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ON CERTAIN VARIANT OF STRONGLY NONLINEAR MULTIDIMENSIONAL INTERPOLATION INEQUALITY

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Dedicated to the memory of Professor Marek Burnat

Abstract. We obtain the inequality

$$\int_\Omega |\nabla u(x)|^p h(u(x))\,dx \leq C(n,p)\int_\Omega \Big(\sqrt{|\nabla^{(2)}u(x)||\mathcal{T}_{h,C}(u(x))|}\Big)^p h(u(x))\,dx,$$

where $\Omega\subset\mathbb{R}^n$ and $n\geq 2$, $u\colon\Omega\to\mathbb{R}$ is in certain subset in second order Sobolev space $W^{2,1}_{\rm loc}(\Omega)$, $\nabla^{(2)}u$ is the Hessian matrix of u, $\mathcal{T}_{h,C}(u)$ is a certain transformation of the continuous function $h(\,\cdot\,)$. Such inequality is the generalization of a similar inequality holding in one dimension, obtained earlier by second author and Peszek.

1. Introduction

The purpose of this paper is to obtain the n-dimensional variant of the following inequality:

$$\int_{(a,b)} |u'(x)|^p h(u(x))\,dx \leq C_p\int_{(a,b)} \Big(\sqrt{|u''(x)\mathcal{T}_h(u(x))|}\Big)^p h(u(x))\,dx, \tag{1.1}$$

where (a,b) is an interval, $u\in W^{2,1}_{\rm loc}((a,b))$ and obeys some additional assumptions, $h(\,\cdot\,)$ the a given continuous function, $\mathcal{T}_h(\,\cdot\,)$ is a certain transform of $h(\,\cdot\,)$.

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When for example $h(t) = t^\alpha$, $\alpha > -1$, then $\mathcal{T}_h(\lambda) = \lambda/(\alpha+1)$, so it is proportional to λ, see Remark 2.4, where we deal with $\mathcal{T}_{h,0}(\lambda)$. The constant C_p does not depend on u.

The above inequality was obtained by the second author and Peszek in [9]. It implies the inequality:

$$\bigg(\int_{\mathbb{R}} |f'|^p h(f)\,dx\bigg)^{2/p} \leq (p-1)\bigg(\int_{\mathbb{R}} |\mathcal{T}_h(f)|^q h(f)\,dx\bigg)^{1/q}\bigg(\int_{\mathbb{R}} |f''|^r h(f)\,dx\bigg)^{1/r},$$

where $q \geq p/2$, $2/p = 1/q + 1/r$. After the substitution of $h \equiv 1$ we obtain the classical Gagliardo–Nirenberg inequality ([7], [14]):

$$\bigg(\int_{\mathbb{R}} |f'|^p\,dx\bigg)^{2/p} \leq C\bigg(\int_{\mathbb{R}} |f|^q\,dx\bigg)^{1/q}\bigg(\int_{\mathbb{R}} |f''|^r\,dx\bigg)^{1/r},$$

where in our case one has to assume that $q \geq p/2$. However, inequality (1.1) within $q \geq p/2$ is more general.

Inequalities of the type (1.1), were generalized later to various settings in [3], [8], [10]. They were applied to results about regularity and asymptotic behavior in nonlinear eigenvalue problems for singular PDE's like for example the Emden–Fowler equation [8], [9]:

$$u''(x) = g(x)u(x)^\alpha, \quad \alpha \in \mathbb{R},\ g(x) \in L^p((a,b)),$$

which appears for example in electricity theory, fluid dynamics or mathematical biology. The pattern equation, with $g(t) = t^{1/2}$, $\alpha = 3/2$, appears in the model found independently in 1927 by Thomas and Fermi to determine the electrical potential in an isolated neutral atom ([6], [17]). Inequalities like (1.1) were also applied to the study of regularity of solutions of the Cucker–Smale equation with singular weight, [16]. In a more theoretical approach, they were used to obtain certain generalization of isoperimetric inequalities and capacitary estimates due to Mazya [10] in the Orlicz setting.

All results obtained in the papers [8]–[10] were dealing only with the case of inequalities for one-variable function and $p \geq 2$. In the most recent paper [3], the authors obtained a generalization of the one-dimensional inequality (1.1) to the case $1 < p < 2$. Unexpectedly, it appears that the new inequality instead of the operator $\mathcal{T}_h(u(x))$ involves a certain nonlocal operator, i.e. a one which uses all values of u in the interval (a,b), not just at x.

In this paper we obtain multiplicative inequalities in the form:

$$\begin{aligned}(1.2)\quad &\int_\Omega |\nabla u(x)|^p h(u(x))\,dx \\ &\qquad \leq C(n,p)\int_\Omega \Big(\sqrt{|\nabla^{(2)}u(x)||\mathcal{T}_{h,C}(u(x))|}\Big)^p h(u(x))\,dx,\end{aligned}$$

where $C(n,p) = (p-1+\sqrt{n-1})^{p/2}$, $\Omega \subset \mathbb{R}^n$ is bounded and $n \geq 2$, $u\colon \Omega \to \mathbb{R}$ belongs to a certain subset in the Sobolev space $W^{2,1}_{\rm loc}(\Omega)$, $\nabla^{(2)}u$ is the Hessian matrix of u and $\mathcal{T}_{h,C}(u)$ is a given transformation of the continuous function $h(\,\cdot\,)$ (see Definition 2.3), under some additional assumptions (see Theorems 3.1 and 3.2).

Let us indicate three earlier sources, where the variants of such inequality could be found. The inspiration for the authors of [9] for inequality (1.1) comes from the estimates due to Mazja [13, Section 8.2.1]:

$$\int_{\operatorname{supp} f'} \bigg(\frac{|f'|}{f^{1/2}}\bigg)^p \, dx \leq \bigg(\frac{p-1}{|1-p/2|}\bigg)^{p/2} \int_{\mathbb{R}} |f''|^p \, dx,$$

where supp f' is the support of f', which are a special case of (1.1) when one considers $h(s) = s^{-p/2}$. They were applied in [13] to the second order izoperimetric inequalities and capacitary estimates in second order Sobolev classes.

Opial obtained inequalities:

$$\int_a^b |yy'| \, dx \leq K \bigg(\int_a^b |y''|^p \, dx \bigg)^{2/p},$$

known as second order Opial inequalities, holding on a compact interval $[a,b]$, where

$$y \in BC_0 = \{y \in W^{2,p}((a,b)) : y(a) = y(b) = y'(b) = 0\},$$

see, e.g. [15] and [1], [2] for further related issues.

In another source [11], the authors obtained the inequality:

$$\int_{\mathbb{R}}^n G(|\nabla u|) \, dx \leq C \int_{\mathbb{R}}^n G(|u||\nabla^{(2)}u|) \, dx,$$

dealing with a convex function G.

Let us mention that the passage from inequality (1.1) to inequality (1.2) is not so direct and it requires quite delicate arguments. For example now we apply integrals over the boundaries of Lipshitz domains and the coarea formulae.

We expect that the new derived inequalities can be applied in the regularity theory for the nonlinear eigenvalue problems in singular elliptic PDE's in the similar way as it was done for functions of one variable.

2. Preliminaries and notation

2.1. Basic notation. In the sequel we assume that $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, is an open domain. We use the standard definition of strongly Lipschitz domain, i.e. of the domain of class $\mathcal{C}^{0,1}$, see, e.g. [13, Section 1.1.9]. We use the standard notation: $C_0^\infty(\Omega)$ to denote smooth functions with compact support, $W^{m,p}(\Omega)$ and $W^{m,p}_{\rm loc}(\Omega)$ to denote the global and local Sobolev functions defined on Ω, respectively, while $C_0^\infty(\Omega,\mathbb{R}^k)$, $W^{m,p}(\Omega,\mathbb{R}^k)$ and $W^{m,p}_{\rm loc}(\Omega,\mathbb{R}^k)$ will denote their vectorial counterparts. By σ we denote the $(n-1)$-dimensional Hausdorff measure. For $p > 1$ we set

$$\Phi_p(\lambda) = \begin{cases} |\lambda|^{p-2}\lambda & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0, \end{cases} \qquad \Phi_p\colon \mathbb{R}^n \to \mathbb{R}^n, \tag{2.1}$$

and we write $p' := p/(p-1)$. When $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, by Du we denote the matrix of its differential $\Big(\frac{\partial u^i(x)}{\partial x_j}\Big)_{i\in\{1,\dots,n\},\, j\in\{1,\dots,m\}}$. When $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$, by $\nabla^{(2)}u(x)$ we denote the Hessian of u, i.e. the matrix $\Big(\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\Big)_{i,j\in\{1,\dots,n\}}$.

If A is a vector or matrix, by $|A|$ we denote its Euclidean norm. The same notation is used to denote Lebesgue's measure of the measurable subset in $\mathbb{R}^n$.

2.2. Properties of absolutely continuous functions. We will need the following variant of Nikodym ACL Characterization Theorem, which can be found, e.g. in [13, Section 1.1.3].

Theorem 2.1 (Nikodym ACL Characterization Theorem).

(a) *Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for every $i \in \{1,\dots,n\}$ and for almost every $a \in \mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{n-i}$, the function*

$$\mathbb{R} \ni t \mapsto u(a + te_i) \tag{2.2}$$

is locally absolutely continuous on $\mathbb{R}$. In particular, for almost every point $x \in \mathbb{R}^n$, the distributional derivative $D_i u(x)$ is the same as the usual derivative at x.

(b) *Assume that $u \in L^1_{\mathrm{loc}}(\Omega)$ and for every $i \in \{1,\dots,n\}$ and for almost every $a \in \mathbb{R}^{n-i-1} \times \{0\} \times \mathbb{R}^i$ the function in (2.2) is locally absolutely continuous on $\mathbb{R}$ and all the derivatives $D_i u$ computed almost everywhere are locally integrable on $\mathbb{R}^n$. Then u belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.*

(c) *Let $1 \le p \le \infty$, $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ be an open subset. Then u belongs to $W^{1,p}(\Omega)$ if and only if $u \in L^p(\Omega)$ and every derivative $D_i u$, computed almost everywhere, belongs to the space $L^p(\Omega)$.*

The following lemma can be easily proven by using the very definition of absolutely continuous functions.

Lemma 2.2.

(a) *If $f\colon [-R, R] \to \mathbb{R}$ is absolutely continuous with values in the interval $[\alpha, \beta]$ and $L\colon [\alpha, \beta] \to \mathbb{R}$ is a Lipschitz function, then the function*

$$(L \circ f)(x) := L(f(x))$$

is absolutely continuous on $[-R, R]$.

(b) *If* $L\colon [-R,R] \to \mathbb{R}$ *is a Lipschitz function with values in the interval* $[\alpha,\beta]$ *and* $f\colon [\alpha,\beta] \to \mathbb{R}$ *is absolutely continuous, then the function*

$$(f \circ L)(x) := f(L(x))$$

is absolutely continuous on $[-R,R]$.

2.3. Transformations of the nonlinear weight. We will use the following definition, which is the special variant of definition introduced in [9].

Definition 2.3 (Transforms of the nonlinear weight h). Let $0 < B \leq \infty$, $h\colon (0,B) \to (0,\infty)$ be a continuous function which is integrable on $(0,\lambda)$ for every $\lambda < B$, $C \in \mathbb{R}$, and let $H_C\colon [0,B) \to \mathbb{R}$ be the locally absolutely continuous primitive of h extended to 0, given by

$$H_C(\lambda) := \int_0^\lambda h(s)\,ds - C, \quad \lambda \in [0,B).$$

We define the transform of h, $\mathcal{T}_{h,C}\colon (0,B) \to (0,\infty)$ by

$$\mathcal{T}_{h,C}(\lambda) := \frac{H_C(\lambda)}{h(\lambda)}, \quad \lambda \in (0,B).$$

In the case when $C = 0$ we omit it from the notation, i.e., we write $H_0 =: H$, $\mathcal{T}_{h,0} =: \mathcal{T}_h$. Note that h and $\mathcal{T}_{h,C}$ might not be defined at 0 or B in case $B < \infty$.

Remark 2.4. Simple examples of the admitted weights $h(\,\cdot\,)$ and their transformations can be found among power weights. Namely, when we choose $h(\lambda) = \lambda^\theta$ where $\theta > -1$, then clearly $\mathcal{T}_{h,0}(\lambda) = (1+\theta)^{-1}\lambda$, so it is proportional to the identity function. Similarly, $\mathcal{T}_{h,C}(\lambda)$ can be estimated by the proportional to the identity function when for example we consider $C = 0$ and we can deduce that $(*)\colon \mathcal{T}_h(\lambda) = H(\lambda)/h(\lambda) \leq A\lambda$, with some general constant A. This is always the case when H is convex, so when h is nondecreasing. Then $(*)$ holds with $A = 1$. The case of power function shows that even when $h(\,\cdot\,)$ is decreasing, the estimate $(*)$ can still hold true.

3. Presentation of main results

Our goal is to prove the following multidimensional variant of inequality (1.1).

Theorem 3.1. *Let* $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *be a bounded domain,* $2 \leq p < \infty$, $0 < B \leq \infty$, $h\colon (0,B) \to (0,\infty)$ *and* $H_C, \mathcal{T}_{h,C}$ *be as in Definition* 2.3. *Moreover, let the following assumptions be satisfied*:

(u) $u \in W^{2,p/2}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ *and* $0 < u < B$ *in* Ω;

(u,Ω) *there exists a sequence* $\{\Omega_k\}_{k\in\mathbf{N}}$ *of bounded subdomains of* Ω *of class* $\mathcal{C}^{0,1}$ *such that* $\overline{\Omega}_k \subset \Omega$, $\bigcup_k \Omega_k = \Omega$ *and*

$$\lim_{k\to\infty} \int_{\partial\Omega_k} \Phi_p(\nabla u(x)) \cdot n(x) H_C(u(x))\,d\sigma(x) \in [-\infty, 0],$$

where $n(x)$ denotes the unit outer normal vector to $\partial\Omega_k$, defined for σ almost all $x \in \partial\Omega_k$.

Then we have

$$\int_\Omega |\nabla u(x)|^p h(u(x))\,dx \le C(n,p)\int_\Omega \Big(\sqrt{|\nabla^{(2)}u(x)||\mathcal{T}_{h,C}(u(x))|}\Big)^p h(u(x))\,dx, \tag{3.1}$$

where $C(n,p) = (p-1+\sqrt{n-1})^{p/2}$.

Our next statement is a special variant of Theorem 3.1, when we assume that Ω is of class $\mathcal{C}^{0,1}$ and we use some additional assumptions on u.

Theorem 3.2. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $2 \le p < \infty$, $0 < B \le \infty$, $h\colon (0,B) \to (0,\infty)$, H_C, $\mathcal{T}_{h,C}$ be as in Definition* 2.3, *$u \in W^{2,p/2}(\Omega) \cap C(\overline{\Omega})$, $0 < u < B$ in Ω and*

$$\int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x) H_C(u(x))\,d\sigma(x) \in [-\infty, 0], \tag{3.2}$$

where $n(x)$ denotes the unit outer normal vector to $\partial\Omega$, defined for σ almost all $x \in \partial\Omega$. Then

$$\int_\Omega |\nabla u(x)|^p h(u(x))\,dx \le C(n,p)\int_\Omega \Big(\sqrt{|\nabla^{(2)}u(x)||\mathcal{T}_{h,C}(u(x))|}\Big)^p h(u(x))\,dx,$$

where $C(n,p) = (p-1+\sqrt{n-1})^{p/2}$.

Remark 3.3. The typical situation when condition (3.2) is satisfied is when $H_C(0) \ge 0$ and we deal with the Dirichlet condition: $u = 0$ on $\partial\Omega$. In this case ∇u is perpendicular to $\partial\Omega$, which is the level set of u and, as u is nonnegative inside Ω, we have $\partial_{n(x)}u(x) = \nabla u(x)\cdot n(x) \le 0$ for σ almost every $x \in \partial\Omega$. This implies that also $\Phi_p(\nabla u(x))\cdot n(x) \le 0$ for σ almost every $x \in \partial\Omega$, and because $H_C(u(x)) = H_C(0) \ge 0$ for σ almost every $x \in \partial\Omega$, condition (3.2) is satisfied.

The interval $(0,B)$ in the assumptions of Theorems 3.1 and 3.2 can be changed to any interval (A,B) where $-\infty < A < B \le \infty$. Our most general result reads as follows.

Theorem 3.4. *Let $n \ge 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain, $2 \le p < \infty$, $-\infty < A < B \le \infty$, $h\colon (A,B) \to (0,\infty)$ is continuous and integrable near A, $C \in \mathbb{R}$,*

$$H_{C,A}(\lambda) := \int_A^{\lambda+A} h(\tau)\,d\tau - C,$$

$$\mathcal{T}_{h,C,A}(\lambda) := \frac{1}{h(\lambda+A)}\bigg(\int_A^{\lambda+A} h(\tau)\,d\tau - C\bigg), \quad \textit{for } \lambda \in (0, B-A),$$

$u \in W^{2,p/2}(\Omega) \cap C(\overline{\Omega})$, $A < u < B$ in Ω. Assume further that one of the following conditions holds:

(a) *there exists a sequence $\{\Omega_k\}_{k\in\mathbf{N}}$ of bounded subdomains of Ω of class $\mathcal{C}^{0,1}$ such that $\overline{\Omega}_k \subset \Omega$, $\bigcup_k \Omega_k = \Omega$ and*
$$\lim_{k\to\infty} \int_{\partial\Omega_k} \Phi_p(\nabla u(x)) \cdot n(x) H_{C,A}(u(x))\, d\sigma(x) \in [-\infty, 0],$$
where $n(x)$ denotes the unit outer normal vector to $\partial\Omega_k$, defined for σ almost all $x \in \partial\Omega_k$;

(b) *$\Omega \subset \mathbb{R}^n$ is Lipschitz and*
$$\int_{\partial\Omega} \Phi_p(\nabla u(x)) \cdot n(x) H_{C,A}(u(x))\, d\sigma(x) \in [-\infty, 0],$$
where $n(x)$ denotes the unit outer normal vector to $\partial\Omega$, defined for σ almost all $x \in \partial\Omega$.

Then
$$\int_\Omega |\nabla u(x)|^p h(u(x))\, dx \le C(n,p) \int_\Omega \Big(\sqrt{|\nabla^{(2)} u(x)||\mathcal{T}_{h,C,A}(u(x))|}\Big)^p h(u(x))\, dx,$$
where $C(n,p) = (p - 1 + \sqrt{n-1})^{p/2}$.

Proof. The proof follows directly from Theorems 3.1 and 3.2. Let us substitute $u_A := u - A \colon \Omega \to (0, B - A)$ instead of u and $h_A(\lambda) := h(\lambda + A)$ instead of $h(\,\cdot\,)$ in Theorems 3.1 and 3.2. Then $h_A(u_A) = h(u)$ and $\mathcal{T}_{h_A,C}(u_A) = \mathcal{T}_{h,C,A}(u)$. In case of condition (a) we apply Theorem 3.1, while in case of condition (b), we apply Theorem 3.2. Verification of their assumptions is left to the reader. □

Remark 3.5. The last statement allows to consider also the situation when the function $h(\,\cdot\,)$ is not integrable near zero. In that case its primitive: $\int_A^s h(\tau)\, d\tau$, where $A > 0$, is well defined and absolutely continuous on segments with left endpoints A.

Our remaining sections are devoted to the proof of Theorems 3.1 and 3.2 and discussion.

4. Auxiliary facts dealing with multiplications and compositions of Sobolev functions

The following two lemmas will be helpful for the proof of Theorem 3.1.

Lemma 4.1 (Multiplicative property of Sobolev vector field). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $\mathcal{C}^{0,1}$, $1 < p < \infty$, $w_1 \in W^{1,1}(\Omega, \mathbb{R}^n) \cap L^{p'}(\Omega, \mathbb{R}^n)$ and $w_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then the vectorfield $v := w_1 w_2$ satisfies: $v \in W^{1,1}(\Omega, \mathbb{R}^n)$ and*

$$\int_\Omega \operatorname{div} v(x)\, dx = \int_{\partial\Omega} v(x) \cdot n(x)\, d\sigma(x), \tag{4.1}$$

where $n(x)$ is the outer normal vector which is defined for σ almost all $x \in \partial\Omega$.

Proof. At first we note that $v \in L^1(\Omega, \mathbb{R}^n)$. We verify that v is absolutely continuous along σ almost all lines parallel to the axes and that the derivatives of the k-th coordinate of v computed almost everywhere read as

$$\frac{\partial v_k}{\partial x_i}(x) = \frac{\partial w_{1,k}}{\partial x_i}(x) w_2(x) + w_{1,k} \frac{\partial w_2}{\partial x_i}(x) \in L^1(\Omega).$$

This together with Theorem 2.1 gives $v \in W^{1,1}(\Omega, \mathbb{R}^n)$. Trace theory (see, e.g. Theorem 6.4.1 in [12]) implies that when $w \in W^{1,1}(\Omega)$ and Ω is of class $\mathcal{C}^{0,1}$, then the restriction of w to $\partial\Omega$ defined in the sense of trace operator belongs to $L^1(\partial\Omega, d\sigma)$. Moreover, formulae (4.1) are a simple consequence of density of $C^1(\overline{\Omega}, \mathbb{R}^n)$ in $W^{1,1}(\Omega, \mathbb{R}^n)$, the trace theorem and differentiation by parts. This finishes the proof of the statement. □

Lemma 4.2 (Compositions with Sobolev mappings). *Let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain of class $\mathcal{C}^{0,1}$, $p \geq 2$, $u \in W^{2,p/2}(\Omega) \cap L^\infty(\Omega)$. Then $\Phi_p(\nabla u) \in W^{1,1}(\Omega, \mathbb{R}^n)$ where $\Phi_p(\,\cdot\,)$ is given by* (2.1).

Proof. We verify that for $p \geq 2$ the function $\Phi_p(\lambda)$ is locally Lipschitz. According to Lemma 2.2 this implies that $(\Phi_p(\nabla u))_l$, where v_l denotes the l-th coordinate of vector v, is locally absolutely continuous for almost all lines parallel to the axes. Applying Theorem 2.1, we only have to check that partial derivatives of $(\Phi_p(\nabla u))_l$, computed almost everywhere, are integrable on Ω. We have for $v = \nabla u$ and almost every x such that $v(x) \neq 0$

$$\frac{\partial}{\partial x_k}(\Phi_p(v))_l = (p-2)|v|^{p-3} \frac{v}{|v|} \cdot \frac{\partial v}{\partial x_k} v_l + |v|^{p-2} \frac{\partial v_l}{\partial x_k}.$$

Consequently

$$\left| \frac{\partial}{\partial x_k}(\Phi_p(v)) \right| \leq (p-1)|v|^{p-2} |\nabla v| \quad \text{almost everywhere.} \tag{4.2}$$

In case $p > 2$ we note that $v \in L^p(\Omega)$ by the classical Gagaliardo–Nirenberg inequality (see [7] and [14]):

$$\|\nabla u\|_p \leq C \|u\|_\infty^{1/2} \|\nabla^{(2)} u\|_{p/2}^{1/2} + \|u\|_\infty. \tag{4.3}$$

As $p/(p-2) = (p/2)'$, we get $|v|^{p-2} \in L^{(p/2)'}(\Omega)$. We deduce from (4.2) that, for all $p \geq 2$, we have

$$\left| \frac{\partial}{\partial x_k}(\Phi_p(v)) \right| \in L^1(\Omega).$$

This finishes the proof of the lemma. □

5. Proof of Theorem 3.1

We start with the following lemma, which is the key tool for our further considerations.

Lemma 5.1 (Precise estimate involving boundary condition). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$, be the bounded domain of class $\mathcal{C}^{0,1}$, $2 \leq p < \infty$, $0 < B \leq \infty$, B, h, H_C, $\mathcal{T}_{h,C}$ be as in Definition 2.3, $u \in W^{2,p/2}(\Omega)$ and there exist a compact interval $[a,b] \subset (0,B)$ such that $u(x) \in [a,b]$ for almost every $x \in \Omega$. Then we have*

$$\begin{aligned}(I(\Omega))^{2/p} &:= \left(\int_\Omega |\nabla u(x)|^p h(u(x))\,dx\right)^{2/p}\\ &\leq (p-1+\sqrt{n-1})\left(\int_\Omega \left(\sqrt{|\nabla^{(2)}u(x)||\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx\right)^{2/p}\\ &\quad + (I(\Omega))^{2/p-1}\int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x) H_C(u(x))\,d\sigma(x).\end{aligned}$$

Moreover, the quantities $I(\Omega)$ and $\int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x)H_C(u(x))\,d\sigma(x)$ are finite.

Before we prove the lemma we recall the following simple fact.

Fact 5.2 ([11]). *For an arbitrary $n \times n$ matrix A and for an arbitrary unit vector $v \in \mathbb{R}^n$, we have*

$$|-v^tAv + \operatorname{tr} A| \leq \sqrt{n-1}|A|.$$

We are now ready to prove Lemma 5.1.

Proof of Lemma 5.1. We consider

$$w_1(x) := \Phi_p(\nabla u(x)), \qquad w_2(x) = H_C(u(x)).$$

Lemma 4.2 shows that $w_1 \in W^{1,1}(\Omega, \mathbb{R}^n)$. We deduce that the pair (w_1, w_2) obeys requirements in Lemma 4.1. Indeed, by (4.3) and the fact that $u \in W^{2,p/2}(\Omega) \cap L^\infty(\Omega)$, we get $u \in W^{1,p}(\Omega)$. Hence $|w_1| \leq |\nabla u|^{p-1} \in L^{p/(p-1)}(\Omega)$, so that $w_1 \in W^{1,1}(\Omega, \mathbb{R}^n) \cap L^{p'}(\Omega, \mathbb{R}^n)$. To verify the properties of w_2 we use Theorem 2.1 and Lemma 2.2. It allows us to deduce that $w_2 \in W^{1,p}(\Omega)$ if we show that the derivatives of w_2 computed almost everywhere belong to $L^p(\Omega)$. For this we observe that $\nabla(H_C(u(x))) = h(u(x))\nabla u(x)$. By our assumptions $h(u)$ is bounded, while $|\nabla u| \in L^p(\Omega)$. As $H_C(u)$ is bounded as well, therefore indeed $w_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Consequently $v := w_1 w_2 \in W^{1,1}(\Omega, \mathbb{R}^n)$ and

$$\int_\Omega \operatorname{div}\big(\Phi_p(\nabla u(x))H_C(u(x))\big)\,dx = \int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x)H_C(u(x))\,d\sigma(x) < \infty. \tag{5.1}$$

At the same time

$$L^1(\Omega) \ni \operatorname{div}\big(\Phi_p(\nabla u(x))H_C(u(x))\big) = \Delta_p u \cdot H_C(u) + |\nabla u|^p h(u), \tag{5.2}$$

where $\Delta_p u = \operatorname{div}(\Phi_p(\nabla u))$ is the p-Laplacian of u. Let us verify that $\Delta_p u \in L^1(\Omega)$. Direct computation gives:

$$\begin{aligned}\Delta_p u &= (p-2)|\nabla u|^{p-2} v^t[\nabla^{(2)} u]v + |\nabla u|^{p-2}\Delta u \\ &= |\nabla u|^{p-2}\big\{(p-1)v^t[\nabla^{(2)}u]v + (-v^t[\nabla^{(2)}u]v + \Delta u)\big\},\end{aligned}$$

almost everywhere, where $v = \nabla u/|\nabla u|$ if $\nabla u \neq 0$ and $v = 0$ otherwise, v^t is the transposition of v. Using Fact 5.2 with $A = \nabla^{(2)}u$, we get:

$$|\Delta_p u| \leq (p-1+\sqrt{n-1})|\nabla u|^{p-2}|\nabla^{(2)}u|. \tag{5.3}$$

As $|\nabla u|^{p-2} \in L^{(p/2)'}(\Omega)$, $|\nabla^{(2)}u| \in L^{p/2}(\Omega)$, it follows that r.h.s. in (5.3) is integrable in Ω, as well as $\Delta_p u \cdot H_C(u)$. Therefore also $|\nabla u|^p h(u)$ is integrable by (5.2). Using (5.1)–(5.3) and noting that $H_C = \mathcal{T}_{h,C} \cdot h$ we obtain

$$\begin{aligned} I(\Omega) &= \int_\Omega |\nabla u|^p h(u)dx \\ &\overset{(5.2)}{=} \int_\Omega \operatorname{div}\big(\Phi_p(\nabla u(x))H_C(u(x))\big)\,dx - \int_\Omega \Delta_p u(x) H_C(u(x))\,dx \\ &\overset{(5.1)}{\leq} \int_{\partial\Omega} \Phi_p(\nabla u(x)) \cdot n(x) H_C(u(x))\,d\sigma(x) \\ &\quad + \int_\Omega |\Delta_p u(x)||\mathcal{T}_{h,C} u(x)| h(u(x))\,dx \\ &\overset{(5.3)}{\leq} (p-1+\sqrt{n-1}) \int_\Omega |\nabla u|^{p-2}|\nabla^{(2)}u||\mathcal{T}_{h,C}(u)| \cdot h(u)\,dx \\ &\quad + \int_{\partial\Omega} \Phi_p(\nabla u(x)) \cdot n(x) H_C(u(x))\,d\sigma(x) =: A + B. \end{aligned} \tag{5.4}$$

Applying the Hölder inequality we get

$$\begin{aligned} A &\leq (p-1+\sqrt{n-1})\bigg(\int_\Omega |\nabla u|^p h(u)\,dx\bigg)^{1-2/p} \\ &\quad \cdot \bigg(\int_\Omega \Big(\sqrt{|\nabla^{(2)}u| \cdot |\mathcal{T}_{h,C}(u)|}\Big)^p h(u)\,dx\bigg)^{2/p} \\ &= (p-1+\sqrt{n-1}) I(\Omega)^{1-2/p} \\ &\quad \cdot \bigg(\int_\Omega \Big(\sqrt{|\nabla^{(2)}u| \cdot |\mathcal{T}_{h,C}(u)|}\Big)^p h(u)\,dx\bigg)^{2/p}. \end{aligned}$$

Now it suffices to divide inequality (5.4) by $I(\Omega)^{1-2/p}$, but for this we must be sure that $I(\Omega)$ is finite. This is clear because we have already realized that $u \in W^{1,p}(\Omega)$ and $h|_{[a,b]}$ is bounded. □

Remark 5.3. In the proof of Lemma 5.1 we have used the fact that $w_2(x) = H_C(u(x))$ belongs to $W^{1,p}(\Omega)$. Note that $\nabla w_2 = h(u)\nabla u$ almost everywhere and we have no information if $h(u)$ is bounded if we only know that $0 < u < B$ almost everywhere. Therefore we had to assume that $u(x) \in [a, b]$ almost everywhere. If we additionally know that h is bounded near zero, the assumption $0 < a < u(x) < b < \infty$ almost everywhere can be relaxed to $0 < u(x) < b < \infty$ almost everywhere.

Now we prove our main statement.

Proof of Theorem 3.1. We may assume that the right hand side in (3.1) is finite and the left hand side in (3.1) is nonzero, as otherwise the inequality holds trivially. We apply Lemma 5.1 to sequence of a subdomains Ω_k as in the property (u, Ω). For this, we note that u restricted to Ω_k obeys assumptions of Lemma 5.1 and we have:

$$(I(\Omega_k))^{2/p} \le (p - 1 + \sqrt{n-1})(D(\Omega_k))^{p/2} + I(\Omega_k)^{2/p-1} B(\Omega_k),$$

where

$$\begin{aligned} I(V) &:= \int_V |\nabla u|^p h(u)\, dx, \\ B(V) &:= \int_{\partial V} \Phi_p(\nabla u(x)) \cdot n(x) H_C(u(x))\, d\sigma(x), \\ D(V) &:= \int_V \left(\sqrt{|\nabla^{(2)} u| \cdot |\mathcal{T}_{h,C}(u)|}\right)^p h(u)\, dx. \end{aligned}$$

As $\lim_{k\to\infty} B(\Omega_k) \in (-\infty, 0]$, having any $s > 0$, we can assume that

$$(I(\Omega_k))^{2/p} \le (p - 1 + \sqrt{n-1})(D(\Omega))^{2/p} + (I(\Omega_k))^{2/p-1} s.$$

This implies that $I(\Omega_k)$ cannot converge to infinity as $k \to \infty$, so that $I(\Omega_k) \to I(\Omega) \in (0, \infty)$. Therefore

$$I(\Omega)^{p/2} \le (p - 1 + \sqrt{n-1}) D(\Omega)^{p/2} + I(\Omega)^{2/p-1} s,$$

which finishes the proof of the statement as we let s converge to zero. □

Following our arguments more carefully, one obtains a more precise statement, which deserves a special attention.

Theorem 5.4. *Let the assumptions of Theorem* 3.1 *be satisfied. Then we have*

$$\left(\int_\Omega |\nabla u(x)|^p h(u(x))\,dx\right)^{2/p} \le (p-2)\left(\int_\Omega \left(\sqrt{|(\Delta_\infty u(x))\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx\right)^{2/p} + \left(\int_\Omega \left(\sqrt{|\Delta u(x)\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx\right)^{2/p},$$

where $\Delta_\infty u$ is the infinity Laplacian:

$$\Delta_\infty u(x) := \begin{cases} v^t\nabla^{(2)}uv, \ \ \textit{where } v = \dfrac{\nabla u(x)}{|\nabla u(x)|} & \textit{if } \nabla u \neq 0, \\ 0 & \textit{if } \nabla u = 0. \end{cases}$$

In particular, when $p = 2$ we have

$$\int_\Omega |\nabla u(x)|^2 h(u(x))\,dx \le \int_\Omega |\Delta u(x)|\mathcal{T}_{h,C}(u(x))|h(u(x))\,dx.$$

Proof. We only have to modify the statement of Lemma 5.1, which reads as follows:

$$\begin{aligned}(I(\Omega))^{2/p} &:= \left(\int_\Omega |\nabla u(x)|^p h(u(x))\,dx\right)^{2/p} \\ &\le (p-2)\left(\int_\Omega \left(\sqrt{|\Delta_\infty u(x)||\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx\right)^{2/p} \\ &\quad + \left(\int_\Omega \left(\sqrt{|\Delta u(x)||\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx\right)^{2/p} \\ &\quad + (I(\Omega))^{2/p-1}\int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x)H_C(u(x))\,d\sigma(x).\end{aligned}$$

To obtain it, instead of inequality (5.3) we use:

$$|\Delta_p u| \le (p-2)|\nabla u|^{p-2}|\Delta_\infty u| + |\Delta u|$$

and substitute it to (5.4). From there the resulting inequality easily follows. □

6. Proof of Theorem 3.2

In this section we prove that if Ω is Lipschitz, we can find a sequence of Lipschitz boundary subdomains $\Omega_k \subset \Omega$ such that $\overline{\Omega}_k \subset \Omega$ and

$$\int_{\partial\Omega_k} \Phi_p(\nabla u(x))\cdot n(x)H_C(u(x))\,d\sigma(x) \xrightarrow{k\to\infty} \int_{\partial\Omega} \Phi_p(\nabla u(x))\cdot n(x)H_C(u(x))\,d\sigma(x).$$

Our main argument is based on the following lemma. Although it seems to be obvious to the specialists, its proof requires several delicate arguments. Therefore we present the proof in detail.

Lemma 6.1. *Assume that* $\Omega \subset \mathbb{R}^n$ *is bounded a domain of class* $\mathcal{C}^{0,1}$, $w \in W^{1,1}(\Omega, \mathbb{R}^n), T \in C(\overline{\Omega})$. *Then there exists a sequence of domains* $\Omega_k \subset \Omega$ *such that* $\overline{\Omega}_k \subset \Omega$, $\Omega_k \in \mathcal{C}^{0,1}$ *and*

$$\int_{\partial\Omega_k} w(x) \cdot n(x) T(x)\, d\sigma(x) \xrightarrow{k\to\infty} \int_{\partial\Omega} w(x) \cdot n(x) T(x)\, d\sigma(x).$$

Proof. The proof follows by steps.

Step 1. Let us fix $t > 0$, $x_0 \in \mathbb{R}^n$ and define the dilation mapping $\mathcal{A}_{x_0,t} \colon \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathcal{A}_{x_0,t}(x) = t(x - x_0) + x_0.$$

We observe that:

(1) $\mathcal{A}_{x_0,t} \circ \mathcal{A}_{x_0,1/t} = \mathrm{Id}$ (and so $\mathcal{A}_{x_0,1/t} \circ \mathcal{A}_{x_0,t} = \mathrm{Id}$), i.e. the mapping $\mathcal{A}_{x_0,1/t}$ is inverse to $\mathcal{A}_{x_0,1/t}$.

(2) When V is any bounded domain which is starshaped with respect to $x_0 \in V$ (i.e. for any $x \in V$ the segment $[x_0, x]$ is contained in V, see, e.g. [13, Section 1.1.8]), we have

$$\overline{\mathcal{A}_{x_0,t}(V)} \subset V \text{ when } t < 1 \text{ and } \overline{V} \subset \mathcal{A}_{x_0,t}(V) \text{ when } t > 1.$$

Indeed, property (1) and the fact that

$$\overline{\mathcal{A}_{x_0,t}(V)} \subset V \text{ when } t < 1 \tag{6.1}$$

follow by simple verification, while the property $\overline{V} \subset \mathcal{A}_{x_0,t}(V)$, when $t > 1$, follows from (6.1) applied to $\mathcal{A}_{x_0,t}(V)$ instead of V as we have:

$$\overline{V} = \overline{\mathcal{A}_{x_0,1/t} \circ \mathcal{A}_{x_0,t}(V)} = \overline{\mathcal{A}_{x_0,1/t}(\mathcal{A}_{x_0,t}(V))} \overset{(6.1)}{\subset} \mathcal{A}_{x_0,t}(V).$$

Step 2. We use the fact that every domain of class $C^{0,1}$ is a finite sum of starshaped domains (see Section 1.1.9 in [13]) and propose formulae for the approximating family of functions.

Let $\Omega_1, \ldots, \Omega_m \subseteq \Omega$ be domains starshaped with respect to balls $B_i = B(x_i, r_i)$ such that $\Omega = \bigcup_{i=1}^m \Omega_i$ and let us choose the appropriate smooth resolution of the unity $\{\phi_k\}_{k=1,\ldots,m}$ subordinated to $\{\Omega_i\}$. In particular we can assume that ϕ_i's are smooth and compactly supported in $\mathbb{R}^n$, moreover $\phi_i|_\Omega \equiv 0$ outside Ω_i, $0 \le \phi_i \le 1$ and $\sum_i \phi_i \equiv 1$. Then we define

$$\Omega_t := \bigcup_{k=1}^m \Omega_{k,t}, \quad \text{where } 0 < t < 1,\ \Omega_{k,t} = \mathcal{A}_{x_i,t}(\Omega_i).$$

For simplicity let us denote $\mathcal{A}_{x_i,t}$ by $A_{i,t}$. Clearly, $\Omega_t \in \mathcal{C}^{0,1}$ and $\overline{\Omega}_t \subset \Omega$, and we have

$$(6.2)\qquad \begin{aligned} I_t &:= \int_{\partial\Omega_t} w(x)\cdot n(x)T(x)\,d\sigma(x) \\ &= \sum_{i=1}^{m} \int_{\partial\Omega_t} \phi_i(x)w(x)\cdot n(x)T(x)\,d\sigma(x) =: \sum_{i=1}^{m} I_{i,t}, \\ I_{i,t} &= \int_{\partial\Omega_t\cap\Omega_i} \phi_i(x)w(x)\cdot n(x)T(x)\,d\sigma(x) =: \int_{\partial\Omega_t\cap\Omega_i} \phi_i(x)\Psi(x)\,d\sigma(x), \\ \Psi(x) &:= w(x)\cdot n(x)\cdot T(x) \quad \text{defined on } \partial\Omega_t, \text{ where } t\in(0,1]. \end{aligned}$$

By the very definition of Ω_t, any given $x \in \partial\Omega_t \cap \Omega_i$ up to σ measure zero comes from: (1) $A_{i,t}(y)$ for some $y \in \partial\Omega_i$, (2) $A_{j,t}(y)$ for some $y \in \partial\Omega_j$, where $j \neq i$. More precisely, we have $\partial\Omega_t \cap \Omega_i = T_{i,t} \cup R_{i,t} \cup S_{i,t}$, where

$$\begin{aligned} T_{i,t} &= A_{i,t}(K_i^0) \quad \text{where } K_i^0 = \partial\Omega_i \setminus \Omega \subseteq \partial\Omega_i, \\ R_{i,t} &= A_{i,t}(\partial\Omega_i \cap \Omega) \cap \partial\Omega_t, \\ S_{i,t} &= \bigcup_{j\neq i} A_{j,t}(\partial\Omega_j \setminus \partial\Omega_i) \cap \Omega_i \cap \partial\Omega_t =: \bigcup_{j\neq i} S_{i,j,t}. \end{aligned}$$

In the following steps we will successively analyze the integrals $\int_K \phi_i\Psi d\sigma$, where $K \in \{T_{i,t}, R_{i,t}, S_{i,t}\}$.

Step 3. We show that

$$\int_{T_{i,t}} \phi_i\Psi\,d\sigma \to \int_{K_i^0} \phi_i\Psi\,d\sigma \quad \text{as } t\to 1.$$

Using the change of variables formula, we obtain

$$(6.3)\qquad \int_{T_{i,t}} \phi_i\Psi\,d\sigma = \int_{A_{i,t}(K_i^0)} \phi_i\Psi\,d\sigma = \int_{K_i^0} (\phi_i\cdot\Psi)(A_{i,t}(x))A_{i,t}^*(d\sigma)(x),$$

where $A_{i,t}^*(d\sigma)(C) := \sigma(A_{i,t}^{-1}(C)) = t^{-(n-1)}\sigma(C)$ is the pullback of the $(n-1)$-dimensional Hausdorff measure under the dilation $A_{i,t}$. Moreover, we have

$$\phi_i \circ A_{i,t} \rightrightarrows \phi_i \quad \text{and} \quad T\circ A_{i,t} \rightrightarrows T \quad \text{as } t\to 1, \text{ uniformly on } \overline{\Omega},$$
$$\Psi(A_{i,t}(x)) = w(A_{i,t}(x))\cdot n_{A_{i,t}(x)}T(A_{i,t}(x)).$$

Applying the telescoping argument it suffices to show that

$$\int_{K_i^0} \phi_i(x)w(A_{i,t}(x))\cdot n_{A_{i,t}(x)}T(x)\,d\sigma(x) \xrightarrow{t\to 1} \int_{K_i^0} \phi_i(x)w(x)\cdot n_x T(x)\,d\sigma(x),$$

where $n_{A_{i,t}(x)}$ is the outer normal to $\partial\Omega_t$ at $A_{i,t}(x)$. We have

$$\begin{aligned} \mathcal{L}(t) &:= \left| \int_{\partial\Omega_i} \phi(x)(w(A_{i,t}(x))\cdot n_{A_{i,t}(x)} - w(x)\cdot n_x)T(x)\,d\sigma(x)\right| \\ &\le \|T\|_{\infty,\overline{\Omega}} \int_{\partial\Omega_i} |w(A_{i,t}(x))\cdot n_{A_{i,t}(x)} - w(x)\cdot n_x|\,d\sigma(x). \end{aligned}$$

Let us consider vectors n_x, the outer normal to $\partial\Omega_i$ at x, and $n_{A_{i,t}(x)}$, the outer normal to $\partial\Omega_t$ at $A_{i,t}(x)$. The second one is the same as the outer normal to $A_{i,t}(\partial\Omega_i)$ at $A_{i,t}(x)$. It is important to note that vectors n_x and $n_{A_{i,t}(x)}$ are the same. Indeed, when the vector w belongs to the tangent space to $\partial\Omega_i$ at x, then the vector $(DA_{i,t})w = (t \cdot \mathrm{Id})w = tw$ belongs to the tangent space at $A_{i,t}(x)$ to $A_{i,t}(\partial\Omega_i)$, because $DA_{i,t} = t\mathrm{Id}$ is the differential of the dilation $A_{i,t}$. This shows that tangent spaces $T_x\partial\Omega_i$ and $T_{A_{i,t}(x)}(A_{i,t}(\partial\Omega_i)) = T_{A_{i,t}(x)}(\partial\Omega_t)$ are the same for σ almost every $x \in K_i^0$. As the dilation mapping $A_{i,t}$ is conformal, it does not change the angles between vectors. Therefore normal spaces to the respective tangent spaces are the same. This together with an easy geometric observation shows that $n_x = n_{A_{i,t}(x)}$. We obtain

$$\begin{aligned}
\mathcal{L}(t) &\le \|T\|_{\infty,\overline{\Omega}} \int_{\partial\Omega_i} |w(A_{i,t}(x)) - w(x)|\, d\sigma(x) \\
&\le \|T\|_{\infty,\overline{\Omega}} \int_{\partial\Omega_i} \int_t^1 |Dw(A_{i,\tau}(x)), x - x_0)|\, d\tau\, d\sigma \\
&\le d_i \int_t^1 \int_{\partial\Omega_i} |Dw(A_{i,\tau}(x))|\, d\sigma\, d\tau \\
&\overset{\text{as in (6.3)}}{=} d_i \int_t^1 \tau^{n-1} \int_{A_{i,\tau}(\partial\Omega_i)} |Dw(x)|\, d\sigma\, d\tau \\
&\le d_i \int_t^1 \int_{A_{i,\tau}(\partial\Omega_i)} |Dw(x)|\, d\sigma\, d\tau \\
&\le d_i C \int_{\Omega_i \setminus \Omega_{i,t}} |Dw(y)|\, dy \xrightarrow{t\to 1} 0,
\end{aligned}$$

where $d_i = \|T\|_{\infty,\overline{\Omega}}\, \mathrm{diam}\Omega_i$. In the last line we have used the Coarea formulae (see [5, Theorem 3.2.12]), which can be applied as far as we prove that $A_{i,\tau}(\partial\Omega_i)$ are level sets of the scalar Lipschitz map $\Pi_2 \colon \overline{\Omega} \setminus \Omega_t \to \mathbb{R}$ constructed below. In the above notation C stands for the Lipschitz constant of Π_2.

It remains to explain that $A_{i,\tau}(\partial\Omega_i)$ are level sets of some Lipschitz map. For simplicity we can assume that $x_i = 0$, $i = 1$, so that $\Omega_i = \Omega$ is star-shaped with respect to some ball centered at the origin. We define the mapping $R \colon \partial\Omega \times [t,1] \to \overline{\Omega} \setminus \Omega_t$ by the formulae $R(y,\tau) = \tau y$, $\tau \in [t,1]$. An easy computation shows that R is Lipschitz. We will recognize that $R^{-1} = (\Pi_1, \Pi_2)$ where $\Pi_1 \colon \overline{\Omega} \setminus \Omega_t \to \partial\Omega$, $\Pi_2 \colon \overline{\Omega} \setminus \Omega_t \to [t,1]$, are Lipschitz. According to the lemma in Section 1.1.8 in [13], when $r = r(\omega)$ is the equation of $\partial\Omega$ in the spherical coordinates, then the function $r(\,\cdot\,)$ is Lipschitz. This implies that the mapping $\Pi_1 \colon \overline{\Omega} \setminus \Omega_t \to \partial\Omega$, given by the formula $\Pi_1(y) := r(y/\|y\|)$, is also Lipschitz, because $y \mapsto 1/\|y\|$ is positive for $y \in \overline{\Omega} \setminus \Omega_t$, and so the internal map $y \mapsto y/\|y\|$

is Lipschitz as well. We construct our final mapping

$$\Pi_2(y) := \frac{\|y\|}{\|\Pi_1(y)\|} : \overline{\Omega} \setminus \Omega_t \to [0,t]$$

and note that $\Pi_2(\,\cdot\,)$ is Lipschitz as well. Moreover, we have $\Pi_2(y) = \tau$ if and only if $y \in A_\tau(\partial\Omega)$. This closes our argument in the proof of Step 3.

Step 4. We show that

$$\int_{R_{i,t}} \phi_i \Psi \, d\sigma \to 0 \quad \text{as } t \to 1.$$

For this, we note that $R_{i,t} \subseteq A_{i,t}(\partial\Omega_i \cap \Omega) =: A_{i,t}(K_i^1)$. The considerations as in the previous step and with the notation as in (6.2), give

$$\bigg| \int_{R_{i,t}} \phi_i \Psi \, d\sigma \bigg| \leq \int_{A_{i,t}(K_i^1)} \phi_i |\Psi| \, d\sigma \xrightarrow{t\to 1} \int_{K_i^1} \phi_i |\Psi| \, d\sigma.$$

The latter one is zero because $\phi_i|_\Omega$ is zero outside Ω_i.

Step 5. We show that

$$\int_{S_{i,j,t}} \phi_i \Psi \, d\sigma \to 0 \quad \text{as } t \to 1,$$

recalling that $S_{i,j,t} := A_{j,t}(\partial\Omega_j \setminus \partial\Omega_i) \cap \Omega_i \cap \partial\Omega_t$. Let us decompose

$$\begin{aligned} S_{i,j,t} &= A_{j,t}(\partial\Omega_j \setminus \partial\Omega_i) \cap A_{j,t} \circ A_{j,t^{-1}}(\Omega_i \cap \partial\Omega_t) \\ &= A_{j,t}((\partial\Omega_j \setminus \partial\Omega_i) \cap W_{j,t}) = A_{j,t}(K_{i,j,t}), \end{aligned}$$

where

$$W_{j,t} := A_{j,t^{-1}}(\Omega_i \cap \partial\Omega_t), \qquad K_{i,j,t} := (\partial\Omega_j \setminus \partial\Omega_i) \cap W_{j,t}.$$

Proceeding as in the proof of Step 4, we verify that

$$\begin{aligned} \int_{S_{i,j,t}} \phi_i \Psi \, d\sigma &= t^{-(n-1)} \int_{K_{i,j,t}} (\phi_i \Psi)(A_{j,t}(x)) \, d\sigma \\ &\leq t^{-(n-1)} \bigg\{ \int_{K_{i,j,t}} |(\phi_i \Psi)(A_{j,t}(x)) - (\phi_i \Psi)(x)| \, d\sigma \\ &\quad + \int_{K_{i,j,t}} |(\phi_i \Psi)(x)| \, d\sigma \bigg\} =: t^{-(n-1)} \{X(t) + Y(t)\}, \end{aligned}$$

and $X(t) \to 0$ as $t \to 1$. We are left with the estimations of the second term. Observe that

$$|Y(t)| \leq \|T\|_{\infty,\overline{\Omega}} \int_{\partial\Omega_j \setminus \partial\Omega_i} |w(x)| \chi_{W_{j,t}}(x) \, d\sigma(x),$$

and we know that $w \in L^1(\partial\Omega_j \setminus \partial\Omega_i, d\sigma)$. Now the fact that $Y(t) \to 0$ as $t \to 1$ follows from Lebesgue's Dominated Convergence Theorem after we show that

$$f_t(x) := \chi_{W_{j,t}}(x) \xrightarrow{t\to 1} 0, \quad \text{for every } x \in \partial\Omega_j \setminus \partial\Omega_i. \tag{6.4}$$

Indeed, let $x \in \partial\Omega_j \setminus \partial\Omega_i$. If (6.4) were not true, we would find a sequence $t_k \to 1$ such that $f_{t_k}(x) = 1$ for every k. Then we would find $y_k \in \Omega_i \cap \partial\Omega_t$ such that $x = x_j + t_k^{-1}(y_k - x_j) = (1 - t_k^{-1})x_j + t_k^{-1}y_k$. Consequently

$$y_k = t_k(x - (1 - t_k^{-1})x_j) \xrightarrow{k\to\infty} x.$$

This is however impossible. Indeed, by the choice of y_k it would imply that $x \in \overline{\Omega}_i \cap \partial\Omega = \partial\Omega_i \cap \partial\Omega$, which contradicts the fact that $x \notin \partial\Omega_i$. This proves (6.4) and completes the arguments of Step 5, closing the whole proof of the statement. □

Proof of Theorem 3.2. We apply Lemma 6.1 to $w(x) = \Phi_p(x), T(x) = H_C(u(x))$ and use Lemma 4.2 and the fact that $u \in C(\overline{\Omega})$ to show that $w \in W^{1,1}(\Omega, \mathbb{R}^n)$ and $T \in C(\overline{\Omega})$. □

Remark 6.2. Note that under the assumptions in Theorem 3.2 we cannot deduce that $H_C(u)$ belongs to $W^{1,p}(\Omega)$. This is because $\nabla H_C(u) = h(u)\nabla u$. It is clear from (4.3) that $|\nabla u| \in L^p(\Omega)$, but $h(u)$ may be unbounded on Ω. Recalling Remark 5.3, even with some extra assumptions on the regularity of Ω and u, the proof of Theorem 3.2 cannot be obtained by simplification of the proof of Theorem 3.1 directly with avoiding the usage of the internal subdomains Ω_k.

7. Perspectives for applications

We are now to discuss the possible applications of our result. In the article [4] the authors described some simple model of electrostatic micromechanical systems (MEMS), which is reduced to the following problem:

$$\begin{cases} u = 0 & \text{on } \partial\Omega, \\ \Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \end{cases}$$

where $\lambda \geq 0$, $f \geq 0$, $u \in C^1(\overline{\Omega}) \cap W^{2,2}(\Omega)$, Ω is open and bounded. Using the analogous techniques as that in [9] (Section 7, Applications to nonlinear eigenvalue problems), we can deduce regularity of solutions, if one only has the inequality

$$\int_\Omega |\nabla u(x)|^p h(u(x))\,dx \leq C(n,p) \int_\Omega \left(\sqrt{|\Delta u(x)||\mathcal{T}_{h,C}(u(x))|}\right)^p h(u(x))\,dx. \tag{7.1}$$

The above inequality is stronger than our inequality (1.2), because instead of the Hesian on the r.h.s. we require now the Laplacian of u. Our next goal is to derive inequality (7.1), which will follow as a consequence of (1.2), then to present applications of the derived inequalities in mathematical models, including the ones inspired by the model of MEMS.

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