a given phase space, it is very interesting to find the critical exponent of growth order for the nonlinear term, which has been done in [1] and [17]. However,

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one does not need to consider this problem for those equations in physics when the nonlinear term is fixed and given explicitly. In this case, an "equivalent" question is to find out the lowest regularity space in which the global attractor exists. The paper is devoted to this direction for the Benjamin–Bona–Mahony (BBM) equations.

Consider the following damped, forced BBM equation on the one-dimensional torus $\mathbb{T} = [0, 2\pi]$:

$$u_t - u_{txx} - u_{xx} + uu_x = f, \quad (x,t) \in \mathbb{T} \times \mathbb{R}^+ \tag{1.1}$$

with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{T}. \tag{1.2}$$

Here the unknown function u is real-valued, $u_{txx} = \partial_t \partial_x^2 u$, and the forcing term f is a given function independent of time.

The model was used to describe the propagation of long waves which incorporates nonlinear dispersive and dissipative effects, see [5], [6]. This equation and also related types of the BBM equation were studied by many authors. The well-posedness and ill-posedness were obtained in [4], [3], [7], [10]. The stability or decay rate of solutions in Sobolev spaces were investigated in [14], [19], [20]. The existence of the global attractor was proved in [21], [22], [25], [27], [34]. Moreover, the higher regularity and finite fractal dimension can be found in [12] and [9], [28], respectively.

Observe that if $u(t,x)$ is a smooth solution of (1.1)–(1.2), then integrating (1.1) on $\mathbb{T}$ yields that

$$\frac{d}{dt} \int_{\mathbb{T}} u\,dx = \int_{\mathbb{T}} f\,dx.$$

If f has zero mean, then we find for all $t > 0$

$$\int_{\mathbb{T}} u(t,x)\,dx = \int_{\mathbb{T}} u_0\,dx.$$

Thus in this article, without loss of generality, we only consider the solution $u(t)$ of mean zero.

The main results in this paper read as follows.

Theorem 1.1. *Assume that $f \in \dot{H}^{s-2,p}(\mathbb{T})$ with $0 \le s \le 1$ and $2 \le p < \infty$. Then, for every $u_0 \in \dot{H}^{s,p}(\mathbb{T})$, problem (1.1)–(1.2) has a unique solution $u \in C([0,T]; \dot{H}^{s,p}(\mathbb{T}))$ for some $T > 0$ depending on u_0 and f. Moreover, the solution map $S(t)\colon u_0 \mapsto u(t)$ is continuous in $\dot{H}^{s,p}(\mathbb{T})$.*

Theorem 1.2. *Assume that $f \in \dot{H}^{s-2,p}(\mathbb{T})$ with $2 \le p < \infty$ and $1/(2p) \le s \le 1$. Then problem (1.1)–(1.2) has a global attractor in $\dot{H}^{s,p}(\mathbb{T})$.*

This is a continuation study of our previous works [29], [30], [31]. The main difference lies in that the force term f is allowed to belong to Sobolev spaces of negative order. The assumption $f \in \dot{H}^{s-2,p}$ is sharp in the sense that it is necessary to obtain the existence of the global attractor in $\dot{H}^{s,p}$. In fact, the solution is not expected to belong to $\dot{H}^{s,p}$ if $f \in \dot{H}^{s'-2,p'}$ with $s' < s$ or $p' < p$. Compared to [29], [31], Theorem 1.1 suggests that the external force does not change the range of s in the local well-posedness. But this is not the case in the global well-posedness, due to the low regularity of the force. By a decomposition, we reduce the equation with a distribution force to an equation with an irregular coefficient. To deal with the irregular coefficient, we need the assumption $s \geq 1/(2p)$ for some technical reasons. It is not clear whether Theorem 1.2 holds for smaller s.

We give some references on similar topics. The existence of the global attractor in L^p type Sobolev spaces are proved in [8] for strongly damped wave equations, and in [11] for Euler equations. When the force belongs to distributional space, the global attractor is obtained for reaction diffusion equations in [24], [33] and damped wave equations in [18], [32].

This paper is organized as follows. Section 2 is devoted to the local well-posedness of problem (1.1)–(1.2) in $\dot{H}^{s,p}(\mathbb{T})$ with $0 \leq s \leq 1$. In Section 3, we obtain the existence of a global attractor in $\dot{H}^{s,p}(\mathbb{T})$ with $1/(2p) \leq s \leq 1$ by the I-method.

2. Local well-posedness

We first recall some definitions. Given a function φ on the torus $\mathbb{T}$, the Fourier coefficient is defined by

$$\widehat{\varphi}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(x) e^{-inx}\, dx, \quad n \in \mathbb{Z}.$$

Then, we have the following Fourier inversion formula:

$$\varphi(x) = \sum_{n \in \mathbb{Z}} \widehat{\varphi}(n) e^{inx}.$$

For $s \in \mathbb{R}$, we define the fractional operator $(1 - \partial_x^2)^{s/2}$ by

$$(1 - \partial_x^2)^{s/2} \varphi(x) = \sum_{n \in \mathbb{Z}} \langle n \rangle^s \widehat{\varphi}(n) e^{inx},$$

where $\langle n \rangle = \sqrt{1 + n^2}$. Moreover, for $1 < p < \infty$, the Bessel potential spaces $H^{s,p}(\mathbb{T})$ are defined as the completion of smooth functions with respect to the norm

$$\|\varphi\|_{H^{s,p}} = \|(1 - \partial_x^2)^{s/2} \varphi\|_{L^p}. \tag{2.1}$$

We denote by $\dot{H}^{s,p}(\mathbb{T})$ the space of functions φ satisfying $\|\varphi\|_{H^{s,p}} < \infty$ and

$$\int_{\mathbb{T}} \varphi(x)\, dx = 0.$$

If φ belongs to $\dot{H}^1(\mathbb{T})$, then $\widehat{\varphi}(0) = 0$,

$$\|\varphi_x\|_{L^2(\mathbb{T})} = \|n\widehat{\varphi}(n)\|_{l^2(dn)} \geq \|\widehat{\varphi}(n)\|_{l^2(dn)} = \|\varphi\|_{L^2(\mathbb{T})}. \tag{2.2}$$

This is the well-known Poincaré inequality on torus. Moreover, the space $\dot{H}^{s,p}(\mathbb{T})$ has the following equivalent norm:

$$\|\varphi\|_{\dot{H}^{s,p}} \sim \bigg\| \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^s \widehat{\varphi}(n) e^{inx} \bigg\|_{L^p}.$$

In particular, let $s = 2$, this implies the bound [1]

$$\|(1 - \partial_x^2)\partial_x^{-2}\|_{\dot{L}^p,\dot{L}^p} \leq C \quad \text{for all } 1 < p < \infty. \tag{2.3}$$

Let $N > 0$. We define the frequency projection operator P_N on low Fourier modes as

$$P_N\varphi = \sum_{|n|\leq N} \widehat{\varphi}(n) e^{inx},$$

and P^N on high Fourier modes

$$P^N\varphi = \sum_{|n|>N} \widehat{\varphi}(n) e^{inx}.$$

It is clear that $\mathrm{Id} = P_N + P^N$.

Throughout, $A \lesssim B$ means $A \leq CB$ for some absolute constant C, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$, and $A \gg B$ means A/B is very big, say $A/B \geq 1000$.

2.1. The decomposition $u = Q + v$. Let $N_1 \geq 1$. Consider the elliptic equation

$$-Q_{xx} + P^{N_1}(QQ_x) = P^{N_1} f, \tag{2.4}$$

where $f \in \dot{H}^{s-2,p}(\mathbb{T})$ with $2 \leq p < \infty$ and $0 \leq s \leq 1$. We shall use the contraction principle to find a solution Q. To this end, we rewrite (2.4) as

$$Q = (-\partial_x^2)^{-1} P^{N_1}(f - QQ_x) := \Gamma_1 Q. \tag{2.5}$$

It is well known [15, Theorem 3.5.7, p. 217] that, for every $\varphi \in H^{s-2,p}(\mathbb{T})$,

$$\lim_{N\to\infty} \|P^N\varphi\|_{H^{s-2,p}} = 0.$$

Thus, for every $0 < \varepsilon < 1$, there exists $N_1 \geq 1$ such that

$$\|P^{N_1}\varphi\|_{H^{s-2,p}} \leq \varepsilon.$$

It follows from (2.3) that

$$\|(-\partial_x^2)^{-1} P^{N_1} f\|_{H^{s,p}} \leq C\|(1 - \partial_x^2)^{-1} P^{N_1} f\|_{H^{s,p}} \leq C\varepsilon. \tag{2.6}$$

To proceed, we need the follow result.

[1] This can also be proved in a similar way as Lemma 2.6 in this paper.

LEMMA 2.1. *Let $s \geq 0$ and $2 \leq p < \infty$. Then*

$$\|\partial_x(1-\partial_x^2)^{-1}(uv)\|_{H^{s,p}} \lesssim \|u\|_{H^s}\|v\|_{H^s}.$$

PROOF. The desired conclusion is equivalent to show that

$$\|\partial_x(1-\partial_x^2)^{-1+s/2}((1-\partial_x^2)^{-s/2}u(1-\partial_x^2)^{-s/2}v)\|_{L^p} \lesssim \|u\|_{L^2}\|v\|_{L^2}. \tag{2.7}$$

Thanks to the fact that $\partial_x(1-\partial_x^2)^{-1/2}$ is bounded in $L^p(\mathbb{T})$ for $2 \leq p < \infty$, and $(1-\partial_x^2)^{-(1/2-1/p)/2}$ is bounded from L^2 to L^p (which follows from the Sobolev embedding $H^{1/2-1/p} \hookrightarrow L^p$), (2.7) follows if one can show that

$$\|(1-\partial_x^2)^{s/2-(1/2+1/p)/2}((1-\partial_x^2)^{-s/2}u(1-\partial_x^2)^{-s/2})v\|_{L^2} \lesssim \|u\|_{L^2}\|v\|_{L^2}. \tag{2.8}$$

Using the Plancherel theorem, (2.8) is reduced to showing

$$\Big\|\sum_{n=n_1+n_2,n_1,n_2\in\mathbb{Z}} \langle n\rangle^{s-(1/2+1/p)}\langle n_1\rangle^{-s}\widehat{u}(n_1)\langle n_2\rangle^{-s}\widehat{u}(n_2)\Big\|_{l^2(dn)} \lesssim \|\widehat{u}\|_{l^2}\|\widehat{v}\|_{l^2}. \tag{2.9}$$

It is easy to check the elementary inequality $\langle n\rangle^s \lesssim \langle n_1\rangle^s\langle n_2\rangle^s$, for $s \geq 0$. Combining this and Cauchy's inequality, we find

$$\begin{aligned}\text{LHS (2.9)} &\lesssim \Big\|\sum_{n=n_1+n_2,n_1,n_2\in\mathbb{Z}} \langle n\rangle^{-(1/2+1/p)}\widehat{u}(n_1)\widehat{u}(n_2)\Big\|_{l^2(dn)} \\ &\lesssim \|\langle n\rangle^{-(1/2+1/p)}\|_{l^2}\Big\|\sum_{n=n_1+n_2,n_1,n_2\in\mathbb{Z}} \widehat{u}(n_1)\widehat{u}(n_2)\Big\|_{l^\infty(dn)} \lesssim \|\widehat{u}\|_{l^2}\|\widehat{v}\|_{l^2}\end{aligned}$$

as desired. □

Consider the set $\mathscr{B}_1 = \{\varphi \in L^1(\mathbb{T}) : \|\varphi\|_{H^{s,p}} \leq 2C\varepsilon\}$. Thanks to Lemma 2.1, using (2.3) again, if $Q \in \mathscr{B}_1$, then

$$\|(-\partial_x^2)^{-1}(QQ_x)\|_{H^{s,p}} \leq C'\|Q\|_{H^{s,p}}^2 \leq 4C^2C'\varepsilon^2.$$

Choose N_1 large enough such that ε is small enough, say $4CC'\varepsilon \leq 1/2$. Then we have

$$\|(-\partial_x^2)^{-1}(QQ_x)\|_{H^{s,p}} \leq C\varepsilon/2.$$

This and (2.6) implies that Γ_1 maps $\mathscr{B}_1$ into $\mathscr{B}_1$. Moreover, if $Q, \widetilde{Q} \in \mathscr{B}_1$, then

$$\begin{aligned}\|\Gamma_1 Q - \Gamma_1\widetilde{Q}\|_{H^{s,p}} &\leq C'\|Q+\widetilde{Q}\|_{H^{s,p}}\|Q-\widetilde{Q}\|_{H^{s,p}} \\ &\leq 4C'C\varepsilon\|Q-\widetilde{Q}\|_{H^{s,p}} \leq \frac{1}{2}\|Q-\widetilde{Q}\|_{H^{s,p}}.\end{aligned}$$

Thus, Γ_1 is a contraction mapping on $\mathscr{B}_1$. This gives the following proposition.

PROPOSITION 2.2. *Let $f \in \dot{H}^{s-2,p}(\mathbb{T})$ with $2 \leq p < \infty$ and $0 \leq s \leq 1$. Then, for every $0 < \varepsilon < 1$, there exists N_1 large enough depending on ε such that the elliptic problem*

$$-Q_{xx} + P^{N_1}(QQ_x) = P^{N_1}f$$

has a unique solution $Q \in \dot{H}^{s,p}(\mathbb{T})$*. Moreover, we have the bound*

$$\|Q\|_{H^{s,p}} \le C\varepsilon.$$

Let Q be the solution of (2.4) defined by Proposition 2.2. If v is a solution of

$$v_t - v_{txx} - v_{xx} + vv_x + (Qv)_x = P_{N_1}(f - QQ_x), \tag{2.10}$$

$$v(0,x) = u_0(x) - Q, \tag{2.11}$$

then $u = v + Q$ is a solution of (1.1)–(1.2).

In the sequel, thanks to Proposition 2.2, we assume that N_1 is large enough to ensure that

$$\|Q\|_{H^{s,p}} \le \min\big\{\|f\|^{1/2}_{H^{s-2,p}}, \|f\|_{H^{s-2,p}}, \varepsilon_0\big\} \tag{2.12}$$

where ε_0 is a small number determined later. We claim that for all $\alpha \ge 0$

$$\|P_{N_1}(f - QQ_x)\|_{H^\alpha} \lesssim (1 + N_1^2)^{1+\alpha/2}\|f\|_{H^{s-2,p}}. \tag{2.13}$$

Indeed, rewrite $P_{N_1}(QQ_x)$ as $P_{N_1}\partial_x(1-\partial_x^2)^{1/2}(1-\partial_x^2)^{-1/2}Q^2/2$, and use the Plancherel theorem, Sobolev embedding, Höler inequality and (2.12), we find

$$\begin{aligned}\|P_{N_1}(QQ_x)\|_{H^\alpha} &\lesssim (1+N_1^2)^{1+\alpha/2}\|(1-\partial_x^2)^{-1/2}Q^2\|_{L^2}\\ &\lesssim (1+N_1^2)^{1+\alpha/2}\|Q^2\|_{L^1} \lesssim (1+N_1^2)^{1+\alpha/2}\|Q\|^2_{L^2}\\ &\lesssim (1+N_1^2)^{1+\alpha/2}\|Q\|^2_{H^{s,p}} \lesssim (1+N_1^2)^{1+\alpha/2}\|f\|_{H^{s-2,p}}.\end{aligned}$$

Similarly, we have $\|P_{N_1}f\|_{H^\alpha} \lesssim (1+N_1^2)^{1+\alpha/2}\|f\|_{H^{s-2,p}}$. Thus the claim (2.13) follows.

2.2. *I*-operator. It is well known that the linear BBM equation does not has a smoothing effect. In fact, if $u(t)$ is the solution of

$$u_t - u_{txx} - u_{xx} = 0, \quad u(0,x) = u_0(x), \tag{2.14}$$

then by Fourier transform, we have

$$\widehat{u}(t,n) = e^{-|n|^2t/(1+|n|^2)}\widehat{u}_0(n), \quad n \in \mathbb{Z}.$$

Clearly, $u(t)$ belongs to $H^s(\mathbb{T})$, $t > 0$, if and only if u_0 belongs to $H^s(\mathbb{T})$. Thus, it is not expected that the solution v of (2.10)–(2.11) belongs to $H^{s+\varepsilon}(\mathbb{T})$ for some $\varepsilon > 0$.

On the other hand, in order to establish the global well-posedness of (2.10)–(2.11), one needs to exploit the cancellation (antisymmetry) property of vv_x. To this end, multiplying both sides of (2.10)–(2.11) with v and integrating, we find

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|v|^2 + |v_x|^2\,dx + \int_{\mathbb{T}}|v_x|^2\,dx + \int_{\mathbb{T}}(Qv)_x v\,dx = \int_{\mathbb{T}}P_{N_1}fv\,dx. \tag{2.15}$$

Equation (2.15) is often referred to the energy equation, which reflects essentially the dissipative property of the original equation. Note that the energy equation holds only if v belongs to $H^1(\mathbb{T})$, which is impossible for solutions v of (2.10)–(2.11) with data $u_0 \in H^s(\mathbb{T})$, $s < 1$.

To overcome this difficulty, for $0 \le s \le 1$ and $N \gg 1$, we introduce (inspired by [13]) the I-operator as follows:

$$I_N \varphi(x) = \sum_{n\in\mathbb{Z}} m_N(n)\widehat{\varphi}(n)e^{inx},$$

where m is a nonnegative smooth decreasing function satisfying

$$m_N(n) = \begin{cases} 1 & \text{if } |n| \le N, \\ \left(\dfrac{|n|}{N}\right)^{s-1} & \text{if } |n| \ge 2N. \end{cases}$$

Lemma 2.3. *If $u \in L^2(\mathbb{T})$, then the inequality*

$$\|I_N \partial_x (1-\partial_x^2)^{-1}(uv)\|_{H^1} \lesssim \|u\|_{L^2}\|I_N v\|_{H^1}$$

holds, where the implicit constant is independent of N.

Proof. The lemma follows if one can show that

$$\|I_N(uI_N^{-1}(1-\partial_x^2)^{-1/2}v)\|_{L^2} \lesssim \|u\|_{L^2}\|v\|_{L^2}. \tag{2.16}$$

The n-th Fourier coefficient of $I_N(uI_N^{-1}(1-\partial_x^2)^{-1/2}v)$ is

$$\sum_{n=n_1+n_2, n_1,n_2\in\mathbb{Z}} m_N(n)\widehat{u}(n_1)\,\frac{\widehat{v}(n_2)}{m_N(n_2)\langle n_2\rangle}.$$

By the Plancherel theorem, we have

$$\|I_N(uI_N^{-1}(1-\partial_x^2)^{-1/2}v)\|_{L^2} = \Bigg\| \sum_{n=n_1+n_2, n_1,n_2\in\mathbb{Z}} m_N(n)\widehat{u}(n_1)\,\frac{\widehat{v}(n_2)}{m_N(n_2)\langle n_2\rangle} \Bigg\|_{l^2(dn)}. \tag{2.17}$$

Rewrite the sum on the right hand side of (2.17), then we find

$$\text{RHS (2.17)} \lesssim S_1 + S_2 + S_3,$$

where

$$S_1 = \Bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|<2N}} m_N(n)\widehat{u}(n_1)\,\frac{\widehat{v}(n_2)}{m_N(n_2)\langle n_2\rangle} \Bigg\|_{l^2(dn)},$$

$$S_2 = \Bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|\ge 2N}} m_N(n)\widehat{u}(n_1)\,\frac{\widehat{v}(n_2)}{m_N(n_2)\langle n_2\rangle} \Bigg\|_{l^2(dn,|n|<2N)},$$

$$S_3 = \Bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|\ge 2N}} m_N(n)\widehat{u}(n_1)\,\frac{\widehat{v}(n_2)}{m_N(n_2)\langle n_2\rangle} \Bigg\|_{l^2(dn,|n|\ge 2N)}.$$

For S_1, since $|n_2| < 2N$, by definition we find $|m_N(n_2)| \gtrsim 1$. Note that we always have $|m_N(n)| \le 1$, then

$$S_1 \le \bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|<2N}} m_N(n)|\widehat{u}(n_1)| \frac{|\widehat{v}(n_2)|}{m_N(n_2)\langle n_2\rangle} \bigg\|_{l^2(dn)} \tag{2.18}$$
$$\lesssim \bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|<2N}} |\widehat{u}(n_1)| \frac{|\widehat{v}(n_2)|}{\langle n_2\rangle} \bigg\|_{l^2(dn)}$$
$$\lesssim \|\widehat{u}(n)\|_{l^2(dn)} \bigg\| \frac{\widehat{v}(n)}{\langle n\rangle} \bigg\|_{l^1(dn)},$$

where in the last step we used Young's inequality for the convolution of sequence. By Cauchy's inequality,

$$\bigg\| \frac{\widehat{v}(n)}{\langle n\rangle} \bigg\|_{l^1(dn)} \le \|\widehat{v}(n)\|_{l^2(dn)} \|\langle n\rangle^{-1}\|_{l^2(dn)} \lesssim \|\widehat{v}(n)\|_{l^2(dn)}. \tag{2.19}$$

Combining (2.18)–(2.19) together and using the Plancherel theorem again, we arrive at

$$S_1 \lesssim \|\widehat{u}(n)\|_{l^2(dn)} \|\widehat{v}(n)\|_{l^2(dn)} = \|u\|_{L^2}\|v\|_{L^2}. \tag{2.20}$$

For S_2, since $|n_2| \ge 2N$, we find

$$m_N(n_2)\langle n_2\rangle = \bigg(\frac{|n_2|}{N}\bigg)^{s-1} \langle n_2\rangle \gtrsim N^{1-s}\langle n_2\rangle^s. \tag{2.21}$$

Using (2.21), the bound $|m_N(n)| \lesssim 1$ and Young's inequality, we obtain

$$S_2 \lesssim \bigg\| \sum_{\substack{n=n_1+n_2, n_1,n_2\in\mathbb{Z}\\ |n_2|\ge 2N}} \frac{m_N(n)}{N^{1-s}\langle n_2\rangle^s} |\widehat{u}(n_1)||\widehat{v}(n_2)| \bigg\|_{l^2(dn,|n|<2N)} \tag{2.22}$$
$$\lesssim \frac{1}{N} \bigg\| \sum_{n=n_1+n_2, n_1,n_2\in\mathbb{Z}} |\widehat{u}(n_1)||\widehat{v}(n_2)| \bigg\|_{l^2(dn,|n|<2N)}$$
$$\lesssim \bigg\| \sum_{n=n_1+n_2, n_1,n_2\in\mathbb{Z}} |\widehat{u}(n_1)||\widehat{v}(n_2)| \bigg\|_{l^\infty(dn)}$$
$$\lesssim \|\widehat{u}(n)\|_{l^2(dn)} \|\widehat{v}(n)\|_{l^2(dn)}.$$

For S_3, note that $m_N(n) = (|n|/N)^{s-1}$, using (2.21) again, we have

$$S_3 \lesssim \bigg\| \sum_{n=n_1+n_2, n_1,n_2\in\mathbb{Z}} \langle n\rangle^{s-1} \widehat{u}(n_1)\widehat{v}(n_2)\langle n_2\rangle^{-s} \bigg\|_{l^2(dn)}. \tag{2.23}$$

By virtue of (2.23), using the Hölder inequality, we get

$$(2.24)\qquad S_3 \lesssim \begin{cases} \|\langle n\rangle^{s-1}\|_{l^2} \Big\| \sum_{n=n_1+n_2} \widehat{u}(n_1)\widehat{v}(n_2)\langle n_2\rangle^{-s} \Big\|_{l^\infty(dn)} & \text{if } 0 \le s < \frac{1}{2}, \\ \|\langle n\rangle^{s-1}\|_{l^4} \Big\| \sum_{n=n_1+n_2} \widehat{u}(n_1)\widehat{v}(n_2)\langle n_2\rangle^{-s} \Big\|_{l^4(dn)} & \text{if } s = \frac{1}{2}, \\ \|\langle n\rangle^{s-1}\|_{l^\infty} \Big\| \sum_{n=n_1+n_2} \widehat{u}(n_1)\widehat{v}(n_2)\langle n_2\rangle^{-s} \Big\|_{l^2(dn)} & \text{if } \frac{1}{2} < s \le 1. \end{cases}$$

Since the $l^p (p = 2, 4, \infty)$ norms of $\langle n\rangle^{s-1}$ in (2.24) are finite, we deduce from Young's inequality that

$$(2.25)\qquad S_3 \lesssim \begin{cases} \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\langle n_2\rangle^{-s}\|_{l^2} & \text{if } 0 \le s < \frac{1}{2}, \\ \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\langle n_2\rangle^{-s}\|_{l^{4/3}} & \text{if } s = \frac{1}{2}, \\ \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\langle n_2\rangle^{-s}\|_{l^1} & \text{if } \frac{1}{2} < s \le 1, \end{cases}$$

$$\lesssim \begin{cases} \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\|_{l^2} \|\langle n_2\rangle^{-s}\|_{l^\infty} & \text{if } 0 \le s < \frac{1}{2}, \\ \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\|_{l^2} \|\langle n_2\rangle^{-s}\|_{l^4} & \text{if } s = \frac{1}{2}, \\ \|\widehat{u}(n_1)\|_{l^2} \|\widehat{v}(n_2)\|_{l^2} \|\langle n_2\rangle^{-s}\|_{l^2}, & \text{if } \frac{1}{2} < s \le 1 \end{cases}$$

$$\lesssim \|u\|_{L^2} \|v\|_{L^2}.$$

Inserting (2.20), (2.22) and (2.25) into (2.17) implies (2.16). □

Corollary 2.4. *The inequality*

$$\|I_N \partial_x (1-\partial_x^2)^{-1}(uv)\|_{H^1} \lesssim \|I_N u\|_{H^1} \|I_N v\|_{H^1}$$

holds with an implicit constant independent of N.

Proof. This is a direct consequence of Lemma 2.3 and the equality

$$\|u\|_{L^2} \lesssim \|I_N u\|_{H^1},$$

which follows from the obvious estimate $1 \lesssim m_N(n)\langle n\rangle$ for $N > 0$. □

2.3. Local well-posedness. Acting with I-operator on both sides of (2.10), (2.11) gives

$$(2.26)\qquad (I_N v)_t - (I_N v)_{txx} - (I_N v)_{xx} + I_N(vv_x) + I_N(Qv)_x = I_N P_{N_1}(f - QQ_x),$$

$$(2.27)\qquad I_N v(0,x) = I_N(u_0(x) - Q).$$

Equation (2.26) is equivalent to

$$(2.28)\qquad (I_N v)_t + A(I_N v) + (1-\partial_x^2)^{-1} I_N(vv_x) + (1-\partial_x^2)^{-1} I_N(Qv)_x = (1-\partial_x^2)^{-1} I_N P_{N_1}(f - QQ_x),$$

where $A = -\partial_x^2(1-\partial_x^2)^{-1}$. Since A is a non-negative operator on $L^2(\mathbb{T})$, we have for every $t > 0$

$$\|e^{-tA}\|_{H^1,H^1} = \|e^{-tA}\|_{L^2,L^2} \leq 1. \tag{2.29}$$

Using the Duhamel principle, we write (2.28) as

$$\begin{aligned} I_N v(t) &= e^{-At} I_N v(0) \\ &\quad + \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1} I_N(P_{N_1}(f - QQ_x) - (Qv)_x - vv_x)\, d\tau. \end{aligned} \tag{2.30}$$

Equation (2.30) can be regarded as an equation of $I_N v$. We define the operator $\Gamma_2\colon L^\infty(0,T;H^1(\mathbb{T})) \to L^\infty(0,T;H^1(\mathbb{T}))$ by

$$\Gamma_2 I_N v = e^{-At} I_N v(0) + \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1} I_N(P_{N_1}(f-QQ_x)-(Qv)_x - vv_x)\, d\tau.$$

We shall show that Γ_2 has a fixed point in the set

$$\mathscr{B}_2 = \Big\{ I_N v \in H^1 : \sup_{0\leq t\leq \delta} \|I_N v(t)\|_{H^1} \leq 2(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}) \Big\}.$$

In fact, thanks to (2.29), we have

$$\|e^{-At} I_N v(0)\|_{H^1} \leq \|I_N v(0)\|_{H^1}, \tag{2.31}$$

and by (2.13)

$$\sup_{0\leq t\leq\delta} \bigg\| \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1} I_N P_{N_1}(f - QQ_x)\, d\tau \bigg\|_{H^1} \leq C_1 \delta \|f\|_{H^{s-2,p}}, \tag{2.32}$$

where C_1 depends on N_1. Moreover, it follows from Lemma 2.3 that

$$\begin{aligned} \sup_{0\leq t\leq\delta} \bigg\| \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1} I_N (Qv)_x\, d\tau \bigg\|_{H^1} \\ \leq C_2 \delta \|Q\|_{H^{s,p}} \sup_{0\leq t\leq\delta} \|I_N v\|_{H^1}. \end{aligned} \tag{2.33}$$

Similarly, by Corollary 2.4

$$\bigg\| \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1} I_N (vv_x)\, d\tau \bigg\|_{H^1} \leq C_2 \delta \sup_{0\leq t\leq\delta} \|I_N v\|_{H^1}^2. \tag{2.34}$$

If we choose $C_3 = \max\{C_1, C_2\}$, then it follows from (2.31)–(2.34) that

$$\begin{aligned} \sup_{0\leq t\leq\delta} \|\Gamma_2 I_N v\|_{H^1} &\leq \|I_N v(0)\|_{H^1} \\ &\quad + C_3\delta(2\|Q\|_{H^{s,p}}(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}) + \|f\|_{H^{s-2,p}}) \\ &\quad + 4C_3\delta(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}})^2, \end{aligned} \tag{2.35}$$

for $I_N v \in \mathscr{B}_2$. Choose first N_1 such that $\|Q\|_{H^{s,p}} \leq \varepsilon_0 \leq 1$ in (2.12) and then $\delta \in (0,1)$ such that

$$4C_3\delta(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}} + 1) \leq 1/2,$$

we find

$$\sup_{0\le t\le\delta} \|\Gamma_2 I_N v\|_{H^1} \le 2(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}).$$

This proves that Γ_2 maps $\mathscr{B}_2$ into $\mathscr{B}_2$. Similarly, with the same choice of δ, we have for $I_N v_1, I_N v_2 \in \mathscr{B}_2$,

$$\sup_{0\le t\le\delta} \|\Gamma_2 I_N v_1 - \Gamma_2 I_N v_1\|_{H^1} \le \sup_{0\le t\le\delta} \frac{1}{2}\|I_N v_1 - I_N v_2\|_{H^1}.$$

Thus, we have proved the following result.

PROPOSITION 2.5. *Let $f \in \dot H^{s-2}(\mathbb{T})$ and $u_0 \in \dot H^s(\mathbb{T})$, $0 \le s \le 1$. Then there is a unique solution of* (2.26)–(2.27)*, such that $I_N v \in L^\infty(0,\delta;\dot H^1(\mathbb{T}))$ and*

$$\sup_{0\le t\le\delta} \|I_N v(t)\|_{H^1} \le 2(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}),$$

where the life span δ satisfies

$$\delta \sim \frac{1}{8C_3(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}} + 1)}.$$

In order to prove Theorem 1.1, we need the following lemma.

LEMMA 2.6. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then the operator $-A = \partial_x^2(1-\partial_x^2)^{-1}$ generates a C_0 semigroup in $\dot H^{s,p}(\mathbb{T})$. Moreover, there exists $\lambda > 0$ such that*

$$\|e^{-At}\|_{\dot H^{s,p},\dot H^{s,p}} \lesssim e^{-\lambda t}, \quad t > 0. \tag{2.36}$$

PROOF. Since A is a bounded operator on $H^{s,p}$, the first statement follows directly. To prove estimate (2.36), note that $(1-\partial_x^2)^{s/2}$ commutes with e^{-At}, it suffices to show that

$$\|e^{-At}\|_{\dot L^p(\mathbb{T}),\dot L^p(\mathbb{T})} \lesssim e^{-\lambda t}, \quad t > 0. \tag{2.37}$$

Thanks to the transference of multipliers, see e.g. [15, Theorem 3.6.7, p. 224], (2.37) follows from

$$\|\chi(D)e^{-At}\|_{L^p(\mathbb{R}),L^p(\mathbb{R})} \lesssim e^{-\lambda t}, \quad t > 0, \tag{2.38}$$

where $\chi(D)e^{-At}$ is a Fourier multiplier defined by

$$\chi(D)e^{-At}\varphi = \mathscr{F}^{-1}(\chi(\xi)e^{-t\xi^2(1+\xi^2)^{-1}}\widehat{\varphi}(\xi))$$

and χ is a smooth function such that $\chi = 0$ near $\xi = 0$ and $\chi = 1$ for $|\xi| \ge 1$. By some elementary calculations, there exists a constant $\lambda > 0$ such that

$$|\partial_\xi^\alpha \chi(\xi)e^{-t\xi^2(1+\xi^2)^{-1}}| \lesssim e^{-\lambda t}\langle\xi\rangle^{-\alpha}$$

for $\alpha = 0$ and 1. Thus, using the Hörmander–Mihlin multiplier theorem, see e.g. [15, Theorem 5.2.7, p. 367], we obtain (2.38). □

Proof of Theorem 1.1. Make the decomposition $u(t) = v(t) + Q$, where Q is the solution given by Proposition 2.2 and v is the solution of (2.10)–(2.11). Since Q is bounded in $H^{s,p}$ and has zero mean, the desired conclusion follows if one can show that, there exists a unique solution $v \in C([0,T];\dot{H}^{s,p}(\mathbb{T}))$ of (2.10)–(2.11), and the map $v(0) \mapsto v(t)$ is continuous in $H^{s,p}(\mathbb{T})$. We divide the proof into three steps.

Step 1. Existence. According to Proposition 2.5, problem (2.10)–(2.11) has a unique solution v such that

$$\sup_{0\le t\le\delta}\|I_N v(t)\|_{H^1} \le 2(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}).$$

Note that, for every $N > 0$ fixed, $\|v\|_{H^s} \le \|I_N v\|_{H^1} \le N^{1-s}\|v\|_{H^s}$, we have

$$\sup_{0\le t\le\delta}\|v(t)\|_{H^s} \le 2(N^{1-s}\|v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}) < \infty. \tag{2.39}$$

Using the Duhamel principle (see (2.30)), we have for $t < \delta$

$$v(t) = e^{-At}v(0) + \int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1}(P_{N_1}(f - QQ_x) - (Qv)_x - vv_x)\,d\tau. \tag{2.40}$$

Thanks to the estimates (2.36) of semigroup and Lemma 2.1, we find

$$\begin{aligned}\|v(t)\|_{H^{s,p}} \lesssim e^{-\lambda t}\|v(0)\|_{H^{s,p}} &+ \int_0^t e^{-\lambda(t-\tau)}(\|P_{N_1}f\|_{H^{s,p}} + \|Q\|^2_{H^{s,p}})\,d\tau \\ &+ \int_0^t e^{-\lambda(t-\tau)}(\|Q\|_{H^{s,p}}\|v(\tau)\|_{H^s} + \|v(\tau)\|^2_{H^s})\,d\tau.\end{aligned} \tag{2.41}$$

In light of (2.39) and (2.12), we have

$$\sup_{0\le t\le\delta}\|v(t)\|_{H^{s,p}} < \infty. \tag{2.42}$$

Taking $T = \delta$, note that v has zero mean, implies the existence.

Step 2. Continuity with respect to time t. Rewrite (2.10) as

$$\begin{aligned}v_t + Av + (1-\partial_x^2)^{-1}(vv_x) + (1-\partial_x^2)^{-1}(Qv)_x \\ = (1-\partial_x^2)^{-1}P_{N_1}(f - QQ_x),\end{aligned} \tag{2.43}$$

where $A = -\partial_x^2(1-\partial_x^2)^{-1}$. Combining (2.42) and Lemma 2.1 implies that

$$\sup_{0\le t\le\delta}\|(1-\partial_x^2)^{-1}(vv_x) + (1-\partial_x^2)^{-1}(Qv)_x\|_{H^{s,p}} < \infty.$$

Moreover, it is easy to see that Av and $(1-\partial_x^2)^{-1}P_{N_1}(f - QQ_x)$ are bounded in $L^\infty(0,\delta;H^{s,p})$. Thus, we obtain

$$\sup_{0\le t\le\delta}\|v_t\|_{H^{s,p}} < \infty.$$

According to the Lions interpolation theorem, we find $v \in C([0,T];H^{s,p})$.

Step 3. Continuity with respect to initial data. Let $\overline{v}(t)$ be the solution of (2.40) with initial data $\overline{v}(0) \in H^{s,p}$. Then

$$v(t) - \overline{v}(t) = e^{-At}(v(0) - \overline{v}(0)) + \int_0^t e^{-A(t-\tau)} \left(Q(v(\tau) - \overline{v}(\tau)) + \frac{v + \overline{v}}{2} (v(\tau) - \overline{v}(\tau)) \right) d\tau.$$

It follows that

$$\|v(t) - \overline{v}(t)\|_{H^{s,p}} \lesssim e^{-\lambda t}\|v(0) - \overline{v}(0)\|_{H^{s,p}} + C(Q, v, \overline{v}) \int_0^t e^{-\lambda(t-\tau)}\|v(\tau) - \overline{v}(\tau)\|_{H^{s,p}}\, d\tau$$

where $C(Q, v, \overline{v}) = \|Q\|_{H^{s,p}} + \sup\limits_{0 \le t \le T} (\|v\|_{H^{s,p}} + \|\overline{v}\|_{H^{s,p}})$. Then an application of Gronwall lemma implies that

$$\|v(t) - \overline{v}(t)\|_{H^{s,p}} \lesssim e^{C(Q,v,\overline{v})t}\|v(0) - \overline{v}(0)\|_{H^{s,p}}.$$

This gives the continuity with respect to initial data. □

3. The global attractor

3.1. Uniform bounds. To obtain global bounds of v, we multiply $I_N v$ on both sides of (2.26), and integrate over $\mathbb{T}$

$$\frac{1}{2} \frac{d}{dt} \|I_N v\|_{H^1}^2 + \|I_N v_x\|^2 + (I_N(vv_x), I_N v) + (I_N(Qv)_x, I_N v) = (I_N P_{N_1}(f - QQ_x), I_N v). \tag{3.1}$$

It is easy to see that $\int_{\mathbb{T}} I_N v\, dx = 0$. Then by the Poincaré inequality (2.2),

$$\|I_N v_x\|^2 \geq \frac{1}{2} \|I_N v\|_{H^1}^2. \tag{3.2}$$

Moreover, recall that (see [30])

$$|(I_N(vv_x), I_N v)| \lesssim N^{-3/2+}\|I_N v\|_{H^1}^3. \tag{3.3}$$

Here and in what follows, we denote by $s+$ that a constant equals s plus a small enough number. Furthermore, using integration by parts and Cauchy's inequality, we find

$$|(I_N(Qv)_x, I_N v)| = |(I_N(Qv), I_N v_x)| \leq \|I_N(Qv)\|_{L^2}\|I_N v_x\|_{L^2} \leq \|Qv\|_{L^2}\|I_N v\|_{H^1}. \tag{3.4}$$

Thanks to the Hölder inequality, we have

$$\|Qv\|_{L^2} \leq \|Q\|_{L^r}\|v\|_{L^q} \tag{3.5}$$

where $1/2 = 1/r + 1/q$. By the Sobolev embedding, the inequalities

$$\|Q\|_{L^r} \lesssim \|Q\|_{H^{s,p}} \quad \text{and} \quad \|v\|_{L^q} \lesssim \|v\|_{H^s} \tag{3.6}$$

hold for s satisfying

$$\frac{1}{r} \geq \frac{1}{p} - s, \qquad \frac{1}{q} \geq \frac{1}{2} - s.$$

Thus, we can choose proper q, r such that (3.5)–(3.6) hold if

$$\frac{1}{2} = \frac{1}{r} + \frac{1}{q} \geq \frac{1}{2} + \frac{1}{p} - 2s.$$

This is always possible if $s \geq 1/(2p)$. Then (3.4) becomes

$$\begin{aligned} |(I_N(Qv)_x, I_N v)| &\leq C\|Q\|_{H^{s,p}}\|v\|_{H^s}\|I_N v\|_{H^1} \\ &\leq C\|Q\|_{H^{s,p}}\|I_N v\|_{H^1}^2 \leq \frac{1}{8}\,\|I_N v\|_{H^1}^2, \end{aligned} \tag{3.7}$$

where the last step is possible if we choose N_1 large enough such that $C\|Q\|_{H^{s,p}} \leq 1/8$, which holds if $\varepsilon_0 \leq 1/(8C)$ in (2.12). Finally, by Cauchy's inequality and (2.13)

$$\begin{aligned} |(I_N P_{N_1}(f - QQ_x), I_N v)| &\leq \|I_N P_{N_1}(f - QQ_x)\|_{L^2}\|I_N v\|_{L^2} \\ &\leq C\|f\|_{H^{s-2,p}}^2 + \frac{1}{8}\,\|I_N v\|_{H^1}^2. \end{aligned} \tag{3.8}$$

Putting (3.2), (3.3), (3.7) and (3.8) into (3.1), we arrive at

$$\frac{d}{dt}\,\|I_N v\|_{H^1}^2 + \frac{1}{2}\,\|I_N v\|_{H^1}^2 \leq CN^{-3/2+}\|I_N v\|_{H^1}^3 + C\|f\|_{H^{s-2,p}}^2. \tag{3.9}$$

Applying the Gronwall lemma to (3.9) over $(0, \delta)$ gives

$$\begin{aligned} \|I_N v(\delta)\|_{H^1}^2 \leq{}& e^{-\delta/4}\|I_N v(0)\|_{H^1}^2 + C\int_0^\delta e^{-\tau/4}\|f\|_{H^{s-2,p}}^2\,d\tau \\ &+ \int_0^\delta e^{-(\delta-\tau)/4}\bigg(CN^{-3/2+}\|I_N v\|_{H^1} - \frac{1}{4}\bigg)\|I_N v\|_{H^1}^2. \end{aligned} \tag{3.10}$$

The integral on right hand side of (3.10) is negative, in view of the estimates of $\|I_N v\|_{H^1}$ in Proposition 2.5, if

$$2CN^{-3/2+}(\|I_N v(0)\|_{H^1} + \|f\|_{H^{s-2,p}}) \leq \frac{1}{2}.$$

This is always possible, since $\|I_N v(0)\|_{H^1} \leq N^{1-s}\|v(0)\|_{H^s}$, if

$$N \sim C(\|v(0)\|_{H^s} + \|f\|_{H^{s-2,p}} + 1)^{2+}.$$

Then

$$\|I_N v(\delta)\|_{H^1}^2 \leq e^{-\delta/4}\|I_N v(0)\|_{H^1}^2 + C\|f\|_{H^{s-2,p}}^2. \tag{3.11}$$

In light of (3.11), we can take $I_N w(\delta)$ as a new data, to obtain a solution on $[\delta, 2\delta]$. Repeat this process, we obtain for all $n \geq 1$

$$\|I_N v(n\delta)\|_{H^1}^2 \leq e^{-\delta/4}\|I_N v((n-1)\delta)\|_{H^1}^2 + C\|f\|_{H^{s-2,p}}^2. \tag{3.12}$$

It follows from (3.12) that

$$\|I_N v(n\delta)\|_{H^1}^2 \le e^{-n\delta/4}\|I_N v(0)\|_{H^1}^2 + \sum_{j=0}^{n-1} Ce^{-j\delta/4}\|f\|_{H^{s-2,p}}^2.$$

Thus, for any $t > 0$,

$$\begin{aligned}(3.13)\quad \|v(t)\|_{H^s}^2 \le \|I_N v(t)\|_{H^1}^2 &\lesssim e^{-t/4}\|I_N v(0)\|_{H^1}^2 + C\|f\|_{H^{s-2,p}}^2 \\ &\lesssim e^{-t/4}(\|v(0)\|_{H^s} + \|f\|_{H^{s-2,p}} + 1)^{2(1-s)+}\|v(0)\|_{H^s}^2 + C\|f\|_{H^{s-2,p}}^2.\end{aligned}$$

Let B be a bounded set in $H^{s,p}$, $u_0 \in B$. Note that $v(0) = u_0 - Q$ is also bounded in $H^{s,p}$. It follows from (3.13) that there exists $T = T(B)$ such that

$$\|v(t)\|_{H^s} \le C\|f\|_{H^{s-2,p}}, \quad t \ge T.$$

Combining the bound, (2.41) and (2.12), we have for $t \ge T$

$$\|v(t)\|_{H^{s,p}} \le C(\|f\|_{H^{s-2,p}} + \|f\|_{H^{s-2,p}}^2).$$

Note that $u = v + Q$ and $\|Q\|_{H^{s,p}} \le \|f\|_{H^{s-2,p}}$, for $t \ge T$ we obtain

$$\|u(t)\|_{H^{s,p}} \le C(\|f\|_{H^{s-2,p}} + \|f\|_{H^{s-2,p}}^2).$$

Thus, we have proved the following result.

Theorem 3.1. *Assume that* $u_0 \in \dot H^{s,p}(\mathbb{T}), f \in \dot H^{s-2,p}(\mathbb{T})$, $2 \le p < \infty$, $1/(2p) \le s < 1$. *Then problem* (1.1)–(1.2) *is global well-posed in* $\dot H^{s,p}(\mathbb{T})$. *Moreover, there is a bounded absorbing set in* $\dot H^{s,p}(\mathbb{T})$ *given by*

$$\mathcal{B} = \{u \in L^2(\mathbb{T}) : \|u\|_{H^{s,p}} \le C(\|f\|_{H^{s-2,p}} + \|f\|_{H^{s-2,p}}^2)\}.$$

Remark 3.2. Compared with the local existence (Theorem 1.1), we need an additional assumption $s \ge 1/(2p)$ in Theorem 3.1. This assumption is used to deal with the term Qv, where the regularity of Q is not good enough for smaller s. The difficulty is of course caused by the distribution force $f \in \dot H^{s-2,p}$. Whether Theorem 3.1 holds for $0 \le s < 1/(2p)$ is open.

3.2. The asymptotic compactness of solution semigroup. We review the definition of the Kuratowski measure of non-compactness. Let X be a Banach space and A be a bounded subset of X. The Kuratowski measure of non-compactness $\kappa(A)$ is defined by

$$\kappa(A) = \inf\{\delta > 0 : A \text{ has a finite open cover of sets of diameter} < \delta\}.$$

It is obvious that Kuratowski measure $\kappa(A)$ depends on the metric of X. Sometimes we shall write $\kappa_X(A)$ instead, to emphasis the metric used, in the following. Some important properties of $\kappa(A)$ are summarized as follows, see e.g. [16] for a proof.

Lemma 3.3. $\kappa(A)$ *satisfies the following properties*:

(a) $\kappa(A) = 0$ *if and only if* $\overline{A}$ *is compact, where* $\overline{A}$ *is the closure of* A,
(b) $\kappa(A) \le d(A)$, $d(A)$ *denotes the diameter of* A,
(c) $\kappa(A + B) \le \kappa(A) + \kappa(B)$ *for any* $A, B \subset X$.

For the convenience of the reader, we recall the following criterion of the existence of a global attractor, see e.g. [23].

PROPOSITION 3.4. *Let* X *be a Banach space and* $\{S(t)\}_{t\ge 0}$ *be a continuous semigroup on* X. *Then* $\{S(t)\}_{t\ge 0}$ *has a global attractor in* X *provided that the following conditions hold*:

(a) $\{S(t)\}_{t\ge 0}$ *has a bounded absorbing set in* X,
(b) *for any bounded subset* B *of* X, *we have* $\kappa_X(S(t)B) \to 0$, *as* $t \to \infty$.

We shall need the following bilinear estimates (compared with Lemma 2.1).

LEMMA 3.5. *Let* $2 \le p < \infty$. *If* $1/(2p) \le s \le 1$, *then*

$$\|\partial_x (1 - \partial_x^2)^{-1}(uv)\|_{H^{s+\sigma,p}} \lesssim \|u\|_{H^{s,p}} \|v\|_{H^{s,p}},$$

where $\sigma > 0$ *is given by*

$$\sigma = \begin{cases} 1 - s & \text{if } \dfrac{1}{2p} \le s < 1, \\ 1 - \dfrac{1}{p} & \text{if } s = 1. \end{cases}$$

PROOF. In the case $1/(2p) \le s < 1$, thanks to the Sobolev embedding $H^{s,p} \hookrightarrow L^{2p}$, it suffices to show

$$\|\partial_x (1 - \partial_x^2)^{-1}(uv)\|_{H^{1,p}} \lesssim \|u\|_{L^{2p}} \|v\|_{L^{2p}}.$$

Using the fact that $\partial_x (1 - \partial_x^2)^{-1/2}$ is bounded in L^p, the Hölder inequality, we find

$$\|\partial_x (1 - \partial_x^2)^{-1}(uv)\|_{H^{1,p}} \lesssim \|\partial_x (1 - \partial_x^2)^{-1/2}(uv)\|_{L^p} \lesssim \|uv\|_{L^p} \lesssim \|u\|_{L^{2p}} \|v\|_{L^{2p}}.$$

In the case $s = 1$, using the Leibniz rule $\nabla(uv) = \nabla uv + u\nabla v$, we only need to show that

$$\|\partial_x (1 - \partial_x^2)^{-1}(\nabla uv)\|_{H^{\sigma,p}} \lesssim \|u\|_{H^{1,p}} \|v\|_{H^{1,p}}.$$

The inequality follows if one can show that

$$\|\partial_x (1 - \partial_x^2)^{-1/2}(1 - \partial_x^2)^{(-1+\sigma)/2}(\nabla uv)\|_{L^p} \lesssim \|u\|_{H^{1,p}} \|v\|_{H^{1,p}}. \tag{3.14}$$

In fact, since $\partial_x (1 - \partial_x^2)^{-1/2}$ is bounded in L^p and $(1 - \partial_x^2)^{(-1+\sigma)/2}$ is bounded from $L^{p/2}$ to L^p, we have

$$\text{LHS (3.14)} \lesssim \|\nabla uv\|_{L^{p/2}} \lesssim \|\nabla u\|_{L^p} \|v\|_{L^p},$$

which gives (3.14) directly. □

Let B be a bounded set in $H^{s,p}$ and $u_0 \in B$. Let $S(t)$ be the solution semigroup of (1.1)–(1.2), namely $S(t)\colon u_0 \mapsto u(t)$. Denote the set

$$S(t)B = \{u(t) = S(t)u_0 : u_0 \in B\}.$$

LEMMA 3.6. *Assume that* $2 \le p < \infty$ *and* $1/(2p) \le s \le 1$. *Then* $S(t)$ *is asymptotic compact in* $H^{s,p}$, *namely*

$$\lim_{t\to\infty} \kappa_{H^{s,p}}(S(t)B) = 0.$$

PROOF. Thanks to Theorem 3.1, we known that problem (1.1)–(1.2) has a bounded absorbing set $\mathcal{B}$ in $H^{s,p}$. Thus it suffices to show

$$\lim_{t\to\infty} \kappa_{H^{s,p}}(S(t)\mathcal{B}) = 0. \tag{3.15}$$

Since $u(t) = v(t) + Q$ and the non-compact measure of single point $\kappa(Q) = 0$, it follows from (c) of Lemma 3.3 that

$$\kappa_{H^{s,p}}(S(t)B) = \kappa_{H^{s,p}}(v(t) : u_0 \in \mathcal{B}).$$

According to (2.40), we have the decomposition $\{v(t) : u_0 \in \mathcal{B}\} = \mathcal{K}_1 + \mathcal{K}_2$, where

$$\begin{aligned}
&\mathcal{K}_1 = \{e^{-At}(u_0 - Q) : u_0 \in \mathcal{B}\},\\
&\mathcal{K}_2 = \bigg\{\int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1}(P_{N_1}(f - QQ_x) - (Qv)_x - vv_x)\,d\tau : u_0 \in \mathcal{B}\bigg\}.
\end{aligned}$$

According to Lemma 2.6, we known that e^{-At} is a C_0 semigroup in $H^{s,p}$ with exponential decay. Then it follows from (a) of Lemma 3.3 that

$$\lim_{t\to\infty} \kappa_{H^{s,p}}(\mathcal{K}_1) = 0. \tag{3.16}$$

Moreover, thanks to Theorem 3.1, $\|v(t)\|_{H^{s,p}} \le C$ for $t \ge T(\mathcal{B})$. Combining this fact and Lemma 3.5, for all $t \ge T(\mathcal{B})$ we have

$$\begin{aligned}
&\bigg\|\int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1}((Qv)_x + vv_x)\,d\tau\bigg\|_{H^{s+\sigma,p}}\\
&\qquad\lesssim \int_0^t e^{-\lambda(t-\tau)}(\|Q\|_{H^{s,p}}\|v\|_{H^{s,p}} + \|v\|^2_{H^{s,p}})\,d\tau \le C.
\end{aligned}$$

Moreover, thanks to (2.13),

$$\begin{aligned}
&\bigg\|\int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1}P_{N_1}(f - QQ_x)\,d\tau\bigg\|_{H^{s+\sigma,p}}\\
&\qquad\lesssim \bigg\|\int_0^t e^{-A(t-\tau)}(1-\partial_x^2)^{-1}P_{N_1}(f - QQ_x)\,d\tau\bigg\|_{H^{s+\sigma+1}}\\
&\qquad\lesssim \int_0^t e^{-\lambda(t-\tau)}\|P_{N_1}(f - QQ_x)\|_{H^{s+\sigma+1}}\,d\tau\\
&\qquad\lesssim (1+N_1^2)^{1+(s+\sigma+1)/2}\|f\|_{H^{s-2,p}}.
\end{aligned}$$

Since $\sigma > 0$, the Sobolev embedding $H^{s+\sigma,p} \hookrightarrow H^{s,p}$ is compact. Thus $\mathcal{K}_2$ is compact in $H^{s,p}$. Then, for all $t > 0$,

$$\kappa_{H^{s,p}}(\mathcal{K}_2) = 0. \tag{3.17}$$

Using (c) of Lemma 3.3 again, it follows from (3.16)–(3.17) that (3.15) holds. □

Proof of Theorem 1.2. This is a direct consequence of Theorem 3.1, Lemma 3.6 and Proposition 3.4. □

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