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A NOTE ON THE 3-D NAVIER–STOKES EQUATIONS

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Abstract. We consider the Navier–Stokes model in a bounded smooth domain $\Omega \subset \mathbb{R}^3$. Assuming a smallness condition on the external force f, which does not necessitate smallness of $\|f\|_{[L^2(\Omega)]^3}$-norm, we show that for any smooth divergence free initial data u_0 there exists $\mathcal{T} = \mathcal{T}(\|u_0\|_{[L^2(\Omega)]^3})$ satisfying

$$\mathcal{T} \to 0 \quad \text{as } \|u_0\|_{[L^2(\Omega)]^3} \to 0 \quad \text{and} \quad \mathcal{T} \to \infty \quad \text{as } \|u_0\|_{[L^2(\Omega)]^3} \to \infty,$$

and such that either a corresponding regular solution ceases to exist until $\mathcal{T}$ or, otherwise, it is globally defined and approaches a maximal compact invariant set $\mathbb{A}$. The latter set $\mathbb{A}$ is a global attractor for the semigroup restricted to initial velocities u_0 in a certain ball of fractional power space $X^{1/4}$ associated with the Stokes operator, which in turn does not necessitate smallness of the gradient norm $\|\nabla u_0\|_{[L^2(\Omega)]^3}$. Moreover, $\mathbb{A}$ attracts orbits of bounded sets in X through Leray–Hopf type solutions obtained as limits of viscous parabolic approximations.

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1. Introduction

Consider the 3-D Navier–Stokes equations

$$(1.1)\qquad \begin{cases} u_t = \nu\Delta u - (u\cdot\nabla)u - \nabla p + f(x), & t>0,\ x\in\Omega,\\ \operatorname{div} u = 0, & x\in\Omega,\\ u = 0, & x\in\partial\Omega,\\ u(0,x) = u_0(x), & x\in\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^3$ is a bounded domain having smooth boundary $\partial\Omega$, $p=p(t,x)$ denotes pressure, $u=(u^1(t,x),u^2(t,x),u^3(t,x))$ velocity, $f=(f_1(x),f_2(x),f_3(x))$ external force, and ν is a constant viscosity coefficient.

Denoting by P the projector onto the closure in $[L^2(\Omega)]^3$ of the space of smooth divergence free vector functions

$$X = \mathrm{cl}_{[L^2(\Omega)]^3}\{\phi\in[C_0^\infty(\Omega)]^3 : \operatorname{div}\phi = 0\}$$

(e.g. [13]) problem (1.1) can be rewritten in the form of an abstract Cauchy problem

$$(1.2)\qquad \begin{cases} u_t + Au = F_0(u) + Pf, & t>0,\\ u(0) = u_0, \end{cases}$$

with the linear and nonlinear parts given by

$$(1.3)\qquad A = -\nu P\Delta \quad\text{and}\quad F_0(u) = -P(u\cdot\nabla)u.$$

We recall that A is a negative generator of an analytic semigroup $\{e^{-At}: t\ge 0\}$ in X and F_0 satisfies

$$(1.4)\qquad \begin{aligned} \|F_0(v)\|_{X^{\min\{2\gamma-5/4,0\}}} &\le L_\gamma\|v\|^2_{X^\gamma},\\ \|F_0(v)-F_0(w)\|_{X^{\min\{2\gamma-5/4,0\}}} &\le L_\gamma\|v-w\|_{X^\gamma}(\|v\|_{X^\gamma}+\|w\|_{X^\gamma}), \end{aligned}$$

for all $v,w\in X^\gamma$, $\gamma\in(1/4,1)$, where

$$(1.5)\qquad X^\alpha = D(A^\alpha)\quad\text{for } \alpha>0 \quad\text{and}\quad X^\alpha = (X^{-\alpha})^*\quad\text{for } \alpha<0$$

(see [17, Lemma 2.2]). We also assume that $f\in[L^2(\Omega)]^3$.

We remark that for $\alpha\ge 0$ we use the norm $\|\cdot\|_{X^\alpha} = \|A^\alpha\cdot\|_X$ whereas for negative α we use the space X^α that can be viewed as the completion of $(X,\|A^{-\alpha}\cdot\|_X)$ (see [1, Chapter V]). We also denote by λ_1 the first positive eigenvalue of the Stokes operator in X and let

$$L := \max\{L_{5/8}, L_{1/2}\},$$

where $L_\gamma, \gamma = 5/8, 1/2$ are the Lipschitz constants in (1.4).

Usage of Hilbert's scale as in (1.5) allows us to derive the a priori bounds on the solution controlling constants precisely (cf. (A.2)).

It is known (see e.g. [19], [17], [29], [3], [11]) that for each $u_0 \in X^{1/4}$ there exists a unique function $u \in C([0,\tau_{u_0}), X^{1/4}) \cap C^1((0,\tau_{u_0}), X) \cap C((0,\tau_{u_0}), X^1)$ satisfying (1.2)–(1.3) until a certain $\tau_{u_0} \in (0,\infty]$. On the other hand recall (see [16], [11]) that (1.2)–(1.3) is also known to be locally well posed for some larger than $X^{1/4}$ phase space of initial data; namely for $u_0 \in P[L^3(\Omega)]^3 \subset X$ provided that $f \in [L^3(\Omega)]^3$. Therefore, we will assume below that $u_0 \in X$ and use the notion of *regular solution* in the sense that $u \in C([0,\tau_{u_0}), X) \cap C^1((0,\tau_{u_0}), X) \cap C((0,\tau_{u_0}), X^1)$ satisfies the first equation in (1.2) on $(0,\tau_{u_0})$ and $u(0) = u_0$.

It is unknown, in general, whether for each $u_0 \in X$ there is a regular solution u as above. However, such u does exists if, for example, $u_0 \in X^{1/4}$ or $u_0 \in P[L^3(\Omega)]^3$. Since at any positive time of its existence a regular solution u takes value in X^1, it can be extended to a maximal interval of existence similarly as in [11, Proof of Theorem 2.6]. Henceforth we will always assume that a regular solution u is maximally defined in the sense that it cannot be extended beyond τ_{u_0} as a function continuous in X^1 and continuously differentiable in X.

With this set-up we first formulate a condition on u_0 and f such that there exists a regular solution with $\tau_{u_0} = \infty$, that is, a global regular solution.

Here and below we assume that $f \in [L^2(\Omega)]^3$. We also denote by $B_{X^{1/4}}(r)$ an open ball in $X^{1/4}$ centered at zero with radius r.

Theorem 1.1. *Assume the smallness condition in $X^{-1/4}$-norm*

$$\|Pf\|_{X^{-1/4}} < \frac{\lambda_1^{1/2}}{4L}. \tag{1.6}$$

If there is a time $t_0 \geq 0$ such that a regular solution u of (1.2)–(1.3) *satisfies*

$$\|u(t_0)\|_{X^{1/4}} < \frac{1}{2L} + \frac{1}{2L}\sqrt{1 - 4\lambda_1^{-1/2} L \|Pf\|_{X^{-1/4}}} =: r_0 \tag{1.7}$$

then $u(t)$ remains in the open ball $B_{X^{1/4}}(r_0)$ after time t_0 for as long as it exists. Actually, u is a global regular solution which satisfies

$$\|u(t)\|^2_{X^{1/2}} \leq \|u(s)\|^2_{X^{1/2}} e^{-2\lambda_1 \mu(t_0)(t-s)} + C, \quad t \geq s > t_0, \tag{1.8}$$

for some constants $C = C(t_0, \lambda_1, f)$ and $\mu(t_0) > 0$.

Note that the *smallness conditions*, as for example in [29, part 3 of Theorem III.4.1, p. 132] or [8], require that $\|Pf\|_X$ is small whereas (1.6) allows for even large $\|Pf\|_X$ (see Appendix). On the other hand note that, for the Navier–Stokes equations in a general domain, including besides bounded domain Ω the case of the whole space $\mathbb{R}^3$ or half space $\mathbb{R}^3_+$, the existence of a unique solution corresponding to small divergence free initial data has been investigated in a wide range of function spaces. This includes, in particular, Lebesgue spaces [17], [18], Besov spaces [5], [7], BMO^{-1} [20], Marcinkiewicz spaces [6], and little Nikol'skiĭ spaces [2], which in turn do not exhaust a vast literature on the subject. Extensive list of references can be found in [2] as well as in the recent work [26].

In the setting of Theorem 1.1 we can estimate, similarly as in [11], the time of a possible blow up of a regular solution.

Theorem 1.2. *Assume* (1.6)*. Then*:

(a) *either a regular solution u of* (1.2)–(1.3) *ceases to exist until time $\mathcal{T}$ given by*

$$\|u_0\|_X^2 = \mathcal{T}\big(\lambda_1^{1/2} r_0^2 - \|Pf\|^2_{X^{-1/2}}\big)e^{\lambda_1 \mathcal{T}/2} \tag{1.9}$$

where r_0 is as in (1.7),

(b) *or, if its interval of existence contains $[0,\mathcal{T}]$, then u is a global regular solution.*

Remark 1.3. (a) Observe that a possible blow up time $\mathcal{T}$ of a regular solution to (1.2)–(1.3) is given in (1.9) in terms of X-norm of u_0, and satisfies

$$\mathcal{T} \to 0 \quad \text{as } \|u_0\|_X \to 0 \quad \text{and} \quad \mathcal{T} \to \infty \quad \text{as } \|u_0\|_X \to \infty.$$

Also quantity $\mathcal{T}$ behaves continuously with respect to $\|u_0\|_X$ and, using the inverse map g^{-1} of $g(\,\cdot\,) = (\,\cdot\,)e^{\lambda_1(\cdot)/2}$, we have that

$$\mathcal{T} = g^{-1}(c_f \|u_0\|_X^2)$$

with constant $c_f := \big(\lambda_1^{1/2} r_0^2 - \|Pf\|^2_{X^{-1/2}}\big)^{-1}$.

(b) This, in turn, gives evaluation of a time interval in which blow up of a regular solution could occur. Namely, if u is a regular solution with initial data u_0 lying on the sphere $S_X(R) = \{\chi \in X : \|\chi\|_X = R\}$ then the blow up, if occurs, must happen in the interval $(0, \mathcal{T}(R)] \subset (0, c_f R^2)$ where $\mathcal{T}(R) = g^{-1}(c_f R^2)$.

Hence we have the following result concerning global existence of regular solutions and its long time behavior.

Theorem 1.4. *Assume* (1.6)*. If u is a regular solution of* (1.2)–(1.3) *whose interval of existence contains $[0,\mathcal{T}]$, where $\mathcal{T}$ is given by* (1.9)*, then u is a global regular solution and, given any $r \in (1/L - r_0, r_0)$,*

$$u(t) \in B_{X^{1/4}}(r)$$

for all t large enough. Furthermore, $u(t)$ tends as $t \to \infty$ in $X^{1/4}$ topology to a maximal compact in $X^{1/4}$ and invariant for (1.2)–(1.3) *set*

$$\mathbb{A} \subset \mathrm{cl}_{X^{1/4}} B_{X^{1/4}}(1/L - r_0).$$

Note that in Theorem 1.4 '$\mathbb{A}$ being invariant for (1.2)–(1.3)' means that each global regular solution originating in $\mathbb{A}$ stays in $\mathbb{A}$ and for each time $t > 0$ any point of $\mathbb{A}$ is a value at time t of a global regular solution originating at a certain point of $\mathbb{A}$.

So far regular solutions of (1.2)–(1.3) were considered which are known to exist for a wide class of initial conditions; in particular for $u_0 \in X^\gamma$ with any $\gamma \geq 1/4$, or for $u_0 \in P[L^r(\Omega)]^3$ with any $r \geq 3$ when $f \in [L^r(\Omega)]^3$. That notion of solution can be generalized using for example *viscous parabolic approximations* as introduced in [22, 23] and considered recently in [12], [11]. In [12] such approximations were recovered and studied, and the convergence to a limiting global weak solution of (1.2) was discussed based on the technique due to J.-L. Lions [23]. The analysis of the 'hyper-viscous Navier–Stokes equation' was of independent interest in [4]. In [11], following the construction of a unique global solution

$$u^\varepsilon \in C([0,\infty), X) \cap C^1((0,\infty), X) \cap C((0,\infty), X^\sigma)$$

of

$$\begin{cases} u^\varepsilon_t + \varepsilon A^\sigma u^\varepsilon + A u^\varepsilon = F_0(u^\varepsilon) + Pf, & t > 0, \\ u^\varepsilon(0) = u_0 \in X, \end{cases} \tag{1.10}$$

with $\varepsilon > 0$, $\sigma \geq 5/4$, the authors obtained the result below concerning global solutions of (1.2)–(1.3) approximated by u^ε as $\varepsilon \to 0$.

Proposition 1.5 ([11, Lemma 3.2 and Theorem 3.6]). *Given any $u_0 \in X$, there exists a map u (not necessarily unique) such that, for every $T > 0$,*

(a) (estimates) $u \in L^\infty((0,T), X) \cap L^2((0,T), X^{1/2})$, $u_t \in L^{4/3}((0,T), X^{-\sigma/2})$,
(b) (continuity) $u\colon [0,T] \to X^{-\sigma/2}$ *is absolutely continuous, weakly continuous in X on $(0,T)$,*
(c) (Duhamel's formula)

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}(F_0(u(s)) + Pf)\, ds \quad \text{in } [0,T],$$

(d) (approximation) *u is a weak limit in $L^2((0,T), X^{1/2})$, weak-$*$ in the space $L^\infty((0,T), X)$, and strong in $L^2((0,T), X^{(1/2)^-})$ of a sequence $\{u^{\varepsilon_n}\}$ of solutions $u^{\varepsilon_n} \in C([0,\infty), X) \cap C^1((0,\infty), X) \cap C((0,\infty), X^\sigma)$ of (1.10) when $\varepsilon = \varepsilon_n \to 0$.*

The symbol $(1/2)^-$ above denotes any number strictly smaller than $1/2$, possibly arbitrarily close to $1/2$.

With the above described properties we get the following result.

Theorem 1.6. *Assume (1.6) and let $\mathbb{A}$ be the maximal compact invariant set for (1.2)–(1.3) in Theorem 1.4. Then any solution u resulting from Proposition 1.5 tends in $X^{1/4}$ to $\mathbb{A}$ as $t \to \infty$. Furthermore, given any set B bounded in X we have that if $\mathcal{U}(u_0)$ denotes a collection of all maps u corresponding to u_0 in Proposition 1.5 then*

$$\sup_{u_0 \in B} \sup_{u \in \mathcal{U}(u_0)} \inf_{a \in \mathbb{A}} \|u(t) - a\|_{X^{1/4}} \to 0 \quad \text{as } t \to \infty. \tag{1.11}$$

The above mentioned theorems are proved in the following Sections 2–5 using the differential inequalities technique. In Appendix we include some technicalities involving fractional powers which indicate that certain initial velocity and external force satisfying $\|Pf\|_{X^{-1/4}} < \lambda_1^{1/2}/(4L)$ and $\|u_0\|_{X^{1/4}} < r_0$ (cf. (1.6) and (1.7)) *can have* arbitrarily large norms $\|\nabla u_0\|_{[L^2(\Omega)]^3}$ and $\|f\|_{[L^2(\Omega)]^3}$, respectively.

2. Global regular solutions of (1.2)–(1.3) for small data

Our *standing hypothesis* throughout the paper is the smallness of $f \in [L^2(\Omega)]^3$ expressed by the condition (1.6), that is,

$$\|Pf\|_{X^{-1/4}} < \frac{\lambda_1^{1/2}}{4L}.$$

We prove that under the above smallness condition a regular solution which enters a certain ball in $X^{1/4}$ has 'life time' $\tau_{u_0} = \infty$.

Note that the notion of regular solution $u \in C([0,\tau_{u_0}), X) \cap C^1((0,\tau_{u_0}), X) \cap C((0,\tau_{u_0}), X^1)$ of (1.2)–(1.3) implicitly includes some more information about the time differentiation of u, namely,

$$u \in C^1((0,\tau_{u_0}), X^\xi) \quad \text{for each } \xi < 1.$$

Indeed, if u is a regular solution then for small enough $t_0 > 0$ we have that $v := u(\,\cdot\, + t_0)$ is, in particular, a γ-solution with $\gamma = 1/2$ in the sense of [11, Definition 2.1] (where we let $r = 2$, $\gamma = \alpha = 1/2$, $\beta = -1/4$ and $\sigma = 1$). Applying [11, Theorem 2.5] we conclude that v is a regular solution and that, for positive times in the interval of existence, v is a C^1 map with values in X^ξ for every $\xi < 1$. Since $t_0 > 0$ can be chosen arbitrarily close to zero, we thus have that u with values in X^ξ is C^1 on $(0,\tau_{u_0})$ for any $\xi < 1$ as stated above.

We proceed with the proof of Theorem 1.1.

Lemma 2.1. *Assume* (1.6). *If there is a time $t_0 \geq 0$ such that a regular solution u of* (1.2)–(1.3) *satisfies*

$$\|u(t_0)\|_{X^{1/4}} < r_0,$$

with r_0 as in (1.7)*, then $u(t)$ does not leave the open ball $B_{X^{1/4}}(r_0)$ after t_0.*

Proof. From (1.2)–(1.3), after multiplying by $A^{1/2}u$ we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/4}} + \|u\|^2_{X^{3/4}} \leq \langle F_0(u), A^{1/2}u\rangle + \langle Pf, A^{1/2}u\rangle,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the scalar product in $[L^2(\Omega]^3$. Applying (1.4) with $\gamma = 1/2$ (see [17]) and moments inequality

$$\|u\|_{X^{1/2}} \leq \|u\|^{1/2}_{X^{1/4}}\|u\|^{1/2}_{X^{3/4}}$$

(see [21, Chapter I, §5.9], also [27, p. 84]) we get

$$\langle F_0(u), A^{1/2}u\rangle = \langle A^{-1/4}F_0(u), A^{3/4}u\rangle \le L\|u\|^2_{X^{1/2}}\|u\|_{X^{3/4}} \le L\|u\|_{X^{1/4}}\|u\|^2_{X^{3/4}}.$$

On the other hand

$$\langle Pf, A^{1/2}u\rangle = \langle A^{-1/4}Pf, A^{3/4}u\rangle \le \|Pf\|_{X^{-1/4}}\|u\|_{X^{3/4}}.$$

Hence we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/4}} + \|u\|^2_{X^{3/4}} \le L\|u\|_{X^{1/4}}\|u\|^2_{X^{3/4}} + \|Pf\|_{X^{-1/4}}\|u\|_{X^{3/4}}. \tag{2.1}$$

We will present further analysis of the differential inequality (2.1), equivalently of (when $\|u\|_{X^{3/4}} \neq 0$)

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/4}} + \left(1 - L\|u\|_{X^{1/4}} - \frac{\|Pf\|_{X^{-1/4}}}{\|u\|_{X^{3/4}}}\right)\|u\|^2_{X^{3/4}} \le 0$$

which, after using the Poincaré inequality $\lambda_1^{1/2}\|u\|_{X^{1/4}} \le \|u\|_{X^{3/4}}$, implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/4}} + \left(1 - L\|u\|_{X^{1/4}} - \frac{\|Pf\|_{X^{-1/4}}}{\lambda_1^{1/2}\|u\|_{X^{1/4}}}\right)\|u\|^2_{X^{3/4}} \le 0.$$

Therefore we analyze

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/4}} + \left(\|u\|_{X^{1/4}} - L\|u\|^2_{X^{1/4}} - \lambda_1^{-1/2}\|Pf\|_{X^{-1/4}}\right)\frac{\|u\|^2_{X^{3/4}}}{\|u\|_{X^{1/4}}} \le 0. \tag{2.2}$$

The expression in the brackets on the left hand side of (2.2) is a second order polynomial of $s = \|u\|_{X^{1/4}}$,

$$g(s) = s - Ls^2 - \lambda_1^{-1/2}\|Pf\|_{X^{-1/4}}, \tag{2.3}$$

and it remains positive when $X^{1/4}$-norm of u belongs to the following interval I

$$I := \left(\frac{1}{2L} - \frac{1}{2L}\sqrt{1 - 4\lambda_1^{-1/2}L\|Pf\|_{X^{-1/4}}},\ \frac{1}{2L} + \frac{1}{2L}\sqrt{1 - 4\lambda_1^{-1/2}L\|Pf\|_{X^{-1/4}}}\right) = \left(\frac{1}{L} - r_0, r_0\right).$$

Therefore, if $u(t)$ at a certain time enters the open ball $B_{X^{1/4}}(r_0)$, where r_0 given in (1.7) is the largest root of the polynomial in (2.3), then u will never leave this ball after that time. □

Next we estimate the $X^{1/2}$-norm of the solution.

Lemma 2.2. *Assume* (1.6). *If there is a time $t_0 \ge 0$ such that the solution u of* (1.2)–(1.3) *satisfies*

$$\|u(t_0)\|_{X^{1/4}} < r_0,$$

where r_0 is as in (1.7)*, then the solution exists globally in time and satisfies for every $t \ge s > t_0$*

$$\|u(t)\|^2_{X^{1/2}} \le \|u(s)\|^2_{X^{1/2}}e^{-2\lambda_1\mu(t_0)(t-s)} + C(t_0, \lambda_1, f)$$

with $C(t_0,\lambda_1,f)$ *given by* (2.5) *and* $\mu(t_0) = L(r_0 - \max\{\|u(t_0)\|_{X^{1/4}}, 1/L - r_0\})$.

Proof. From (1.2)–(1.3), after multiplying by Au, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/2}} + \|u\|^2_{X^1} \le \langle F_0(u), Au\rangle + \langle Pf, Au\rangle.$$

Applying (1.4) with $\gamma = 5/8$ and moments inequality (A.2)

$$\|u\|_{X^{5/8}} \le \|u\|^{1/2}_{X^{1/4}}\|u\|^{1/2}_{X^1},$$

we get

$$\langle F_0(u), Au\rangle \le \|F_0(u)\|_X\|u\|_{X^1} \le L\|u\|^2_{X^{5/8}}\|u\|_{X^1} \le L\|u\|_{X^{1/4}}\|u\|^2_{X^1}.$$

On the other hand, for nonzero Pf,

$$\langle Pf, Au\rangle \le \frac{\|Pf\|^2_X}{2\Big(1-\sqrt{1-4\lambda_1^{-1/2}L\|Pf\|_{X^{-1/4}}}\Big)} + \bigg(\frac{1}{2} - \frac{1}{2}\sqrt{1-4\lambda_1^{-1/2}L\|Pf\|_{X^{-1/4}}}\bigg)\|u\|^2_{X^1}.$$

Hence, for r_0 as in (1.7), we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{X^{1/2}} + (Lr_0 - L\|u\|_{X^{1/4}})\|u\|^2_{X^1} \le \frac{\|Pf\|^2_X}{2(1-Lr_0)}. \tag{2.4}$$

Since $\|u(t_0)\|_{X^{1/4}} < r_0$, from (2.2) we have that

$$\|u(t)\|_{X^{1/4}} \le \max\bigg\{\|u(t_0)\|_{X^{1/4}}, \frac{1}{L} - r_0\bigg\} =: \theta(t_0) \quad \text{for } t > t_0,$$

and we observe that

$$L(r_0 - \|u\|_{X^{1/4}}) \ge L(r_0 - \theta(t_0)) =: \mu(t_0) > 0.$$

Consequently, using the Poincaré inequality, we obtain

$$\frac{d}{dt}\|u\|^2_{X^{1/2}} + 2\lambda_1\mu(t_0)\|u\|^2_{X^{1/2}} \le \frac{\|Pf\|^2_X}{1-Lr_0},$$

where the right hand side above is zero if Pf is zero. Due to Gronwall's inequality we get the result with

$$\begin{aligned} C(t_0,\lambda_1,f) &= \frac{\|Pf\|^2_X}{2(1-Lr_0)\lambda_1\mu(t_0)} \quad && \text{if } Pf \ne 0,\\ C(t_0,\lambda_1,f) &= 0 && \text{if } Pf = 0. \end{aligned} \tag{2.5}$$

□

Observe that due to Lemmas 2.1 and 2.2 the proof of Theorem 1.1 is complete.

3. Global regular solutions for small external forces

In this section we prove Theorem 1.2. Hence u denotes here a regular solution of (1.2)–(1.3) with a maximal existence time τ_{u_0}.

LEMMA 3.1. *Assume* (1.6). *If* $\|u(t_0)\|_{X^{1/4}} \geq r_0$ *for some* t_0, *where* r_0 *is as in* (1.7), *then*

$$\|u(t_0)\|_{X^{1/2}} \geq 2\|Pf\|_{X^{-1/2}}$$

and

(a) $\|u\|_X$ *is strictly decreasing as long as* $\|u\|_{X^{1/2}} > \|Pf\|_{X^{-1/2}}$ *after* t_0,
(b) $\|u\|_X^2 \leq e^{-\lambda_1(t-t_0)}\|u(t_0)\|_X^2$ *as long as* $\|u\|_{X^{1/2}} \geq 2\|Pf\|_{X^{-1/2}}$ *after* t_0.

PROOF. First note that, since (1.6) holds and $\|u(t_0)\|_{X^{1/4}} \geq r_0 \geq 1/(2L)$, using the Poincaré inequality we get

$$\|u(t_0)\|_{X^{1/2}} \geq \lambda_1^{1/4}\|u(t_0)\|_{X^{1/4}} \geq \frac{\lambda_1^{1/4}}{2L} \geq 2\lambda_1^{-1/4}\|Pf\|_{X^{-1/4}} \geq 2\|Pf\|_{X^{-1/2}}.$$

Then, observe from (1.2)–(1.3) that after multiplying by u, since $\langle F_0(u), u\rangle = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_X^2 \leq (\|Pf\|_{X^{-1/2}} - \|u\|_{X^{1/2}})\|u\|_{X^{1/2}}.$$

Hence $d\|u\|_X^2/dt$ is negative as long as $\|u\|_{X^{1/2}} > \|Pf\|_{X^{-1/2}}$ after t_0 and we get (a).

On the other hand, as long as $\|u\|_{X^{1/2}} \geq 2\|Pf\|_{X^{-1/2}}$ after t_0, we have that

$$\frac{d}{dt}\|u\|_X^2 \leq 2(\|Pf\|_{X^{-1/2}} - \|u\|_{X^{1/2}})\|u\|_{X^{1/2}} \leq -\|u\|_{X^{1/2}}^2 \leq -\lambda_1\|u\|_X^2, \quad t \geq t_0,$$

which leads to the exponential bound in (b). □

LEMMA 3.2. *Assume* (1.6). *If* $\tau_{u_0} < \infty$ *and* r_0 *is as in* (1.7) *then*

$$\|u(t)\|_{X^{1/4}} \geq r_0 \quad \textit{for all } t \in (0, \tau_{u_0}) \tag{3.1}$$

and $\|u(t)\|_X$*-norm is strictly decreasing, satisfying also the exponential bound*

$$\|u(t)\|_X \leq e^{-\lambda_1 t/2}\|u_0\|_X \quad \textit{for all } t \in [0, \tau_{u_0}). \tag{3.2}$$

PROOF. Observe first that, since u ceases to exist at time τ_{u_0}, we must have (3.1) or, otherwise, the solution is global due to Theorem 1.1. Having proved (3.1), from parts (a) and (b) of Lemma 3.1 we get the remaining conclusions. □

LEMMA 3.3. *Assume* (1.6). *If* $\tau_{u_0} < \infty$ *then* τ_{u_0} *is not larger than the time* $\mathcal{T}$ *satisfying*

$$\|u_0\|_X^2 = \frac{\mathcal{T}}{2}\big(\lambda_1^{1/2}r_0^2 - \|Pf\|_{X^{-1/2}}^2\big)e^{\lambda_1\mathcal{T}/2}. \tag{3.3}$$

Proof. Observe that after multiplying (1.2) by u and using the estimate

$$\langle Pf, u\rangle \le \frac{1}{2}\,\|Pf\|^2_{X^{-1/2}} + \frac{1}{2}\|u\|^2_{X^{1/2}}$$

we get

$$\frac{d}{dt}\,\|u\|^2_X + \|u\|^2_{X^{1/2}} \le \|Pf\|^2_{X^{-1/2}}. \tag{3.4}$$

Using Lemma 3.2 observe also that (3.1) and (3.2) hold.

Integrating (3.4) over any interval $[t/2, t] \subset (0, \tau_{u_0})$, and applying the mean value theorem for the integral on the left hand side and part (a) of Lemma 3.1 on the right side, we obtain

$$\begin{aligned}\frac{t}{2}\,\lambda_1^{1/2}\|u(s)\|^2_{X^{1/4}} &\le \frac{t}{2}\,\|u(s)\|^2_{X^{1/2}} \le \int_{t/2}^{t} \|u(z)\|^2_{X^{1/2}}\,dz \\ &< \left\|u\left(\frac{t}{2}\right)\right\|^2_X + \frac{t}{2}\,\|Pf\|^2_{X^{-1/2}} \le \|u_0\|^2_X e^{-\lambda_1 t/2} + \frac{t}{2}\,\|Pf\|^2_{X^{-1/2}},\end{aligned} \tag{3.5}$$

for some $s \in [t/2, t]$. Observe from (3.5) that

$$\frac{t}{2}\,\lambda_1^{1/2}\|u(s)\|^2_{X^{1/4}} < \|u_0\|^2_X e^{-\lambda_1 t/2} + \frac{t}{2}\,\|Pf\|^2_{X^{-1/2}} \quad \text{for some } s \in \left[\frac{t}{2}, t\right],$$

which holds for every positive $t < \tau_{u_0}$. Hence, if $\mathcal{T} < \tau_{u_0}$, where $\mathcal{T}$ is given by

$$\|u_0\|^2_X e^{-\lambda_1 \mathcal{T}/2} + \frac{\mathcal{T}}{2}\,\|Pf\|^2_{X^{-1/2}} = \frac{\mathcal{T}}{2}\,\lambda_1^{1/2} r_0^2,$$

then $\|u(s)\|_{X^{1/4}} < r_0$ for a certain $s \in [\mathcal{T}/2, \mathcal{T}]$ and hence u is a global regular solution (see Theorem 1.1).

We remark that due to (1.6) and the Poincaré inequality the expression in the brackets in (3.3) is positive. □

Due to Lemmas 3.1–3.3 the proof of Theorem 1.2 is complete.

4. Long time behavior of global regular solutions

In this section we complete the proof of Theorem 1.4. As previously u is a regular solution of (1.2)–(1.3). The notions used below, like *positive invariance*, *bounded dissipativeness*, *global attractor*,..., are understood in the standard formulation, e.g. as in [9].

Lemma 4.1. *Assume* (1.6). *If $\mathcal{T}$ satisfies* (3.3) *and* $[0, \mathcal{T}]$ *is contained in the interval of existence* $[0, \tau_{u_0})$ *of u, then u is a global regular solution and, given any $r \in (1/L - r_0, r_0)$,*

$$u(t) \in B_{X^{1/4}}(r) \tag{4.1}$$

for all t large enough. Also, $B_{X^{1/4}}(r)$ is positively invariant.

Proof. Observe first that if u is a regular solution of (1.2)–(1.3) and its interval of existence contains $[0,\mathcal{T}]$ then, due to Lemma 3.3, u is a global regular solution.

Note that it is not possible to have $\|u\|_{X^{1/4}} \geq r_0$ in $(0,\mathcal{T}]$. This is because via Lemma 3.1 we have $\|u\|_{X^{1/2}} \geq 2\|Pf\|_{X^{-1/2}}$ in $(0,\mathcal{T}]$ and $\|u(t)\|_X \leq e^{-\lambda_1 t/2}\|u_0\|_X$ in $[0,\mathcal{T}]$. Consequently, we can write (3.5) with $t=\mathcal{T}$ to deduce that $u(s) \in B_{X^{1/4}}(r_0)$ for some $s \in [\mathcal{T}/2,\mathcal{T}]$, which in turn contradicts that $\|u\|_{X^{1/4}} \geq r_0$ in $(0,\mathcal{T}]$. Therefore $\|u(t_0)\|_{X^{1/4}} < r_0$ for some $t_0 \in (0,\mathcal{T}]$ and now, given any $r \in (1/L - r_0, r_0)$, we observe that

- if $\|u(t_0)\|_{X^{1/4}} < r$ then inequality (2.2) ensures that $u \in B_{X^{1/4}}(r)$ for all $t \geq t_0$,
- if $r \leq \|u(t_0)\|_{X^{1/4}} < r_0$ then inequality (2.2) ensures that $\|u\|_{X^{1/4}}$ is strictly decreasing for $t \geq t_0$ until u enters inside $B_{X^{1/4}}(r)$ remaining then in this open ball for ever.

Hence we get (4.1) for all t large enough and using (2.2) we observe that $B_{X^{1/4}}(r)$ is positively invariant. □

Lemma 4.2. *Assume* (1.6) *and let* $u_0 \in B_{X^{1/4}}(r)$ *for some* $r \in (1/L - r_0, r_0)$. *Then a global regular solution u satisfies*

$$\|u(t)\|^2_{X^{1/2}} \leq \|u_0\|^2_X e^{-2\lambda_1 L(r_0-r)} + D \quad \text{for all } t \geq 1,$$

for some constant D independent of u_0.

Proof. Similarly as in Lemma 2.2 we show that, for any $t \geq s > 0$,

$$\|u(t)\|^2_{X^{1/2}} \leq \|u(s)\|^2_{X^{1/2}} e^{-2\lambda_1 L(r_0-r)(t-s)} + c(r,\lambda_1,f) \tag{4.2}$$

because, using the last sentence of Lemma 4.1, we have that (2.4) holds for $t>0$ with

$$L(r_0 - \|u\|_{X^{1/4}}) \geq L(r_0 - r) > 0.$$

On the other hand, integrating (3.4) over $(0,1)$ we infer that

$$\|u(s)\|^2_{X^{1/2}} \leq \int_0^1 \|u(z)\|^2_{X^{1/2}}\,dz \leq \|u_0\|^2_X + \|Pf\|^2_{X^{-1/2}} \quad \text{for some } s \in (0,1).$$

Combining the above estimates we get the result. □

Remark 4.3. Observe that the constant $c(r,\lambda_1,f)$ in (4.2) is proportional to $\|Pf\|_X$ which can be large (see Appendix). Hence constant D in Lemma 4.2 can be large.

Lemma 4.4. *Fix $r \in (1/L - r_0, r_0)$ and let V be the closure of $B_{X^{1/4}}(r)$ in $X^{1/4}$, so that V is a complete metric space with metric inherited from $X^{1/4}$. Then V is positively invariant for* (1.2)–(1.3) *and there is a semigroup of global regular solutions on V which has a compact global attractor $\mathbb{A}$.*

Proof. Inequality (2.2) ensures that V is positively invariant for the semigroup of global regular solutions which exists on V via Theorem 1.1. Since V is bounded and positively invariant, this semigroup is bounded dissipative. It is also asymptotically compact, because if $t_n \to \infty$ and $\{u_{0n}\} \subset V$ then, due to Lemma 4.2, almost all elements of the sequence $\{u(t_n, u_{0n})\}$ are bounded in $X^{1/2}$ by the constant independent of n, whereas the embedding $X^{1/2} \hookrightarrow X^{1/4}$ is compact. This proves the existence of a global attractor in a complete metric space V. □

Lemma 4.5. *$\mathbb{A}$ in Lemma 4.4 satisfies the inclusion*

$$\mathbb{A} \subset \mathrm{cl}_{X^{1/4}} B_{X^{1/4}}\left(\frac{1}{L} - r_0\right)$$

and $\mathbb{A}$ is a maximal compact invariant set for (1.2)–(1.3).

Proof. To prove maximality, let $\widetilde{\mathbb{A}}$ be compact and invariant for (1.2)–(1.3). Then, by invariance, $\widetilde{\mathbb{A}}$ consists of points belonging to full regular solutions $u \in C^1(\mathbb{R}, X) \cap C(\mathbb{R}, X^1)$. Then no point of $\widetilde{\mathbb{A}}$ can stay outside $B_{X^{1/4}}(r_0)$ because each such point is a value, at arbitrarily chosen time, of a certain regular solution lying on $\widetilde{\mathbb{A}}$ whereas, as shown in the proof of Lemma 4.1, any regular solution enters $B_{X^{1/4}}(r_0)$ not later than at time $\mathcal{T}$, which in turn depends on the X-norm of initial data and thus can be chosen uniform for all points of $\widetilde{\mathbb{A}}$. Hence $\widetilde{\mathbb{A}} \subset B_{X^{1/4}}(r_0)$ which, due to compactness of $\widetilde{\mathbb{A}}$, implies that $\widetilde{\mathbb{A}} \subset B_{X^{1/4}}(r)$ for some $r \in (1/L - r_0, r_0)$.

Now, if there is a point v_0 of $\widetilde{\mathbb{A}}$ such that $v_0 \in B_{X^{1/4}}(r) \backslash \mathrm{cl}_{X^{1/4}} B_{X^{1/4}}(1/L - r_0)$ then a full regular solution u on $\widetilde{\mathbb{A}}$ with $u(0) = v_0$ satisfies, for each $n \in \mathbb{N}$ and any $t \in [-n, 0]$, the inequality (2.2), which leads to the estimate

$$\|u(0)\|_{X^{1/4}} \leq \|u(-n)\|_{X^{1/4}} - c_* n, \tag{4.3}$$

where $c_* = \lambda_1 \inf_{\|v_0\|_{X^{1/4}} \leq s \leq r} g(s)$ and g is as in (2.3). This however leads to a contradiction because $\|u(-n)\|_{X^{1/4}}$ in (4.3) is bounded by r uniformly for $n \in \mathbb{N}$. Therefore we have $\widetilde{\mathbb{A}} \subset \mathrm{cl}_{X^{1/4}} B_{X^{1/4}}(1/L - r_0) \subset V$ and we conclude that $\widetilde{\mathbb{A}} \subset \mathbb{A}$ because $\widetilde{\mathbb{A}}$ is invariant and, due to Lemma 4.4, $\widetilde{\mathbb{A}}$ is attracted by $\mathbb{A}$. □

Due to Lemmas 4.1–4.5, Theorem 1.4 is proved. We remark that any global regular solution u of (1.2)–(1.3) must enter $V = B_{X^{1/4}}(r)$ (see (4.1)) and hence u tends to $\mathbb{A}$, because $\mathbb{A}$ attracts points of V.

5. Asymptotics of solutions obtained via viscous parabolic approximations

In this section we prove Theorem 1.6. The following two lemmas will be useful in that.

Lemma 5.1. *Assume* (1.6). *Given any* $r \in (\max\{1/(4L), 1/L - r_0\}, r_0)$ *and* $u_0 \in X$, *there is a time* $T = T(\|u_0\|_X, \|Pf\|_{X^{-1/2}})$ *specified in* (5.7) *such that, for each* $\varepsilon > 0$, *the solution* $u^\varepsilon \in C([0,\infty), X) \cap C^1((0,\infty), X) \cap C((0,\infty), X^\sigma)$ *of* (1.10) *satisfies the estimate*

$$\|u^\varepsilon(t)\|^2_{X^{1/2}} \le \left(\|u_0\|^2_X + \left(1 + \frac{1}{\lambda_1}\right)\|Pf\|^2_{X^{-1/2}}\right) e^{-2\lambda_1 L(r_0 - r)(t-T-1)} + c \tag{5.1}$$

for all $t \ge T+1$ *with constant* $c = \|Pf\|^2_X/(2\lambda_1 L(r - r_0)(1 - Lr_0))$.

Proof. Recall that, due to [11, Lemma 3.2], the problem (1.10) has a unique solution u^ε belonging to $C([0,\infty), X) \cap C^1((0,\infty), X) \cap C((0,\infty), X^\sigma)$ and that a similar argument as in the second paragraph of Section 2 ensures that $u^\varepsilon \in C^1((0, \tau_{u_0}), X^\xi)$ for each $\xi < \sigma$.

Observe that after multiplying (1.10) by u^ε we get, similarly as in Lemma 3.3,

$$\frac{d}{dt}\|u^\varepsilon\|^2_X + \|u^\varepsilon\|^2_{X^{1/2}} \le \|Pf\|^2_{X^{-1/2}}, \tag{5.2}$$

where using the Poincaré inequality we have

$$\lambda_1\|u^\varepsilon\|^2_X \le \|u^\varepsilon\|^2_{X^{1/2}}, \tag{5.3}$$

and using (1.6) we also have

$$\|Pf\|^2_{X^{-1/2}} \le \frac{\lambda_1^{1/2}}{(4L)^2}. \tag{5.4}$$

Given any $t > 0$ and using (5.2), (5.3) we thus get

$$\|u^\varepsilon(t)\|^2_X \le \|u_0\|^2_X e^{-\lambda_1 t} + \frac{1}{\lambda_1}\|Pf\|^2_{X^{-1/2}}. \tag{5.5}$$

On the other hand, by (5.5),

$$\begin{aligned} \frac{t}{2}\lambda_1^{1/2}\|u^\varepsilon(s)\|^2_{X^{1/4}} &\le \frac{t}{2}\|u^\varepsilon(s)\|^2_{X^{1/2}} \\ &\le \int_{t/2}^t \|u^\varepsilon(z)\|^2_{X^{1/2}}\,dz \le \left\|u^\varepsilon\left(\frac{t}{2}\right)\right\|^2_X + \frac{t}{2}\|Pf\|^2_{X^{-1/2}} \\ &\le \|u_0\|^2_X e^{-\lambda_1 t/2} + \left(\frac{1}{\lambda_1} + \frac{t}{2}\right)\|Pf\|^2_{X^{-1/2}} \end{aligned} \tag{5.6}$$

for a certain $s \in [t/2, t]$. Furthermore, choosing $r \in (1/(4L), r_0)$, there exists a unique positive time T such that

$$\frac{2}{T\lambda_1^{1/2}}\|u_0\|^2_X e^{-\lambda_1 T/2} + \left(\frac{2}{T\lambda_1^{3/2}} + \frac{1}{\lambda_1^{1/2}}\right)\|Pf\|^2_{X^{-1/2}} = r^2, \tag{5.7}$$

because due to (5.4) when time T increases from 0 to ∞ the left hand side in (5.7) decreases from ∞ to a value smaller or equal $1/(4L)^2$. We thus conclude from (5.6) and (5.7) that if $r \in (1/(4L), r_0)$ then

$$\|u^\varepsilon(s)\|_{X^{1/4}} \le r < r_0 \quad \text{for a certain } s \in [T/2, T]. \tag{5.8}$$

Now, proceeding as in Lemma 2.1, we obtain that u^ε satisfies

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|^2_{X^{1/4}} + g(\|u^\varepsilon\|_{X^{1/4}})\frac{\|u^\varepsilon\|^2_{X^{3/4}}}{\|u^\varepsilon\|_{X^{1/4}}} \le 0, \tag{5.9}$$

where g is a second order polynomial given in (2.3) which remains positive in the interval $I = (1/L - r_0, r_0)$. As a consequence of (5.9), if

$$r \in \left(\max\left\{\frac{1}{4L}, \frac{1}{L} - r_0\right\}, r_0\right),$$

inequality (5.8) strengthens to

$$\sup_{t\ge s}\|u^\varepsilon(t)\|_{X^{1/4}} \le r < r_0 \quad \text{for a certain } s \in [T/2, T]. \tag{5.10}$$

Proceeding next as in the proof of Lemma 2.2 (see, in particular, (2.4)) we get

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon\|^2_{X^{1/2}} + \lambda_1(Lr_0 - L\|u^\varepsilon\|_{X^{1/4}})\|u^\varepsilon\|^2_{X^{1/2}} \le \frac{\|Pf\|^2_X}{2(1-Lr_0)}. \tag{5.11}$$

Formulas (5.10), (5.11) together imply that

$$\|u^\varepsilon(t)\|^2_{X^{1/2}} \le \|u^\varepsilon(\xi)\|^2_{X^{1/2}} e^{-2\lambda_1 L(r_0-r)(t-\xi)} + \frac{\|Pf\|^2_X}{2\lambda_1 L(r-r_0)(1-Lr_0)} \tag{5.12}$$

for all $t \ge \xi \ge T$, whereas integrating (5.2) over $[T, T+1]$, we have

$$\|u^\varepsilon(\xi)\|^2_{X^{1/2}} \le \int_T^{T+1}\|u^\varepsilon(z)\|^2_{X^{1/2}}\,dz \le \|u^\varepsilon(T)\|^2_X + \|Pf\|^2_{X^{-1/2}} \tag{5.13}$$

for some $\xi \in [T, T+1]$. Connecting (5.12), (5.13) and (5.5), we obtain (5.1). □

Lemma 5.2. *A map u in Proposition* 1.5 *satisfies*

$$\|u\|_{L^\infty((T+1,\infty),X^{1/2})} \le \left(\|u_0\|^2_X + \left(1 + \frac{1}{\lambda_1}\right)\|Pf\|^2_{X^{-1/2}} + c\right)^{1/2}, \tag{5.14}$$

where T and c are as in Lemma 5.1. *Consequently, after choosing time $t_0 > T+1$ (arbitrarily close to $T+1$) we have that $u(\cdot + t_0)$ is a global regular solution of* (1.2)–(1.3) *with u_0 replaced by $u(t_0)$.*

Proof. Note that (5.14) follows from (5.1) because u in Proposition 1.5 is then a weak-$*$ limit of $\{u^{\varepsilon_n}\}$ in $L^\infty((T+1,\infty), X^{1/2})$. Hence we can choose $t_0 \ge T+1$, arbitrarily close to $T+1$, such that $u(t_0) \in X^{1/2}$ and, due to Proposition 1.5, $v(t) := u(t+t_0)$ satisfies Duhamel's formula

$$v(t) = e^{-At}v_0 + \int_0^t e^{-A(t-s)}(F_0(v(s)) + Pf)\,ds, \quad t > 0,$$

and $v(0) = u(t_0) =: v_0 \in X^{1/2}$. Then v is a mild γ-solution with $\gamma = 1/2$ of

$$v_t + Av = F_0(v) + Pf, \quad t > 0, \qquad v(0) = v_0 \tag{5.15}$$

in the sense of [11, Definition 2.1] (where we let $r = 2$, $\gamma = \alpha = 1/2$, $\beta = -1/4$ and $\sigma = 1$) and applying [11, Theorem 2.5] we conclude that v is a regular solution of (5.15). □

PROOF OF THEOREM 1.6. Since due Lemma 5.2 a map u in Proposition 1.5 regularizes after time t_0 (which is bigger but can be chosen arbitrarily close to $T + 1$), $u(\cdot + t_0)$ is a global regular solution of (1.2)–(1.3) with u_0 replaced by $u(t_0)$. Consequently, Theorem 1.4 yields the convergence of $u(\cdot + t_0)$ to $\mathbb{A}$.

Concerning (1.11) observe first that, due to (5.10), u in Proposition 1.5 can also be viewed as weak-$*$ limit of $\{u^{\varepsilon_n}\}$ in $L^\infty((T,\infty), X^{1/4})$ and, consequently,

$$\|u\|_{L^\infty((T,\infty),X^{1/4})} \leq r < r_0. \tag{5.16}$$

Also note that the time T in Lemma 5.1, and hence in (5.16), can be chosen uniform for u_0 in bounded subsets of X. Using this and recalling regularization property of u after time $T + 1$ described in Lemma 5.2 we infer that, if B is bounded in X, then for t_0 large enough the set $\mathcal{B}(t_0) := \{u(t_0) : u \in \mathcal{U}(u_0), u_0 \in B\}$ is contained in $B_{X^{1/4}}(r)$ with $r < r_0$. Due to Lemma 4.4, $\mathcal{B}(t_0)$ is thus attracted by $\mathbb{A}$ with respect to the Hausdorff semidistance in $X^{1/4}$, which in turn leads to (1.11). □

6. Closing remarks

The solutions u in Proposition 1.5 are not known to be unique. Hence, in general, the problem falls into a formalism of multivalued semigroups or processes as reported in the recent monograph [24]. We do not pursue this here using instead regularizing properties of the equation and thus observe that $\mathbb{A}$ in Theorem 1.4 attracts orbits of bounded subsets of X under solutions of Proposition 1.5. Note that in [11, Lemmas 3.2–3.4] the solutions of Proposition 1.5 are shown to be Leray–Hopf type solutions.

For zero external force any solution in Proposition 1.5 satisfies, for all t large enough, the estimate

$$\|u\|_{X^{1/2}} \leq \frac{1}{\sqrt{t}}\,\|u_0\|_X e^{-\lambda_1 t/2}$$

(see [11, Lemma 4.3] from which this follows). In particular, when $f = 0$, the set $\mathbb{A}$ in Theorem 1.4 consists of a single trivial equilibrium.

It remains an open problem whether $\mathbb{A}$ is a single equilibrium under the assumption on f as in Theorem 1.1. This in turn is true when f is small in some better than $X^{-1/4}$-norm (see [10]; also [9, Theorem 8.3.1]).

We finally remark that a discussion can be found in the literature, smallness of which quantities is sufficient for the global in time solvability of the 3-D Navier–Stokes equations (see e.g. [25]).

Appendix A. Some technicalities concerning fractional powers in Hilbert spaces

Recall that A is a strictly positive self-adjoint operator in X (see [14], [28]). Observe also that, due to Rellich's theorem and characterization of fractional power spaces in [15], A^{-1} is compact. Hence A in X has an increasing to infinity sequence of positive eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$ and the corresponding eigenfunctions, denoted $e_1, \ldots, e_n, \ldots$, constitute an orthonormal basis for X. In particular, $Pe_j = e_j$ for any $j \in \mathbb{N}$ and *fractional powers* of A can be expressed as

$$A^\alpha \phi = \sum_{n=1}^{\infty} \lambda_n^\alpha \langle \phi, e_n \rangle e_n, \quad \phi \in X^\alpha, \tag{A.1}$$

where $\langle \cdot\,, \cdot \rangle$ denotes a scalar product in X (see [27, Sections 3.9 and 3.10]). We also mention the *moments inequality* in Hilbert scale of fractional powers corresponding to A in X

$$\|\phi\|_{X^\beta} \leq \|\phi\|_{X^\gamma}^{(\beta-\alpha)/(\gamma-\alpha)} \|\phi\|_{X^\alpha}^{(\gamma-\beta)/(\gamma-\alpha)} \tag{A.2}$$

valid for any $\alpha < \beta < \gamma$ and $\phi \in X^\gamma$ (see [21], [27]). Note that the corresponding moments inequality in Banach spaces requires a constant $c > 0$ on the right hand side (see again [21]). On the other hand note that after using (A.1) the following Poincaré inequality holds

$$\lambda_1^{\beta-\alpha} \|\phi\|_{X^\alpha} \leq \|\phi\|_{X^\beta} \quad \text{for } \beta > \alpha.$$

Let us mention that

$$\|u_0\|_{X^{1/2}} = \nu^{1/2} \|\nabla u_0\|_{[L^2(\Omega)]^3} \quad \text{for} \quad u_0 \in X^{1/2} \tag{A.3}$$

because, since P and fractional powers of A are selfadjoint in X, we have

$$\begin{aligned}\|u_0\|_{X^{1/2}}^2 &= \langle A^{1/2} u_0, A^{1/2} u_0 \rangle = \langle A u_0, u_0 \rangle \\ &= -\nu \langle \Delta u_0, P u_0 \rangle = -\nu \langle \Delta u_0, u_0 \rangle = \nu \|\nabla u_0\|_{[L^2(\Omega)]^3}^2,\end{aligned}$$

which gives (A.3).

Finally, observe that if for arbitrarily fixed $r_1, r_2 > 0$ and $\varepsilon \in (0, 1/4)$ we consider particular data in (1.2) of the form

$$u_0 = r_1 \lambda_j^{-1/4-\varepsilon} e_j, \qquad f = r_2 \lambda_j^\varepsilon e_j,$$

then, as $j \to \infty$, we have on the one hand

$$\|u_0\|_{X^{1/4}} = r_1 \lambda_j^{-\varepsilon} \to 0, \qquad \|f\|_{X^{-1/4}} = r_2 \lambda_j^{-1/4+\varepsilon} \to 0$$

so that (1.6)–(1.7) hold and, on the other hand,

$$\begin{aligned}\|\nabla u_0\|_{[L^2(\Omega)]^3} &= \nu^{-1/2} \|u_0\|_{X^{1/2}} = \nu^{-1/2} r_1 \lambda_j^{1/4-\varepsilon} \to \infty, \\ \|f\|_{[L^2(\Omega)]^3} &= \|f\|_X = r_2 \lambda_j^\varepsilon \to \infty.\end{aligned}$$

We thus emphasize that while (1.6) and (1.7) necessitate that some "weaker" norms of u_0 and f respectively are small, then the "stronger" norms $\|\nabla u_0\|_{[L^2(\Omega)]^3}$ and $\|f\|_{[L^2(\Omega)]^3}$ can be very large.

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