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SCHAUDER'S THEOREM AND THE METHOD OF A PRIORI BOUNDS

Andrzej Granas — Marlene Frigon

This article is dedicated to the memory of Professeur Marek Burnat

Abstract. We first recall simple proofs relying on the Schauder Fixed Point Theorem of the Nonlinear Alternative, the Leray–Schauder Alternative and the Coincidence Alternative for compact maps on normed spaces. We present also an alternative for compact maps defined on convex subsets of normed spaces. Those alternatives permit to apply the method of a priori bounds to obtain results establishing the existence of solutions to differential equations. Using those alternatives, we present some new proofs of existence results for first order differential equations.

1. Introduction

In this note, we first recall simple proofs relying on the Schauder Fixed Point Theorem of the Nonlinear Alternative, the Leray–Schauder Alternative and the Coincidence Alternative for compact maps on normed spaces [4], [5]. The avantage of those proofs is that they avoid the use of more sophisticated theories such as the topological degree theory, the topological transversality theory or the coincidence degree theory due to Mawhin [9]. To our knowledge, the first result in this direction was obtained by Schaefer [11].

In Section 2, we present some examples of applications of those alternatives with the method of a priori bounds to differential equations. First, we recall a generalization of a theorem due to S. Bernstein for second order differential

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equations [7], [8], [6]. Then, we present a new proof of a result establishing the existence of solutions to periodic problems with, as right member, a first order, nonlinear differential operator. It is worth to point out that this proof relies on the Coincidence Alternative and it involves a non-invertible, nonlinear operator. In particular, it does not require a modification of the problem in order to obtain a linear, invertible operator as it is done usually.

Finally, we present a generalization of the Nonlinear Alternative to compact maps defined on arbitrary closed, convex sets. Using this result, we obtain a new proof of the existence of solutions to problems for first order differential equations under an assumption of existence of an ordered pair of lower and upper solutions. This proof does not require to consider a modified problem as it is done in the classical proof.

2. Nonlinear Alternative for compact maps

In what follows, E and F denote normed spaces, $B_\rho = \{x \in E : \|x\| \leq \rho\}$ and $S_\rho = \partial B_\rho$ are respectively the closed ball and the sphere of radius $\rho > 0$ in E. The *standard retraction* of E on B_ρ noted $r_\rho \colon E \to B_\rho$ is defined by

$$r_\rho(x) = \begin{cases} x & \text{if } x \in B_\rho, \\ \dfrac{\rho x}{\|x\|} & \text{otherwise.} \end{cases}$$

Let $X \subset E$. A map $f \colon X \to F$ is *compact* (resp. *completely continuous*) if it is continuous and $\overline{f(X)}$ is compact (resp. $\overline{f(A)}$ is compact for every bounded set $A \subset X$).

Definition 2.1. Let $X \subset E$ and $f \colon X \to E$. An *ejectable point of* f is an element $x_0 \in X$ such that $x_0 = \lambda f(x_0)$ for some $\lambda \in (0,1)$.

Let $X \subset E$ and $f \colon X \to E$. In what follows, we will use the following notations:

$$\begin{aligned} E(f) &= \{x \in X : x \text{ is an ejectable point of } f\}, \\ E_\rho(f) &= E(f) \cap S_\rho, \\ \operatorname{Fix}(f) &= \{x \in X : x = f(x)\}, \\ \operatorname{Fix}_\rho(f) &= \operatorname{Fix}(f) \cap B_\rho. \end{aligned}$$

One can observe that $\operatorname{Fix}(f) \cap E(f) = \emptyset$.

Here is a simple proof of the Nonlinear Alternative relying on the Schauder Fixed Point Theorem, see [4] and [5].

Theorem 2.2 (Nonlinear Alternative). *Let $f \colon B_\rho \to E$ be a compact map. Then, one of the following statements holds*:

(a) $\operatorname{Fix}(f) \neq \emptyset$;

(b) $E_\rho(f) \neq \emptyset$.

Moreover, $\mathrm{Fix}(r_\rho \circ f) = \mathrm{Fix}_\rho(f) \cup E_\rho(f) \neq \emptyset$.

Proof. The Schauder Fixed Point Theorem implies that the compact map $r_\rho \circ f\colon B_\rho \to B_\rho$ has a fixed point x_0. If $f(x_0) \in B_\rho$, then $x_0 = r_\rho(f(x_0)) = f(x_0)$, and so, (a) is true. Otherwise, $\lambda = \rho/\|f(x_0)\| < 1$ and $x_0 = r_\rho(f(x_0)) = \lambda f(x_0)$. In that case, (b) is verified. □

As a corollary, we deduce the Leray–Schauder Alternative, see [4], [5].

Theorem 2.3 (Leray–Schauder Alternative). *Let $f\colon E \to E$ be a completely continuous map. Then, one of the following statements holds*:

(a) $\mathrm{Fix}(f) \neq \emptyset$;

(b) $E(f)$ *is not bounded.*

Proof. We assume that $E(f)$ is bounded and included in $B_\rho \setminus S_\rho$ for some $\rho > 0$. The previous theorem applied to $f\colon B_\rho \to E$ implies that $\mathrm{Fix}_\rho(f) \neq \emptyset$. □

Now, we consider continuous maps $\phi\colon E \to F$ and $f\colon X \to F$ with $X \subset E$. A *coincidence point* between ϕ and f is an element $x_0 \in X$ such that

$$\phi(x_0) = f(x_0).$$

Definition 2.4. A continuous function $\phi\colon E \to F$ is *invertible from the right* if there exists a continuous map $\psi\colon F \to E$ such that $\phi \circ \psi = \mathrm{id}|_F$.

We introduce the following notations:

$$\begin{aligned}
E(\phi, f) &= \{x \in X : \text{there exists } \lambda \in (0,1) \text{ such that } \phi(x) = \lambda f(x)\},\\
E_\rho(\phi, f) &= E(\phi, f) \cap S_\rho,\\
\mathrm{Coin}(\phi, f) &= \{x \in X : \phi(x) = f(x)\},\\
\mathcal{DI}(E, F) &= \{\phi\colon E \to F : \phi \text{ is continuous, invertible from the right}\\
&\qquad \text{and such that } \forall\, (\mu, x) \in (0,1) \times E,\ \exists\, \lambda \in (0,1)\\
&\qquad \text{such that } \phi(\mu x) = \lambda \phi(x)\}.
\end{aligned}$$

The Nonlinear Alternative can be generalized by the following coincidence result.

Theorem 2.5 (Coincidence Alternative). *Let $\phi \in \mathcal{DI}(E, F)$ and $f\colon B_\rho \to F$ be a compact map. Then, one of the following statements holds*:

(a) $\mathrm{Coin}(\phi, f) \neq \emptyset$;

(b) $E_\rho(\phi, f) \neq \emptyset$.

Proof. Let $\psi\colon F \to E$ be a continuous map such that $\phi \circ \psi = \mathrm{id}|_F$. The map $r_\rho \circ \psi \circ f\colon B_\rho \to B_\rho$ is compact. The Schauder Fixed Point Theorem insures the existence of $x_0 \in B_\rho$ such that $x_0 = r_\rho(\psi(f(x_0))) = r_\rho(y_0)$, where $y_0 = \psi(f(x_0))$.

If $y_0 \in B_\rho$, one has $x_0 = y_0$ and

$$\phi(x_0) = \phi(y_0) = \phi(\psi(f(x_0))) = f(x_0).$$

Hence, $x_0 \in \operatorname{Coin}(\phi, f)$.

Otherwise, $x_0 \in S_\rho$ and there exists $\mu \in (0,1)$ such that $x_0 = \mu y_0 = r_\rho(y_0)$. Since $\phi \in \mathcal{DI}(E,F)$, there exists $\lambda \in (0,1)$ such that

$$\phi(x_0) = \phi(\mu y_0) = \lambda\phi\big(\psi(f(x_0))\big) = \lambda f(x_0).$$

Therefore, $x_0 \in E_\rho(\phi, f)$. □

Corollary 2.6. *Let E, F be Banach spaces, $\phi \in \mathcal{L}(E,F)$ a surjective, continuous, linear application and let $f\colon B_\rho \to F$ be a compact map. Then, one of the following statements holds*:

(a) $\operatorname{Coin}(\phi, f) \neq \emptyset$;
(b) $E_\rho(\phi, f) \neq \emptyset$.

Proof. A theorem due to Bartle and Graves [1] implies that $\phi \in \mathcal{DI}(E,F)$. The conclusion follows from Theorem 2.5. □

An analogue of the Leray–Schauder Alternative can also be obtained.

Theorem 2.7. *Let $\phi \in \mathcal{DI}(E,F)$ and $f\colon E \to F$ be a completely continuous map. Then, one of the following statements holds:*

(a) $\operatorname{Coin}(\phi, f) \neq \emptyset$;
(b) *$E(\phi, f)$ is not bounded.*

An analogous argument permits to obtain similar alternatives and coincidence theorems for compact, u.s.c., multi-valued maps relying on the Kakutani Fixed Point Theorem. The interested reader is referred to [4].

3. The method of a priori bounds

In this section, we present some examples of applications of the Nonlinear Alternative to problems of existence of solutions to differential equations illustrating what it is called the *method of a priori bounds*. To simplify the presentation, we chose to consider differential equations whose right member is a continuous function. Our results are still valid for Carathéodory functions.

We denote $I = [0,T]$ and $C(I)$ (resp. $C^k(I)$) the space of continuous functions defined on I (resp. k-time continuously differentiable) endowed with the usual norm $\|\cdot\|_0$ (resp. $\|\cdot\|_k$).

We consider the following second order differential equation with the homogenous Dirichlet boundary conditions:

$$
\begin{aligned}
&u''(t) = g(t, u(t), u'(t)) \quad \text{for every } t \in I = [0, T], \\
&u(0) = u(T) = 0.
\end{aligned} \tag{3.1}
$$

Here is a first existence result which illustrates the method of a priori bounds. The interested reader may find in the literature more general results, see for example [2].

Theorem 3.1. *Let $g\colon I \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous map satisfying the following conditions*:

(a) *There exists a constant $M \ge 0$ such that $g(t, M, 0) \ge 0 \ge g(t, -M, 0)$ for every $t \in I$.*
(b) *There exist $c \ge 0$, $l\colon I \to [0, \infty)$ and $\omega\colon (0, \infty) \to (0, \infty)$ continuous maps such that*

$$
\frac{1}{\omega} \in L^1_{\text{loc}}([0, \infty)), \qquad \int_0^\infty \frac{ds}{\omega(s)} > 2cM + \|l\|_{L^1}
$$

and

$$
|g(t, x, y)| \le \omega(|y|)(l(t) + c|y|) \quad \textit{for every } (t, x, y) \in I \times [-M, M] \times \mathbb{R} \setminus \{0\}.
$$

Then, there exists $u \in C^2(I)$ a solution of (3.1) *such that $\|x\|_0 \le M$.*

Proof. Let $C^2_D(I) = \{u \in C^2(I) : u(0) = u(T) = 0\}$. We consider $L\colon C^2_D(I) \to C(I)$ the linear, continuous, invertible operator defined by $L(u)(t) = u''(t)$ and $i\colon C^2_D(I) \to C^1(I)$ the canonical injection.

We define $\widehat{g}\colon I \times \mathbb{R}^2 \to \mathbb{R}$ and $G\colon C^1(I) \to C(I)$ by

$$
\widehat{g}(t, x, y) =
\begin{cases}
g(t, x, y) & \text{if } |x| \le M, \\
g(t, M, y) + x - M & \text{if } x > M, \\
g(t, -M, y) + x + M & \text{if } x < -M;
\end{cases}
$$

and

$$
G(u)(t) = \widehat{g}(t, u(t), u'(t)).
$$

The map $\widehat{g}$ is continuous and the operator $f\colon C^1(I) \to C^1(I)$ defined by $f = i \circ L^{-1} \circ G$ is completely continuous, see [5].

It is easy to verify that the fixed points of λf for $\lambda \in [0, 1]$ are solutions of the problem:

$$
\begin{aligned}
&u''(t) = \lambda \widehat{g}(t, u(t), u'(t)) \quad \text{for every } t \in I = [0, T], \\
&u(0) = u(T) = 0.
\end{aligned} \tag{3.2}
$$

It follows from the Leray–Schauder Alternative (Theorem 2.3) that

$$
\operatorname{Fix}(f) \ne \emptyset \quad \text{or} \quad E(f) \text{ is not bounded.} \tag{3.3}
$$

We claim that, for every $u \in E(f) \cup \mathrm{Fix}(f)$,

$$\|u\|_0 \le M. \tag{3.4}$$

Indeed, we assume that $J = \{t \in I : u(t) > M\} \neq \emptyset$. Let

$$u(t_0) = \max\{u(t) : t \in I\}.$$

Since $u(0) = u(T) = 0$, one has that $t_0 \in (0,T)$, $u'(t_0) = 0$ and $u''(t_0) \le 0$. One deduces from (a) and the fact that $u \in E(f) \cup \mathrm{Fix}(f)$ that, for some $\lambda \in [0,1]$,

$$0 \ge u''(t_0) = \lambda \widehat{g}(t_0, u(t_0), u'(t_0)) = \lambda g(t_0, M, 0) + u(t_0) - M > 0.$$

This is a contradiction. Hence $J = \emptyset$. Similarly, one can show that $u(t) \ge -M$ for every $t \in I$.

We assume that $t_2 \in I$ is such that $|u'(t_2)| = \max\{|u'(t)| : t \in I\} > 0$. It follows from Rolle's Theorem and the continuity of u' that there exists $t_1 \in (0,T)$ such that

$$u'(t_1) = 0 \quad \text{and} \quad |u'(t)| > 0 \quad \text{for every } t \in (\min\{t_1,t_2\}, \max\{t_1,t_2\}).$$

Without loss of generality, we assume that $t_1 < t_2$ and $u'(t) > 0$ for every $t \in (t_1, t_2)$. It follows from (b) and (3.4) that

$$u''(t) \le \omega(u'(t))(l(t) + cu'(t)) \quad \text{for every } t \in (t_1, t_2).$$

So,

$$2cM + \|l\|_{L^1} \ge \int_{t_1}^{t_2} \frac{u''(t)}{\omega(u'(t))}\, dt = \int_0^{u'(t_2)} \frac{ds}{\omega(s)}.$$

This implies that $u'(t_2) \le K$, where $K > 0$ is a constant such that

$$\int_0^K \frac{ds}{\omega(s)} > 2cM + \|l\|_{L^1}.$$

Such a constant K exists by (b). One concludes that

$$\|u'\|_0 \le K \quad \text{for every } u \in E(f) \cup \mathrm{Fix}(f). \tag{3.5}$$

Combining (3.3)–(3.5), one deduces that there exists $u \in \mathrm{Fix}(f)$. Again, (3.4) and the definition of $\widehat{g}$ imply that u is a solution of (3.1). □

As a corollary, let us recall the following result due to Granas, Guenther and Lee [7] which generalizes a theorem obtained by S. Bernstein in 1912.

Corollary 3.2. *Let $g\colon I \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous map satisfying the following conditions*:

(a) *There exists a constant $M \ge 0$ such that $xg(t,x,0) > 0$ for every $|x| \ge M$.*
(b) *There exist $a, b\colon I \times \mathbb{R} \to (0,\infty)$ continuous maps such that*

$$|g(t,x,y)| \le a(t,x)y^2 + b(t,x) \quad \textit{for every } (t,x,y) \in I \times \mathbb{R}^2.$$

Then, there exists $u \in C^2(I)$ *a solution of* (3.1) *such that* $\|x\|_0 \le M$.

Proof. The proof follows from the previous theorem applied to $\omega(s) = (1+s^2)/s$, $l(t) = 0$ and $c = \max\{\{a(t,x), b(t,x)\} : (t,x) \in I \times [-M,M]\}$. □

Now, we fix $p \ge 1$ and consider the following periodic problem involving a nonlinear, first order differential operator (it is linear only for $p = 1$):

$$\begin{aligned} &\frac{d}{dt}\,(|u(t)|^{p-1}u(t)) = g(t,u(t)) \quad \text{for every } t \in I = [0,T],\\ &u(0) = u(T). \end{aligned} \tag{3.6}$$

We present a new proof of the following result relying on the Coincidence Alternative. The interested reader is referred to [3] for results on this type of problems relying on the fixed point index.

Theorem 3.3. *Let* $g\colon I \times \mathbb{R} \to \mathbb{R}$ *be a continuous map. We assume that there exists* $M \ge 0$ *such that* $g(t,M) \le 0 \le g(t,-M)$ *for every* $t \in I$. *Then, there exists* $u \in C^1(I)$ *a solution of* (3.6) *such that* $\|u\|_0 \le M$.

Proof. We consider the Banach spaces

$$E = \{u \in C(I) : u(0) = u(T)\} \quad \text{and} \quad F = \bigg\{u \in C(I) : \int_0^T u(s)\,ds = 0\bigg\}.$$

We define $\phi\colon E \to F$ by $\phi(u)(t) = |u(t)|^{p-1}u(t) - \overline{u^p}$, where

$$\overline{u^p} = \frac{1}{T}\int_0^T |u(s)|^{p-1}u(s)\,ds.$$

Observe that ϕ is continuous, invertible from the right and $\phi(\mu x) = \mu^p\phi(x)$ for every $(\mu,x) \in (0,1) \times E$.

We define $\widetilde{g}\colon I \times \mathbb{R} \to \mathbb{R}$ and $f\colon E \to F$ by

$$\widetilde{g}(t,x) = \begin{cases} g(t,x) & \text{if } |x| \le M,\\ g(t,M) + M - x & \text{if } x > M,\\ g(t,-M) - M - x & \text{if } x < -M; \end{cases}$$

and

$$f(u)(t) = \int_0^t \widetilde{g}(s,u(s))\,ds - \frac{1}{T}\int_0^T\int_0^\tau \widetilde{g}(s,u(s))\,ds\,d\tau.$$

Since $\widetilde{g}$ is continuous, it is well known that f is completely continuous.

We remark that if $u \in \operatorname{Coin}(\phi,f) \cup E(\phi,f)$, then

$$\int_0^T \widetilde{g}(s,u(s))\,ds = 0 \tag{3.7}$$

and, for some $\lambda \in (0,1]$, u is a solution of the problem

$$\begin{aligned} &\frac{d}{dt}(|u(t)|^{p-1}u(t)) = \lambda\widetilde{g}(t,u(t)) \quad \text{for every } t \in I = [0,T],\\ &u(0) = u(T). \end{aligned} \tag{3.8}$$

Indeed, for some $\lambda \in (0,1]$, $\phi(u) = \lambda f(u)$. So, for every $t \in I$,

$$|u(t)|^{p-1}u(t) - \overline{u^p} = \lambda\bigg(\int_0^t \widetilde{g}(s,u(s))\,ds - \frac{1}{T}\int_0^T\int_0^\tau \widetilde{g}(s,u(s))\,ds\,d\tau\bigg). \tag{3.9}$$

In particular, for $t = 0$ and $t = T$, we obtain

$$|u(0)|^{p-1}u(0) - \overline{u^p} = -\frac{\lambda}{T}\int_0^T\int_0^\tau \widetilde{g}(s,u(s))\,ds\,d\tau$$

and

$$|u(T)|^{p-1}u(T) - \overline{u^p} = \lambda\bigg(\int_0^T \widetilde{g}(s,u(s))\,ds - \frac{1}{T}\int_0^T\int_0^\tau \widetilde{g}(s,u(s))\,ds\,d\tau\bigg).$$

Knowing that $u(0) = u(T)$, we deduce (3.7). Also, by taking the derivative of (3.9), one has that u is a solution of (3.8).

Now, we assume that $\|u\|_0 \not\leq M$. Without loss of generality, $J = \{t \in I : u(t) > M\} \neq \emptyset$. Let

$$t_0 = \max\Big\{t \in J : u(t) = \max_{s\in I} u(s)\Big\}.$$

Since $u(0) = u(T)$, $t_0 > 0$ and $u'(t_0) \geq 0$. Using (3.8) and the definition of $\widetilde{g}$, we obtain

$$0 \leq \frac{d}{dt}\,(|u|^{p-1}u)(t_0) = \lambda\widetilde{g}(t_0,u(t_0)) = \lambda(g(t_0,M) + M - u(t_0)) < 0.$$

This is a contradiction. Hence, $J = \emptyset$. Similarly, we can show that $u(t) \geq -M$ for every $t \in I$. We conclude that

$$\|u\|_0 \leq M \quad \text{for every } u \in E(\phi,f) \cup \operatorname{Coin}(\phi,f). \tag{3.10}$$

Corollary 2.6 implies that there exists $u \in \operatorname{Coin}(\phi,f)$. From (3.8), (3.10) and the definition of $\widetilde{g}$, we conclude that u is a solution of (3.6). □

Remark 3.4. It is interesting to realize that, in the previous proof, we used the non-invertible and non-linear operator ϕ (linear only in the case $p = 1$). This proof shows that it is not necessary to make a change of variables and to modify the problem in order to obtain a problem of the form $L(v) = h(v)$ with L a linear invertible operator.

4. Alternative with a non-standard retraction

Let $C \subset E$ be a closed, convex set. In this section, we extend the Nonlinear Alternative to compact maps defined on C. It is well known that there exists a continuous retraction $r\colon E \to C$. Such a retraction may be not unique.

We fix $r\colon E \to C$ a continuous retraction.

Definition 4.1. Let $f\colon C \to E$. An *r-ejectable point of f* is an element $x_0 \notin \operatorname{Fix}(f)$ such that $f(x_0) \in r^{-1}(x_0)$.

We denote $E_{C,r}(f) = \{x \in C : x \text{ is an } r\text{-ejectable point of } f\}$. By an argument analogous to the one in the proof of the Nonlinear Alternative, we obtain the following result.

Theorem 4.2. *Let $f\colon C \to E$ be a compact map. Then, one of the following statements holds*:

(a) $\mathrm{Fix}(f) \neq \emptyset$;
(b) $E_{C,r}(f) \neq \emptyset$.

We apply the previous theorem to present a new proof of an existence result for first order differential equations involving the method of upper and lower solutions [10]. Again, to simplify the presentation, we consider differential equations with as right-member a continuous function. Our result is still valid for Carathéodory maps.

We consider the following problem:

$$\begin{aligned} &u'(t) = g(t, u(t)) \quad \text{for every } t \in I = [0, T], \\ &u(0) = x_0. \end{aligned} \tag{4.1}$$

Theorem 4.3. *Let $g\colon I \times \mathbb{R} \to \mathbb{R}$ be a continuous map. We assume that there exist $\alpha, \beta \in C^1(I)$ such that*

(a) *$\alpha(t) \le \beta(t)$ for every $t \in I$;*
(b) *$\alpha(0) \le x_0 \le \beta(0)$;*
(c) *$\alpha'(t) \le g(t, \alpha(t))$ and $\beta'(t) \ge g(t, \beta(t))$ for every $t \in I$.*

Then, there exists $u \in C^1(I)$ a solution of (4.1) such that $\alpha(t) \le u(t) \le \beta(t)$ for every $t \in I$.

Proof. We consider the convex set $C = \{u \in C(I) : \alpha(t) \le u(t) \le \beta(t)$ for every $t \in I\}$, and the continuous retraction $r\colon C(I) \to C$ defined by

$$r(u)(t) = \begin{cases} \beta(t) & \text{if } u(t) > \beta(t), \\ u(t) & \text{if } \alpha(t) \le u(t) \le \beta(t), \\ \alpha(t) & \text{if } u(t) < \alpha(t). \end{cases}$$

We consider the map $f\colon C(I) \to C(I)$ defined by

$$f(u)(t) = x_0 + \int_0^t g(s, u(s))\, ds. \tag{4.2}$$

It is well known that f is completely continuous. Therefore, its restriction to C, $f\colon C \to C(I)$ is a compact map since C is bounded.

Theorem 4.2 implies that

$$\mathrm{Fix}(f) \cup E_{C,r}(f) \neq \emptyset.$$

If there exists $u \in E_{C,r}(f)$, then $u = r(f(u))$ and $f(u) \notin C$. We assume that

$$J = \{t \in I : f(u)(t) > \beta(t)\} \neq \emptyset.$$

Let $t_1 \in J$ and $t_0 = \sup [0, t_1] \setminus J$. It follows from (b) and the continuity of $f(u)$ and β that $0 \leq t_0 < t_1$, $(t_0, t_1] \subset J$,

$$f(u)(t_0) = \beta(t_0) \quad \text{and} \quad f(u)(t) > \beta(t) \quad \text{for every } t \in (t_0, t_1]. \tag{4.3}$$

Therefore,

$$u(t) = r(f(u))(t) = \beta(t) \quad \text{for every } t \in [t_0, t_1]. \tag{4.4}$$

From (4.2), one has

$$f(u)(t) = f(u)(t_0) + \int_{t_0}^{t} g(s, u(s))\, ds \quad \text{for every } t \in [t_0, t_1]. \tag{4.5}$$

Combining (c) with (4.3)–(4.5), one deduces that

$$\begin{aligned} \beta(t_1) < f(u(t_1)) &= f(u(t_0)) + \int_{t_0}^{t_1} g(s, u(s))\, ds \\ &= \beta(t_0) + \int_{t_0}^{t_1} g(s, \beta(s))\, ds \leq \beta(t_0) + \int_{t_0}^{t_1} \beta'(s)\, ds = \beta(t_1). \end{aligned}$$

This is a contradiction. Thus, $J = \emptyset$. Similarly, one can show that $f(u)(t) \geq \alpha(t)$ for every $t \in I$. Thus, $f(u) \in C$.

One concludes that $E_{C,r}(f) = \emptyset$ and there exists $u \in \mathrm{Fix}(f)$ a solution of (4.1). □

Remark 4.4. Let us point out that the previous proof shows that, contrarily to the standard proof, one can avoid to consider the modified problem:

$$\begin{aligned} &u'(t) = g(t, r(u(t))) \quad \text{for every } t \in I = [0, T], \\ &u(0) = x_0. \end{aligned}$$

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Andrzej Granas, Professor Emeritus
University of Montréal
Montréal, CANADA

E-mail address: baragranas@hotmail.com

Marlene Frigon
Department of Mathematics and Statistic
University of Montréal
Montréal, CANADA

E-mail address: frigon@dms.umontreal.ca