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SIGN CHANGING SOLUTIONS OF p-FRACTIONAL EQUATIONS WITH CONCAVE-CONVEX NONLINEARITIES

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Abstract. We study the existence of sign changing solutions to the following p-fractional problem with concave-critical nonlinearities:

$$\begin{aligned} (-\Delta)^s_p u &= \mu |u|^{q-1}u + |u|^{p^*_s-2}u && \text{in } \Omega, \\ u &= 0 && \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $s \in (0,1)$ and $p \geq 2$ are fixed parameters, $0 < q < p-1$, $\mu \in \mathbb{R}^+$ and $p^*_s = Np/(N-ps)$. Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > ps$.

1. Introduction

Let us consider the following fractional p-Laplace equation with concave-critical nonlinearities:

$$(\mathcal{P}_\mu) \qquad \begin{cases} (-\Delta)^s_p u = \mu |u|^{q-1}u + |u|^{p^*_s-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $p > 1$ are fixed, $N > ps$, Ω is an open, bounded domain in $\mathbb{R}^N$ with smooth boundary, $0 < q < p-1$, $p^*_s = Np/(N-ps)$ and $\mu \in \mathbb{R}^+$. The

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non-local operator $(-\Delta)_p^s$ is defined as follows:

$$(1.1)\qquad (-\Delta)_p^s u(x) = 2\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{N+ps}}\,dy,$$

for $x\in\mathbb{R}^N$. For $p\geq 1$, we denote the usual fractional Sobolev space by $W^{s,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \bigg(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\bigg)^{1/p}.$$

We set $Q:=\mathbb{R}^{2N}\setminus(\Omega^{\rm c}\times\Omega^{\rm c})$ with $\Omega^{\rm c}=\mathbb{R}^N\setminus\Omega$ and define

$$X := \bigg\{u\colon \mathbb{R}^N\to\mathbb{R} \text{ measurable } : u|_\Omega\in L^p(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy<\infty\bigg\}.$$

The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \bigg(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\bigg)^{1/p}.$$

Then, we define $X_0:=\{u\in X : u=0 \text{ almost everywhere in } \mathbb{R}^N\setminus\Omega\}$ or equivalently, as $\overline{C_{\rm c}^\infty(\Omega)}^X$ and for any $p>1$, X_0 is a uniformly convex Banach space (see [16]) endowed with the norm

$$\|u\|_{X_0} = \bigg(\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\bigg)^{1/p}.$$

Since $u=0$ in $\mathbb{R}^N\setminus\Omega$, the above integral can be extended to all of $\mathbb{R}^N$. The embedding $X_0\hookrightarrow L^r(\Omega)$ is continuous for any $r\in[1,p_s^*]$ and compact for $r\in[1,p_s^*)$. For further details on X_0 and its properties we refer to [14].

Definition 1.1. We say that $u\in X_0$ is a weak solution of $(\mathcal{P}_\mu)$ if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy = \mu\int_\Omega |u(x)|^{q-1}u(x)\phi(x)\,dx + \int_\Omega |u(x)|^{p_s^*-2}u(x)\phi(x)\,dx,$$

for all $\phi\in X_0$.

The Euler–Lagrange energy functional associated to $(\mathcal{P}_\mu)$ is

$$(1.2)\qquad \begin{aligned} I_\mu(u) &= \frac{1}{p}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy - \frac{\mu}{q+1}\int_\Omega |u|^{q+1}\,dx - \frac{1}{p_s^*}\int_\Omega |u|^{p_s^*}\,dx \\ &= \frac{1}{p}\,\|u\|_{X_0}^p - \frac{\mu}{q+1}\,|u|_{L^{q+1}(\Omega)}^{q+1} - \frac{1}{p_s^*}\,|u|_{L^{p_s^*}(\Omega)}^{p_s^*}. \end{aligned}$$

We define the best fractional critical Sobolev constant S as

$$S := \inf_{v \in W^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy}{\left(\displaystyle\int_{\mathbb{R}^N} |v(x)|^{p_s^*}\,dx\right)^{p/p_s^*}}, \tag{1.3}$$

which is positive by fractional Sobolev inequality. Since the embedding $X_0 \hookrightarrow L^{p_s^*}$ is not compact, I_μ does not satisfy the Palais–Smale condition globally, but that holds true when the energy level falls inside a suitable range related to S. As it was mentioned in [13], the main difficulty in dealing with critical fractional case with $p \neq 2$, is the lack of an explicit formula for minimizers of S which is very often a key tool to handle the estimates leading to the compactness range of I_μ. This difficulty has been tactfully overcome in [13] and [20] by the optimal asymptotic behavior of minimizers, which was recently obtained in [9]. Using the same optimal asymptotic behavior of minimizer of S, we will establish suitable compactness range.

Thanks to the continuous Sobolev embedding $X_0 \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$, I_μ is a well-defined C^1 functional on X_0. It is well known that there exists a one-to-one correspondence between the weak solutions of $(\mathcal{P}_\mu)$ and the critical points of I_μ on X_0.

A classical topic in nonlinear analysis is the study of existence and multiplicity of solutions for nonlinear equations. In past few years there has been considerable interest in studying the following general fractional p-Laplacian problem:

$$\begin{aligned} (-\Delta)_p^s u &= f(u) && \text{in } \Omega, \\ u &= 0 && \text{in } \mathbb{R}^N \setminus \Omega. \end{aligned}$$

In [19], the eigenvalue problem associated with $(-\Delta)_p^s$ has been studied. Some results about the existence of solutions have been considered in [17]–[19], see also the references therein.

On the other hand, the fractional problems for $p = 2$ have been investigated by many researchers, see for example [22] for the subcritical case, [3], [5], [23] for the critical case. In [6] the authors studied the nonlocal equation involving a concave-convex nonlinearity in the subcritical case. In [12] the existence of multiple positive solutions to $(\mathcal{P}_\mu)$ for both the subcritical and critical case were obtained. Existence of infinitely many nontrivial solutions to $(\mathcal{P}_\mu)$ in both subcritical and critical cases and existence of at least one sign-changing solution have been established in [5]. In the local case $s = 1$ equations with concave-convex nonlinearities were studied by many authors, to mention few, see [2], [1], [4], [10]. When $s = 1$ and $p = 2$, existence of sign changing solutions was studied in [11].

In [16], Goyal and Sreenadh studied the existence and multiplicity of non-negative solutions of p-fractional equations with subcritical concave-convex non-linearities. In [13], Chen and Squassina have studied the concave-critical system of equations with the p-fractional Laplace operator. More precisely, they studied:

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{q-1}u + \dfrac{2\alpha}{\alpha+\beta}\, |u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)_p^s v = \lambda |v|^{q-1}u + \dfrac{2\beta}{\alpha+\beta}\, |v|^{\beta-2}v|u|^{\alpha} & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\alpha+\beta = p_s^*$, $0 < q < p-1$, $\alpha, \beta > 1$, λ, μ are two positive parameters. When $(N(p-2)+ps)/(N-ps) \le q < p-1$ and $N > p^2 s$, they have proved that there exists $\lambda_* > 0$ such that for $0 < \lambda^{p/(p-q)} + \mu^{p/(p-q)} < \lambda_*$, the above system of equations admits at least two nontrivial solutions.

Note that, if we set $\lambda = \mu$, $\alpha = \beta = p_s^*/2$ and $u = v$ then the above system reduces to $(\mathcal{P}_\mu)$. Therefore, it follows that when $(N(p-2)+ps)/(N-ps) \le q < p-1$ and $N > p^2 s$, problem $(\mathcal{P}_\mu)$ admits two nontrivial solutions for $\mu \in (0, \mu_*)$, for some $\mu_* > 0$. It can be shown that the nontrivial solutions obtained in [13] are actually positive solutions of $(\mathcal{P}_\mu)$ (see Remark 2.2 in Section 2).

The main result of this article is the following:

Theorem 1.2. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$. Let $s \in (0,1)$, $p \ge 2$. Then there exist $\mu^* > 0$, $N_0 \in \mathbb{N}$ and $q_0 \in (0, p-1)$ such that for all $\mu \in (0, \mu^*)$, $N > N_0$ and $q \in (q_0, p-1)$, problem $(\mathcal{P}_\mu)$ has at least one sign changing solution, where N_0 is given by the following relation:*

$$N_0 := \begin{cases} sp(p+1) & \textit{when } 2 \le p < \dfrac{3+\sqrt{5}}{2}, \\ sp(p^2-p+1) & \textit{when } p \ge \dfrac{3+\sqrt{5}}{2}. \end{cases}$$

Notations. Throughout this paper C denotes the generic constant which may vary from line to line. For a Banach space X, we denote by X', the dual space of X. $|u|_{L^p(\Omega)}$ denotes $\|u\|_{L^p(\Omega)}$.

2. Existence of sign-changing solutions

Define the Nehari manifold N_μ by

$$N_\mu := \{u \in X_0 \setminus \{0\} : \langle I_\mu'(u), u\rangle_{X_0} = 0\}.$$

The Nehari manifold N_μ is closely linked to the behavior of the fibering map $\varphi_u \colon (0,\infty) \to \mathbb{R}$ defined by

$$\varphi_u(r) := I_\mu(ru) = \frac{r^p}{p}\, \|u\|_{X_0}^p - \frac{\mu r^{q+1}}{q+1}\, |u|_{L^{q+1}(\Omega)}^{q+1} - \frac{r^{p_s^*}}{p_s^*}\, |u|_{L^{p_s^*}(\Omega)}^{p_s^*},$$

which was first introduced by Drábek and Pohozaev in [15].

LEMMA 2.1. *For any $u \in X_0 \setminus \{0\}$, we have $ru \in N_\mu$ if and only if $\varphi_u'(r) = 0$.*

PROOF. We note that for $r > 0$, $\varphi_u'(r) = \langle I_\mu'(ru), u\rangle_{X_0} = \frac{1}{r}\langle I_\mu'(ru), ru\rangle_{X_0}$. Hence, $\varphi_u'(r) = 0$ if and only if $ru \in N_\mu$. □

Therefore, we can conclude that the elements in N_μ correspond to the stationary points of the map φ_u. Observe that

$$\varphi_u'(r) = r^{p-1}\|u\|_{X_0}^p - \mu r^q |u|_{L^{q+1}(\Omega)}^{q+1} - r^{p_s^*-1}|u|_{L^{p_s^*}(\Omega)}^{p_s^*}, \tag{2.1}$$

$$\varphi_u''(r) = (p-1)r^{p-2}\|u\|_{X_0}^p - q\mu r^{q-1}|u|_{L^{q+1}(\Omega)}^{q+1} - (p_s^*-1)r^{p_s^*-2}|u|_{L^{p_s^*}(\Omega)}^{p_s^*}. \tag{2.2}$$

By Lemma 2.1, we note that $u \in N_\mu$ if and only if $\varphi_u'(1) = 0$. Hence for $u \in N_\mu$, using (2.1) and (2.2), we obtain that

$$\begin{aligned} \varphi_u''(1) &= (p-1)\|u\|_{X_0}^p - q\mu|u|_{L^{q+1}(\Omega)}^{q+1} - (p_s^*-1)|u|_{L^{p_s^*}(\Omega)}^{p_s^*} \\ &= (p-p_s^*)|u|_{L^{p_s^*}(\Omega)}^{p_s^*} + (1-q)\mu|u|_{L^{q+1}(\Omega)}^{q+1} \\ &= (p-1-q)\|u\|_{X_0}^p - (p_s^*-1-q)|u|_{L^{p_s^*}(\Omega)}^{p_s^*} \\ &= (p-p_s^*)\|u\|_{X_0}^p + (p_s^*-1-q)\mu|u|_{L^{q+1}(\Omega)}^{q+1}. \end{aligned} \tag{2.3}$$

Therefore, we split the manifold into three parts corresponding to local minima, maxima and points of inflection

$$N_\mu^+ := \{u \in N_\mu : \varphi_u''(1) > 0\}, \qquad N_\mu^- := \{u \in N_\mu : \varphi_u''(1) < 0\},$$
$$N_\mu^0 := \{u \in N_\mu : \varphi_u''(1) = 0\}.$$

REMARK 2.2. From [13], it follows that $\inf_{u\in N_\mu^+} I_\mu(u)$ and $\inf_{u\in N_\mu^-} I_\mu(u)$ are achieved and those two infimum points are two critical points of I_μ. Now, if we define I_μ^+ as follows:

$$I_\mu^+(u) := \frac{1}{p}\,\|u\|_{X_0}^p - \frac{\mu}{q+1}\,|u^+|_{L^{q+1}(\Omega)}^{q+1} - \frac{1}{p_s^*}\,|u^+|_{L^{p_s^*}(\Omega)}^{p_s^*} \tag{2.4}$$

and

$$\widetilde{\alpha}_\mu^+ := \inf_{u\in N_\mu^+} I_\mu^+(u) \quad \text{and} \quad \widetilde{\alpha}_\mu^- := \inf_{u\in N_\mu^-} I_\mu^+(u), \tag{2.5}$$

then repeating the same analysis as in [13] for I_μ^+, it can be shown that there exists $\mu_* > 0$ such that for $\mu \in (0, \mu_*)$, there exist two non-trivial critical points $w_0 \in N_\mu^+$ and $w_1 \in N_\mu^-$ of I_μ^+. It is not difficult to see that w_0 and w_1 are nonnegative in $\mathbb{R}^N$. Indeed,

$$
\begin{aligned}
(2.6)\quad 0 &= \langle (I_\mu^+)'(w_0), w_0^- \rangle \\
&= \int_{\mathbb{R}^{2N}} \frac{|w_0(x)-w_0(y)|^{p-2}(w_0(x)-w_0(y))(w_0^-(x)-w_0^-(y))}{|x-y|^{N+sp}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \frac{|w_0(x)-w_0(y)|^{p-2}((w_0^-(x)-w_0^-(y))^2+2(w_0^-(x)w_0^+(y)))}{|x-y|^{N+sp}}\,dx\,dy \\
&\geq \int_{\mathbb{R}^{2N}} \frac{|w_0^-(x)-w_0^-(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = \|w_0^-\|_{X_0}^p.
\end{aligned}
$$

Thus, $\|w_0^-\|_{X_0}=0$ and hence, $w_0=w_0^+$. Similarly we can show that $w_1=w_1^+$. Using the maximum principle [7, Theorem A.1], we conclude that both w_0,w_1 are positive almost everywhere in Ω. Hence $(\mathcal{P}_\mu)$ has at least two positive solutions.

Set

$$
(2.7)\quad \widetilde{\mu} = \bigg(\frac{p-1-q}{p_s^*-q-1}\bigg)^{(p-1-q)/(p_s^*-p)} \cdot \frac{p_s^*-p}{p_s^*-q-1}\,|\Omega|^{(q+1-p_s^*)/(p_p^*)}S^{N(p-1-q)/p^2s+(q+1)/p}.
$$

Next we prove three elementary lemmas.

Lemma 2.3. *Let $\mu\in(0,\widetilde{\mu})$. For every $u\in X_0$, $u\neq 0$, there exists unique*

$$
t^-(u) < t_0(u) = \bigg(\frac{(p-1-q)\|u\|_{X_0}^p}{(p_s^*-1-q)|u|_{L^{p_s^*}(\Omega)}^{p_s^*}}\bigg)^{(N-ps)/p^2s} < t^+(u),
$$

such that

$$
\begin{aligned}
&t^-(u)u\in N_\mu^+ \quad \textit{and} \quad I_\mu(t^-u)=\min_{t\in[0,t_0]} I_\mu(tu),\\
&t^+(u)u\in N_\mu^- \quad \textit{and} \quad I_\mu(t^+u)=\max_{t\geq t_0} I_\mu(tu).
\end{aligned}
$$

Proof. For $t\geq 0$,

$$
I_\mu(tu)=\frac{t^p}{p}\,\|u\|_{X_0}^p-\frac{\mu t^{q+1}}{q+1}\,|u|_{L^{q+1}(\Omega)}^{q+1}-\frac{t^{p_s^*}}{p_s^*}\,|u|_{L^{p_s^*}(\Omega)}^{p_s^*}.
$$

Therefore

$$
\frac{\partial}{\partial t}\,I_\mu(tu)=t^q\big(t^{p-1-q}\|u\|_{X_0}^p-t^{p_s^*-q-1}|u|_{L^{p_s^*}(\Omega)}^{p_s^*}-\mu|u|_{L^{q+1}(\Omega)}^{q+1}\big).
$$

Define

$$
(2.8)\qquad \psi(t)=t^{p-1-q}\|u\|_{X_0}^p-t^{p_s^*-q-1}|u|_{L^{p_s^*}(\Omega)}^{p_s^*}.
$$

By a straight forward computation, it follows that ψ attains maximum at the point

$$t_0 = t_0(u) = \left(\frac{(p-1-q)\|u\|_{X_0}^p}{(p_s^*-1-q)|u|_{L^{p_s^*}(\Omega)}^{p_s^*}} \right)^{1/(p_s^*-p)}. \tag{2.9}$$

Thus

$$\psi'(t_0) = 0, \qquad \psi'(t) > 0 \quad \text{if } t < t_0, \qquad \psi'(t) < 0 \quad \text{if } t > t_0. \tag{2.10}$$

Moreover,

$$\psi(t_0) = \left(\frac{p-1-q}{p_s^*-1-q} \right)^{(p-1-q)/(p_s^*-p)} \left(\frac{p_s^*-p}{p_s^*-1-q} \right) \left(\frac{\|u\|_{X_0}^{p(p_s^*-1-q)}}{|u|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1-q)}} \right)^{(N-ps)/p^2 s}.$$

Therefore, using Sobolev embedding, we have

$$\psi(t_0) \geq \left(\frac{p-1-q}{p_s^*-1-q} \right)^{(p-1-q)(N-2s)/(4s)} \cdot \left(\frac{p_s^*-p}{p_s^*-1-q} \right) S^{N(p-1-q)/p^2 s} \|u\|_{X_0}^{q+1}. \tag{2.11}$$

Using the Hölder inequality followed by the Sobolev inequality, and the fact that μ in $(0, \widetilde{\mu})$, we obtain

$$\mu \int_\Omega |u|^{q+1} dx \leq \mu \|u\|_{X_0}^{q+1} S^{-(q+1)/p} |\Omega|^{(p_s^*-q-1)/p_s^*} \leq \widetilde{\mu} \|u\|_{X_0}^{q+1} S^{-(q+1)/p} |\Omega|^{(p_s^*-q-1)/p_s^*} \leq \psi(t_0),$$

where in the last inequality we have used expression of $\widetilde{\mu}$ (see (2.7)) and (2.11). Hence, there exists $t^+(u) > t_0 > t^-(u)$ such that

$$\psi(t^+) = \mu \int_\Omega |u|^{q+1} = \psi(t^-) \quad \text{and} \quad \psi'(t^+) < 0 < \psi'(t^-). \tag{2.12}$$

This in turn, implies $t^+u \in N_\mu^-$ and $t^-u \in N_\mu^+$. Moreover, using (2.10) and (2.12) in the expression of $\partial I_\mu(tu)/\partial t$, we have

$$\begin{aligned} \frac{\partial}{\partial t} I_\mu(tu) > 0 &\quad \text{when } t \in (t^-, t^+), \\ \frac{\partial}{\partial t} I_\mu(tu) < 0 &\quad \text{when } t \in [0, t^-) \cup (t^+, \infty), \\ \frac{\partial}{\partial t} I_\mu(tu) = 0 &\quad \text{when } t = t^\pm. \end{aligned}$$

We note that $I_\mu(tu) = 0$ at $t = 0$ and strictly negative when $t > 0$ is small enough. Therefore it is easy to conclude that

$$\max_{t \geq t_0} I_\mu(tu) = I_\mu(t^+u) \quad \text{and} \quad \min_{t \in [0, t_0]} J_\mu(tu) = I_\mu(t^-u). \qquad \square$$

Repeating the same argument as in Lemma 2.3, we can also prove that the following lemma holds:

Lemma 2.4. *Let $\mu \in (0, \widetilde{\mu})$, where $\widetilde{\mu}$ is defined as in* (2.7). *For every $u \in X_0$, $u \neq 0$, there exists unique*

$$\widetilde{t}^-(u) < \widetilde{t}_0(u) = \bigg(\frac{(p-1-q)\|u\|^p_{X_0}}{(p^*_s - 1 - q)|u^+|^{p^*_s}_{L^{p^*_s}(\Omega)}}\bigg)^{(N-ps)/p^2 s} < \widetilde{t}^+(u)$$

such that

$$\begin{aligned} \widetilde{t}^-(u)u \in N^+_\mu \quad &\text{and} \quad I^+_\mu(\widetilde{t}^- u) = \min_{t \in [0, t_0]} I^+_\mu(tu), \\ \widetilde{t}^+(u)u \in N^-_\mu \quad &\text{and} \quad I^+_\mu(\widetilde{t}^+ u) = \max_{t \geq t_0} I^+_\mu(tu), \end{aligned}$$

where I^+_μ is defined as in (2.4).

Lemma 2.5. *Let $\widetilde{\mu}$ be defined as in* (2.7). *Then $\mu \in (0, \widetilde{\mu})$, implies $N^0_\mu = \emptyset$.*

Proof. Suppose not. Then there exists $w \in N^0_\mu$ such that $w \neq 0$ and

$$(p-1-q)\|w\|^p_{X_0} - (p^*_s - q - 1)|w^+|^{p^*_s}_{L^{p^*_s}(\Omega)} = 0. \tag{2.13}$$

The above expression combined with the Sobolev inequality yields

$$\|w\|_{X_0} \geq S^{N/p^2 s}\bigg(\frac{p-1-q}{p^*_s - 1 - q}\bigg)^{(N-ps)/p^2 s}. \tag{2.14}$$

As $w \in N^0_\mu \subseteq N_\mu$, using (2.13) and the Hölder inequality followed by the Sobolev inequality, we get

$$\begin{aligned} 0 &= \|w\|^p_{X_0} - |w|^{p^*_s}_{L^{p^*_s}(\Omega)} - \mu |w|^{q+1}_{L^{q+1}(\Omega)} \\ &\geq \|w\|^p_{X_0} - \bigg(\frac{p-1-q}{p^*_s - q - 1}\bigg)\|w\|^p_{X_0} - \mu|\Omega|^{1-(q+1)/p^*_s} S^{-(q+1)/p}\|w\|^{q+1}_{X_0}. \end{aligned}$$

Combining the above inequality with (2.14) and using $\mu < \widetilde{\mu}$, we have

$$\begin{aligned} 0 \geq \|w\|^{q+1}_{X_0}\bigg[\bigg(\frac{p^*_s - p}{p^*_s - q - 1}\bigg)\bigg(\frac{p-1-q}{p^*_s - q - 1}\bigg)^{(N-ps)(p-1-q)/p^2 s} &S^{N(p-1-q)/p^2 s} \\ - \mu|\Omega|^{1-(q+1)/p^*_s} S^{-(q+1)/p}\bigg] &> 0, \end{aligned}$$

which is a contradiction. □

Lemma 2.6. *Let $\widetilde{\mu}$ be defined as in* (2.7) *and $\mu \in (0, \widetilde{\mu})$. Given $u \in N^-_\mu$, there exist $\rho_u > 0$ and a differentiable function $g_{\rho_u} \colon B_{\rho_u}(0) \to \mathbb{R}^+$ satisfying the* *following:*

$$g_{\rho_u}(0)=1, \qquad (g_{\rho_u}(w))(u+w)\in N_\mu^- \quad \textit{for all } w\in B_{\rho_u}(0),$$

$$\langle g'_{\rho_u}(0),\phi\rangle = \frac{pA(u,\phi)-p_s^*\int_\Omega |u|^{p_s^*-2}u\phi-(q+1)\mu\int_\Omega |u|^{q-1}u\phi}{(p-1-q)\|u\|_{X_0}^p-(p_s^*-q-1)|u|_{L^{p_s^*}(\Omega)}^{p_s^*}}$$

for all $\phi\in B_{\rho_u}(0)$, *where*

$$A(u,\phi)=\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy.$$

PROOF. Define $E\colon \mathbb{R}\times X_0\to\mathbb{R}$ as follows:

$$E(r,w)=r^{p-1-q}\|u+w\|_{X_0}^p-r^{p_s^*-q-1}|(u+w)|_{L^{p_s^*}(\Omega)}^{p_s^*}-\mu|(u+w)|_{L^{q+1}(\Omega)}^{q+1}.$$

We note that $u\in N_\mu^-\subset N_\mu$ implies

$$E(1,0)=0 \quad\text{and}\quad \frac{\partial E}{\partial r}(1,0)=(p-1-q)\|u\|_{X_0}^p-(p_s^*-q-1)|u|_{L^{p_s^*}(\Omega)}^{p_s^*}<0.$$

Therefore, by the implicit function theorem, there exist a neighbourhood $B_{\rho_u}(0)\subset N_\mu$ for some $\rho_u>0$ and a C^1 function $g_{\rho_u}\colon B_{\rho_u}(0)\to\mathbb{R}^+$ such that

(i) $g_{\rho_u}(0)=1$,
(ii) $E(g_{\rho_u}(w),w)=0$, for all $w\in B_{\rho_u}(0)$,
(iii) $E_r(g_{\rho_u}(w),w)<0$, for all $w\in B_{\rho_u}(0)$,
(iv) $\langle g'_{\rho_u}(0),\phi\rangle=-\Big\langle\dfrac{\partial E}{\partial w}(1,0),\phi\Big\rangle\Big/\dfrac{\partial E}{\partial r}(1,0)$.

Multiplying (ii) by $(g_{\rho_u}(w))^{q+1}$, it follows that $g_{\rho_u}(w)(u+w)\in N_\mu$. In fact, simplifying (iii), we obtain

$$(p-1-q)g_{\rho_u}(w)^p\|u+w\|_{X_0}^p-(p_s^*-q-1)g_{\rho_u}(w)^{p_s^*}|(u+w)|_{p_s^*}^{p_s^*}<0$$

for all $w\in B_{\rho_u}(0)$. Thus $(g_{\rho_u}(w))(u+w)\in N_\mu^-$, for every $w\in B_{\rho_u}(0)$. The last assertion of the lemma follows from (iv). □

Let S be as in (1.3). From [9], we know that for $1<p<\infty$, $s\in(0,1)$, $N>ps$, there exists a minimizer for S, and for every minimizer U, there exist $x_0\in\mathbb{R}^N$ and a constant sign monotone function $u\colon\mathbb{R}\to\mathbb{R}$ such that $U(x)=u(|x-x_0|)$. In the following, we shall fix a radially symmetric nonnegative decreasing minimizer $U=U(r)$ for S. Multiplying U by a positive constant if necessary, we may assume that

$$(-\Delta)_p^s U=U^{p_s^*-1}\quad\text{in }\mathbb{R}^n. \tag{2.15}$$

For any $\varepsilon>0$ we note that the function

$$U_\varepsilon(x)=\frac{1}{\varepsilon^{(N-sp)/p}}U\bigg(\frac{|x|}{\varepsilon}\bigg) \tag{2.16}$$

is also a minimizer for S satisfying (2.15). From [20], we also have the following asymptotic estimates for U.

Lemma 2.7 ([20]). *Let U be the solution of* (2.15). *Then, there exist $c_1, c_2 > 0$ and $\theta > 1$ such that, for all $r \geq 1$,*

$$\frac{c_1}{r^{(N-sp)/(p-1)}} \leq U(r) \leq \frac{c_2}{r^{(N-sp)/(p-1)}} \tag{2.17}$$

and

$$\frac{U(r\theta)}{U(r)} \leq \frac{1}{2}. \tag{2.18}$$

Proof. See [20, Lemma 2.2]. □

Therefore we have

$$c_1 \frac{\varepsilon^{(N-sp)/(p(p-1))}}{|x|^{(N-sp)/(p-1)}} \quad \text{for } |x| > \varepsilon. \tag{2.19}$$

We consider a cut-off function $\psi \in C_0^\infty(\Omega)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in Ω_δ, $\psi \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, where $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$. Define

$$u_\varepsilon(x) = \psi(x) U_\varepsilon(x). \tag{2.20}$$

We need the following lemmas in order to prove Theorem 1.2.

Lemma 2.8. *Suppose w_1 is a positive solution of (P_μ) and u_ε is defined as in* (2.20). *Then, for every $\varepsilon > 0$ small enough,*

(a) $A_1 := \displaystyle\int_\Omega w_1^{p_s^*-1} u_\varepsilon \, dx \leq k_1 \varepsilon^{(N-ps)/(p(p-1))}$,

(b) $A_2 := \displaystyle\int_\Omega w_1^{q} u_\varepsilon \, dx \leq k_2 \varepsilon^{(N-ps)/(p(p-1))}$,

(c) $A_3 := \displaystyle\int_\Omega w_1 u_\varepsilon^{q} \, dx \leq k_3 \varepsilon^{(N-ps)q/(p(p-1))}$,

(d) $A_4 := \displaystyle\int_\Omega w_1 u_\varepsilon^{p_s^*-1} \, dx \leq k_4 \varepsilon^{(N(p-1)+ps)/(p(p-1))}$.

Proof. Applying the Moser iteration technique (see [8, Theorem 3.3]), it can be shown that any positive solution of (P_μ) is in $L^\infty(\Omega)$. Let $R, M > 0$ be such that $\Omega \subset B(0,R)$ and $|w_1|_{L^\infty(\Omega)} < M$.

(a)

$$\begin{aligned}
A_1 &= \int_\Omega w_1^{p_s^*-1} u_\varepsilon \, dx \\
&\le C\bigg[\int_{\Omega\cap\{|x|\le\varepsilon\}} U_\varepsilon(x)\,dx + \varepsilon^{(N-sp)/(p(p-1))}\int_{\Omega\cap\{|x|>\varepsilon\}} \frac{dx}{|x|^{(N-sp)/(p-1)}}\bigg] \\
&\le C\bigg[\varepsilon^{N-(N-sp)/p}\int_{\{|x|<1\}} U(x)\,dx \\
&\qquad + \varepsilon^{(N-sp)/(p(p-1))}\int_{B(0,R)} \frac{dx}{|x|^{(N-sp)/(p-1)}}\,dx\bigg] \\
&\le C\bigg[\varepsilon^{N-(N-sp)/p} + \varepsilon^{(N-sp)/(p(p-1))}\int_0^R r^{N-1-(N-sp)/(p-1)}\,dr\bigg] \\
&\le k_1\varepsilon^{(N-sp)/(p(p-1))}.
\end{aligned}$$

The proof of (b) is similar to (a).

(c)

$$\begin{aligned}
A_3 &= \int_\Omega w_1 u_\varepsilon^q \, dx \\
&\le C\bigg[\int_{\Omega\cap\{|x|\le\varepsilon\}} U_\varepsilon^q(x)\,dx + \varepsilon^{(N-sp)q/(p(p-1))}\int_{\Omega\cap\{|x|>\varepsilon\}} \frac{dx}{|x|^{(N-sp)q/(p-1)}}\bigg] \\
&\le C\bigg[\varepsilon^{N-(N-sp)q/p}\int_{\{|x|<1\}} U(x)^q\,dx \\
&\qquad + \varepsilon^{(N-sp)q/p(p-1)}\int_{B(0,R)} \frac{dx}{|x|^{(N-sp)q/(p-1)}}\,dx\bigg] \\
&\le C\bigg[\varepsilon^{N-(N-sp)q/p} + \varepsilon^{(N-sp)q/(p(p-1))}\int_0^R r^{N-1-(N-sp)q/(p-1)}\,dr\bigg] \\
&\le k_3\varepsilon^{(N-ps)q/(p(p-1))},
\end{aligned}$$

since $0 < q < p-1 < N(p-1)/(N-sp)$.

(d) can be proved similarly to (c). □

LEMMA 2.9. *Let u_ε be defined as in* (2.20), $0 < q < p-1$ *and* $N > p^2 s$. *Then, for every $\varepsilon > 0$ small enough,*

$$\int_\Omega |u_\epsilon|^{q+1}\,dx \ge \begin{cases} k_5\varepsilon^{(N-ps)(q+1)/(p(p-1))} & \text{if } 0 < q < \dfrac{N(p-2)+ps}{N-ps}, \\[2mm] k_6\varepsilon^{N/p}|\ln\varepsilon| & \text{if } q = \dfrac{N(p-2)+ps}{N-ps}, \\[2mm] k_7\varepsilon^{N-(N-ps)(q+1)/p} & \text{if } \dfrac{N(p-2)+ps}{N-ps} < q < p-1. \end{cases}$$

Proof. We recall that $R' > 0$ was chosen so that $B(0,R') \subset \Omega_\delta$. Therefore, for $\varepsilon > 0$ small enough, we have

$$\begin{aligned}(2.21)\qquad \int_\Omega |u_\varepsilon|^{q+1}\,dx &\geq \int_{B(0,R')} |u_\varepsilon|^{q+1}\,dx = \int_{B(0,R')} U_\varepsilon^{q+1}(x)\,dx \\ &= C\varepsilon^{N-(N-sp)(q+1)/p} \int_{B(0,R'/\varepsilon)} U^{q+1}(y)\,dy \\ &\geq C\varepsilon^{N-(N-ps)(q+1)/p} \int_{B(0,R'/\varepsilon)\setminus B(0,1)} U^{q+1}(y)\,dy \\ &\geq C\varepsilon^{N-(N-ps)(q+1)/p} \int_1^{R'/\varepsilon} r^{N-1-(N-ps)(q+1)/(p-1)}\,dr.\end{aligned}$$

Case 1. $0 < q \leq (N(p-2)+ps)/(N-ps)$. We note that

$$(2.22)\qquad \int_1^{R'/\varepsilon} r^{(N-1)-(N-ps)(q+1)/(p-1)}\,dr \geq C_1\varepsilon^{-N+(N-ps)(q+1)/(p-1)} - C_2.$$

Thus substituting back in (2.17), we obtain

$$\begin{aligned}\int_\Omega |u_\varepsilon|^{q+1}\,dx &\geq C\varepsilon^{N-(N-ps)(q+1)/p}\big[C_1\varepsilon^{-N+(N-ps)(q+1)/(p-1)} - C_2\big] \\ &= C_3\varepsilon^{(N-ps)(q+1)/(p(p-1))} - C_4\varepsilon^{N-(N-ps)(q+1)/p} \\ &\geq k_5\varepsilon^{(N-ps)(q+1)/(p(p-1))}.\end{aligned}$$

Case 2. $q = (N(p-2)+ps)/(N-ps)$. In this case it follows that

$$\int_1^{R'/\varepsilon} r^{N-1-(N-ps)(q+1)/(p-1)}\,dr \geq C|\ln\varepsilon|.$$

Plugging back in (2.17), we obtain

$$\int_\Omega |u_\varepsilon|^{q+1}dx \geq k_6\varepsilon^{N-(N-ps)(q+1)/p}|\ln\varepsilon| = k_6\varepsilon^{N/p}|\ln\varepsilon|.$$

Case 3. $(N(p-2)+ps)/(N-ps) < q < p-1$.

$$\begin{aligned}(2.23)\qquad \text{RHS of (2.16)} &\geq k_7\varepsilon^{N-(N-sp)(q+1)/p}\int_{B(0,1)} U^{q+1}(x)\,dx \\ &\geq k_7\varepsilon^{N-(N-sp)(q+1)/p}.\end{aligned}$$

Hence the lemma follows. □

Definition 2.10. We say $\{u_n\}$ is a Palais–Smale (PS) sequence of I_μ at level c (in short $(\mathrm{PS})_c$) if $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$ in $(X_0)'$. Furthermore, we say I_μ satisfies the Palais–Smale condition at level c if for all $\{u_n\} \subset X_0$ with $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$ in $(X_0)'$, implies, up to a subsequence, u_n converges strongly in X_0.

Let us define

$$M := \frac{(pN-(N-ps)(q+1))(p-1-q)}{p^2(q+1)} \cdot \bigg(\frac{(p-1-q)(N-sp)}{p^2 s}\bigg)^{(q+1)/(p_s^*-q-1)} |\Omega|. \tag{2.24}$$

Lemma 2.11. *Let M be as in* (2.24). *For any $\mu>0$ and for*

$$c<\frac{s}{N}\,S^{N/sp}-M\mu^{p_s^*/(p_s^*-q-1)},$$

I_μ satisfies the $(\mathrm{PS})_c$ *condition.*

Proof. Let $\{u_k\}\subset X_0$ be a $(\mathrm{PS})_c$ sequence for I_μ, that is, we have $I_\mu(u_k)\to c$ and $I'_\mu(u_k)\to 0$ in $(X_0)'$ as $k\to\infty$. By the standard method it is not difficult to see that $\{u_k\}$ is bounded in X_0. Then up to a subsequence, still denoted by u_k, there exists $u_\infty\in X_0$ such that

$$\begin{aligned}
&u_k\rightharpoonup u_\infty && \text{weakly in } X_0 && \text{as } k\to\infty,\\
&u_k\rightharpoonup u_\infty && \text{weakly in } L^{p_s^*}(\mathbb{R}^N) && \text{as } k\to\infty,\\
&u_k\to u_\infty && \text{strongly in } L^r(\mathbb{R}^N) && \text{for any } 1\le r<p_s^* \text{ as } k\to\infty,\\
&u_k\to u_\infty && \text{a.e. in } \mathbb{R}^N && \text{as } k\to\infty.
\end{aligned}$$

As $0<q<p-1$, we have

$$\int_\Omega |u_k|^{q+1}(x)\,dx\to\int_\Omega |u_\infty|^{q+1}(x)\,dx \quad \text{as } k\to\infty.$$

Using above properties it can be shown that $\langle I'_\mu(u_\infty),\varphi\rangle_{X_0}=0$ for any $\varphi\in X_0$. Indeed, for any $\varphi\in X_0$,

$$\begin{aligned}
\langle I'_\mu(u_k),\varphi\rangle &-\langle I'_\mu(u_\infty),\varphi\rangle\\
&=\int_{\mathbb{R}^{2N}}\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad-\int_{\mathbb{R}^{2N}}\frac{|u_\infty(x)-u_\infty(y)|^{p-2}(u_\infty(x)-u_\infty(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad-\mu\bigg(\int_\Omega |u_k|^{q-1}u_k\varphi\,dx-\int_\Omega |u_\infty|^{q-1}u_\infty\varphi\,dx\bigg)\\
&\quad-\bigg(\int_\Omega |u_k|^{p_s^*-2}u_k\varphi\,dx-\int_\Omega |u_\infty|^{p_s^*-2}u_\infty\varphi\,dx\bigg).
\end{aligned}$$

As

$$\bigg\{\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{(N+sp)/p'}}\bigg\}_{k\ge 1}$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$, where $p'=p/(p-1)$, up to a subsequence

$$\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{|x-y|^{(N+sp)/p'}}\rightharpoonup\frac{|u_\infty(x)-u_\infty(y)|^{p-2}(u_\infty(x)-u_\infty(y))}{|x-y|^{(N+sp)/p'}}$$

weakly in $L^{p'}(\mathbb{R}^{2N})$, $u_k \rightharpoonup u_\infty$ weakly in $L^{p_s^*}(\mathbb{R}^N)$ and $u_k \to u_\infty$ strongly in $L^{q+1}(\mathbb{R}^N)$ as $k \to \infty$.

Combining these we have $\langle I'_\mu(u_k), \varphi\rangle - \langle I'_\mu(u_\infty), \varphi\rangle \to 0$ as $k \to \infty$. But as $I'_\mu(u_k) \to 0$ in X'_0 as $k \to \infty$, we have $\langle I'_\mu(u_\infty), \varphi\rangle_{X_0} = 0$, for any $\varphi \in X_0$. Hence, in particular $\langle I'_\mu(u_\infty), u_\infty\rangle_{X_0} = 0$.

Furthermore, by the Brezis–Lieb lemma, as $k \to \infty$, we get

$$\int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy = \int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{|u_\infty(x) - u_\infty(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + o(1)$$

and

$$\int_\Omega |u_k(x)|^{p_s^*}\,dx = \int_\Omega |(u_k - u_\infty)(x)|^{p_s^*}\,dx + \int_\Omega |u_\infty(x)|^{p_s^*}\,dx + o(1).$$

Now

$$\begin{aligned}\langle I'_\mu(u_k), u_k\rangle_{X_o} &= \int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_k(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad - \mu \int_\Omega |u_k(x)|^{q+1}\,dx - \int_\Omega |u_k(x)|^{p_s^*}\,dx \\ &= \int_{\mathbb{R}^{2N}} \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad - \int_\Omega |u_k(x) - u_\infty(x)|^{p_s^*}\,dx + \langle I'_\mu(u_\infty), u_\infty\rangle_{X_0} + o(1).\end{aligned}$$

Since, as $\langle I'_\mu(u_\infty), u_\infty\rangle_{X_0} = 0$ and $\langle I'_\mu(u_k), u_k\rangle_{X_0} \to 0$ as $k \to \infty$, we have that there exists $b \in \mathbb{R}$ with $b \geq 0$ such that

$$\|u_k - u_\infty\|^p_{X_0} = \int_Q \frac{|u_k(x) - u_\infty(x) - u_k(y) + u_\infty(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \to b$$

and

$$\int_\Omega |(u_k - u_\infty)(x)|^{p_s^*}\,dx \to b \quad \text{as } k \to \infty.$$

If $b = 0$ we are done. Suppose $b > 0$. Moreover, using the Sobolev inequality, we have

$$\|u_k - u_\infty\|^p_{X_0} \geq S\biggl(\int_\Omega (|u_k - u_\infty)(x)|^{p_s^*}\,dx\biggr)^{p/p_s^*}.$$

Therefore, $b \geq Sb^{p/p_s^*}$, and this implies $b \geq S^{N/sp}$. On the other hand, since $\langle I'_\mu(u_\infty), u_\infty\rangle_{X_0} = 0$, we obtain

$$\begin{aligned}I_\mu(u_\infty) &= I_\mu(u_\infty) - \frac{1}{p}\langle I'_\mu(u_\infty), u_\infty\rangle_{X_0} \\ &= \frac{s}{N}\int_\Omega |u_\infty(x)|^{p_s^*}\,dx + \mu\biggl(\frac{1}{p} - \frac{1}{q+1}\biggr)\int_\Omega |u_\infty(x)|^{q+1}\,dx.\end{aligned} \tag{2.25}$$

Using (2.25) and $\langle I'_\mu(u_k), u_k\rangle_{X_0} \to 0$ as $k \to \infty$, we get

$$\begin{aligned}
c &= \lim_{k\to\infty} I_\mu(u_k) = \lim_{k\to\infty} \bigg[I_\mu(u_k) - \frac{1}{p} \langle I'_\mu(u_k), u_k\rangle_{X_0} \bigg] \\
&= \lim_{k\to\infty} \bigg[\frac{s}{N} \int_\Omega |(u_k - u_\infty)(x)|^{p_s^*}\,dx + \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*}\,dx \\
&\quad + \mu\bigg(\frac{1}{p} - \frac{1}{q+1}\bigg) \int_\Omega |u_k(x)|^{q+1}\,dx \bigg] \\
&= \frac{s}{N}\, b + \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*} dx + \mu\left(\frac{1}{p} - \frac{1}{q+1}\right) \int_\Omega |u_\infty(x)|^{q+1}\,dx \\
&\geq \frac{s}{N}\, S^{N/sp} + \frac{s}{N} \int_\Omega |u_\infty(x)|^{p_s^*}\,dx + \mu\bigg(\frac{1}{p} - \frac{1}{q+1}\bigg) \int_\Omega |u_\infty(x)|^{q+1}\,dx \\
&= \frac{s}{N}\, S^{N/sp} + I_\mu(u_\infty).
\end{aligned} \tag{2.26}$$

By assumption we have $c < sS^{N/sp}/N$, the last inequality implies $I_\mu(u_\infty) < 0$. In particular, $u_\infty \not\equiv 0$ and

$$0 < \frac{1}{p}\, \|u_\infty\|_{X_0}^p < \frac{\mu}{q+1} \int_\Omega (u_\infty(x))^{q+1}\,dx + \frac{1}{p_s^*} \int_\Omega (u_\infty(x))^{p_s^*}\,dx.$$

Moreover, by the Hölder inequality, we have

$$\int_\Omega |u_\infty(x)|^{q+1}\,dx \leq |\Omega|^{(p_s^*-(q+1))/p_s^*} \bigg(\int_\Omega |u_\infty(x)|^{p_s^*}\,dx \bigg)^{(q+1)/p_s^*}.$$

Thus, from (2.26)

$$\begin{aligned}
c \geq&\, \frac{s}{N}\, S^{N/sp} + \frac{s}{N} \int_\Omega |u_\infty|^{p_s^*}\,dx \\
&+ \mu\bigg(\frac{1}{p} - \frac{1}{q+1}\bigg) |\Omega|^{(p_s^*-(q+1))/p_s^*} \bigg(\int_\Omega |u_\infty(x)|^{p_s^*}\,dx \bigg)^{(q+1)/p_s^*} \\
:=&\, \frac{s}{N}\, S^{N/sp} + h(\eta),
\end{aligned}$$

where

$$h(\eta) = \frac{s}{N}\, \eta^{p_s^*} + \mu\bigg(\frac{1}{p} - \frac{1}{q+1}\bigg) |\Omega|^{(p_s^*-(q+1))/p_s^*} \eta^{q+1}$$

with

$$\eta = \bigg(\int_\Omega |u_\infty(x)|^{p_s^*}\,dx \bigg)^{1/p_s^*}.$$

By elementary analysis, we can show that h attains its minimum at

$$\eta_0 = \bigg(\frac{\mu(p-1-q)(N-sp)}{p^2 s} \bigg)^{1/(p_s^*-(q+1))} |\Omega|^{1/p_s^*}$$

and

$$h(\eta_0) = \frac{s}{N}\left(\frac{\mu(p-1-q)(N-sp)}{p^2 s}\right)^{p_s^*/(p_s^*-(q+1))}|\Omega| - \frac{\mu(p-1-q)}{p(q+1)}\,|\Omega|^{(p_s^*-(q+1))/p_s^*} \cdot \left(\frac{\mu(p-1-q)(N-sp)}{p^2 s}\right)^{(q+1)/(p_s^*-(q+1))}|\Omega|^{(q+1)/p_s^*} = -\,M\mu^{p_s^*/(p_s^*-(q+1))},$$

with M given in (2.24). This in turn implies

$$c \geq \frac{s}{N}\,S^{N/sp} - M\mu^{p_s^*/(p_s^*-(q+1))}$$

and that gives a contradiction to our hypothesis. Hence $b = 0$. This concludes that $u_k \to u_\infty$ strongly in X_0.

Lemma 2.12. *Let $N \in \mathbb{N}$ be such that $N > sp[p+1+\sqrt{(p+1)^2-4}]/2$ and $q \in (q_1, p-1)$, where*

$$q_1 := \frac{N^2(p-1)}{(N-sp)(N-s)} - 1. \tag{2.27}$$

Then, there exist $\widetilde{\mu}_1 > 0$ and $u_0 \in X_0$ such that

$$\sup_{t\geq 0} I_\mu^+(tu_0) < \frac{s}{N}\,S^{N/(sp)} - M\mu^{p_s^*/(p_s^*-q-1)}, \tag{2.28}$$

for $\mu \in (0, \widetilde{\mu}_1)$. In particular,

$$\widetilde{\alpha}_\mu^- < \frac{s}{N}\,S^{N/(sp)} - M\mu^{p_s^*/(p_s^*-q-1)}, \tag{2.29}$$

where I_μ^+ is defined as in (2.4) and α_μ^- and M are given as in (2.5) and (2.24), respectively.

Proof. Let u_ε be defined as in (2.20). Then we claim

$$|u_\varepsilon^+|_{L^{p_s^*}} = |u_\varepsilon|_{L^{p_s^*}}^{p_s^*} \geq S^{N/(sp)} + o(\varepsilon^{N/(p-1)}). \tag{2.30}$$

To see this,

$$|u_\varepsilon|_{L^{p_s^*}(\Omega)}^{p_s^*} = \int_\Omega |u_\varepsilon|^{p_s^*}\,dx \geq \int_{\Omega_\delta} |u_\varepsilon|^{p_s^*}\,dx = \int_{\Omega_\delta} |U_\varepsilon(x)|^{p_s^*}\,dx = \int_{\mathbb{R}^N} |U_\varepsilon(x)|^{p_s^*}\,dx - \int_{\mathbb{R}^N\setminus\Omega_\delta} |U_\varepsilon(x)|^{p_s^*}\,dx. \tag{2.31}$$

Moreover,

$$\int_{\mathbb{R}^N\setminus\Omega_\delta} |U_\varepsilon(x)|^{p_s^*}\,dx \leq \int_{\mathbb{R}^N\setminus B(0,R')} |U_\varepsilon(x)|^{p_s^*}\,dx = \frac{1}{\varepsilon^N}\int_{\mathbb{R}^N\setminus B(0,R')} U^{p_s^*}\left(\frac{x}{\varepsilon}\right)dx \leq C\int_{R'/\varepsilon}^{\infty} r^{N-1-Np/(p-1)}\,dr \leq C\varepsilon^{N/(p-1)}.$$

Therefore substituting back to (2.31) we obtain

$$|u_\varepsilon|^{p_s^*}_{L^{p_s^*}(\Omega)} \geq S^{N/(sp)} - C\varepsilon^{N/(p-1)}.$$

Furthermore, a similar analysis as in [23, Proposition 21] (see also [20, Lemma 2.7]) yields, for $\varepsilon > 0$ small enough ($0 < \varepsilon < \delta/2$) we have,

$$\|u_\varepsilon\|^p_{X_0} \leq S^{N/(sp)} + o(\varepsilon^{(N-ps)/(p-1)}). \tag{2.32}$$

Define

$$J(u) := \frac{1}{p}\,\|u\|^p_{X_0} - \frac{1}{p_s^*}\,|u^+|^{p_s^*}_{L^{p_s^*}}, \quad u \in X_0,$$

and choose $\varepsilon_0 > 0$ small enough such that (2.32) and (2.30) hold and Lemma 2.9 is satisfied. Let $\varepsilon \in (0, \varepsilon_0)$. Then, consider corresponding $u_0 := u_{\varepsilon_0}$. Let us consider the function $h\colon [0,\infty) \to \mathbb{R}$ defined by $h(t) = J(tu_0)$ for all $t \geq 0$. It can be shown that h attains its maximum at

$$t = t_* = \left(\frac{\|u_0\|^p_{X_0}}{|u_0^+|^{p_s^*}_{L^{p_s^*}}}\right)^{1/(p^*-p)}$$

and

$$\sup_{t\geq 0} J(tu_0) = \frac{s}{N}\left(\frac{\|u_0\|^p_{X_0}}{|u_0^+|^p_{L^{p_s^*}}}\right)^{N/(sp)}.$$

Using (2.32) and (2.30) a straight forward computation yields,

$$\sup_{t\geq 0} J(tu_0) \leq \frac{s}{N}\,S^{N/(sp)} + o(\varepsilon^{(N-sp)/(p-1)}). \tag{2.33}$$

Since $I^+_\mu(tu_0) < 0$ for t small, we can find $t_0 \in (0,1)$ such that

$$\sup_{0\leq t\leq t_0} I^+_\mu(tu_0) \leq \frac{s}{N}\,S^{N/(sp)} - M\mu^{p_s^*/(p_s^*-q-1)},$$

for $\mu > 0$ small enough. Hence, we are left to estimate $\sup\limits_{t_0\leq t} I^+_\mu(tu_0)$.

$$\begin{aligned}
&\sup_{t\geq t_0} I^+_\mu(tu_0) = \sup_{t\geq t_0}\left[J(tu_0) - \frac{t^{q+1}}{q+1}\,|u_0^+|^{q+1}_{L^{q+1}}\right]\\
&\leq \frac{s}{N}\,S^{N/(sp)} + o(\varepsilon^{(N-sp)/(p-1)}) - \frac{t^{q+1}}{q+1}\,|u_0|^{q+1}_{L^{q+1}}\\
&\leq \begin{cases}
\dfrac{s}{N}\,S^{N/(sp)} + c_1\varepsilon^{(N-ps)/(p-1)} - c_2\mu\varepsilon^{(N-ps)(q+1)/(p(p-1))}, & 0 < q < \dfrac{N(p-2)+ps}{N-sp},\\[2mm]
\dfrac{s}{N}\,S^{N/(sp)} + c_1\varepsilon^{(N-ps)/(p-1)} - c_2\mu\varepsilon^{N/p}|\ln\varepsilon|, & q = \dfrac{N(p-2)+ps}{N-sp},\\[2mm]
\dfrac{s}{N}\,S^{N/(sp)} + c_1\varepsilon^{(N-ps)/(p-1)} - c_2\mu\varepsilon^{N-(N-sp)(q+1)/p}, & \dfrac{N(p-2)+ps}{N-sp} < q < p-1.
\end{cases}
\end{aligned}$$

Choose $\varepsilon \in (0, \delta/2)$ such that $\varepsilon^{(N-sp)/(p-1)} = \mu^{(p_s^*)/(p_s^*-q-1)}$. Then, for $(N(p-2) +ps)/(N-sp) < q < p-1$, the term

$$\frac{s}{N} S^{N/(sp)} + c_1 \varepsilon^{(N-ps)/(p-1)} - c_2 \mu \varepsilon^{N-(N-sp)(q+1)/p}$$

reduces to

$$\frac{s}{N} S^{N/(sp)} + c_1 \mu^{p_s^*/(p_s^*-q-1)} - c_2 \mu \big(\mu^{p^*/(p^*-q-1)}\big)^{(N-(N-sp)(q+1)/p)((p-1)/(N-ps))}.$$

Now, note that we can make

$$c_1 \mu^{p_s^*/(p_s^*-q-1)} - c_2 \mu \big(\mu^{p^*/(p^*-q-1)}\big)^{(N-(N-sp)(q+1)/p)((p-1)/(N-ps))} < -M \mu^{p_s^*/(p_s^*-q-1)},$$

for $\mu > 0$ small enough if we further choose

$$\bigg(\frac{p_s^*}{p_s^*-q-1}\bigg)\bigg(\frac{p-1}{p}\bigg)\bigg[\frac{Np}{N-ps} - (q+1)\bigg] < \frac{p_s^*}{p_s^*-q-1} - 1,$$

i.e., if

$$q+1 > \frac{N^2(p-1)}{(N-sp)(N-s)}.$$

This proves (2.28). It is easy to see that (2.29) follows by combining (2.28) along with Lemma 2.4. $\square$

2.1. Sign changing critical points of I_μ. Define

$$\mathcal{N}_{\mu,1}^- := \{u \in N_\mu : u^+ \in N_\mu^-\}. \qquad \mathcal{N}_{\mu,2}^- := \{u \in N_\mu : -u^- \in N_\mu^-\},$$

We set

$$\beta_1 = \inf_{u \in \mathcal{N}_{\mu,1}^-} I_\mu(u) \quad \text{and} \quad \beta_2 = \inf_{u \in \mathcal{N}_{\mu,2}^-} I_\mu(u). \tag{2.34}$$

Theorem 2.13. *Let $p \geq 2$, $N > sp[p+1+\sqrt{(p+1)^2-4}]/2$ and $q_1 < q < p-1$, where q_1 is defined as in* (2.27). *Assume $0 < \mu < \min\{\widetilde{\mu}, \widetilde{\mu}_1, \mu_*, \mu_1\}$, where $\widetilde{\mu}, \widetilde{\mu}_1$ and μ_1 are as in* (2.7), *Lemmas* 2.12 *and* A.1, *respectively. μ_* is chosen so that $\widetilde{\alpha}_\mu^-$ is achieved in $(0, \mu_*)$. Let $\beta_1, \beta_2, \widetilde{\alpha}_\mu^-$ be defined as in* (2.34) *and* (2.5), *respectively.*

(a) *Let $\beta_1 < \widetilde{\alpha}_\mu^-$. Then, there exists a sign changing critical point $\widetilde{w}_1$ of I_μ such that $\widetilde{w}_1 \in \mathcal{N}_{\mu,1}^-$ and $I_\mu(\widetilde{w}_1) = \beta_1$.*

(b) *If $\beta_2 < \widetilde{\alpha}_\mu^-$, then there exists a sign changing critical point $\widetilde{w}_2$ of I_μ such that $\widetilde{w}_2 \in \mathcal{N}_{\mu,1}^-$ and $I_\mu(\widetilde{w}_2) = \beta_2$.*

Proof. (a) Let $\beta_1 < \widetilde{\alpha}_\mu^-$. We prove the theorem in several steps.

Step 1. $\mathcal{N}_{\mu,1}^-$ and $\mathcal{N}_{\mu,2}^-$ are closed sets. To see this, let $\{u_n\} \subset \mathcal{N}_{\mu,1}^-$ be such that $u_n \to u$ in X_0. It is easy to note that $|u_n|, |u| \in X_0$ and $|u_n| \to |u|$ in X_0.

This in turn implies $u_n^+ \to u^+$ in X_0 and $L^\gamma(\mathbb{R}^N)$ for $\gamma \in [1, p_s^*]$ (by the Sobolev inequality). Since, $u_n \in \mathcal{N}_{\mu,1}^-$, we have $u_n^+ \in N_\mu^-$. Therefore

$$\|u_n^+\|_{X_0}^p - |u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} - \mu |u_n^+|_{L^{q+1}(\Omega)}^{q+1} = 0 \tag{2.35}$$

and

$$(p-1-q)\|u_n^+\|_{X_0}^p - (p_s^* - q - 1)|u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} < 0 \quad \text{for all } n \geq 1. \tag{2.36}$$

Passing to the limit as $n \to \infty$, we obtain $u^+ \in N_\mu$ and

$$(p-1-q)\|u^+\|_{X_0}^p - (p_s^* - q - 1)|u^+|_{L^{p_s^*}(\Omega)}^{p_s^*} \leq 0.$$

But, from Lemma 2.5, we know that $N_\mu^0 = \emptyset$. Therefore $u^+ \in N_\mu^-$ and hence $\mathcal{N}_{\mu,1}^-$ is closed. Similarly it can be shown that $\mathcal{N}_{\mu,2}^-$ is also closed. Hence Step 1 follows.

By the Ekeland Variational Principle there exists a sequence $\{u_n\} \subset \mathcal{N}_{\mu,1}^-$ such that

$$I_\mu(u_n) \to \beta_1 \quad \text{and} \quad I_\mu(z) \geq I_\mu(u_n) - \frac{1}{n}\|u_n - z\|_{X_0} \quad \text{for all } z \in \mathcal{N}_{\mu,1}^-. \tag{2.37}$$

Step 2. $\{u_n\}$ is uniformly bounded in X_0. To see this, we notice that $u_n \in \mathcal{N}_{\mu,1}^-$ implies $u_n \in N_\mu$ and this in turn implies $\langle I_\mu'(u_n), u_n\rangle = 0$, that is,

$$\|u_n\|_{X_0}^p = |u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} + \mu |u_n|_{L^{q+1}(\Omega)}^{q+1}.$$

Since $I_\mu(u_n) \to \beta_1$, using the above equality in the expression of $I_\mu(u_n)$, we get, for n large enough

$$\frac{s}{N}\|u_n\|_{X_0}^p \leq \beta_1 + 1 + \left(\frac{1}{q+1} - \frac{1}{p_s^*}\right)\mu |u_n|_{L^{q+1}(\Omega)}^{q+1} \leq C\big(1 + \|u_n\|_{X_0}^{q+1}\big).$$

As $p > q+1$, the above implies that $\{u_n\}$ is uniformly bounded in X_0.

We note that, for any $u \in X_0$, we have

$$\begin{aligned}
\|u\|_{X_0}^p &= \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy = \int_{\mathbb{R}^{2N}} \frac{(|u(x)-u(y)|^2)^{p/2}}{|x-y|^{N+ps}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \frac{\big(|(u^+(x)-u^+(y)) - (u^-(x)-u^-(y))|^2\big)^{p/2}}{|x-y|^{N+ps}}\,dx\,dy \\
&= \int_{\mathbb{R}^{2N}} \bigg(\frac{(u^+(x)-u^+(y))^2 + (u^-(x)-u^-(y))^2}{|x-y|^{N+ps}} \\
&\qquad + \frac{2u^+(x)u^-(y) + 2u^+(y)u^-(x)\big)^{p/2}}{|x-y|^{N+ps}}\bigg)\,dx\,dy \\
&\geq \int_{\mathbb{R}^{2N}} \frac{\big((u^+(x)-u^+(y))^2 + (u^-(x)-u^-(y))^2\big)^{p/2}}{|x-y|^{N+ps}}\,dx\,dy
\end{aligned} \tag{2.38}$$

$$\geq \int_{\mathbb{R}^{2N}} \frac{\big((u^+(x)-u^+(y))^2\big)^{p/2}}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{\big((u^-(x)-u^-(y))^2\big)^{p/2}}{|x-y|^{N+ps}}\,dx\,dy = \|u^+\|_{X_0}^p + \|u^-\|_{X_0}^p.$$

By a simple calculation, it follows that

$$|u|_{L^{p_s^*}(\Omega)}^{p_s^*} = |u^+|_{L^{p_s^*}(\Omega)}^{p_s^*} + |u^-|_{L^{p_s^*}(\Omega)}^{p_s^*}, \qquad |u|_{L^{q+1}(\Omega)}^{q+1} = |u^+|_{L^{q+1}(\Omega)}^{q+1} + |u^-|_{L^{q+1}(\Omega)}^{q+1}. \tag{2.39}$$

Combining (2.38) and (2.39), we obtain

$$I_\mu(u) \geq I_\mu(u^+) + I_\mu(u^-) \quad \text{for all } u \in X_0. \tag{2.40}$$

Step 3. There exists $b > 0$ such that $\|u_n^-\|_{X_0} \geq b$ for all $n \geq 1$. Suppose this is not true. Then, for each $k \geq 1$, there exists u_{n_k} such that

$$\|u_{n_k}^-\|_{X_0} < \frac{1}{k} \quad \text{for all } k \geq 1. \tag{2.41}$$

Therefore, $\|u_{n_k}^-\|_{X_0} \to 0$ as $k \to \infty$ and by the Sobolev inequality

$$|u_{n_k}^-|_{L^{p_s^*}(\Omega)} \to 0, \quad |u_{n_k}^-|_{L^{q+1}(\Omega)} \to 0, \quad \text{as } k \to \infty.$$

Consequently, $I_\mu(u_{n_k}^-) \to 0$ as $k \to \infty$. As a result, using (2.40) we have

$$\beta_1 = I_\mu(u_{n_k}) + o(1) \geq I_\mu(u_{n_k}^+) + I_\mu(u_{n_k}^-) + o(1) = I_\mu^+(u_{n_k}^+) + o(1) \geq \widetilde{\alpha}_\mu^- + o(1).$$

This is a contradiction to the hypothesis. Hence Step 3 follows.

Step 4. $I'_\mu(u_n) \to 0$ in $(X_0)'$ as $n \to \infty$. Since $u_n \in \mathcal{N}_{\mu,1}^-$, we have $u_n^+ \in N_\mu^-$. Thus, by Lemma 2.6 applied to the element u_n^+, there exist

$$\rho_n := \rho_{u_n^+} \quad \text{and} \quad g_n := g_{\rho_{u_n^+}}, \tag{2.42}$$

such that

$$g_n(0) = 1, \quad (g_n(w))(u_n^+ + w) \in N_\mu^- \quad \text{for all } w \in B_{\rho_n}(0). \tag{2.43}$$

Choose $0 < \widetilde{\rho}_n < \rho_n$ such that $\widetilde{\rho}_n \to 0$. Let $v \in X_0$ with $\|v\|_{X_0} = 1$. Define

$$v_n := -\,\widetilde{\rho}_n[v^+\chi_{\{u_n \geq 0\}} - v^-\chi_{\{u_n \leq 0\}}]$$
$$z_{\widetilde{\rho}_n} := (g_n(v_n^-))(u_n - v_n) =: z_{\widetilde{\rho}_n}^1 - z_{\widetilde{\rho}_n}^2,$$

where

$$z_{\widetilde{\rho}_n}^1 := (g_n(v_n^-))(u_n^+ + \widetilde{\rho}_n v^+\chi_{\{u_n \geq 0\}}) \quad \text{and} \quad z_{\widetilde{\rho}_n}^2 := (g_n(v_n^-))(u_n^- + \widetilde{\rho}_n v^-\chi_{\{u_n \leq 0\}}).$$

Note that $v_n^- = \widetilde{\rho}_n v^+\chi_{\{u_n \geq 0\}}$. So, $\|v_n^-\|_{X_0} \leq \widetilde{\rho}_n\|v\|_{X_0} \leq \widetilde{\rho}_n$. Hence, taking $w = v_n^-$ in (2.43), we have $z_{\widetilde{\rho}_n}^+ = z_{\widetilde{\rho}_n}^1 \in N_\mu^-$ so $z_{\widetilde{\rho}_n} \in N_{\mu,1}^-$. Hence,

$$I_\mu(z_{\widetilde{\rho}_n}) \geq I_\mu(u_n) - \frac{1}{n}\,\|u_n - z_{\widetilde{\rho}_n}\|_{X_0}.$$

This implies,

$$
\begin{aligned}
(2.44)\qquad \frac{1}{n}\,\|u_n - z_{\widetilde{\rho}_n}\|_{X_0} \geq & I_\mu(u_n) - I_\mu(z_{\widetilde{\rho}_n}) \\
= & \langle I'_\mu(u_n), u_n - z_{\widetilde{\rho}_n}\rangle + o(1)\|u_n - z_{\widetilde{\rho}_n}\|_{X_0} \\
= & -\langle I'_\mu(u_n), z_{\widetilde{\rho}_n}\rangle + o(1)\|u_n - z_{\widetilde{\rho}_n}\|_{X_0},
\end{aligned}
$$

as $\langle I'_\mu(u_n), u_n\rangle = 0$ for all n. Let $w_n = \widetilde{\rho}_n v$. Then

$$
\begin{aligned}
(2.45)\qquad \frac{1}{n}\,\|u_n - z_{\widetilde{\rho}_n}\|_{X_0} \geq & -\langle I'_\mu(u_n), w_n + z_{\widetilde{\rho}_n}\rangle \\
& + \langle I'_\mu(u_n), w_n\rangle + o(1)\|u_n - z_{\widetilde{\rho}_n}\|_{X_0}.
\end{aligned}
$$

Now, $\langle I'_\mu(u_n), w_n\rangle = \langle I'_\mu(u_n), \widetilde{\rho}_n v\rangle = \widetilde{\rho}_n\langle I'_\mu(u_n), v\rangle$. Define

$$
\overline{v_n} := v^+\chi_{\{u_n\geq 0\}} - v^-\chi_{\{u_n\leq 0\}}.
$$

So, $z_{\widetilde{\rho}_n} = g_n(v_n^-)(u_n - \widetilde{\rho}_n\overline{v_n})$. Hence we have

$$
\begin{aligned}
(2.46)\qquad \langle I'_\mu(u_n), w_n + z_{\widetilde{\rho}_n}\rangle &= \langle I'_\mu(u_n), w_n + g_n(v_n^-)(u_n - \widetilde{\rho}_n\overline{v_n})\rangle \\
&= \langle I'_\mu(u_n), \widetilde{\rho}_n v - g_n(v_n^-)\widetilde{\rho}_n\overline{v_n}\rangle = \widetilde{\rho}_n\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\rangle.
\end{aligned}
$$

Using (2.46) in (2.45), we have

$$
\begin{aligned}
(2.47)\qquad \frac{1}{n}\,\|u_n - z_{\widetilde{\rho}_n}\|_{X_0} \geq & -\widetilde{\rho}_n\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\rangle \\
& + \widetilde{\rho}_n\langle I'_\mu(u_n), v\rangle + o(1)\|u_n - z_{\widetilde{\rho}_n}\|_{X_0}.
\end{aligned}
$$

First we will estimate $\langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\rangle$. For this,

$$
\begin{aligned}
v - g_n(v_n^-)\overline{v_n} &= v^+ - v^- - g_n(v_n^-)[v^+\chi_{\{u_n\geq 0\}} - v^-\chi_{\{u_n\leq 0\}}] \\
&= v^+[g_n(0) - g_n(v_n^-)\chi_{\{u_n\geq 0\}}] - v^-[g_n(0) - g_n(v_n^-)\chi_{\{u_n\leq 0\}}] \\
&= -v^+[\langle g'_n(0), v_n^-\rangle + o(1)\|v_n^-\|_{X_0}] + v^-[\langle g'_n(0), v_n^-\rangle + o(1)\|v_n^-\|_{X_0}] \\
&= -v^+\widetilde{\rho}_n[\langle g'_n(0), v^+\rangle + o(1)\|v^+\|_{X_0}] + v^-\widetilde{\rho}_n[\langle g'_n(0), v^+\rangle + o(1)\|v^+\|_{X_0}] \\
&= -\widetilde{\rho}_n\big[\langle g'_n(0), v^+\rangle + o(1)\|v^+\|_{X_0}\big]v.
\end{aligned}
$$

Therefore

$$
(2.48)\qquad \langle I'_\mu(u_n), v - g_n(v_n^-)\overline{v_n}\rangle = -\widetilde{\rho}_n\big(\langle g'_n(0), v^+\rangle + o(1)\|v^+\|\big)\langle I'_\mu(u_n), v\rangle.
$$

CLAIM. $g_n(v_n^-)$ is uniformly bounded in X_0.

To see this, we observe that from (2.43) we have, $g_n(v_n^-)(u_n^+ + v_n^-) \in N_\mu^- \subset N_\mu$, which implies

$$
\|c_n\widetilde{\psi}_n\|_{X_0}^p - \mu|c_n\widetilde{\psi}_n|_{L^{q+1}(\Omega)}^{q+1} - |c_n\widetilde{\psi}_n|_{L^{p_s^*}(\Omega)}^{p_s^*} = 0,
$$

where $c_n := g_n(v_n^-)$ and $\widetilde{\psi}_n := u_n^+ + v_n^-$. Dividing by $c_n^{p^*}$ we have,

$$
(2.49)\qquad c_n^{p-p^*}\|\widetilde{\psi}_n\|_{X_0}^p - \mu c_n^{q+1-p^*}|\widetilde{\psi}_n|_{L^{q+1}(\Omega)}^{q+1} = |\widetilde{\psi}_n|_{L^{p_s^*}(\Omega)}^{p_s^*}.
$$

Note that $\|\widetilde{\psi}_n\|_{X_0}$ is uniformly bounded above as $\|u_n\|_{X_0}$ is uniformly bounded and $\widetilde{\rho}_n = o(1)$. Also, $\|\widetilde{\psi}_n\|_{X_0} \geq \|u_n^+\|_{X_0} - \widetilde{\rho}_n\|v\|_{X_0}$. Note that $\|u_n^+\|_{X_0} \geq \widetilde{b}$ for large enough n. If not, $\|u_n^+\|_{X_0} \to 0$ as $n \to \infty$. As $u_n \in N_{\mu,1}^-$, so $u_n^+ \in N_\mu^-$. Now, N_μ^- is a closed set and $0 \notin N_\mu^-$ and therefore $\|u_n^-\|_{X_0} \not\to 0$ as $n \to \infty$. Thus there exists $\widetilde{b} \geq 0$ such that $\|u_n^+\|_{X_0} \geq \widetilde{b} > 0$. This in turn implies that $\|\widetilde{\psi}_n\|_{X_0} \geq C$, for some $C > 0$, by choosing $\widetilde{\rho}_n$ small enough. Consequently, if c_n is not uniformly bounded, we obtain that LHS of (2.49) converges to 0 as $n \to \infty$.

On the other hand,

$$|\widetilde{\psi}_n|_{L^{p_s^*}(\Omega)} \geq |u_n^+|_{L^{p_s^*}(\Omega)} - \widetilde{\rho}_n|v|_{L^{p_s^*}(\Omega)} > c,$$

for some positive constant c as $\rho_n = o(1)$ and $u_n^+ \in N_\mu^-$ implies

$$(p_s^* - 1 - q)|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} > (p - 1 - q)\|u_n^+\|^p_{X_0} > (p - 1 - q)\widetilde{b}^p.$$

Hence, the claim follows.

Now using the fact that $g_n(0) = 1$ and the above claim we obtain

$$\begin{aligned}\|u_n - z_{\widetilde{\rho}_n}\|_{X_0} \leq &\|u_n\|_{X_0}|1 - g_n(v_n^-)| + \widetilde{\rho}_n\|\overline{v_n}\|_{X_0} g_n(v_n^-)\\ \leq &\|u_n\|_{X_0}[|\langle g_n'(0), v_n^-\rangle| + o(1)\|\overline{v_n}\|_{X_0}] + \widetilde{\rho}_n\|v\|_{X_0} g_n(v_n^-)\\ \leq &\widetilde{\rho}_n[\|u_n\|_{X_0}\langle g_n'(0), \overline{v_n}^+\rangle + o(1)\|v\|_{X_0} + \|v\|_{X_0} g_n(v_n^-)] \leq \widetilde{\rho}_n C.\end{aligned}$$

Substituting this and (2.48) in (2.47) yields

$$\widetilde{\rho}_n(\langle g_n'(0), v^+\rangle + o(1)\|v^+\|_{X_0})\langle I_\mu'(u_n), v\rangle + \langle I_\mu'(u_n), v\rangle\widetilde{\rho}_n + \widetilde{\rho}_n o(1) \leq \widetilde{\rho}_n \cdot \frac{C}{n}.$$

This implies

$$[(\langle g_n'(0), v^+\rangle + o(1)\|v^+\|_{X_0}) + 1]\langle I_\mu'(u_n), v\rangle \leq \frac{C}{n} + o(1) \quad \text{for all } n \geq n_0.$$

Since $|\langle g_n'(0), v^+\rangle|$ is uniformly bounded (see Lemma A.1 in Appendix), letting $n \to \infty$ we have $I_\mu'(u_n) \to 0$ in $(X_0)'$. Hence Step 4 follows.

Therefore $\{u_n\}$ is a (PS) sequence of I_μ at level $\beta_1 < \widetilde{\alpha}_\mu^-$. From Lemma 2.12, it follows that

$$\widetilde{\alpha}_\mu^- < \frac{s}{N} S^{N/(ps)} - M\mu^{p_s^*/(p_s^* - q - 1)} \quad \text{for } \mu \in (0, \widetilde{\mu}_1),$$

where

$$M = \frac{(pN - (N - ps)(q + 1))(p - 1 - q)}{p^2(q + 1)}\bigg(\frac{(p - 1 - q)(N - ps)}{p^2 s}\bigg)^{(q+1)/(p_s^* - q - 1)}|\Omega|.$$

Thus

$$\beta_1 < \widetilde{\alpha}_\mu^- < \frac{s}{N} S^{N/(ps)} - M\mu^{p_s^*/(p_s^* - q - 1)}.$$

On the other hand, it follows from Lemma 2.11 that I_μ satisfies (PS) at level c for

$$c < \frac{s}{N} S^{N/(ps)} - M\mu^{p_s^*/(p_s^* - q - 1)},$$

this yields that there exists $u \in X_0$ such that $u_n \to u$ in X_0. By doing a simple calculation we get $u_n^- \to u^-$ in X_0. Consequently, by Step 3, $\|u^-\|_{X_0} \geq b$. As $\mathcal{N}_{\mu,1}^-$ is a closed set and $u_n \to u$, we obtain $u \in \mathcal{N}_{\mu,1}^-$, that is $u^+ \in N_\mu^-$ and $u^+ \neq 0$. Therefore u is a solution of $(\mathcal{P}_\mu)$ with u^+ and u^- both nonzero. Hence, u is a sign-changing solution of $(\mathcal{P}_\mu)$. Define $\widetilde{w}_1 := u$. This completes the proof of part (a) of the theorem.

The proof of part (b) is similar to part (a) and we omit it. □

THEOREM 2.14. *Let $\beta_1, \beta_2 \geq \widetilde{\alpha}_\mu^-$ where $\beta_1, \beta_2, \widetilde{\alpha}_\mu^-$ are defined as in* (2.34) *and* (2.5) *respectively. Then, there exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$,*

(a) *for $p \geq (3+\sqrt{5})/2$, there exists $q_2 := Np/(N-sp) - p/(p-1)$ such that when $q > q_2$ and $N > sp(p^2-p+1)$, I_μ has a sign changing critical point,*

(b) *for $2 \leq p < (3+\sqrt{5})/2$, there exists $q_3 := N(p-1)/(N-sp) - (p-1)/p$ such that when $q > q_3$ and $N > sp(p+1)$, I_μ has a sign changing critical point.*

We need the following proposition to prove the above Theorem 2.14.

PROPOSITION 2.15. *Assume $0 < \mu < \min\{\mu_*, \widetilde{\mu}, \widetilde{\mu_1}\}$, where $\widetilde{\mu}$ is defined as in* (2.7) *and $\mu_* > 0$ is chosen so that $\widetilde{\alpha}_\mu^-$ is achieved in $(0, \mu_*)$ and $\widetilde{\mu_1}$ is as in Lemma* 2.12*. Then, for $p \geq (3+\sqrt{5})/2$, there exists $q_2 := Np/(N-sp) - p/(p-1)$ such that when $q > q_2$ and $N > sp(p^2-p+1)$ we have*

$$\sup_{a \geq 0,\ b \in \mathbb{R}} I_\mu(aw_1 - bu_\varepsilon) < \widetilde{\alpha}_\mu^- + \frac{s}{N} S^{N/(ps)},$$

for $\varepsilon > 0$ sufficiently small, where w_1 is a positive solution of $(\mathcal{P}_\mu)$ and u_ε is as in (2.20)*. Furthermore, when $2 \leq p < (3+\sqrt{5})/2$, there exists $q_3 := N(p-1)/(N-sp) - (p-1)/p$ such that, when $q > q_3$ and $N > sp(p+1)$, it holds*

$$\sup_{a \geq 0,\, b \in \mathbb{R}} I_\mu(aw_1 - bu_\varepsilon) < \widetilde{\alpha}_\mu^- + \frac{s}{N} S^{N/(ps)},$$

for $\varepsilon > 0$ sufficiently small.

To prove the above proposition, we need the following lemmas.

LEMMA 2.16. *Let w_1 and μ be as in Proposition* 2.15*. Then*

$$\sup_{s>0} I_\mu(sw_1) = \widetilde{\alpha}_\mu^-.$$

PROOF. By the definition of $\widetilde{\alpha}_\mu^-$, we have $\widetilde{\alpha}_\mu^- = \inf_{u \in N_\mu^-} I_\mu^+(u) = I_\mu^+(w_1) = I_\mu(w_1)$. In the last equality we have used the fact that $w_1 > 0$. Define $g(s) := I_\mu(sw_1)$. From the proof of Lemma 2.3, it follows that there exist only two critical points of g, namely $t^+(w_1)$ and $t^-(w_1)$ and $\max_{s>0} g(s) = g(t^+(w_1))$. On the other hand, $\langle I'_\mu(w_1), v\rangle = 0$ for every $v \in X_0$. Therefore $g'(1) = 0$ which implies either $t^+(w_1) = 1$ or $t^-(w_1) = 1$.

Claim. $t^-(w_1) \neq 1$.

To see this, we note that $t^-(w_1) = 1$ implies $t^-(w_1)w_1 \in N_\mu^-$ as $w_1 \in N_\mu^-$. Using Lemma 2.3, we know that $t^-(w_1)w_1 \in N_\mu^+$. Thus $N_\mu^+ \cap N_\mu^- \neq \emptyset$, which is a contradiction. Hence we have the claim.

Therefore $t^+(w_1) = 1$ and this completes the proof. □

Lemma 2.17. *Let u_ε be as in* (2.20) *and μ be as in Proposition* 2.15*. Then, for $\varepsilon > 0$ sufficiently small, we have*

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) = \frac{s}{N}\, S^{N/(ps)} + C\varepsilon^{(N-ps)/(p-1)} - k_8 |u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}.$$

Proof. Define $\widetilde{\phi}(t) = t^p \|u_\varepsilon\|^p_{X_0}/p - t^{p_s^*}|u_\varepsilon|^{p_s^*}_{L^{p_s^*}(\Omega)}/p_s^*$. Thus $I_\mu(tu_\varepsilon) = \widetilde{\phi}(t) - \mu t^{q+1}|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}/(q+1)$. On the other hand, applying the analysis done in Lemma 2.3 to u_ε, we obtain that there exists

$$(t_0)_\varepsilon = \left(\frac{(p-1-q)\|u_\varepsilon\|^p_{X_0}}{(p_s^*-1-q)|u_\varepsilon|^{p_s^*}_{L^{p_s^*}(\Omega)}}\right)^{(N-ps)/(p^2 s)} < t_\varepsilon^+$$

such that

$$\begin{aligned}\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) &= \sup_{t\geq 0} I_\mu(tu_\varepsilon) = I_\mu(t_\varepsilon^+ u_\varepsilon)\\ &= \widetilde{\phi}(t_\varepsilon^+) - \mu\,\frac{(t_\varepsilon^+)^{q+1}}{q+1}\,|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)} \leq \sup_{t\geq 0}\widetilde{\phi}(t) - \mu\,\frac{(t_0)_\varepsilon^{q+1}}{q+1}\,|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}.\end{aligned}$$

Substituting the value of $(t_0)_\varepsilon$ and using the Sobolev inequality, we have

$$\mu\,\frac{(t_0)_\varepsilon^{q+1}}{q+1} \geq \frac{\mu}{q+1}\left(\frac{p-1-q}{p_s^*-q-1} S\right)^{(N-ps)(q+1)/(p^2 s)} = k_8.$$

Consequently,

$$\sup_{t\in\mathbb{R}} I_\mu(tu_\varepsilon) \leq \sup_{t\geq 0}\widetilde{\phi}(t) - k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}. \tag{2.50}$$

Using elementary analysis, it is easy to check that $\widetilde{\phi}$ attains its maximum at the point $\widetilde{t_0} = (\|u_\varepsilon\|^p_{X_0}/|u_\varepsilon|^{p_s^*}_{L^{p_s^*}(\Omega)})^{1/(p_s^*-p)}$ and

$$\widetilde{\phi}(t_0) = \frac{s}{N}\left(\frac{\|u_\varepsilon\|^p_{X_0}}{|u_\varepsilon|^p_{L^{p_s^*}(\Omega)}}\right)^{N/(ps)}.$$

Moreover, using (2.32) and (2.30), we can deduce as in (2.33) that

$$\widetilde{\phi}(t_0) \leq \frac{s}{N}\, S^{N/(ps)} + C\varepsilon^{(N-ps)/(p-1)}. \tag{2.51}$$

Substituting back (2.51) into (2.50), completes the proof. □

PROOF OF PROPOSITION 2.15. Note that, for fixed a and b,

$$I_\mu\big(\eta(aw_1 - bu_{\varepsilon,\delta})\big) \to -\infty \quad \text{as } |\eta| \to \infty.$$

Therefore $\sup_{a\geq 0,\, b\in\mathbb{R}} I_\mu(aw_1 - bu_{\varepsilon,\delta})$ exists and supremum will be attained in $a^2 + b^2 \leq R^2$, for some large $R > 0$. Thus it is enough to estimate $I_\mu(aw_1 - bu_{\varepsilon,\delta})$ in $\{(a,b) \in \mathbb{R}^+ \times \mathbb{R} : a^2 + b^2 \leq R^2\}$. Using elementary inequality, there exists $d(m) > 0$ such that

$$|a+b|^m \geq |a|^m + |b|^m - d(|a|^{m-1}|b| + |a||b|^{m-1}) \quad \text{for all } a,\ b \in \mathbb{R},\ m > 1. \tag{2.52}$$

Define, $f(v) := \|v\|^p_{X_0}$. Then using Taylor's theorem

$$\begin{aligned} f(aw_1 - bu_{\varepsilon,\delta}) &= f(aw_1) - \langle f'(aw_1), bu_\varepsilon\rangle + o(\|bu_{\varepsilon,\delta}\|^2_{X_0}) \leq \|aw_1\|^p_{X_0} \\ &- p\int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ c\|bu_{\varepsilon,\delta}\|^2_{X_0}, \end{aligned}$$

where $c > 0$ is small enough. We also note that from the definition of $u_{\varepsilon,\delta}$, it follows that $\|u_{\varepsilon,\delta}\|_{X_0}$ is bounded away from 0. Therefore, since $p \geq 2$ we have $c\|bu_{\varepsilon,\delta}\|^2_{X_0} \leq \|bu_{\varepsilon,\delta}\|^p_{X_0}$, for $c > 0$ small enough. Hence

$$\begin{aligned} \|aw_1 - bu_{\varepsilon,\delta}\|^p_{X_0} &= \|aw_1\|^p_{X_0} \\ &- p\int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \|bu_{\varepsilon,\delta}\|^p_{X_0}. \end{aligned}$$

Consequently, $a^2 + b^2 \leq R^2$ implies

$$\begin{aligned} I_\mu(aw_1 - bu_{\varepsilon,\delta}) &\leq \frac{1}{p}\,\|aw_1\|^p_{X_0} \\ &- \int_{\mathbb{R}^{2N}} \frac{|aw_1(x) - aw_1(y)|^{p-2}(aw_1(x) - aw_1(y))(bu_{\varepsilon,\delta}(x) - bu_{\varepsilon,\delta}(y))}{|x-y|^{N+ps}}\,dx\,dy \\ &+ \frac{1}{p}\,\|bu_{\varepsilon,\delta}\|^p_{X_0} - \frac{1}{p^*_s}\int_\Omega |aw_1|^{p^*_s}\,dx - \frac{1}{p^*_s}\int_\Omega |bu_{\varepsilon,\delta}|^{p^*_s}\,dx \\ &- \frac{\mu}{q+1}\int_\Omega |aw_1|^{q+1}\,dx - \frac{\mu}{q+1}\int_\Omega |bu_{\varepsilon,\delta}|^{q+1}\,dx \\ &+ C\bigg(\int_\Omega |aw_1|^{p^*_s-1}|bu_{\varepsilon,\delta}|\,dx + \int_\Omega |aw_1||bu_{\varepsilon,\delta}|^{p^*_s-1}\,dx\bigg) \\ &+ C\bigg(\int_\Omega |aw_1|^{q}|bu_{\varepsilon,\delta}|\,dx + \int_\Omega |aw_1||bu_{\varepsilon,\delta}|^{q}\,dx\bigg) \\ &= I_\mu(aw_1) + I_\mu(bu_{\varepsilon,\delta}) - a^q b\mu\int_\Omega |w_1|^{q-1}w_1u_{\varepsilon,\delta}\,dx \\ &- a^{p^*_s} b\int_\Omega |w_1|^{p^*_s-2}w_1u_{\varepsilon,\delta}\,dx \end{aligned}$$

$$+ C\biggl(\int_\Omega |w_1|^{p_s^*-1}|u_{\varepsilon,\delta}|\,dx + \int_\Omega |w_1||u_{\varepsilon,\delta}|^{p_s^*-1}\,dx\biggr)$$
$$+ C\biggl(\int_\Omega |w_1|^{q}|u_{\varepsilon,\delta}|dx + \int_\Omega |w_1||u_{\varepsilon,\delta}|^{q}\,dx\biggr).$$

Using Lemmas 2.8, 2.16 and 2.17 we estimate in $a^2+b^2\le R^2$,

$$I_\mu(aw_1 - bu_{\varepsilon,\delta}) \le \widetilde{\alpha}_\mu^- + \frac{s}{N}\,S_s^{N/(ps)} - k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)} + C\bigl(\varepsilon^{(N-ps)/(p-1)}$$
$$+ \varepsilon^{(N-ps)}/(p(p-1)) + \varepsilon^{(N-ps)/q}p(p-1) + \varepsilon^{(N(p-1)+ps)/(p(p-1))}\bigr).$$

For the term $k_8|u_\varepsilon|^{q+1}_{L^{q+1}(\Omega)}$, we invoke Lemma 2.9. Therefore when

$$\frac{N(p-2)+ps}{N-ps} < q < p-1,$$

we have

$$\text{(2.53)}\quad aI_\mu(aw_1 - bu_{\varepsilon,\delta}) \le \widetilde{\alpha}_\mu^- + \frac{s}{N}\,S_s^{N/(ps)} - k_9\varepsilon^{N-(N-ps)(q+1)/p}$$
$$+C\bigl(\varepsilon^{(N-ps)/(p-1)}+\varepsilon^{(N-ps)/(p(p-1))}+\varepsilon^{(N-ps)q/(p(p-1))}+\varepsilon^{(N(p-1)+ps)/(p(p-1))}\bigr).$$

We will choose q in such a way that the term $k_9\varepsilon^{N-(N-ps)(q+1)/p}$ dominates the other term involving ε. Note that among the terms in the bracket, $\varepsilon^{(N-ps)/(p(p-1))}$ and $\varepsilon^{(N-ps)q/(p(p-1))}$ dominate the others.

This in turn implies that we have to choose q so that

$$\text{(2.54)}\qquad N - \frac{(N-ps)(q+1)}{p} < \frac{N-ps}{p(p-1)}$$

and

$$\text{(2.55)}\qquad N - \frac{(N-ps)(q+1)}{p} < \frac{(N-ps)q}{p(p-1)}.$$

(2.54) and (2.55) imply $q > q_2$ and $q > q_3$ respectively, where

$$\text{(2.56)}\qquad q_2 := \frac{Np}{N-sp} - \frac{p}{p-1} \quad\text{and}\quad q_3 := \frac{N(p-1)}{N-sp} - \frac{p-1}{p}.$$

Case 1. $p \ge 3+\sqrt{5}/2$. In this case by straightforward calculation it follows that $q_2 > q_3$. So in this case, we choose $q > q_2$. Moreover, since $q < p-1$, to make the interval $(q_2, p-1) \ne \emptyset$, we have to take $N > sp(p^2-p+1)$.

Case 2. $2 \le p < 3+\sqrt{5}/2$. In this case again by simple calculation it follows that $q_3 > q_2$. Thus, in this case, we choose $q > q_3$. Furthermore, as $q < p-1$, to make the interval $(q_3, p-1) \ne \emptyset$, we have to take $N > sp(p+1)$.

Hence in both the cases taking $\varepsilon > 0$ to be small enough in (2.53), we obtain

$$\sup_{a\ge 0,\, b\in\mathbb{R}} I_\mu(aw_1 - bu_{\varepsilon,\delta}) < \widetilde{\alpha}_\mu^- + \frac{s}{N}\,S_s^{N/(ps)}. \qquad \square$$

Proof of Theorem 2.14. Define $\mu_0 := \min\{\widetilde{\mu}, \mu_*\}$,

$$\mathcal{N}_*^- := \mathcal{N}_{\mu,1}^- \cap \mathcal{N}_{\mu,2}^-, \tag{2.57}$$

$$c_2 := \inf_{u \in \mathcal{N}_*^-} I_\mu(u). \tag{2.58}$$

Let $\mu \in (0, \mu_0)$. Using the Ekeland Variational Principle and similarly to the proof of Theorem 2.13, we obtain a sequence $\{u_n\} \in \mathcal{N}_*^-$ satisfying

$$I_\mu(u_n) \to c_2, \quad I'_\mu(u_n) \to 0 \quad \text{in } (X_0)'.$$

Thus $\{u_n\}$ is a (PS) sequence at level c_2. From Lemma 2.18, given below, it follows that there exist $a > 0$ and $b \in R$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$. Therefore Proposition 2.15 yields

$$c_2 < \widetilde{\alpha}_\mu^- + \frac{s}{N} S^{N/(ps)}. \tag{2.59}$$

Claim 1. There exist two positive constants c and C such that $0 < c \le \|u_n^\pm\|_{X_0} \le C$.

To see this, we note that $\{u_n\} \subset \mathcal{N}_*^- \subset \mathcal{N}_{\mu,1}^-$. Thus using (2.38), Steps 2 and 3 of the proof of Theorem 2.13, we have $\|u_n^\pm\|_{X_0} \le C$ and $\|u_n^-\|_{X_0} \ge c$. To show $\|u_n^+\|_{X_0} \ge a$ for some $a > 0$, we use the method of contradiction. Assume, up to a subsequence, $\|u_n^+\|_{X_0} \to 0$ as $n \to \infty$. This together with Sobolev embedding implies $|u_n^+|_{L^{p_s^*}(\Omega)} \to 0$. On the other hand, $u_n^+ \in N_\mu^-$ implies

$$(p-1-q)\|u_n^+\|_{X_0}^p - (p_s^* - q - 1)|u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} < 0.$$

Therefore, by the Sobolev inequality, we have

$$S \le \frac{\|u_n^+\|_{X_0}^p}{|u_n^+|_{L^{p_s^*}(\Omega)}^p} < \frac{p_s^* - q - 1}{p - 1 - q} |u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^* - p},$$

which is a contradiction to the fact that $|u_n^+|_{L^{p_s^*}(\Omega)} \to 0$. Hence the claim follows.

Going to a subsequence if necessary we have

$$u_n^+ \rightharpoonup \eta_1, \quad u_n^- \rightharpoonup \eta_2 \quad \text{in } X_0. \tag{2.60}$$

Claim 2. $\eta_1 \not\equiv 0$, $\eta_2 \not\equiv 0$.

Suppose not, that is $\eta_1 \equiv 0$. Then by compact embedding, $u_n^+ \to 0$ in $L^{q+1}(\Omega)$. Moreover, $u_n^+ \in N_\mu^- \subset N_\mu$, implies $\langle I'_\mu(u_n^+), u_n^+ \rangle = 0$. Consequently,

$$\|u_n^+\|_{X_0}^p - |u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} = \mu |u_n^+|_{L^{q+1}(\Omega)}^{q+1} = o(1).$$

So we have $|u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*} = \|u_n^+\|_{X_0}^p + o(1)$. This together with $\|u_n^+\|_{X_0} \ge c$ implies

$$\frac{|u_n^+|_{L^{p_s^*}(\Omega)}^{p_s^*}}{\|u_n^+\|_{X_0}^p} \ge 1 + o(1).$$

This along with Sobolev embedding gives $|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} \geq S^{N/(ps)} + o(1)$. Thus we have

$$I_\mu(u_n^+) = \frac{1}{p}\,\|u_n^+\|^p_{X_0} - \frac{1}{p_s^*}\,|u_n^+|^{p_s^*}_{L^{p_s^*}(\Omega)} + o(1) \geq \frac{s}{N}\,S^{N/(ps)} + o(1). \tag{2.61}$$

Moreover, $u_n \in \mathcal{N}_*^-$ implies $-u_n^- \in N_\mu^-$. Therefore using the given condition on β_2, we get

$$I_\mu(-u_n^-) \geq \beta_2 \geq \widetilde{\alpha}_\mu^-. \tag{2.62}$$

Also it follows that

$$I_\mu(u_n^+) + I_\mu(-u_n^-) \leq I_\mu(u_n) = c_2 + o(1)$$

(see (2.40)). Combining this along with (2.62) and (2.59), we obtain

$$I_\mu(u_n^+) \leq c_2 - \widetilde{\alpha}_\mu^- + o(1) < \frac{s}{N}\,S_s^{N/(ps)},$$

which is a contradiction to (2.61). Therefore $\eta_1 \neq 0$. Similarly $\eta_2 \neq 0$ and this proves the claim.

Set $w_2 := \eta_1 - \eta_2$.

Claim 3. $w_2^+ = \eta_1$ and $w_2^- = \eta_2$ almost everywhere.

To see the claim we observe that $\eta_1\eta_2 = 0$ almost everywhere in Ω. Indeed,

$$\begin{aligned}\left|\int_\Omega \eta_1\eta_2\,dx\right| &= \left|\int_\Omega (u_n^+ - \eta_1)u_n^-\,dx + \int_\Omega \eta_1(u_n^- - \eta_2)\,dx\right| \\ &\leq |u_n^+ - \eta_1|_{L^p(\Omega)}|u_n^-|_{L^{p'}(\Omega)} + |\eta_1|_{L^{p'}(\Omega)}|u_n^- - \eta_2|_{L^p(\Omega)},\end{aligned} \tag{2.63}$$

where $1/p + 1/p' = 1$. By compact embedding we have $u_n^+ \to \eta_1$ and $u_n^- \to \eta_2$ in $L^p(\Omega)$. As $p \geq 2N/(N+s)$, then $p' \leq p_s^*$. Therefore, using Claim 1, we pass to the limit in (2.63) and obtain

$$\int_\Omega \eta_1\eta_2\,dx = 0.$$

Moreover, by (2.60), $\eta_1, \eta_2 \geq 0$ almost everywhere. Hence $\eta_1\eta_2 = 0$ almost everywhere in Ω. We have $w_2^+ - w_2^- = w_2 = \eta_1 - \eta_2$. It is easy to check that $w_2^+ \leq \eta_1$ and $w_2^- \leq \eta_2$. To show that equality holds almost everywhere we apply the method of contradiction. Suppose, there exists $E \subset \Omega$ such that $|E| > 0$ and $0 \leq w_2^+(x) < \eta_1(x)$ for all $x \in E$. Therefore $\eta_2 = 0$ almost everywhere in E by the observation that we made. Hence $w_2^+(x) - w_2^-(x) = \eta_1(x)$ almost everywhere in E. Clearly $w_2^-(x) \not> 0$ almost everywhere, otherwise $w_2^+(x) = 0$ almost everywhere and that would imply $\eta_1(x) = -w_2^-(x) < 0$ almost everywhere, which is not possible since $\eta_1 > 0$ in E. Thus $w_2^-(x) = 0$. Hence $\eta_1(x) = w_2^+(x)$ almost everywhere in E, which is a contradiction. Hence the claim follows.

Therefore w_2 is sign changing in Ω and $u_n \rightharpoonup w_2$ in X_0. Moreover, $I'_\mu(u_n) \to 0$ in $(X_0)'$ implies

$$\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}\,dx\,dy \\ - \mu\int_\Omega |u_n|^{q-1}u_n\phi\,dx - \int_\Omega |u_n|^{p_s^*-2}u_n\phi\,dx = o(1)$$

for every $\phi \in X_0$. Passing to the limit using Vitali's convergence theorem via Hölder's inequality we obtain $\langle I'_\mu(w_2), \phi\rangle = 0$. Hence w_2 is a sign changing weak solution to $(\mathcal{P}_\mu)$. □

Lemma 2.18. *Let $u_{\varepsilon,\delta}$ be defined as in* (2.20) *and w_1 be a positive solution of $(\mathcal{P}_\mu)$ for which $\widetilde{\alpha}_\mu^-$ is achieved, when $\mu \in (0,\mu_*)$. Then there exist $a,b \in \mathbb{R}$, $a \geq 0$ such that $aw_1 - bu_\varepsilon \in \mathcal{N}_*^-$, where $\mathcal{N}_*^-$ is defined as in* (2.57).

This lemma can be proved in the spirit of [5, Lemma 4.8], for the convenience of the reader we sketch the proof in Appendix.

Proof of Theorem 1.2. Define $\mu^* = \min\{\mu_*, \widetilde{\mu}, \widetilde{\mu}_1, \mu_0, \mu_1\}$, where μ_* is chosen such that $\widetilde{\alpha}_\mu^-$ is achieved in $(0,\mu_*)$. $\widetilde{\mu}, \widetilde{\mu}_1, \mu_0$ and μ_1 are as in (2.7), Lemma 2.12, Theorem 2.14 and Lemma A.1, respectively. Furthermore, define q_0 and N_0 as follows:

$$q_0 := \begin{cases} \max\{q_1,q_2\} & \text{when } p \geq \dfrac{3+\sqrt{5}}{2}, \\ \max\{q_1,q_3\} & \text{when } 2 \leq p < \dfrac{3+\sqrt{5}}{2}, \end{cases}$$

$$N_0 := \begin{cases} sp(p^2-p+1) & \text{when } p \geq \dfrac{3+\sqrt{5}}{2}, \\ sp(p+1) & \text{when } 2 \leq p < \dfrac{3+\sqrt{5}}{2}. \end{cases}$$

Note that $N_0 > sp[p+1+\sqrt{(p+1)^2-4}]/2$, where the RHS appeared in Theorem 2.13. Hence combining Theorems 2.13 and 2.14, we complete the proof of this theorem for $\mu \in (0,\mu^*)$, $q > q_0$ and $N > N_0$. □

Appendix A

Lemma A.1. *Let g_n be as in* (2.42) *in Theorem* 2.13 *and $v \in X_0$ be such that $\|v\|_{X_0} = 1$. Then there exists $\mu_1 > 0$ such that $\mu \in (0,\mu_1)$ implies $\langle g_n'(0), v^+\rangle$ is uniformly bounded in X_0.*

Proof. In view of Lemma 2.6 we have

$$\langle g_n'(0), v^+\rangle = \frac{pA(u_n,v^+) - p_s^*\displaystyle\int_\Omega |u_n|^{p_s^*-p}u_nv^+ - (q+1)\mu\int_\Omega |u_n|^{q-1}u_nv^+}{(p-1-q)\|u_n\|_{X_0}^p - (p_s^*-q-1)|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*}}.$$

Using Claim 2 in Theorem 2.13, there exists $C>0$ such that $||u_n\|_{X_0} \le C$ for all $n \ge 1$. Therefore, applying the Hölder inequality followed by the Sobolev inequality, we have

$$|\langle g_n'(0), v^+\rangle| \le \frac{C\|v\|_{X_0}}{\big|(p-1-q)\|u_n\|_{X_0}^p - (p_s^*-q-1)|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*}\big|}.$$

Hence it is enough to show that

$$\big|(p-1-q)\|u_n\|_{X_0}^p - (p_s^*-q-1)|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*}\big| > C,$$

for some $C>0$ and n large enough. Suppose it does not hold. Then, up to a subsequence,

$$(p-1-q)\|u_n\|_{X_0}^p - (p_s^*-q-1)|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} = o(1) \quad \text{as } n \to \infty.$$

Hence,

$$\|u_n\|_{X_0}^p = \frac{p^*-q-1}{p-1-q}\,|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} + o(1) \quad \text{as } n \to \infty. \tag{A.1}$$

Combining the above expression along with the fact that $u_n \in N_\mu$, we obtain

$$\mu|u_n|_{L^{q+1}(\Omega)}^{q+1} = \frac{p_s^*-p}{p-1-q}\,|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} + o(1) = \frac{p_s^*-p}{p_s^*-1-q}\,\|u_n\|_{X_0}^p + o(1). \tag{A.2}$$

After applying the Hölder inequality and followed by the Sobolev inequality, expression (A.2) yields

$$\|u_n\|_{X_0} \le \bigg(\mu\,\frac{p_s^*-q-1}{p_s^*-p}\,|\Omega|^{(p_s^*-q-1)/p_s^*} S^{-(q+1)/p}\bigg)^{1/(p-1-q)} + o(1). \tag{A.3}$$

Combining (2.38) and Claim 3 in the proof of Theorem 2.13, we have $\|u_n\|_{X_0} \ge b$, for some $b>0$. Therefore from (A.1) we get

$$|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} \ge C \quad \text{for some constant } C>0, \text{ and } n \text{ large enough.} \tag{A.4}$$

Define $\psi_\mu \colon N_\mu \to \mathbb{R}$ as follows:

$$\psi_\mu(u) = k_0\bigg(\frac{\|u\|_{X_0}^{p(p_s^*-1)}}{|u|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\bigg)^{1/(p_s^*-p)} - \mu|u|_{L^{q+1}(\Omega)}^{q+1},$$

where

$$k_0 = \bigg(\frac{p-1-q}{p_s^*-q-1}\bigg)^{(p_s^*-1)/(p_s^*-p)}\bigg(\frac{p_s^*-p}{p-1-q}\bigg).$$

Simplifying $\psi_\mu(u_n)$ using (A.2), we obtain

$$\begin{aligned}\psi_\mu(u_n) = k_0\Bigg[\bigg(\frac{p_s^*-q-1}{p-1-q}\bigg)^{p_s^*-1}\frac{|u_n|_{L^{p_s^*}(\Omega)}^{(p_s^*-1)p_s^*}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\Bigg]^{1/(p_s^*-p)}& \\ - \frac{p_s^*-p}{p-1-q}\,|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*} + o(1) = o(1).&\end{aligned} \tag{A.5}$$

On the other hand, using the Hölder inequality in the definition of $\psi_\mu(u_n)$, we obtain

$$\begin{aligned}
\text{(A.6)}\quad \psi_\mu(u_n) &= k_0\left(\frac{\|u_n\|_{X_0}^{p(p_s^*-1)}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\right)^{1/(p_s^*-p)} - \mu|u_n|_{L^{q+1}(\Omega)}^{q+1} \\
&\geq k_0\left(\frac{\|u_n\|_{X_0}^{p(p_s^*-1)}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\right)^{1/(p_s^*-p)} - \mu|\Omega|^{(p_s^*-q-1)/p_s^*}|u_n|_{L^{p_s^*}(\Omega)}^{q+1} \\
&= |u_n|_{L^{p_s^*}(\Omega)}^{q+1}\left\{k_0\left(\frac{\|u_n\|_{X_0}^{p(p_s^*-1)}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\right)^{1/(p_s^*-p)}\frac{1}{|u_n|_{L^{p_s^*}(\Omega)}^{q+1}} - \mu|\Omega|^{(p_s^*-q-1)/p_s^*}\right\}.
\end{aligned}$$

Using Sobolev embedding and (A.3), we simplify the term

$$\left(\frac{\|u_n\|_{X_0}^{p(p_s^*-1)}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\right)^{1/(p_s^*-p)}\frac{1}{|u_n|_{L^{p_s^*}(\Omega)}^{q+1}}$$

and obtain

$$\begin{aligned}
\text{(A.7)}\quad &\left(\frac{\|u_n\|_{X_0}^{p(p_s^*-1)}}{|u_n|_{L^{p_s^*}(\Omega)}^{p_s^*(p-1)}}\right)^{1/(p_s^*-p)}\frac{1}{|u_n|_{L^{p_s^*}(\Omega)}^{q+1}} \geq S^{(p_s^*-1)/(p_s^*-p)}|u_n|_{L^{p_s^*}(\Omega)}^{-q} \\
&\geq S^{(p_s^*-1)/(p_s^*-p)+q/p}\|u_n\|_{X_0}^{-q} \\
&\geq S^{(p_s^*-1)/(p_s^*-p)+q/p}\left(\mu\,\frac{p_s^*-q-1}{p_s^*-p}\,|\Omega|^{(p^*-q-1)p_s^*}S^{-(q+1)/p}\right)^{-q/(p-1-q)}.
\end{aligned}$$

Substituting back (A.7) into (A.6) and using (A.4), we obtain

$$\begin{aligned}
\psi_\mu(u_n) \geq C^{q+1}\bigg[&k_0 S^{(p_s^*-1)/(p_s^*-p)+q/(p-1-q)}\mu^{-q/(p-1-q)} \\
&\cdot\left(\frac{p_s^*-q-1}{p_s^*-p}\,|\Omega|^{(p_s^*-q-1)/p_s^*}\right)^{-q/(p-1-q)} - \mu|\Omega|^{(p_s^*-q-1)/p_s^*}\bigg] \geq d_0,
\end{aligned}$$

for some $d_0 > 0$, n large and $\mu < \mu_1$, where $\mu_1 = \mu_1(k_0, s, q, N, |\Omega|)$. This is a contradiction to (A.5). Hence the lemma follows. □

PROOF OF LEMMA 2.18. We will show that there exist $a > 0$, $b \in \mathbb{R}$ such that

$$a(w_1 - bu_\varepsilon)^+ \in N_\mu^- \quad \text{and} \quad -a(w_1 - bu_\varepsilon)^- \in N_\mu^-.$$

Let us denote $\overline{r}_1 = \inf_{x\in\Omega} w_1(x)/u_\varepsilon(x)$, $\overline{r}_2 = \sup_{x\in\Omega} w_1(x)/u_\varepsilon(x)$. As both w_1 and u_ε are positive in Ω, we have $\overline{r}_1 \geq 0$ and $\overline{r}_2$ can be $+\infty$. Let $r \in (\overline{r}_1, \overline{r}_2)$. Then $w_1, u_\varepsilon \in X_0$ implies $(w_1 - ru_\varepsilon) \in X_0$ and $(w_1 - ru_\varepsilon)^+ \not\equiv 0$. Otherwise, $(w_1 - ru_\varepsilon)^+ \equiv 0$ would imply $\overline{r}_2 \leq r$, which is not possible. Define $v_r := w_1 - ru_\varepsilon$. Hence $0 \not\equiv v_r^+ \in X_0$ (since for any $u \in X_0$, we have $|u| \in X_0$). Similarly $0 \not\equiv v_r^- \in X_0$. Therefore, by Lemma 2.3, there exist $0 < s^+(r) < s^-(r)$ such that $s^+(r)v_r^+ \in N_\mu^-$, and $-s^-(r)(v_r^-) \in N_\mu^-$. Let us consider the functions $s^\pm\colon \mathbb{R} \to (0,\infty)$ defined as above.

Claim. The functions $r \mapsto s^\pm(r)$ are continuous and

$$\lim_{r\to\overline{r}_1^+} s^+(r) = t^+(v_{\overline{r}_1}^+) \quad \text{and} \quad \lim_{r\to\overline{r}_2^-} s^+(r) = +\infty,$$

where the function t^+ is the same as defined in Lemma 2.3.

To see the claim, choose $r_0 \in (\overline{r}_1, \overline{r}_2)$ and $\{r_n\}_{n\geq 1} \subset (\overline{r}_1, \overline{r}_2)$ such that $r_n \to r_0$ as $n \to \infty$. We need to show that $s^+(r_n) \to s^+(r_0)$ as $n \to \infty$. Corresponding to r_n and r_0, we have $v_{r_n}^+ = (w_1 - r_n u_\varepsilon)^+$ and $v_{r_0}^+ = (w_1 - r_0 u_\varepsilon)^+$. By Lemma 2.3 we note that $s^+(r) = t^+(v_r^+)$. Let us define the function

$$\begin{aligned} F(s,r) &:= s^{p-1-q}\|(w_1 - ru_\varepsilon)^+\|_{X_0}^p \\ &\qquad - s^{p_s^*-q-1}|(w_1 - ru_\varepsilon)^+|_{L^{p_s^*}(\Omega)}^{p_s^*} - \mu|(w_1 - ru_\varepsilon)^+|_{L^{q+1}(\Omega)}^{q+1} \\ &= \phi(s,r) - \mu|(w_1 - ru_\varepsilon)^+|_{L^{q+1}(\Omega)}^{q+1}, \end{aligned}$$

where $\phi(s,r) := s^{p-1-q}\|(w_1 - ru_\varepsilon)^+\|_{X_0}^p - s^{p_s^*-q-1}|(w_1 - ru_\varepsilon)^+|_{L^{p_s^*}(\Omega)}^{p_s^*}$.

Doing a similar calculation as in Lemma 2.3, we obtain that for any fixed r, the function $F(s,r)$ has only two zeros $s = t^+(v_r^+)$ and $s = t^-(v_r^+)$. Consequently, $s^+(r)$ is the largest 0 of $F(s,r)$ for any fixed r. As $r_n \to r_0$ we have $v_{r_n}^+ \to v_{r_0}^+$ in X_0. Indeed, by a straightforward computation it follows that $v_{r_n} \to v_{r_0}$ in X_0. Therefore $|v_{r_n}| \to |v_{r_0}|$ in X_0. This in turn implies that $v_{r_n}^+ \to v_{r_0}^+$ in X_0. Hence $\|v_{r_n}^+\|_{X_0} \to \|v_{r_0}^+\|_{X_0}$. Moreover, by the Sobolev inequality, we have $|v_{r_n}^+|_{L^{p_s^*}(\Omega)} \to |v_{r_0}^+|_{L^{p_s^*}(\Omega)}$ and $|v_{r_n}^+|_{L^{q+1}(\Omega)} \to |v_{r_0}^+|_{L^{q+1}(\Omega)}$. As a result, we have $F(s,r_n) \to F(s,r_0)$ uniformly. Therefore an elementary analysis yields $s^+(r_n) \to s^+(r_0)$.

Moreover, $\overline{r}_2 \geq w_1/u_\varepsilon$ implies $w_1 - \overline{r}_2 u_\varepsilon \leq 0$. As a consequence $r \to \overline{r}_2^-$ implies $(w_1 - ru_\varepsilon)^+ \to 0$ pointwise. Moreover, since $|(w_1 - ru_\varepsilon)^+|_{L^\infty(\Omega)} \leq |w_1|_{L^\infty(\Omega)}$, using the dominated convergence theorem we have $|(w_1 - ru_\varepsilon)^+|_{L^{p_s^*}(\Omega)} \to 0$. From the analysis in Lemma 2.3, for any r, we also have $s^+(r) > t_0(v_r^+)$, where the function t_0 is defined as in Lemma 2.3, which is the maximum point of $\phi(\,\cdot\,,r)$. Therefore it is enough to show that $\lim_{r\to\overline{r}_2^-} t_0(v_r^+) = \infty$. Applying the Sobolev inequality in the definition of $t_0(v_r^+)$ we get

$$t_0(v_r^+) = \left(\frac{(p-1-q)\|v_r^+\|_{X_0}^p}{(p_s^*-1-q)|v_r^+|_{L^{p_s^*}(\Omega)}^{p_s^*}}\right)^{1/(p_s^*-p)} \geq \left(\frac{S(p-1-q)}{p_s^*-1-q}\right)^{1/(p_s^*-p)} |v_r^+|_{L^{p_s^*}(\Omega)}^{-1}.$$

Hence $\lim_{r\to\overline{r}_2^-} t_0(v_r^+) = \infty$.

Proceeding similarly, we can show that if $r \to \overline{r}_1^-$ then $v_r^+ \to v_{\overline{r}_1}$ and $\lim_{r\to\overline{r}_1^+} s^+(r) = t^+(v_{\overline{r}_1}^+)$, and

$$\lim_{r\to r_1^+} s^-(r) = +\infty, \qquad \lim_{r\to r_2^-} s^-(r) = t^+(v_r^-) < +\infty.$$

The continuity of $s^\pm$ implies that there exists $b \in (\overline{r}_1, \overline{r}_2)$ such that $s^+(r) = s^-(r) = a > 0$. Therefore, $a(w_1 - bu_\varepsilon)^+ \in N_\mu^-$ and $-a(w_1 - bu_\varepsilon)^- \in N_\mu^-$, that is, the function $a(w_1 - bu_\varepsilon) \in \mathcal{N}_*^-$ and this completes the proof. □

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