A NOTE ON DIMENSIONAL ENTROPY
FOR AMENABLE GROUP ACTIONS

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ABSTRACT. In this short note, for countably infinite amenable group actions, we provide topological proofs for the following results: Bowen topological entropy (dimensional entropy) of the whole space equals the usual topological entropy along tempered Følner sequences; the Hausdorff dimension of an amenable subshift (for certain metric associated to some Følner sequence) equals its topological entropy. This answers questions by Zheng and Chen [10] and Simpson [9].

1. Introduction

Let (X, G) be a G-action topological dynamical system, where X is a compact Hausdorff space and G a topological group. Throughout this paper, G is always assumed to be a countably infinite amenable group, i.e. there exists a sequence of nonempty finite subsets $\{F_n\}$ of G (a *Følner sequence*) such that

$$\lim_{n \to +\infty} \frac{|F_n \triangle gF_n|}{|F_n|} = 0, \quad \text{for all } g \in G.$$

Let A and K be two nonempty finite subsets of G. For $\delta > 0$, the set A is said to be (K, δ)-*invariant* if

$$\frac{|B(A, K)|}{|A|} < \delta,$$

where $B(A, K)$, the K-*boundary* of A, is defined by

$$B(A, K) = \{g \in G : Kg \cap A \neq \emptyset \text{ and } Kg \cap (G \setminus A) \neq \emptyset\}.$$

Another equivalent condition for the sequence of finite subsets $\{F_n\}$ of G to be a Følner sequence is that $\{F_n\}$ becomes more and more invariant, i.e. for any $\delta > 0$ and any finite subset K of G, F_n is (K, δ)-invariant for sufficiently large n. For more information on amenable groups and their actions one may refer to [2], [5], [7].

For the case $G = \mathbb{Z}$, by resembling the definition of Hausdorff dimension, Bowen [1] introduced a definition of topological entropy on subsets. This definition is also known as dimensional entropy and has plenty of applications to thermodynamical formulism, fractal geometry, hyperbolic systems, multi-fractal analysis and so on (see, for example, [3] and [8]).

For the case that G is a general countably infinite amenable group, Bowen's dimensional entropy was recently introduced in [10] in the following way.

Let $\mathcal{U}$ be a finite open cover of X. For a subset F of G, denote by $\mathcal{W}_F(\mathcal{U})$ the collection of families $\mathbf{U} = \{U_g\}_{g \in F}$ with $U_g \in \mathcal{U}$ (we also call $\mathbf{U}$ a $\mathcal{U}$-word or a $(\mathbf{U}, F)$-name). For $\mathbf{U} \in \mathcal{W}_F(\mathcal{U})$ we call the integer $m(\mathbf{U}) = |F|$ the length of $\mathbf{U}$ and the set F the domain of $\mathbf{U}$ (denoted by $\mathrm{dom}(\mathbf{U})$). For any $\mathcal{U}$-word $\mathbf{U}$, define

$$\mathbf{X}(\mathbf{U}) := \bigcap_{g \in \mathrm{dom}(\mathbf{U})} g^{-1} U_g = \{x \in X : gx \in U_g \text{ for } g \in \mathrm{dom}(\mathbf{U})\}.$$

Now let $\{F_n\}$ be a Følner sequence in G. For $Z \subset X$, we say that a collection of $(\mathbf{U}, F_n)$-names $\Lambda \subset \bigcup_{n \geq 1} \mathcal{W}_{F_n}(\mathcal{U})$ covers Z if $\bigcup_{\mathbf{U} \in \Lambda} \mathbf{X}(\mathbf{U}) \supset Z$. For $s \geq 0$, define

$$\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) = \inf_{\Lambda} \left\{ \sum_{\mathbf{U} \in \Lambda} \exp(-sm(\mathbf{U})) \right\},$$

where the infimum is taken over all $\Lambda \subset \bigcup_{j \geq N} \mathcal{W}_{F_j}(\mathcal{U})$ that covers Z. It is not hard to see that $\mathcal{M}(\cdot, \mathcal{U}, N, s, \{F_n\})$ is a finite outer measure on X. As $\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\})$ increases when N increases, then define

$$\mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\})$$

and

$$h_{\mathrm{top}}^B(\mathcal{U}, Z, \{F_n\}) = \inf\{s \geq 0 : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = 0\}$$
$$= \sup\{s \geq 0 : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = +\infty\}.$$

Set

$$h_{\text{top}}^B(Z, \{F_n\}) = \sup_{\mathcal{U}} h_{\text{top}}^B(\mathcal{U}, Z, \{F_n\}),$$

where $\mathcal{U}$ runs over finite open covers of X. We call $h_{\text{top}}^B(Z, \{F_n\})$ the Bowen topological entropy or dimensional entropy of Z (w.r.t. the Følner sequence $\{F_n\}$).

Recall that a Følner sequence $\{F_n\}$ in G is said to be *tempered* if there exists a constant $C > 0$ which is independent of n such that

$$(1.1) \qquad \left| \bigcup_{k<n} F_k^{-1} F_n \right| \leq C|F_n|, \quad \text{for any } n.$$

In [10], Zheng and Chen proved that for any tempered Følner sequence $\{F_n\}$ in G with the increasing condition $\lim_{n \to +\infty} |F_n|/\log n = \infty$, the Bowen topological entropy $h_{\text{top}}^B(X, \{F_n\})$ coincides with the usual topological entropy. This generalized Bowen's classical result in [1] to discrete countable amenable group actions.

In contrast to Bowen's original topological proof, Zheng and Chen used tools in ergodic theory — they employed SMB theorem, Brin–Katok's formula for local entropy and variation principle for topological entropy. They also asked in [10] whether there exists a pure topological proof.

In this note, we will give a direct proof for this result. More precisely, we will prove the following

THEOREM 1.1 (Main theorem). *Let (X, G) be a G-action topological dynamical system where X is a compact Hausdorff space and G is a discrete countable amenable group, then for any tempered Følner sequence $\{F_n\}$ in G, we have*

$$h_{\text{top}}^B(X, \{F_n\}) = h_{\text{top}}(X, G),$$

where $h_{\text{top}}(X, G)$ is the topological entropy of (X, G) defined through open covers.

The itinerary of our proof uses Simpson's idea for proving the relations between entropy and Hausdorff dimension of $\mathbf{Z}^d$-subshifts (the idea was originally by Furstenberg [4] and it was also used in Bowen [1]). But there exists essential difficulty for amenable group actions. The key point is that we need a Vitali type covering lemma. To this aim we employ a more general covering lemma developed by Lindenstrauss [6] (for proving the amenable version of the pointwise ergodic theorem). In Section 2 we will give the detailed proof.

2. Proof of the main theorem

We first recall the covering lemma by Lindenstrauss.

LEMMA 2.1 (Lindenstrauss [6, Corollary 2.7]). *For any $\delta \in (0, 1/100)$, $C > 0$ and finite $D \subset G$, let $M \in \mathbb{N}$ be sufficiently large (depending only on δ, C and D).*

Let $F_{i,j}$ be an array of finite subsets of G where $i = 1, \ldots, M$ and $j = 1, \ldots, N_i$, such that

(a) *For every i, $\overline{F}_{i,*} = \{F_{i,j}\}_{j=1}^{N_i}$ satisfies*

$$\left| \bigcup_{k' < k} F_{i,k'}^{-1} F_{i,k} \right| \leq C|F_{i,k}|, \quad for \ k = 2, \ldots, N_i.$$

Denote $F_{i,*} = \bigcup \overline{F}_{i,*}$.

(b) *The finite set sequences $F_{i,*}$ satisfy that for every $1 < i \leq M$ and every $1 \leq k \leq N_i$,*

$$\left| \bigcup_{i' < i} DF_{i',*}^{-1} F_{i,k} \right| \leq (1 + \delta)|F_{i,k}|.$$

Assume that $A_{i,j}$ is another array of finite subsets of G with $F_{i,j} A_{i,j} \subset F$ for some finite subset F of G. Let $A_{i,*} = \bigcup_j A_{i,j}$ and

$$\alpha = \frac{\min_{1 \leq i \leq M} |DA_{i,*}|}{|F|}.$$

Then the collection of subsets of F,

$$\widetilde{\mathcal{F}} = \{F_{i,j} a : 1 \leq i \leq M, \ 1 \leq j \leq N_i \ and \ a \in A_{i,j}\}$$

has a subcollection $\mathcal{F}$ that is $10\delta^{1/4}$-disjoint such that

$$\left| \bigcup \mathcal{F} \right| \geq (\alpha - \delta^{1/4})|F|.$$

We note here that a collection $\widetilde{\mathcal{F}}$ of finite subsets of G is said to be δ-disjoint if for every $A \in \widetilde{\mathcal{F}}$ there exists an $A' \subset F$ such that $|A'| \geq (1 - \delta)|A|$ and such that $A \cap B = \emptyset$ for every $A \neq B \in \widetilde{\mathcal{F}}$.

Now we turn to proving of our main theorem. The proof of the upper bound is straight-forward and has been shown in Section 4 of [10]. We omit it here.

In the following we will give the proof of the lower bound.

For any $\varepsilon > 0$, let $0 < \delta < \min\{\varepsilon, 1/100\}$ be small enough such that

$$(2.1) \quad -(1 - 2\delta - 11\delta^{1/4})\log(1 - 2\delta - 11\delta^{1/4})$$
$$-(2\delta + 11\delta^{1/4})\log(2\delta + 11\delta^{1/4}) < \varepsilon.$$

Let $D = \{e_G\} \subset G$ and let $C > 0$ be the constant in the tempered condition (1.1) for the Følner sequence $\{F_n\}$. Let $M > 0$ be large enough to satisfy the requirement of Lemma 2.1 corresponding to δ, D and C.

Let $\mathcal{U}$ be any finite open cover of X and $s > 0$ such that $\mathcal{M}(X, \mathcal{U}, s, \{F_n\}) = 0$. We will show $h_{\text{top}}(G, \mathcal{U}) \leq s$ and then it follows that

$$h_{\text{top}}^B(X, \mathcal{U}, \{F_n\}) \geq h_{\text{top}}(G, \mathcal{U}).$$

Since $\mathcal{M}(X, \mathcal{U}, s, \{F_n\}) = \lim\limits_{N \to +\infty} \mathcal{M}(X, \mathcal{U}, N, s, \{F_n\}) = 0$, for each $i = 1, 2, \ldots$, there exist $0 < p_i$ and $\Lambda_i \subset \bigcup\limits_{j \geq p_i} \mathcal{W}_{F_j}(\mathcal{U})$ such that Λ_i covers X and

$$\sum_{\mathbf{U} \in \Lambda_i} \exp(-sm(\mathbf{U})) < 2^{-i}.$$

As X is compact, we may let each Λ_i be finite. In addition, we let the sequence $\{p_i\}$ be increasing and

$$(2.2) \qquad |F_j| > \frac{1}{\delta(1 - 10\delta^{1/4})}, \quad \text{for all } j \geq p_1.$$

Denote $\Lambda_\infty = \bigcup\limits_{i=1}^{\infty} \Lambda_i$. Then

$$\sum_{\mathbf{U} \in \Lambda_\infty} \exp(-sm(\mathbf{U})) < \sum_{i=1}^{\infty} 2^{-i} = 1.$$

Hence

$$\sum_{k=1}^{\infty} \sum_{\mathbf{U}_1, \ldots, \mathbf{U}_k \in \Lambda_\infty} \exp(-s(m(\mathbf{U}_1) + \ldots + m(\mathbf{U}_k)))$$
$$= \sum_{k=1}^{\infty} \left(\sum_{\mathbf{U} \in \Lambda_\infty} \exp(-sm(\mathbf{U})) \right)^k = S < \infty.$$

For each $i = 1, \ldots, M$, let $\{F_{n_{i,1}}, \ldots, F_{n_{i,N_i}}\} = \{\text{dom}(\mathbf{U}) : \mathbf{U} \in \Lambda_i\}$ with $n_{i,1} < \ldots < n_{i,N_i}$. Then let $\{F_{i,1}, \ldots, F_{i,N_i}\}$ in Lemma 2.1 be as

$$\{F_{i,1}, \ldots, F_{i,N_i}\} := \{F_{n_{i,1}}, \ldots, F_{n_{i,N_i}}\}.$$

For any $x \in X$ and sufficiently large n (independent on x), let

$$A_{i,j} = \{a \in F_n : F_{i,j}a \subset F_n \text{ and there exists } \mathbf{U} \in \Lambda_i$$
$$\text{such that } \text{dom}(\mathbf{U}) = F_{i,j} \text{ and } ax \in \mathbf{X}(\mathbf{U})\}.$$

We note here that $A_{i,j}$ depends on x. Denote by e_G the identity element of G. For any $g \in F_n \setminus B(F_n, F_{i,*} \cup \{e_G\})$, we have $F_{i,*}g \subset F_n$. Hence for all $1 \leq j \leq N_i$, $F_{i,j}g \subset F_n$. Since Λ_i covers X, there exists some $\mathbf{U} \in \Lambda_i$ which contains gx. Suppose $\text{dom}(\mathbf{U}) = F_{i,j}$. We then have $g \in A_{i,j}$. This implies that

$$A_{i,*} \supset F_n \setminus B(F_n, F_{i,*} \cup \{e_G\}).$$

Hence when n is large enough such that F_n is $(F_{i,*} \cup \{e_G\}, \delta)$-invariant for all $1 \leq i \leq M$ and $1 \leq j \leq N_i$,

$$\alpha = \frac{\min\limits_{1 \leq i \leq M} |DA_{i,*}|}{|F_n|} > 1 - \delta.$$

The first requirement for the array $\{F_{i,j}\}$ in Lemma 2.1 comes from the tempered condition of $\{F_n\}$. To ensure the second requirement, we can make p_i be large enough compared with $n_{i-1,N_{i-1}}$ for every $2 \leq i \leq M$.

Now we can apply Lemma 2.1 to the collection of subsets of F_n,

$$\widetilde{\mathcal{F}} = \{F_{i,j}a : 1 \leq i \leq M, \ 1 \leq j \leq N_i \text{ and } a \in A_{i,j}\}$$

to obtain a subcollection $\mathcal{F}$ that is $10\delta^{1/4}$-disjoint such that

$$\left| \bigcup \mathcal{F} \right| \geq (1 - \delta - \delta^{1/4})|F_n|.$$

For each element in $\mathcal{F}$, we will fix one way to write it as the form $F_{i,j}a$. Denote by $\overline{A}$ the collection of a's such that $F_{i,j}a$ occurs in $\mathcal{F}$. The cardinality of $\overline{A}$ is no more than the cardinality of the subcollection $\mathcal{F}$. Since $\mathcal{F}$ is $10\delta^{1/4}$-disjoint, we have that

$$\sum_{F_{i,j}a \in \mathcal{F}} |F_{i,j}a| \leq \frac{1}{1 - 10\delta^{1/4}} \left| \bigcup \mathcal{F} \right| \leq \frac{1}{1 - 10\delta^{1/4}} |F_n|.$$

Hence, by (2.2),

$$|\overline{A}| \leq |\mathcal{F}| \leq \frac{1}{\min |F_{i,j}|} \cdot \frac{1}{1 - 10\delta^{1/4}} |F_n| \leq \delta|F_n|.$$

We note that for each $a \in \overline{A}$, there may exist more than one $F_{i,j}$'s with $F_{i,j}a \in \mathcal{F}$. Denote by $n(a)$ the total number of such $F_{i,j}$'s.

We now construct $\mathbf{U}(x) = \{U_g(x)\}_{g \in F_n} \in \mathcal{W}_{F_n}(\mathcal{U})$, a $(\mathbf{U}, F_n)$-name associated to x, in the following way:

1. For each g that lies in exactly one element of $\mathcal{F}$, say $F_{i,j}a$, assume that $ax \in \mathbf{X}(\mathbf{U})$ for $\mathbf{U} = \{U_g\}_{g \in F_{i,j}} \in \Lambda_i$. We then let $U_g(x) = U_{ga^{-1}}$.
2. For each g that lies in either $F_n \setminus \bigcup \mathcal{F}$ or the overlapping part of elements in $\mathcal{F}$, we choose any $U \in \mathcal{U}$ that contains gx and then let $U_g(x) = U$.

Clearly $x \in \mathbf{X}(\mathbf{U}(x)) \in \mathcal{U}^{F_n}$ and hence $\{\mathbf{X}(\mathbf{U}(x))\}_{x \in X}$ forms a subcover of $\mathcal{U}^{F_n}$. In the following we will estimate the cardinality of the subcover $\{\mathbf{X}(\mathbf{U}(x))\}_{x \in X}$. Let $\Lambda = \bigcup_{x \in X} \{\mathbf{U}(x)\}$. An element $\mathbf{U} = \{U_g\}_{g \in F_n}$ in Λ is determined by two parts:

1. the U_g's for the g's that lie in exactly one element of $\mathcal{F}$;
2. the U_g's for the g's that lies in either $F_n \setminus \bigcup \mathcal{F}$ or the overlapping part of elements in $\mathcal{F}$.

The first part corresponds to the choice of $\mathcal{F}$ and the choice of the sequence of $\mathbf{U}_1, \ldots, \mathbf{U}_k$ with $m(\mathbf{U}_1) + \ldots + m(\mathbf{U}_k) \leq |F_n|/(1 - 10\delta^{1/4})$ (here k is the cardinality of $\mathcal{F}$). When $\mathbf{U}_1, \ldots, \mathbf{U}_k$ are fixed, if we know $\overline{A}$ and the number $n(a)$ for each $a \in \overline{A}$, $\mathcal{F}$ is then determined. Hence the first part of $\mathbf{U} \in \Lambda$ can be determined by $\overline{A}$, $\{n(a)\}_{a \in \overline{A}}$ with $\sum_{a \in \overline{A}} n(a) = k$ and the sequence of $\mathbf{U}_1, \ldots, \mathbf{U}_k$

with $m(\mathbf{U}_1)+\ldots+m(\mathbf{U}_k)\leq|F_n|/(1-10\delta^{1/4})$. The second part corresponds to the choice of U_g's for g in either $F_n\setminus\bigcup\mathcal{F}$ or the overlapping part of $\bigcup\mathcal{F}$.

Since $|\overline{A}|\leq\delta|F_n|$, the total number of the choices of $\overline{A}$'s is bounded from above by $\sum_{m=1}^{\lfloor\delta|F_n|\rfloor}\binom{|F_n|}{m}$. Applying the Stirling formula

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n},\qquad\frac{1}{12n+1}<\alpha_n<\frac{1}{12n},$$

we have that

$$
\begin{aligned}
\sum_{m=1}^{\lfloor\delta|F_n|\rfloor}\binom{|F_n|}{m}\leq{}&\delta|F_n|\cdot\frac{\sqrt{2\pi|F_n|}e^{\alpha_{|F_n|}-\alpha_{|F_n|-\lfloor\delta|F_n|\rfloor}-\alpha_{\lfloor\delta|F_n|\rfloor}}}{\sqrt{2\pi(|F_n|-\lfloor\delta|F_n|\rfloor)}\sqrt{2\pi\lfloor\delta|F_n|\rfloor}}\\
&\cdot\left(\frac{|F_n|}{|F_n|-\lfloor\delta|F_n|\rfloor}\right)^{|F_n|-\lfloor\delta|F_n|\rfloor}\cdot\left(\frac{|F_n|}{\lfloor\delta|F_n|\rfloor}\right)^{\lfloor\delta|F_n|\rfloor}\\
\leq{}&\delta|F_n|\cdot\frac{\sqrt{2\pi|F_n|}e^{\alpha_{|F_n|}-\alpha_{|F_n|-\lfloor\delta|F_n|\rfloor}-\alpha_{\lfloor\delta|F_n|\rfloor}}}{\sqrt{2\pi(|F_n|-\lfloor\delta|F_n|\rfloor)}\sqrt{2\pi\lfloor\delta|F_n|\rfloor}}\\
&\cdot\left(\frac{1}{1-\delta}\right)^{(1-\delta)|F_n|+1}\cdot\left(\frac{1}{\delta}\right)^{\delta|F_n|}\cdot\left(\frac{\delta|F_n|}{\lfloor\delta|F_n|\rfloor}\right)^{\lfloor\delta|F_n|\rfloor}\\
\leq{}&\delta|F_n|\cdot\frac{\sqrt{2\pi|F_n|}e^{\alpha_{|F_n|}-\alpha_{|F_n|-\lfloor\delta|F_n|\rfloor}-\alpha_{\lfloor\delta|F_n|\rfloor}}}{\sqrt{2\pi(|F_n|-\lfloor\delta|F_n|\rfloor)}\sqrt{2\pi\lfloor\delta|F_n|\rfloor}}\\
&\cdot\frac{1}{1-\delta}\cdot e\cdot\left(\frac{1}{1-\delta}\right)^{(1-\delta)|F_n|}\cdot\left(\frac{1}{\delta}\right)^{\delta|F_n|}\\
={}&Q(|F_n|)\cdot\exp((-(1-\delta)\log(1-\delta)-\delta\log\delta)|F_n|),
\end{aligned}
$$

where

$$Q(n)=\frac{\delta}{1-\delta}\cdot n\cdot\sqrt{\frac{n}{2\pi(n-\lfloor\delta n\rfloor)\lfloor\delta n\rfloor}}\cdot e^{\alpha_n-\alpha_{n-\lfloor\delta n\rfloor}-\alpha_{\lfloor\delta n\rfloor}+1}.$$

We can choose n sufficiently large to make $Q(|F_n|)\leq\exp(\varepsilon|F_n|)$. Together with (2.1), we can obtain that the total number of the choices of $\overline{A}$'s will not exceed $\exp(2\varepsilon|F_n|)$ when n is large enough.

The total number of the choices of the sequence $\{n(a)\}_{a\in\overline{A}}$ is no larger than $\binom{|\mathcal{F}|}{|A|}$, which can be bounded by $\exp(\varepsilon|F_n|)$.

The cardinality of the union of $F_n\setminus\bigcup\mathcal{F}$ and the overlapping part of $\bigcup\mathcal{F}$ is no more than $(\delta+\delta^{1/4}+10\delta^{1/4})|F_n|$. Also note that for each g of such case, U_g has $\#\mathcal{U}$ many choices. Hence the cardinality of Λ is no more than $\exp((\delta+11\delta^{1/4})|F_n|\log\#\mathcal{U})\cdot\exp(2\varepsilon|F_n|)\cdot\exp(\varepsilon|F_n|)$ times the total number of the sequences of $\mathbf{U}_1,\ldots,\mathbf{U}_k\in\Lambda_\infty$ with $m(\mathbf{U}_1)+\ldots+m(\mathbf{U}_k)\leq|F_n|/(1-10\delta^{1/4})$.

Finally,

$$N(\mathcal{U}^{F_n})2^{-s|F_n|/(1-10\delta^{1/4})}$$

$$\leq \exp((\delta + 11\delta^{1/4})|F_n|\log \#\mathcal{U}) \cdot \exp(3\varepsilon|F_n|)$$

$$\cdot \sum_{m(\mathbf{U}_1)+...+m(\mathbf{U}_k)\leq|F_n|/(1-10\delta^{1/4})} 2^{-s|F_n|/(1-10\delta^{1/4})}$$

$$\leq \exp(((\delta + 11\delta^{1/4})\log \#\mathcal{U} + 3\varepsilon)|F_n|)$$

$$\cdot \sum_{\mathbf{U}_1,...,\mathbf{U}_k\in\Lambda_\infty} 2^{-s(m(\mathbf{U}_1)+...+m(\mathbf{U}_k))}$$

$$\leq \exp(((\delta + 11\delta^{1/4})\log \#\mathcal{U} + 3\varepsilon)|F_n|)S.$$

Letting ε tend to 0, it holds that $h_{\mathrm{top}}(G,\mathcal{U}) \leq s$.

Remark. In [9], Simpson discussed the relation between the entropy and Hausdorff dimension for $\mathbf{Z}^d$ subshifts and showed that they are the same for some "standard metric". He also asked (Question 5.4 of [9], in part) whether it also holds for wider classes of groups or semigroups such as an amenable group. We remark here that the same argument can be applied to show that the entropy and Hausdorff dimension for amenable subshifts (for some metric associated with some tempered Følner sequence) coincide. We can also show this by applying Theorem 1.1.

Recall that when X is a metric space with metric d, the dimensional entropy for the system (X,G) can be defined in the following alternative way [10].

For a finite subset F in G, denote $B_F(x,\varepsilon) := \{y \in X : d(gx, gy) < \varepsilon$, for any $g \in F\}$. For $Z \subseteq X$, $s \geq 0$, $N \in \mathbf{N}$, $\{F_n\}$ a Følner sequence in G and $\varepsilon > 0$, define

$$\mathcal{M}(Z, N, \varepsilon, s, \{F_n\}) = \inf \sum_i \exp(-s|F_{n_i}|),$$

where the infimum is taken over all countable families $\{B_{F_{n_i}}(x_i, \varepsilon)\}$ such that $x_i \in X$, $n_i \geq N$ and $\bigcup_i B_{F_{n_i}}(x_i, \varepsilon) \supseteq Z$. Then let

$$\mathcal{M}(Z, \varepsilon, s, \{F_n\}) = \lim_{N\to+\infty} \mathcal{M}(Z, N, \varepsilon, s, \{F_n\}),$$

$$\mathcal{M}(Z, s, \{F_n\}) = \lim_{\varepsilon\to 0} \mathcal{M}(Z, \varepsilon, s, \{F_n\}).$$

Bowen topological entropy $h_{\mathrm{top}}^B(Z, \{F_n\})$ can be equivalently defined as the critical value of the parameter s, where $\mathcal{M}(Z, s, \{F_n\})$ jumps from $+\infty$ to 0.

Let A be a finite set. Consider the left action of G on the product space $A^G = \{(x_g)_{g\in G} : x_g \in A\}$:

$$g'(x_g)_{g\in G} = (x_{gg'})_{g\in G}, \quad \text{for all } g' \in G \text{ and } (x_g)_{g\in G} \in A^G.$$

Let X be a closed G-invariant subset of A^G (we call such X a *subshift*). Fix any tempered Følner sequence $\{F_n\}$ of G with $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_n F_n = G$. We can then define a metric d on A^G by the following: $d(x,y) = 1$ if x and y are not equal on F_1; otherwise $d(x,y) = e^{-|F_n|}$ where n is the maximal number such that $x_g = y_g$ for all $g \in F_n$. The Hausdorff measure of X is defined as

$$\mu_s(X) = \lim_{\varepsilon \to 0} \inf_{\mathcal{E}} \sum_{E \in \mathcal{E}} \mathrm{diam}(E)^s,$$

where $\mathcal{E}$ takes over (open) coverings of X such that $\mathrm{diam}(E) \leq \varepsilon$ for all $E \in \mathcal{E}$. It is easy to see that the covering $\mathcal{E}$ can be also taken from the cylinders of A^G with the form $[\omega]_{F_n} := \{x \in A^G : x_g = \omega_g, \text{ for every } g \in F_n\}$, $\omega \in A^G$. Hence

$$\mu_s(X) = \lim_{N \to \infty} \inf_{\mathcal{E}} \sum_{E \in \mathcal{E}} \mathrm{diam}(E)^s,$$

where $\mathcal{E} = \{[\omega^i]_{F_{n_i}}\}$ is taken over countable families of cylinders that cover X with $n_i \geq N$. The Hausdorff dimension of X is then defined by

$$\dim(X) = \inf\{s \geq 0 : \mu_s(X) = 0\} = \sup\{s \geq 0 : \mu_s(X) = \infty\}.$$

It is easy to see that $\dim(X) = h_{\mathrm{top}}^B(X, \{F_n\})$ by comparing the definition of the Bowen topological entropy and Hausdorff dimension. By Theorem 1.1, we have that $\dim(X) = h_{\mathrm{top}}(X, G)$ for the metric d associated with any tempered Følner sequence $\{F_n\}$ of G with $F_1 \subseteq F_2 \subseteq \ldots$ and $\bigcup_n F_n = G$.

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