Topological Methods in Nonlinear Analysis
Volume 51, No. 2, 2018, 545–563
DOI: 10.12775/TMNA.2017.054

DYNAMICS ON SENSITIVE AND EQUICONTINUOUS FUNCTIONS

Jie Li — Tao Yu — Tiaoying Zeng

Abstract. The notions of sensitive and equicontinuous functions under semigroup action are introduced and intensively studied. We show that a transitive system is sensitive if and only if it has a sensitive pair if and only if it has a sensitive function. While there exists a minimal non-weakly mixing system such that every non-constant continuous function is sensitive, and a topological dynamical system is weakly mixing if and only if it is sensitive consistently with respect to (at least) any two non-constant continuous functions. We also get a dichotomy result for minimal systems — every continuous function is either sensitive or equicontinuous.

1. Introduction

By a *topological dynamical system* (t.d.s. for short) we mean a pair (X,T) where X is a compact metric space with metric d and $T\colon X\to X$ is a continuous map. The collection of all continuous real-valued functions on a given t.d.s. (X,T) is denoted by $C(X,\mathbb{R})$.

In [7], Glasner and Weiss first discovered the link between ℓ_1-structure via coordinate density for elements of $C(X,\mathbb{R})$ and the topological entropy of (X,T). Later Kerr and Li [8], [9] completely characterized this connection. In particular,

2010 *Mathematics Subject Classification.* Primary: 37B05; Secondary: 54H20.

Key words and phrases. Sensitivity; sensitive pairs; sensitive functions; equicontinuous functions; weak mixing.

Research of Jie Li was supported by NSF of Jiangsu Province (BK20170225), Science Foundation of Jiangsu Normal University (17XLR011), NNSF of China (11701231) and China Postdoctoral Science Foundation (2017M611026).

Tao Yu was supported by NNSF of China (11371339 and 11571335).

they showed that for a given t.d.s. (X,T) with T a homeomorphism and for any $f \in C(X,\mathbb{R})$, f has an ℓ_1-isomorphism set of positive density if and only if there exists an entropy pair $(x,y) \in X \times X$ with $f(x) \neq f(y)$.

The notion of sensitivity was first used by Ruelle [13]. According to Auslander and Yorke [4], a t.d.s. (X,T) is called *sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every neighbourhood U_x of x, there exist $y \in U_x$ and $n \in \mathbb{N}$ with $d(T^n x, T^n y) > \delta$. Using ideas from the local entropy theory, Ye and Zhang [17] introduced the notion of *sensitive pair* and showed that a transitive t.d.s. is sensitive if and only if there exists a sensitive pair. They also proved that under the transitivity assumption each entropy pair is a sensitive pair. Now we naturally ask, given $f \in C(X,\mathbb{R})$, what happens if there exists a sensitive pair $(x,y) \in X \times X$ with $f(x) \neq f(y)$?

In [6], Glasner and Megrelishvili studied the opposite side of sensitivity and its functional version, particularly in which the characteristics of the corresponding Ellis semigroup are mostly involved. Inspired by this we naturally investigate the dynamics of a sensitive t.d.s. associated to a real continuous function. To be precise, for an $f \in C(X,\mathbb{R})$ we say that a t.d.s. (X,T) is *f-sensitive* (or equivalently f is a *sensitive function* for (X,T)) if there is $\delta > 0$ such that for every non-empty open subset $U \subset X$ there are $x, y \in U$ and $n \in \mathbb{Z}_+$ such that $d_f(T^n x, T^n y) > \delta$, where d_f is a pseudometric on X given by $d_f(x,y) = |f(x) - f(y)|$ for all $x, y \in X$. Note that recently Achigar, Artigue and Monteverde in [1] discussed the observability of expansive maps (which are apparently sensitive ones) associated to continuous real functions in the same manner, where and in references therein embedding properties are mainly involved.

We point out that the notion of sensitive function is obviously weaker than the classical sensitivity, see Examples 4.4 ([1]) and 4.5 for explicit illustrations. But under transitive assumption things may differ (Proposition 4.2). Particularly we show that for any $f \in C(X,\mathbb{R})$, given the transitivity property, (X,T) is f-sensitive if and only if there exists a sensitive pair (x_1, x_2) such that $f(x_1) \neq f(x_2)$ (Theorem 4.1), which answers the question raised above.

Now we propose another natural question: given a transitive t.d.s. (X,T), what happens if for any non-constant function $f \in C(X,\mathbb{R})$ there is a sensitive pair (x_1, x_2) such that f separates them? This is equivalent to ask when a transitive t.d.s. (X,T) is f-sensitive for all non-constant functions $f \in C(X,\mathbb{R})$? Indeed we can also find a similar motivation for this question in the papers of Kerr and Li [8], [9] regarding a characterization of completely positive entropy and in the paper of Ye and Zhang [17] concerning connection between entropy

([1]) In this paper all results are stated in a general semigroup context, which contains the classical $\mathbb{Z}_+$ action (generated by a single continuous map) as a particular case.

and sensitive pairs. Another motivation comes from the work of García-Ramos and Marcus [5]. In particular, they proved that an ergodic measurable preserving transformation (X, μ, T) is measurably weakly mixing if and only if it is μ-f-mean sensitive for all non-constant functions $f \in L^2(X, \mu)$. Note that measurably weak mixing captures the idea of asymptotic independence in a mean sense, and it is clear that if additionally (X, μ, T) can be realized by a t.d.s. (X, T) with an invariant probability Borel measure μ on X then the dynamics on the support of μ is (topologically) weakly mixing (i.e. $(\mathrm{supp}(X) \times \mathrm{supp}(X), T \times T)$ is transitive). These enlighten us to further conjecture that (X, T) is weakly mixing if and only if it is f-sensitive for all non-constant functions $f \in C(X, \mathbb{R})$.

Unfortunately we show that this is not an equivalent characterization in general, even if (X, T) is strengthened to a P-system (i.e. transitive with dense periodic points), see Examples 4.7 and 4.9. But for some special cases (by adding extra conditions such as minimality (Proposition 4.10), extreme f-sensitivity (Proposition 4.11) and multi-sensitive functions (Theorem 5.4)) the conjecture is eventually true. To prove Theorem 5.4, we introduce the notion of multi-variants of sensitive functions. In Section 5 we provide suitable examples to distinguish all different levels of the multi-forms, see Examples 5.5 and 5.6 for details.

In the remaining part of this paper we also consider the opposite side of sensitive functions. Unlike the concerns of Glasner and Megrelishvili [6], we mainly focus on the dynamics of equicontinuous and almost equicontinuous functions. Some classical results in this line, such as dichotomy theorem (Theorem 6.5), are naturally generalized to this function-dependent case.

There is a rather abundant research on the sensitivity for $\mathbb{Z}_+$-actions, see Section 3 in a recent survey [11] on this topic. To make our study work in a more general frame, we would like to proceed with the main part under the semigroup actions. By a general *topological dynamical system* (also t.d.s. for short) we mean a triple (X, S, π) where X is a compact metric space with metric d, S is a discrete topological semigroup with an identity and $\pi\colon S \times X \to X$, $(s, x) \mapsto sx$ is a continuous action on X with the property that $(s_1 s_2)x = s_1(s_2 x)$ for every $s_1, s_2 \in S$ and $x \in X$. For simplicity we use the pair (X, S) to denote the t.d.s. and it is clear that each element s in S can be viewed as a continuous map from X to itself. When $S = \{T^n : n \in \mathbb{Z}_+\}$ and $T\colon X \to X$ is a continuous map, (X, S) is the same as the classical t.d.s. (X, T). Since a general semigroup may be uncountable and the order relation may vanish, hence traditional methods are not always workable. The recent work of Wang, Chen and Fu [15] and the work of Kontorovich and Megrelishvili [10] not only serve as our motivation of considering semigroup action but also provide us with some useful notions and techniques that we will need.

This paper is organized as follows. In Section 2 some basic definitions and results under semigroup actions are described. In Section 3, we introduce the concept of sensitive tuples for semigroup actions and show that a transitive system is n-sensitive if and only if it has an n-sensitive tuple (Proposition 3.1). Section 4 is devoted to studying sensitive functions. We show that for transitive systems a function is sensitive if and only if it separates sensitive pairs (Theorem 4.1). In Section 5, we study multi-sensitive functions and show that a topological dynamical system is weakly mixing if and only if it is sensitive consistently with respect to at least any two non-constant continuous functions (Theorem 5.4). The final Section 6 takes care of the corresponding equicontinuous functions and we obtain dichotomy results between sensitive functions and equicontinuous functions (Theorem 6.5).

2. Preliminaries

Throughout this paper, the sets of integers, nonnegative integers, natural numbers and real numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{R}$, respectively.

2.1. Topological dynamics. Let (X,S) be a t.d.s. as explained in the introduction. Fix $n\in\mathbb{N}$, we write (X^n,S) as the n-fold product t.d.s. $(X\times\ldots\times X,S)$ defined by $s(x_1,\ldots,x_n)=(sx_1,\ldots,sx_n)$ for all $(x_1,\ldots,x_n)\in X^n$ and $s\in S$. Set $\Delta_n(X)=\{(x,\ldots,x)\in X^n\colon x\in X\}$. For $x\in X$, non-empty subsets $U,V\subset X$ and $s\in S$ we denote the orbit of x by $Sx=\{sx\colon s\in S\}$ and define

$$s^{-1}U=\{x\in X: sx\in U\},\qquad sU=\{sx: x\in U\},$$
$$N(U,V)=\{s\in S: U\cap s^{-1}V\neq\emptyset\}.$$

We say that (X,S) is *point-transitive* if $\overline{Sx}=X$ for some $x\in X$ and at this time the point x is referred to be a *transitive point*. Denote by Tran_S the collection of all transitive points in X. If every point of X is transitive, we call (X,S) a *minimal t.d.s.* and meanwhile each point in it is called a *minimal point*. We say that a point $x\in X$ is *recurrent* if for every neighbourhood U of x there exist infinitely many $s\in S$ such that $sx\in U$ and *periodic* if x is recurrent and Sx is finite.

A t.d.s. (X,S) is said to be (*topologically*) *transitive* if for any non-empty open subsets U,V in X we have $N(U,V)\neq\emptyset$. It is (*topologically*) *weakly mixing* if the product t.d.s. $(X\times X,S)$ is transitive. By standard discussion one can easily see that if (X,S) is transitive then it is point-transitive. If further assumiing that S is a commutative semigroup we have the following observation. The easy verification is omitted.

Lemma 2.1. *Let (X, S) be a transitive t.d.s. with S being a commutative semigroup. Then, for each $n \in \mathbb{N}$, the set*

$$W_n = \{(s_1 x, \dots, s_n x) : x \in \mathrm{Trans}_S,\ s_1, \dots, s_n \in S\}$$

is dense in X^n and each element of it is a recurrent point.

We will need the following characterization of weak mixing [12].

Lemma 2.2. *Let (X, S) be a t.d.s. and S be a commutative semigroup. Then the following are equivalent*:

(a) *(X, S) is weakly mixing.*
(b) *For all non-empty open subsets $U, V \subset X$, $N(U, U) \cap N(U, V) \neq \emptyset$.*
(c) *For all non-empty open subsets $U_i, V_i \subset X$ $(i = 1, 2)$, there exist non-empty open subsets $U_3, V_3 \subset X$ such that*

$$N(U_1, V_1) \cap N(U_2, V_2) \supset N(U_3, V_3) \neq \emptyset.$$

(d) *(X^n, S) is weakly mixing for any $n \in \mathbb{N}$.*

We say that a t.d.s. (X, S) is a/an

- *P-system* if it is transitive and the set of periodic points is dense;
- *M-system* if it is transitive and the set of minimal points is dense;
- *E-system* if it is transitive and there exists an S-invariant probability measure with full support.

Let (X, S) and (Y, S) be two t.d.s.s and a map $\pi\colon X \to Y$ be given. We say that π is a *factor map* if π is continuous onto and satisfies that $\pi \circ S = S \circ \pi$.

2.2. Symbolic dynamics. In this paper many examples relying on the symbolic dynamics are illustrated. To make it easier for readers we collect the needed notions in this subsection.

Let (Σ_2^+, σ) be the *full shift*, where $\Sigma_2^+ = \{0, 1\}^{\mathbb{Z}_+}$ is equipped with Cantor product topology and a compatible metric on it is defined by $d(x, y) = 0$ if $x = y$ otherwise $d(x, y) = 1/(i + 1)$ with $i = \min\{j \in \mathbb{Z}_+ : x_j \neq y_j\}$; the continuous *shift map* $\sigma\colon \Sigma_2^+ \to \Sigma_2^+$ is given by $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{Z}_+$. It is clear that Σ_2^+ is compact. In common use we refer to its every compact σ-invariant subspace X together with the shift map σ as a *subshift*.

Given $n \in \mathbb{N}$, we call $w \in \{0, 1\}^n$ a *word of length n* and write $|w| = n$. For any two words $u = u_0 \dots u_n$ and $v = v_0 \dots v_m$, we define the *concatenation* of u, v by $uv = u_0 \dots u_n v_0 \dots v_m$. In the same manner we define by u^m for some $m \in \mathbb{N}$ the concatenation of m copies of u, and by u^∞ the infinite concatenation of u. Let X be a subshift of Σ_2^+ and $x = x_0 x_1 \ldots \in X$, for any $i, j \in \mathbb{Z}_+$, we denote $x_{[i,i+j]} = x_i x_{i+1} \dots x_{i+j}$. A word $w = w_0 w_1 \dots w_n$ is said to appear in x at position i if $x_{[i,i+n]} = w$. By $\mathcal{L}_n(X)$ we mean the set of all words of length n in X. For any word $u \in \mathcal{L}_n(X)$ its *cylinder set* is defined by $[u] = \{x \in X :$

$x_0 \dots x_{n-1} = u\}$. Note that all cylinder sets $\left\{[u]u \in \bigcup_{n\in\mathbb{N}} \mathcal{L}_n(X)\right\}$ form a basis of the topology of X.

3. Sensitive tuples for semigroup actions

A t.d.s. (X, S) is said to be *sensitive* if there exists $\delta > 0$ such that for each $x \in X$ and each $\varepsilon > 0$ there are $y \in X$ with $d(x, y) < \varepsilon$ and $s \in S$ such that $d(sx, sy) > \delta$. Given $n \geq 2$, we call a t.d.s. (X, S) is *n-sensitive* if there exists $\delta > 0$ such that for any non-empty open subset U of X there exist $x_1, \dots, x_n \in U$ and $s \in S$ such that $d(sx_i, sx_j) > \delta$ for $1 \leq i < j \leq n$. A tuple $(x_1, \dots, x_n)$ in X^n is referred to be *sensitive* if $(x_1, \dots, x_n)$ is not on the diagonal $\Delta_n(X)$ and for each $\varepsilon > 0$, each non-empty open subset U of X there exist $y_1, y_2, \dots, y_n \in U$ and $s \in S$ such that $d(x_i, sy_i) < \varepsilon$ for $i = 1, 2, \dots, n$. Denote by $S_n(X, S)$ the collection of all n-sensitive tuples. We say $(x_1, \dots, x_n) \in S_n(X, S)$ is *essential* if $x_i \neq x_j$ for each $1 \leq i < j \leq n$ and denote all such tuples by $S_n^{\mathrm{e}}(X, S)$ at this time.

Note that for the case of $S = \mathbb{Z}_+$ the notion of n-sensitivity property was first generalized by Xiong in [16] and the case $n = 2$ is the classical sensitivity property. Later using ideas from the local entropy theory Ye and Zhang [17] further introduced the notion of sensitive tuples. Particularly they showed that a transitive t.d.s. (X, T) is n-sensitive if and only if $S_n^{\mathrm{e}}(X, T) \neq \emptyset$. Next we aim to extend this result to semigroup actions.

To do so, we need to put extra restriction on the semigroup S, that is S is assumed to be a C-semigroup. Following [10], a semigroup S is said to be a *C-semigroup* if for every $s_0 \in S$, the closure of $S \setminus Ss_0$ is compact. Since we have assumed that the topology of S is discrete, then it is a C-semigroup if and only if the subset $S \setminus Ss_0$ is finite for every $s_0 \in S$ (otherwise, there is a limit element a in $S \setminus Ss_0$ and then $S \setminus (Ss_0 \cup \{a\})$ is not closed, contradicting the fact that each subset of a discrete space is both open and closed). It is clear that every topological group is a C-semigroup.

Proposition 3.1. *Let (X, S) be a transitive t.d.s. with S being a C-semigroup and $n \in \mathbb{N}$. Then (X, S) is n-sensitive if and only if $S_n^{\mathrm{e}}(X, S) \neq \emptyset$.*

Proof. Assume (X, S) is n-sensitive and $x \in \mathrm{Tran}_S$. Let U_m be a sequence of open neighbourhoods of x with $\mathrm{diam}(U_m) < 1/m$ for $m \in \mathbb{N}$. Then there is $\delta > 0$ such that for each $m \in \mathbb{N}$, there exist $x_1^m, \dots, x_n^m \in U_m$ and $s_m \in S$ satisfying $d(s_m x_i^m, s_m x_j^m) > \delta$ for $1 \leq i < j \leq n$. Without loss of generality we assume that $s_m x_i^m \to x_i$ $(m \to \infty)$ for $i = 1, \dots, n$. It is easy to see that the limit points $x_1, \dots, x_n$ are pairwise distinct. Also, we can check the cardinality of the set $\{s_m : m \in \mathbb{N}\}$ is infinite.

Now we show that the tuple $(x_1,\dots,x_n)$ is sensitive. Let $\varepsilon>0$ and $U\subset X$ be any non-empty open subset of X. By transitivity there is $t\in S$ such that $tx\in U$. Since t as a map from X to X is continuous, there is $m\in\mathbb{N}$ such that $tU_m\subset U$. For this t, by the definition of a C-semigroup we have that $S\setminus St$ is finite. This means that we can find suitable $m\in\mathbb{N}$ such that there are $x_1^m,\dots,x_n^m\in U_m$ and $s_m\in St$ satisfying $d(x_i,s_mx_i^m)<\varepsilon$ for all $1\le i\le n$. Let $s'_m\in S$ be such that $s_m=s'_mt$. This implies that for any $\varepsilon>0$, there are $tx_i^m\in U$ and $s'_m\in S$ such that $d(x_i,s'_m(tx_i^m))<\varepsilon$, $i=1,\dots,n$. So $(x_1,\dots,x_n)\in S_n^e(X,S)$, proving the necessity.

For the sufficiency side, assume there is an n-tuple $(x_1,\dots,x_n)\in S_n^e(X,S)$. Put $\delta=\Big(\min\limits_{1\le i<j\le n}d(x_i,x_j)\Big)/2$. Let ε be such that $0<\varepsilon<\delta/2$ and U be a non-empty open subset of X, then there exist $y_1,\dots,y_n\in U$ and $s\in S$ such that $d(x_i,sy_i)<\varepsilon$, $i=1,\dots,n$. This implies that $d(sy_i,sy_j)>\delta$, $1\le i\ne j\le n$. That is, (X,S) is n-sensitive. □

For $n\ge 2$, an n-tuple $(x_1,\dots,x_n)\in(X^n,S)$ is called *regionally proximal* if for every $\varepsilon>0$ there exist $y_1,y_2,\dots,y_n\in X$ and $s\in S$ such that $\sup\limits_{1\le i\le n}d(x_i,y_i)<\varepsilon$ and $\sup\limits_{1\le i,j\le n}d(sy_i,sy_j)<\varepsilon$. Let $Q_n(X,S)$ denote the collection of all n-regionally proximal tuples. Using the same techniques as in [14] and [17] (where $S=\mathbb{Z}_+$ action considered), we have the following characterization.

Theorem 3.2. *Let (X,S) be a t.d.s. and S a commutative C-semigroup.*

(a) *If (X,S) is transitive, then $S_n(X,S)\subset Q_n(X,S)$ for every $n\ge 2$.*

(b) *If (X,S) is minimal then $S_n(X,S)=Q_n(X,S)\setminus\Delta_n(X)$ for every $n\ge 2$.*

Proof. (a) Let $(x_1,\dots,x_n)\in S_n(X,S)$ and U_i be the neighbourhood of x_i, respectively. Then, for any non-empty open subset $U\subset X$, there is $s_0\in S$ such that $V_i:=U\cap s_0^{-1}U_i\ne\emptyset$. Observe that $V_1\times\ldots\times V_n\subset X^n$ is open, hence by Lemma 2.1 there is a recurrent point $(y_1,\dots,y_n)\in U\times\ldots\times U$ such that $s_0y_i\in U_i$ for all $1\le i\le n$. Since S is a C-semigroup, from the definition of recurrence there is $s_1\in Ss_0$ such that $s_1y_i\in U$ for each $1\le i\le n$. Let s_2 be such that $s_1=s_2s_0$ and then $s_2(s_0y_i)\in U$ for all $1\le i\le n$. This implies that $(x_1,\dots,x_n)\in Q_n(X,S)$.

(b) Now assume that (X,S) is minimal and $(x_1,\dots,x_n)\in Q_n(X,S)\setminus\Delta_n(X)$. For any $m\in\mathbb{N}$, there are recurrent points $(y_1^m,\dots,y_n^m)\in B(x_1,1/m)\times\ldots\times B(x_n,1/m)$ (by Lemma 2.1) and $s_m\in S$ such that $\sup\limits_{1\le i,j\le n}d(s_my_i^m,s_my_j^m)<1/m$. Without loss of generality let $\lim\limits_{m\to\infty}s_my_i^m=x$ for any $1\le i\le n$ and some $x\in X$. Since X is minimal, x is a transitive point. Then, for any non-empty open subset U of X, there is $t\in S$ such that $tx\in U$. And so for large enough m we have $ts_my_i\in U$ for all $1\le i\le n$. Since S is a C-semigroup, there is $s' \in S$ such that $s'(ts_m y_i) \in B(x_i, 1/m)$, $i = 1, \ldots, n$. This shows that $(x_1, \ldots, x_n) \in S_n(X, S)$. □

The following result was first observed in [17]. With the help of Lemma 2.2 we can strengthen it to the semigroup action.

Proposition 3.3. *Let (X, S) be a t.d.s. with S being a commutative semigroup. Then it is weakly mixing if and only if $S_n(X, S) = X^n \setminus \Delta_n(X)$.*

Proof. We only give the details for $n = 2$. The general case is similar. Assume that (X, S) is weakly mixing. It is clear that $S_2(X, S) \subset X^2 \setminus \Delta_2(X)$. Let $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ and $\varepsilon > 0$. For any non-empty open subset $U \subset X$, there is $s \in S$ such that $U \cap s^{-1}B(x_1, \varepsilon) \neq \emptyset$ and $U \cap s^{-1}B(x_2, \varepsilon) \neq \emptyset$. This implies that $(x_1, x_2) \in S_2(X, S)$ and hence $S_2(X, S) = X^2 \setminus \Delta_2(X)$.

Now assume $S_2(X, S) = X^2 \setminus \Delta_2(X)$. If there are two non-empty open subsets $U, V \subset X$ such that $N(U, U) \cap N(U, V) = \emptyset$, then $U \times V \cap S_2(X, S) = \emptyset$ which is impossible. Hence by Lemma 2.2 (X, S) is weakly mixing. □

4. Sensitive functions for semigroup actions

Let (X, S) be a t.d.s. and $f \in C(X, \mathbb{R})$. We say that (X, S) is *f-sensitive* (or equivalently f is a *sensitive function* for (X, S)) if there is $\delta > 0$ such that for every non-empty open subset $U \subset X$ there are $x, y \in U$ and $s \in S$ such that

$$d_f(sx, sy) = |f(sx) - f(sy)| > \delta.$$

Such constant δ is called an *f-sensitive constant*. It is clear that a constant function is not sensitive.

First we have the following characterization of sensitive functions.

Theorem 4.1. *Let (X, S) be a transitive t.d.s. with S being a C-semigroup and $f \in C(X, \mathbb{R})$. Then f is sensitive for (X, S) if and only if there exists a sensitive pair (x_1, x_2) such that $f(x_1) \neq f(x_2)$.*

Proof. Assume that there exists a sensitive pair (x_1, x_2) such that $f(x_1) \neq f(x_2)$. Put $\delta = |f(x_1) - f(x_2)|/2$ and let $\varepsilon \in (0, 1)$ be such that $0 < \varepsilon < \delta/2$. Since $f \in C(X, \mathbb{R})$ there is $\delta' > 0$ such that if $d(x, y) < \delta'$ then $|f(x) - f(y)| < \varepsilon$. Let U be a non-empty open subset of X. Since (x_1, x_2) is a sensitive pair, there are $y_1, y_2 \in U$ and $s \in S$ such that $sy_1 \in B(x_1, \delta')$ and $sy_2 \in B(x_2, \delta')$. This implies that

$$|f(sy_1) - f(sy_2)| \geq |f(x_1) - f(x_2)| - |f(x_1) - f(sy_1)| - |f(x_2) - f(sy_2)| > \delta,$$

which ends the proof of sufficiency.

For the necessity side, we assume that $f \in C(X, \mathbb{R})$ is a sensitive function with an f-sensitivity constant $\delta > 0$. Let U be a non-empty open subset of X. Then there are $x, y \in U$ and $s \in S$ with $|f(sx) - f(sy)| > \delta$. For this δ, since

$f \in C(X,\mathbb{R})$, there is $\delta' > 0$ such that for any $z_1, z_2 \in X$ if $|f(z_1) - f(z_2)| > \delta$ then $d(z_1, z_2) > \delta'$. This implies that $d(sx, sy) > \delta'$. Now let $x \in \mathrm{Tran}_S$ and U_m ($m \in \mathbb{N}$) be a sequence of open neighbourhoods of x with $\mathrm{diam}(U_m) < 1/m$. For each $m \in \mathbb{N}$, there then exist $x_1^m, x_2^m \in U_m$ and $s_m \in S$ such that $|f(s_m x_1^m) - f(s_m x_2^m)| > \delta$ and $d(s_m x_1^m, s_m x_2^m) > \delta'$. Without loss of generality let $s_m x_1^m \to x_1$ and $s_m x_2^m \to x_2$ when $n \to \infty$. Then following arguments similar to those in Proposition 3.1, we have that (x_1, x_2) is a sensitive pair. Meanwhile it is also clear that $f(x_1) \neq f(x_2)$. This means that the necessity holds. □

Proposition 4.2. *Let (X,S) be a transitive t.d.s. with S being a C-semigroup. Then (X,S) is sensitive if and only if there exists a sensitive function.*

Proof. Assume that $f \in C(X,\mathbb{R})$ is a sensitive function. Let U be a non-empty open subset of X. Then there is $\delta > 0$ such that there are $x, y \in U$ and $s \in S$ with $|f(sx) - f(sy)| > \delta$. For this δ, since $f \in C(X,\mathbb{R})$, there is $\delta' > 0$ such that for any $z_1, z_2 \in X$ if $|f(z_1) - f(z_2)| > \delta$ then $d(z_1, z_2) > \delta'$. This implies that $d(sx, sy) > \delta'$, and hence (X,S) is sensitive.

Now assume that (X,S) is sensitive. By Proposition 3.1, there exists a sensitive pair $(x_1, x_2) \in X^2$. Put $f(x) = d(x, x_1)$. Then $f(x_1) \neq f(x_2)$. By Theorem 4.1, we have that f is a sensitive function. □

Remark 4.3. Note that in the proofs of sufficiency of Theorem 4.1 and Proposition 4.2 we do not need the transitivity condition and the restriction of C-semigroup action.

In the following we will provide several examples to show that there are sensitive t.d.s. such that it is not f-sensitive for some non-constant $f \in C(X,\mathbb{R})$.

Example 4.4. There is a transitive system (X,S) such that it is sensitive but there exists non-constant $f \in C(X,\mathbb{R})$ such that it is not f-sensitive.

Proof. Let (Y,S) be a weakly mixing system (e.g., for full shift with $S = \mathbb{Z}_+$). Set $X = Y \times \{0,1\}$ with the sup metric. Define the action by $s(x,a) = (sx, \bar{a})$, where $s \in S$ and $\bar{a} = 1 - a$ with $a \in \{0,1\}$. Then it is clear that (X,S) is transitive and sensitive. Choose a continuous function f such that $f(x,a) = a$ for all $x \in X$ and $a \in \{0,1\}$. By Proposition 3.3 it is not hard to see that $S_2(X,S) = \{((x,a),(y,a)) : x \neq y \in X,\ a \in \{0,1\}\}$. Then using Theorem 4.1 we have that (X,S) is not f-sensitive. □

Example 4.5. There is a minimal system (X,S) such that it is sensitive but there exists non-constant $f \in C(X,\mathbb{R})$ such that it is not f-sensitive.

Proof. Let the unit circle be $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$, $\alpha \in (0,1) \in \mathbb{R} \setminus \mathbb{Q}$. Let $T_\alpha \colon \mathbb{T}^1 \to \mathbb{T}^1$ be $T_\alpha(x) = x + \alpha \pmod 1$. It is easy to see that $(\mathbb{T}^1, T_\alpha)$ is a minimal equicontinuous system. Fix any point $x_0 \in \mathbb{T}^1$, consider $\mathrm{Orb}(x_0) := \{T_\alpha^n(x_0),\ n \in \mathbb{Z}\}$

and stretch $T_\alpha^n(x_0)$ into an arc I_n ($I_n = [a_n, b_n]$ is an arc going from a_i to b_i anticlockwise) such that

(1) $l(I_{m+1}) = l(I_m)/2$ and $l(I_{-m}) = l(I_m)$ for any $m \in \mathbb{N} \cup \{0\}$,

(2) $\sum_{n\in\mathbb{Z}} l(I_n)=1$, $l(I)$ is the length of I.

Using the method we can get a larger circle Y. The metric d_0 on Y is defined as follows $d_0(x,y) = \min\{l([x,y]), l([y,x])\}$, $x,y \in Y$. We can find a homeomorphic monotonic map $h\colon Y \to Y$ such that

$$h(a_k) = a_{k+1}, \quad h(b_k) = b_{k+1}, \quad h(I_k) = I_{k+1}$$

and

$$h(x) = T_\alpha(x), \quad \text{for all } x \in Y \setminus \bigcup_{k\in\mathbb{Z}} I_k.$$

Let $X = Y \setminus \bigcup_{i=-\infty}^{+\infty} (a_i, b_i)$ and define $T\colon X \to X$ to be the restriction of h on X. We call the system (X,T) the *Denjoy system*. If $\varphi\colon X \to \mathbb{T}^1$ satisfies $\varphi(a_i) = \varphi(b_i) = T_\alpha^i(x_0)$ and $\varphi(x) = x$ for $x \in X \setminus \{a_i, b_i\}_{i=-\infty}^{+\infty}$, we can see that φ is an almost 1–1 extension. It is not hard to see that $S_2(X,T) = \{(a_n,b_n)\colon n \in \mathbb{Z}\}$ and $\overline{S_2(X,T)} = S_2(X,T) \cup \Delta_2(X)$. Let $f \in C(\mathbb{T}^1, \mathbb{R})$ be a non-constant function and define $\widetilde{f}(x) = f(\varphi(x))$, $x \in X$. It is clear that $\widetilde{f} \in C(X,\mathbb{R})$ and $\widetilde{f}$ is non-constant too. By Theorem 4.1 we know that X is not $\widetilde{f}$-sensitive. $\square$

Next, as mentioned in the introduction, we address the question whether (X,S) is weakly mixing if and only if it is f-sensitive for all non-constant $f \in C(X,\mathbb{R})$. From Proposition 3.3 and Theorem 4.1 we know that if (X,S) is weakly mixing, then for every non-constant f, it is f-sensitive. For the converse side, we will show by particular examples that it is false. To do so, we first provide a criterion for a given t.d.s. (X,S) to be f-sensitive for all non-constant $f \in C(X,\mathbb{R})$.

Proposition 4.6. *Let (X,S) be a t.d.s. If there is a non-empty subset $Y \subset X$ such that $Y \times Y \setminus \Delta_2(X) \subset S_2(X,S)$ and, for any $x \in X \setminus Y$, there is $y \in Y$ such that $(x,y) \in S_2(X,S)$, then (X,S) is f-sensitive for every non-constant $f \in C(X,\mathbb{R})$.*

Proof. Let $f \in C(X,\mathbb{R})$ be a non-constant function. There exist $x_1 \neq x_2 \in X$ such that $f(x_1) \neq f(x_2)$. If $(x_1,x_2) \in S_2(X,S)$ then using Theorem 4.1 and Remark 4.3, (X,S) is f-sensitive and this case is complete. If $(x_1,x_2) \notin S_2(X,S)$, by assumption there are $y_1,y_2 \in Y$ such that $(x_1,y_1) \in S_2(X,S)$ and $(x_2,y_2) \in S_2(X,S)$. If $y_1 \neq y_2$, by assumption $(y_1,y_2) \in S_2(X,S)$. Then it is not hard to see that there exist at least two distinct values in the set $\{f(x_1), f(y_1), f(y_2)\}$ or the set $\{f(x_2), f(y_1), f(y_2)\}$. By simple discussion and combining Theorem 4.1 and Remark 4.3 again we can show (X,S) is f-sensitive. If $y_1 = y_2$ then either

$f(x_1) \neq f(y_1)$ or $f(x_2) \neq f(y_1)$, and in either case we can get by previous arguments that (X, S) is f-sensitive. $\square$

Note that the weakly mixing t.d.s. satisfies all the conditions of Proposition 4.6. Next we present an alternative example for which the set of sensitive pairs is extremely meager.

Example 4.7. There is a transitive system (X, T) which is neither minimal nor weakly mixing, such that (X, T) is f-sensitive for every non-constant $f \in C(X, \mathbb{R})$.

Proof. We will construct such a t.d.s. resorting to the theory of symbolic dynamics. The idea comes from Example 6.5 in [17].

Let $A_1 = 1010$, $k_1 = |A_1| = 4$ and $A_2 = A_1 0^{k_1} A_1$. Recursively we put $k_n = |A_n|$ and

$$A_{n+1} = A_n 0^{k_n} A_n, \quad n \geq 1.$$

It is clear that $x = \lim_{n\to\infty} A_n$ is a recurrent point. Let $(X, \sigma) = (\overline{\mathrm{Orb}(x, \sigma)}, \sigma)$, where σ is the shift. Then (X, σ) is transitive, not minimal as the fixed point $\{\mathbf{0} = (0, 0, \ldots)\}$ is in X. It is also not weakly mixing, since $N([1010], [1010]) \cap N([1010], [0100]) = \emptyset$.

We claim that $S_2(X, \sigma) = \{(y, \mathbf{0}) : y \in X \setminus \{\mathbf{0}\}\} \cup \{(\mathbf{0}, y) : y \in X \setminus \{\mathbf{0}\}\}$. Note that for the including relation it suffices to show for the above transitive point x, $(x, \mathbf{0}) \in S_2(X, \sigma)$. This is clear since for any non-empty open neighbourhoods U, V of $x, \mathbf{0}$ respectively there are $n \geq 1$ and points $x_1^n = A_n 0^{k_n} A_n \ldots, x_2^n = A_n 0^{k_{n+1}} \ldots$ such that $x_1^n, x_2^n \in [A_n] \subset U$, and so $T^{2k_n}(x_1^n) \in U$ and $T^{2k_n}(x_2^n) \in V$. Now we show the included relation. Let $y_1 \neq y_2 \in X \setminus \{\mathbf{0}\}$. Without loss of generality there are $i < j \in \mathbb{Z}_+$ such that $1 = y_1(i) \neq y_2(i) = 0$ and $0 = y_1(j) \neq y_2(j) = 1$. Let $n \in \mathbb{N}$ be such that $k_n > j$. Observe that for each pair $(z_1, z_2) \in [A_n] \times [A_n]$, it has the form of $z_1 = A_n 0^{k_n} B_1 0^{k_n} B_2 0^{k_n} \ldots$ and $z_2 = A_n 0^{k_n} C_1 0^{k_n} C_2 0^{k_n} \ldots$, where $B_s, C_s \in \{A_n, 0^{k_n}\}$ for all $s \in \mathbb{N}$. Then if there are $m \in \mathbb{N}$ and a point $z_1 \in [A_n]$ such that $\sigma^m z_1 \in [(y_1)_{[0,j]}]$, we can find some $s \in \mathbb{N}$ such that $B_s = A_n$. By the construction, if $(y_1, y_2) \in S_2(X, \sigma)$ then for any point $z_2 \in [A_n]$ with $\sigma^m z_2 \in [(y_2)_{[0,j]}]$ the corresponding word C_s must be 0^{k_n}. While $(y_2)_{[0,j]}$ is a subword of $C_s 0^{k_n}$ since $k_n > j$. This leads to a contradiction with $y_2(j) = 1$. So the claim holds.

It is easy to check that this system satisfies the conditions in Proposition 4.6 (with $Y = \{\mathbf{0}\}$), and so it is f-sensitive for every non-constant $f \in C(X, \mathbb{R})$. $\square$

Remark 4.8. Example 4.7 motivates the construction of a t.d.s. satisfying the conditions in Proposition 4.6 with Y being any finite set. For example, let $A_1 = 1010$, $k_1 = |A_1| = 4$, $A_2 = A_1 0^{k_1} A_1$ and $A_3 = A_2 1^{k_2} A_2$. Recursively put

$k_n = |A_n|$,

$$A_{2n} = A_{2n-1} 0^{k_{2n-1}} A_{2n-1} \quad \text{and} \quad A_{2n+1} = A_{2n} 1^{k_{2n}} A_{2n}, \quad n \geq 1.$$

Let $x = \lim_{n\to\infty} A_n$ and $(X,\sigma) = (\overline{\mathrm{Orb}(x,\sigma)},\sigma)$ with $Y = \{\mathbf{0},\mathbf{1}\}$. We leave the similar verifications to the reader.

It is easy to see that the system in Example 4.7 is indeed proximal with $\{\mathbf{0} = (0,0,\ldots)\}$ as the unique minimal point. We wonder if such a system can be improved to an M-system (or further to a P-system) with all the properties retained. Note that Ye and Zhang showed in [17] that an M-system (X,T) is finitely sensitive (i.e. n-sensitive for any $n \in \mathbb{N}$), and recently Wang, Chen and Fu [15] generalized and proved this result for commutative semigroup actions. Among other things, Ye and Zhang [17] also presented an example of E-system which has a unique minimal point and is not 5-sensitive. We point out that the sensitive pairs in such E-system contain the set $\{(y,\mathbf{0}) : y \in X\setminus\{\mathbf{0}\}\} \cup \{(\mathbf{0},y) : y \in X\setminus\{\mathbf{0}\}\}$ and so it keeps apparently all the properties listed in Example 4.7.

In the following we shall answer the above question positively by modifying the construction in Example 4.7.

Example 4.9. There is a P-system (X,T) which is neither minimal nor weakly mixing, such that (X,T) is f-sensitive for every non-constant $f \in C(X,\mathbb{R})$.

Proof. Let $A_1 = 1011$, $k_1 = |A_1| = 4$. Recursively for $n \geq 1$ we put $k_n = |A_n|$ and

$$A_{n+1} = A_n 0^{k_n} A_n A_1^{l_n^1} A_2^{l_n^2} \ldots A_{n-1}^{l_n^{n-1}} A_n,$$

where $|A_1|^{l_n^1} = |A_2|^{l_n^2} = \ldots = |A_{n-1}|^{l_n^{n-1}} = |A_n| = k_n$. It is easy to see that for each $m \in \mathbb{N}$, $l_n^m \to \infty$ when $n \to \infty$.

Let $x = \lim_{n\to\infty} A_n$ and $X = \overline{\mathrm{Orb}(x,\sigma)}$. It is clear that under the shift map σ, the pair (X,σ) is a transitive t.d.s. and has a dense set of periodic points and therefore not minimal. It is also not weakly mixing, since $N([1011],[1011]) \cap N([1011],[0110]) = \emptyset$.

We claim that $\{(y,\mathbf{0}) : y \in X\setminus\{\mathbf{0}\}\} \cup \{(\mathbf{0},y) : y \in X\setminus\{\mathbf{0}\}\} \subset S_2(X,\sigma) \subsetneq X \times X$. For the first included relation it suffices to show for the above transitive point x, $(x,\mathbf{0}) \in S_2(X,\sigma)$. This is clear since for any non-empty open neighbourhoods U, V of $x,\mathbf{0}$ respectively there are $n \geq 1$ and points $x_1^n = A_n 0^{k_n} A_n \ldots$, $x_2^n = A_n 0^{k_{n+1}} \ldots$ such that $x_1^n, x_2^n \in [A_n] \subset U$, and so $T^{2k_n}(x_1^n) \in U$ and $T^{2k_n}(x_2^n) \in V$. The second included relation follows from its non-weak mixing property along Proposition 3.3.

By the same reason this system meets the conditions in Proposition 4.6 (with $Y = \{\mathbf{0}\}$), and so it is f-sensitive for every non-constant $f \in C(X,\mathbb{R})$.

Examples 4.7 and 4.9 tell us generally that f-sensitiveness of t.d.s. for all non-constant $f \in C(X, \mathbb{R})$ cannot ensure its weak mixing property. Nonetheless in some particular cases this condition is enough.

Proposition 4.10. *Let (X, S) be a minimal t.d.s. with S being a commutative group. Then X is weakly mixing if and only if it is f-sensitive for every non-constant $f \in C(X, \mathbb{R})$.*

Proof. It suffices to show the sufficiency. Assume that (X, S) is not weakly mixing. Since (X, S) is a minimal and S is a commutative group (and so a commutative C-semigroup), then by Theorem 3.2 and Proposition 3.3 we have $Q(X, S) = S_2(X, S) \cup \Delta_2(X) \subsetneq X \times X$. A remarkable result on the regionally proximal relation $Q(X, S)$ says that it is an equivalence relation for commutative group actions [3].

Now let $\pi\colon X \to X_{\mathrm{eq}} := X/Q(X, S)$. So (X_{eq}, S) is not a singleton. Let $\overline{f} \in C(X_{\mathrm{eq}})$ be a non-constant function and write $f(x) = \overline{f}(\pi(x))$. Then f is also a non-constant function and $f \in C(X, \mathbb{R})$. We also have $f(x_1) = f(x_2)$ for every pair $(x_1, x_2) \in S_2(X, S)$. By Theorem 4.1, (X, S) is not f-sensitive, a contradiction. $\square$

Inspired by Examples 4.7 and 4.9, to ensure weak mixing property we may try to put extra conditions on the collection of all sensitive functions. With this idea in mind we introduce the notion of extreme f-sensitivity. To be specific, for a given $f \in C(X, \mathbb{R})$, we say that a t.d.s. (X, S) is *extremely f-sensitive* if for every $0 < \varepsilon < \sup\{|f(x) - f(y)| : x \neq y\}$, it is f-sensitive with ε being an f-sensitive constant.

Proposition 4.11. *Let (X, S) be a t.d.s. with S being a commutative semigroup. Then (X, S) is weakly mixing if and only if for every non-constant $f \in C(X, \mathbb{R})$, it is extremely f-sensitive.*

Proof. It suffices to show the sufficiency. Let $x_1 \neq x_2 \in X$. Define a function

$$f(x) = \frac{d(x, x_1)}{d(x, x_1) + d(x, x_2)}.$$

It is clear that f is continuous. Moreover, we have $f(x) = 0$ if and only if $x = x_1$, $f(x) = 1$ if and only if $x = x_2$, and $\sup\{|f(x) - f(y)| \colon x \neq y\} = 1$. By assumption, (X, S) is extremely f-sensitive, we claim that (x_1, x_2) is a sensitive pair. Indeed, for any $0 < \varepsilon < 1$ and any non-empty open subset $U \subset X$, there are $y_1, y_2 \in U$ and $s \in S$ such that $|f(sy_1) - f(sy_2)| > 1 - \varepsilon$. Observe the property of f, without loss of generality we assume $sy_1 \in B(x_1, \varepsilon)$ and $sy_2 \in B(x_2, \varepsilon)$. This implies that (x_1, x_2) is a sensitive pair. Hence $S_2(X, S) \cup \Delta_2(X) = X^2$ and so (X, S) is weakly mixing by Proposition 3.3.

5. Multi-sensitive functions for semigroup actions

In this section we add another restriction on sensitive functions by defining its multi-variants forms.

Fix $n \in \mathbb{N}$ and non-constant functions $f_1, \ldots, f_n \in C(X, \mathbb{R})$. We say that a t.d.s. (X, S) is $(f_1, \ldots, f_n)$-*sensitive* (or equivalently $(f_1, \ldots, f_n)$ is an n-*sensitive function tuple* for (X, S)) if there exists $\delta > 0$ such that for every non-empty open subset U of X, there exist $x, y \in U$ and $s \in S$ with $|f_i(sx) - f_i(sy)| > \delta$ for all $1 \leq i \leq n$. It is clear that if (X, S) is $(f_1, \ldots, f_{n+1})$-sensitive for some $n \in \mathbb{N}$ then it is $(f_1, \ldots, f_n)$-sensitive.

Similarly to the proof of Theorem 4.1 we have a characterization of multi-sensitive functions. The details are left to the reader.

Proposition 5.1. *Let (X, S) be a transitive t.d.s. with S being a C-semigroup, $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in C(X, \mathbb{R})$. Then (X, S) is $(f_1, \ldots, f_n)$-sensitive if and only if there exists a sensitive pair $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ such that $f_k(x_1) \neq f_k(x_2)$ for every $1 \leq k \leq n$.*

With Proposition 5.1 in hand we can present a characterization of sensitive pair as follows. For comparison we remind that the systems at this time are not needed to be transitive and S can be adapted to any semigroup.

Theorem 5.2. *Let (X, S) be a t.d.s. A pair $(x_1, x_2) \in X \times X$ is sensitive if and only if for every $f_1, f_2 \in C(X, \mathbb{R})$ with $f_j(x_1) \neq f_j(x_2)$, $j = 1, 2$, (X, S) is (f_1, f_2)-sensitive.*

Proof. Necessity follows from Proposition 5.1. For emphasis that it does not need extra restrictions on X and S, we provide the details. Assume a pair (x_1, x_2) is sensitive and $f_j \in C(X, \mathbb{R})$ with $f_j(x_1) \neq f_j(x_2)$, $j = 1, 2$. Put $\delta = \min\{|f_1(x_1) - f_1(x_2)|, |f_2(x_1) - f_2(x_2)|\}/2$. Let $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < \delta/2$. By the compactness of X there is $\eta > 0$ such that if $d(x, y) < \eta$ then $|f_j(x) - f_j(y)| < \varepsilon$ for $j = 1, 2$. For this η, since (x_1, x_2) is sensitive, then for any non-empty open subset U of X, there are $y_1, y_2 \in U$ and $s \in S$ such that $d(x_i, sy_i) < \eta$, $i = 1, 2$. This implies that

$$|f_j(sy_1) - f_j(sy_2)| > \delta, \quad j = 1, 2,$$

and then (X, S) is (f_1, f_2)-sensitive.

For sufficiency, we assume that (X, S) is (f_1, f_2)-sensitive for every f_1, f_2 in $C(X, \mathbb{R})$ with $f_1(x_1) \neq f_1(x_2)$ and $f_2(x_1) \neq f_2(x_2)$. Apparently then $x_1 \neq x_2$. We shall show that (x_1, x_2) is a sensitive pair. Let U be a non-empty open subset of X and $\varepsilon \in \mathbb{R}$ be such that $0 < \varepsilon < d(x_1, x_2)/2$. By the Urysohn lemma we can choose $f_1 \in C(X, \mathbb{R})$ such that $f_1(x) = 1$ when $x = x_1$ and $f_1(x) = 0$ when $x \in X \setminus B(x_1, \varepsilon)$. Similarly choose $f_2 \in C(X, \mathbb{R})$ such that $f_2(x) = 1$ when $x = x_2$ and $f_2(x) = 0$ when $x \in X \setminus B(x_2, \varepsilon)$. For f_1 and f_2, by assumption there is

$\delta > 0$ such that there are $y_1, y_2 \in U$ and $s \in S$ such that $|f_1(sy_1) - f_1(sy_2)| > \delta$ and $|f_2(sy_1) - f_2(sy_2)| > \delta$. From the former we know that at least one of sy_1 and sy_2 is in $B(x_1, \varepsilon)$. By the same reason one of them belongs also to $B(x_2, \varepsilon)$. Without loss of generality we assume $sy_i \in B(x_i, \varepsilon)$ for $i = 1, 2$, and this implies that (x_1, x_2) is a sensitive pair. $\square$

Repeating the same arguments we can conclude the following

PROPOSITION 5.3. *Let (X, S) be a t.d.s. and $n \geq 2$. A tuple $(x_1, \ldots, x_n) \in X^n \setminus \Delta_n(X)$ is sensitive if and only if for every $f_1, \ldots, f_n \in C(X, \mathbb{R})$ with $f_k(x_i) \neq f_k(x_j)$ for every $1 \leq i \neq j \leq n$ and every $1 \leq k \leq n$, (X, S) is $(f_1, \ldots, f_n)$-sensitive.*

Now we are ready to show the main characterization of this section.

THEOREM 5.4. *Let (X, S) be a t.d.s. with S being a commutative semigroup. Then (X, S) is weakly mixing if and only if for every $n \in \mathbb{N}$ and every non-constant $f_1, \ldots, f_n \in C(X, \mathbb{R})$, it is $(f_1, \ldots, f_n)$-sensitive.*

PROOF. Sufficiency follows from Lemma 2.2 and Propositions 3.3 and 5.3. For necessity, by Proposition 5.1, it suffices to show that for every $n \in \mathbb{N}$ and all non-constant $f_1, \ldots, f_n \in C(X, \mathbb{R})$ there is a sensitive pair $(x_1, x_2) \in X^2 \setminus \Delta_2(X)$ such that $f_k(x_1) \neq f_k(x_2)$ for every $1 \leq k \leq n$. This is clear and for completeness we give the details for $n = 2$. That is, if there is a sensitive pair (x_1, x_2) such that $f_1(x_1) = f_1(x_2)$ and $f_2(x_1) \neq f_2(x_2)$, choose $x_3 \in X \setminus \{x_1, x_2\}$ such that $f_1(x_1) \neq f_1(x_3)$ and then either we have $f_2(x_1) \neq f_2(x_3)$ or $f_2(x_2) \neq f_2(x_3)$. Since $(x_1, x_3), (x_2, x_3) \in S_2(X, T)$ by Proposition 3.3, in any case (X, S) is (f_1, f_2)-sensitive. $\square$

As pointed out, for each $n \in \mathbb{N}$ and non-constant functions $f_1, \ldots, f_n \in C(X, \mathbb{R})$, $(f_1, \ldots, f_{n+1})$-sensitivity implies $(f_1, \ldots, f_n)$-sensitivity. In the following we provide examples to show that the converse is not true.

EXAMPLE 5.5. There is a transitive system (X, T) such that it is f-sensitive but not (f, g)-sensitive. Here $f, g \in C(X, \mathbb{R})$ are non-constant and there are $x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ and $g(x_1) \neq g(x_2)$.

PROOF. Consider the t.d.s. (X, T) constructed by Ye and Zhang in [17, Example 6.5]. That is, let $A_1 = 1010$, $k_1 = |A_1| = 4$ and $A_2 = A_1 0^{k_1} A_1 0^{k_1}$. Put recursively $k_n = |A_n|$ and $A_{n+1} = A_n 0^{k_n} A_n 0^{k_n}$ for $n \geq 1$. Define $x = \lim_{n\to\infty} A_n$ and (X, T) as its orbit closure under the shift map. It is known that (X, T) is transitive but neither minimal nor weakly mixing, and in particular $S_2(X, T) = \{(y, \mathbf{0}) : y \in X \setminus \{\mathbf{0}\}\} \cup \{(\mathbf{0}, y) \colon y \in X \setminus \{\mathbf{0}\}\}$. Choose $(x_1, x_2) \in X \times X \setminus (S_2(X, T) \cup \Delta_2(X))$, then by Theorem 5.2, it is not (f, g)-sensitive for some $f, g \in C(X, \mathbb{R})$ with $f(x_1) \neq f(x_2)$ and $g(x_1) \neq g(x_2)$. By Proposition 4.6, (X, S) is f-sensitive for any non-constant $f \in C(X, \mathbb{R})$. $\square$

Example 5.6. Fix $n \geq 3$. There is a minimal system (X,T) such that it is $(f_1,\dots,f_{n-1})$-sensitive but not $(f_1,\dots,f_n)$-sensitive. Here $f_1,\dots,f_n \in C(X,\mathbb{R})$ and there are $x_1,\dots,x_n \in X$ such that $f_k(x_i) \neq f_k(x_j)$ for every $1 \leq i \neq j \leq n$ and every $1 \leq k \leq n$.

Proof. Let (X,T) be a minimal system which is $(n-1)$-sensitive but not n-sensitive. We refer the reader to [17] for explicit examples. By Propositions 3.1 and 5.3, we know that such t.d.s. (X,T) is the suitable one satisfying all the properties as stated. □

6. Equicontinuous functions for semigroup actions

In this section, we consider the opposite of sensitive functions — equicontinuous functions. A dichotomy result between the sensitive and equicontinuous sides will be studied.

Let (X,S) be a t.d.s. and $f \in C(X,\mathbb{R})$. Following [6], a point $x \in X$ is said to be *f-equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $y \in X$ with $d(x,y) < \delta$, we have $|f(sx) - f(sy)| < \varepsilon$ for each $s \in S$. We say that (X,S) is an *f-equicontinuous system* (or equivalently say that f is an *equicontinuous function* for (X,S)) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x,y) < \delta$, $|f(sx) - f(sy)| < \varepsilon$ for each $s \in S$. (X,S) is an *f-almost equicontinuous system* (or f is an *almost equicontinuous function* for (X,S)) if the collection of all f-equicontinuous points is residual (i.e. contains a dense G_δ subset) in X.

We remark that when we omit the action of f and replace $|f(sx) - f(sy)|$ by $d(sx,sy)$ in the definitions above, we get the classical concepts of equicontinuous point, equicontinuous system and almost equicontinuous system correspondingly.

By compactness of X, it is easy to see that (X,S) is f-equicontinuous if and only if every point in X is f-equicontinuous. It is also clear that if f is a constant function then (X,S) is f-equicontinuous.

Proposition 6.1. *Let (X,S) be a t.d.s. and $x \in X$. The point $x \in X$ is equicontinuous if and only if it is f-equicontinuous for all $f \in C(X,\mathbb{R})$.*

Proof. Let $x \in X$ be an equicontinuous point, $f \in C(X,\mathbb{R})$ and $\varepsilon > 0$. By compactness there is $0 < \delta' < \varepsilon$ such that if $d(x,y) < \delta'$ then $|f(x) - f(y)| < \varepsilon$. For this δ', using the equicontinuity of x, there is $\delta > 0$ such that $d(sx,sy) < \delta'$ for all $s \in S$ whenever $d(x,y) < \delta$. This implies that $|f(sx) - f(sy)| < \varepsilon$ for all $s \in S$ whenever $d(x,y) < \delta$, proving necessity.

As for sufficiency, assume $x \in X$ is not an equicontinuous point, then there is $\delta > 0$ such that for any $n \in \mathbb{N}$ there are x_n with $d(x,x_n) < 1/n$ and $s_n \in S$ such that $d(s_n x, s_n x_n) > 3\delta$. Without loss of generality let $y_1 = \lim\limits_{n\to\infty} s_n x$ and $y_2 = \lim\limits_{n\to\infty} s_n x_n$. It is clear that $d(y_1,y_2) \geq 3\delta$. By the Urysohn lemma choose $f \in C(X, \mathbb{R})$ such that $f(x) = 1$ when $x \in B(y_1, \delta)$ and $f(x) = 0$ when $x \in B(y_2, \delta)$. We have $|f(s_n x) - f(s_n x_n)| = 1$ for n large enough, and which in turn implies that x is not an f-equicontinuous point, a contradiction. □

Consequently we have the followings:

Corollary 6.2. *A t.d.s. (X, S) is equicontinuous if and only if it is f-equicontinuous for all $f \in C(X, \mathbb{R})$.*

Corollary 6.3. *A t.d.s. (X, S) is almost equicontinuous if and only if it is f-almost equicontinuous for all $f \in C(X, \mathbb{R})$.*

Let $C_{\mathrm{eq}}(X)$ be the collection of all equicontinuous functions for (X, S). The following result reveals that it is a closed subspace of the Banach space $C(X, \mathbb{R})$ with the supremum norm $\| \cdot \|_\infty$.

Proposition 6.4. *$C_{\mathrm{eq}}(X)$ is a closed subspace of $C(X, \mathbb{R})$.*

Proof. First it is easy to show that if $f, g \in C_{\mathrm{eq}}(X)$ then for all real numbers α, β, $\alpha f + \beta g$ is also in $C_{\mathrm{eq}}(X)$. To show that $C_{\mathrm{eq}}(X)$ is closed, we need to show $C(X, \mathbb{R}) \setminus C_{\mathrm{eq}}(X)$ is open. For each $f \in C(X, \mathbb{R}) \setminus C_{\mathrm{eq}}(X)$. Assume x is a non-f-equicontinuous point. Then there is $\delta > 0$ such that for any open neighbourhood U of x, there are $y \in U$ and $s \in S$ such that $|f(sx) - f(sy)| > \delta$. Let $g \in C(X, \mathbb{R})$ with $\sup_{x \in X} |f(x) - g(x)| < \delta/3$. Then

$$\begin{aligned} |g(sx) - g(sy)| &\geq |f(sx) - f(sy)| - |f(sx) - g(sx)| - |f(sy) - g(sy)| \\ &> \delta - \delta/3 - \delta/3 = \delta/3. \end{aligned}$$

This implies that $B_{\|\cdot\|_\infty}(f, \delta/3) \subset C(X, \mathbb{R}) \setminus C_{\mathrm{eq}}(X)$ and so $C_{\mathrm{eq}}(X)$ is closed. □

For a given $f \in C(X, \mathbb{R})$ and $k \in \mathbb{N}$, we write

$$G_k^f = \Big\{ x \in X : \text{there exists } \delta > 0 \text{ such that} \\ \sup_{s \in S} |f(sx) - f(sy)| < 1/k \text{ whenever } d(x, y) < \delta \Big\}.$$

It is clear that G_k^f is open and $G^f = \bigcap_{k=1}^{\infty} G_k^f$ is the collection of all f-equicontinuous points.

We have the following dichotomy theorem. Note that the classical dichotomy result without the influence of continuous functions was first proved under $\mathbb{Z}_+$ action by Auslander and Yorke [4] and Akin, Auslander and Berg [2] and recently generalized to C-semigroup actions by Kontorovich and Megrelishvili [10].

Theorem 6.5. *Let (X, S) be a t.d.s. with S being a C-semigroup and $f \in C(X, \mathbb{R})$.*

(a) *If (X, S) is transitive then it is either f-almost equicontinuous or f-sensitive.*

(b) *If (X, S) is minimal then it is either f-equicontinuous or f-sensitive.*

Proof. (a) If (X, S) is not f-sensitive, then for any $\varepsilon > 0$ there exists a nonempty open subset U_ε such that $|f(sx_1) - f(sx_2)| < \varepsilon$ for any $x_1, x_2 \in U_\varepsilon$ and $s \in S$. Let $x \in \mathrm{Tran}_S$. Then there is $s_\varepsilon \in S$ such that $s_\varepsilon x \in U_\varepsilon$ and so the open set $V_\varepsilon := s_\varepsilon^{-1} U_\varepsilon$ is a neighbourhood of x. This yields that $|f(ss_\varepsilon y_1) - f(ss_\varepsilon y_2)| < \varepsilon$ for any $y_1, y_2 \in V_\varepsilon$ and $s \in S$. Since S is a C-semigroup the set $S \setminus Ss_\varepsilon$ is finite. So we can shrink the set V_ε (still denoting it by V_ε) so that $|f(sy_1) - f(sy_2)| < \varepsilon$ holds for all $y_1, y_2 \in V_\varepsilon$ and $s \in S$. This means that $x \in \bigcap_{k=1}^{\infty} G_k^f$. Since the transitive points are dense and the set G_k^f is open, G^f is residual and therefore (X, S) is f-almost equicontinuous. (b) follows from (a) and the fact that $\mathrm{Tran}_S = X$. $\square$

Acknowledgments. This work was carried out when the first and second authors visited Shantou University. They acknowledge the hospitality of Department of Mathematics at the University. The authors are also deeply grateful to Jian Li for his insightful discussions and valuable comments.

References

[1] M. Achigar, A. Artigue and I. Monteverde, *Observing expansive maps*, arXiv:1611.08488v1, 2016.

[2] E. Akin, J. Auslander and E. Glasner, *The topological dynamics of Ellis actions*, Mem. Amer. Math. Soc. **195** (2008), no. 913, vi+152 pp.

[3] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Mathematics Studies, vol. 153, Amsterdam, North-Holland, 1988.

[4] J. Auslander and J. Yorke, *Interval maps, factors of maps and chaos*, Tôhoku Math. J. (2) **32** (1980), no. 2, 177–188.

[5] F. García-Ramos and B. Marcus, *Mean sensitive, mean equicontinuous and almost periodic functions for dynamical systems*, arXiv:1509.05246v3, 2015.

[6] E. Glasner and M. Megrelishvili, *Linear representations of hereditarily non-sensitive dynamical systems*, Colloq. Math. **104** (2006), no. 2, 223–283.

[7] E. Glasner and B. Weiss, *Quasi-factors of zero-entropy systems*, J. Amer. Math. Soc. **8** (1995), 665–686.

[8] D. Kerr and H. Li, *Dynamical entropy in banach spaces*, Invent. Math. **162** (2005), no. 3, 649–686.

[9] D. Kerr and H. Li, *Independence in topological and C^*-dynamics*, Math. Ann. **338** (2007), 869–926.

[10] E. Kontorovich and M. Megrelishvili, *A note on sensitivity of semigroup actions*, Semigroup Forum **76** (2008), no. 1, 133–141.

[11] J. Li and X. Ye, *Recent development of chaos theory in topological dynamics*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 1, 83–114.

[12] K. Petersen, *Disjointness and weak mixing of minimal sets*, Proc. Amer. Math. Soc. **24** (1970), 278–280.

[13] D. Ruelle, *Dynamical systems with turbulent behavior*, Mathematical Problems in Theoretical Physics, Proc. Internat. Conf., Univ. Rome, Rome, 1977, Lecture Notes in Phys., vol. 80, Springer, Berlin, New York, 1978, pp. 341–360.

[14] S. Shao, X. Ye and R. Zhang, *Sensitivity and regionally proximal relation in minimal systems*, Sci. China Math. **51** (2008), 987–994.

[15] H. Wang, Z. Chen and H. Fu, *M-systems and scattering systems of semigroup actions*, Semigroup Forum **91** (2015), no. 3, 699–717.

[16] J. Xiong, *Chaos in a topologically transitive system*, Sci. China Ser. A **48** (2005), 929–939.

[17] X. Ye and R. Zhang, *On sensitive sets in topological dynamics*, Nonlinearity **21** (2008), 1601–1620.

Manuscript received April 11, 2017

accepted August 8, 2017

Jie Li
School of Mathematics and Statistics
Jiangsu Normal University
Xuzhou, Jiangsu, 221116, P.R. CHINA
and
Academy of Mathematics and Systems Science
Chinese Academy of Sciences
Beijing, 100190, P.R. CHINA

E-mail address: jiel0516@mail.ustc.edu.cn

Tao Yu (corresponding author)
Shanghai Center for Mathematical Sciences
Fudan University
Shanghai, 200433, P.R. CHINA
and
Wu Wen-Tsun Key Laboratory of Mathematics
USTC, Chinese Academy of Sciences
Department of Mathematics
University of Science and Technology of China
Hefei, Anhui, 230026, P.R. CHINA

E-mail address: ytnuo@mail.ustc.edu.cn

Tiaoying Zeng
Department of Mathematics
Shantou University
Shantou, Guangdong 515063, P.R. CHINA
and
School of Mathematics
Jiaying University
Meizhou, Guangdong 514015, P.R. CHINA

E-mail address: zengtiaoying@126.com