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TOPOLOGICAL SHADOWING AND THE GROBMAN–HARTMAN THEOREM

Piotr Zgliczyński

Abstract. We give geometric proofs for the Grobman–Hartman theorem for diffeomorphisms and ODEs. Proofs use covering relations and cone conditions for maps and isolating segments and cone conditions for ODEs. We establish topological versions of the Grobman–Hartman theorem as the existence of some semiconjugaces.

1. Introduction

The goal of this paper is to give a new geometric proof of the Grobman–Hartman theorem [8]–[10] for diffeomorphisms and ODEs in finite dimension. By the 'geometric proof' we understand the proof which works in the phase space of the system under consideration and uses concepts of qualitative geometric nature.

We focus on the global version of the Grobman–Hartman theorem, which in the case maps states that, if $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a hyperbolic linear isomorphism and if $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$g(x) = Ax + h(x),$$

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where $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bounded C^1 function such that $\|Dh(x)\| \leq \varepsilon$ for $x \in \mathbb{R}^n$, then if ε is sufficiently small, A and g are conjugated by a continuous homeomorphism.

There are many of proofs of the Grobman–Hartman theorem in the literature. An exemplary geometric proof can be found in the Katok–Hasselblatt book [13]. This proof is placed in the context of the hyperbolicity, where it is shown that dynamics of g is hyperbolic on the whole $\mathbb{R}^n$ and the conjugating homeomorphism is constructed geometrically by considering the stable and unstable leaves of points to obtain the linearizing coordinates.

The other family of proofs of the Grobman–Hartman theorem uses tools from the functional analysis. The standard functional analysis proof [15], [17], [2], which is now a textbook proof (see for example [1], [5], [16], [24]), studies the conjugacy problem in some abstract Banach space of maps. The original proof by P. Hartman [10]–[12] also belongs to this category, but it lacks the simplicity of the contemporary approach, because to solve the conjugacy problem Hartman required first to introduce new coordinates which straighten the invariant manifolds of the hyperbolic fixed point. The standard functional analysis proof, whose idea apparently comes from the paper by Moser [14] (see also [15], [17]), in a current form is a straightforward application of the Banach contraction principle. The whole effort is to choose the correct Banach space and a contraction, whose fixed point will give us the conjugacy.

In this paper we would like to give a new geometric proof the global version of the Grobman–Hartman theorem (Theorem 2.1). The geometric idea behind our approach can be seen as shadowing of δ-pseudo orbit, with δ not small. This is accomplished using covering relations and the cone condition [26], [25] in the case of diffeomorphisms; and for ODEs the notion of the isolating segment [18]–[21], [23] and the cone conditions have been used. Compared to the geometric proof in [13] we stress more the topological aspects. As the byproduct of our approach we obtain two topological variants of the Grobman–Hartman theorem:

- if we drop the assumption that $\|Dh\|$ is small, but demand instead that g is a homeomorphism, then we show that there exists a semiconjugacy between A and g, see Theorem 2.2 for the precise statement,
- if we drop the assumption that $\|Dh\|$ is small, then we show that there exists a semiconjugacy between A restricted to the unstable subspace and g, see Theorem 2.3 for the precise statement.

Let us comment about the relation between our proofs of the theorem for maps and for ODEs. The standard approach would be to derive the ODE case from the map case, by considering the time shift by one time unit and then arguing that we can obtain from it the conjugacy for all times (see [10], [15]–[17]). Here, we provide a proof for ODEs which is independent from the map case in order to illustrate the power of the concept of the isolating segment with the aim to obtain a clean ODE-type proof. For another clean ODE-type proof using the functional analysis type arguments see [6].

Regarding the regularity of the conjugating homeomorphism in the global Grobman–Hartman theorem, there is a nice argument of geometric nature in Katok and Hasselblatt's book [13] that shows that this conjugacy has to be Hölder. However, no effort is made there to estimate the Hölder exponent. Using our shadowing ideas we estimate this exponent. We obtain the same estimate for the Hölder exponent as in the works by Barreira and Valls [2], Belitskiĭ [3], Belitskiĭ and Rayskin [4] which apparently are the best results in this direction (see [2] and references given there). In these papers the functional analysis type of reasoning was used and results are valid also in the Banach space.

The organization of this paper can be described as follows. Section 2 contains the geometric proof of the global version of the Grobman–Hartman theorem. In Section 3 we show the Hölder regularity of the conjugacy in the Grobman–Hartman theorem.

Section 4 contains a geometric proof of the Grobman–Hartman theorem for flows, which is independent from the proof for maps.

At the end of this paper we included two appendices, which contain relevant definitions and theorems about the covering relations and the isolating segments.

1.1. Notation. If $A \in \mathbb{R}^{d_1 \times d_2}$ is a matrix, then by A^t we will denote its transpose. By $B(x,r)$ we will denote the open ball centered at x and with radius r. For maps depending on some parameters $h\colon P \times X \to X$ by $h_p\colon X \to X$ we will denote the map $h_p(x) = h(p,x)$.

In this note we will work in $\mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^s$. According to this decomposition we will often represent points $z \in \mathbb{R}^n$ as $z = (x,y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. On $\mathbb{R}^n$ we assume the standard scalar product $(u,v) = \sum_i u_i v_i$. This scalar product induces the norm on $\mathbb{R}^u$ and $\mathbb{R}^s$. We will use the following norm on $\mathbb{R}^n$, $\|(x,y)\|_{\max} = \max(\|x\|, \|y\|)$ and we will usually drop the subscript max.

We will use also projections π_x and π_y, so that $\pi_x(x,y) = x$ and $\pi_y(x,y) = y$.

2. Global version of the Grobman–Hartman theorem for maps

In this section we will give a geometric proof of the Grobman–Hartman theorem for maps and its topological variants.

We will consider a map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$g(z) = A(z) + h(z).$$

We will have the following set of assumptions on A and h, which we will refer to as the *standard conditions*:

- We assume that $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism of the form

$$A(x,y) = (A_u x, A_s y),$$

where $n = u+s$, $A_u : \mathbb{R}^u \to \mathbb{R}^u$ and $A_s : \mathbb{R}^s \to \mathbb{R}^s$ are linear isomorphisms such that

$$\|A_u x\| \geq c_u \|x\|, \quad c_u > 1 \qquad \text{for all } x \in \mathbb{R}^u,$$
$$\|A_s y\| \leq c_s \|y\|, \quad 0 < c_s < 1, \quad \text{for all } y \in \mathbb{R}^s.$$

- The map $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists M such that

$$\|h(x)\| \leq M, \quad \text{for all } x \in \mathbb{R}^n.$$

Theorem 2.1. *Assume the standard conditions. Additionally assume that h is of class C^1 and such that there exists ε such that*

$$\|Dh(x)\| \leq \varepsilon, \quad \textit{for all } x \in \mathbb{R}^n.$$

Then there exists $\varepsilon_0 = \varepsilon_0(A) > 0$ such that if $\varepsilon < \varepsilon_0(A)$, then there exists a homeomorphism $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\sigma \circ g = A \circ \sigma. \tag{2.1}$$

Comment. Observe that there is no bound on M, we also do not assume that $h(0) = 0$.

In the next theorem we drop the assumption that h is C^1 with small Dh, but we keep the requirement that g is an injective map.

Theorem 2.2. *Assume the standard conditions. Assume the map g is an injection. Then there exists a continuous surjective map $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\sigma \circ g = A \circ \sigma. \tag{2.2}$$

In the next theorem we will drop the assumption that g is an injection. Then we no longer have a unique full trajectory through a point for the map h.

Theorem 2.3. *Assume the standard conditions. Then there exists a continuous surjective map $\sigma_u\colon \mathbb{R}^n \to \mathbb{R}^u$ such that*

$$\sigma_u \circ g = A_u \circ \sigma_u. \tag{2.3}$$

Before the proofs of Theorems 2.1–2.3 we need first to develop some technical tools. The basic steps and constructions used in the proofs are given in Section 2.5. We invite the reader to jump first to this section to see the overall picture of the proofs and then consult other more technical sections when necessary.

We will use the following notation: $g_\lambda = A+\lambda h$ for $\lambda \in [0,1]$. In this notation we have $g = g_1$.

2.1. g_λ are onto.

LEMMA 2.4. *Assume standard conditions. Then g_λ are onto, i.e.*

$$g_\lambda(\mathbb{R}^n) = \mathbb{R}^n.$$

PROOF. The surjectivity of g_λ follows from the following observation: a bounded continuous perturbation of a linear isomorphism is a surjection – the proof is based on the local Brouwer degree (see for example Appendix in [26] for the definition and properties). Details are as follows.

For a fixed $y \in \mathbb{R}^n$ we consider the equation $y = g_\lambda(x)$, which is equivalent to $x + \lambda A^{-1}h(x) = A^{-1}y = \widetilde{y}$. Let us define a map

$$F_\lambda(x) = x + \lambda A^{-1}h(x) - \widetilde{y}. \tag{2.4}$$

Observe that if $\|x - \widetilde{y}\| > \|A^{-1}\|M$, then $F_\lambda(x) \neq 0$.

This shows that $\deg(F_\lambda, B(\widetilde{y}, \|A^{-1}\|M), 0)$ (the local Brouwer degree of F_λ on the set $B(\widetilde{y}, \|A^{-1}\|M)$ at 0) is defined and

$$\deg(F_\lambda, B(\widetilde{y}, \|A^{-1}\|M), 0) = \deg(F_0, B(\widetilde{y}, \|A^{-1}\|M), 0),$$

for all $\lambda \in [0, 1]$. But for $\lambda = 0$ we have $F_0(x) = x - \widetilde{y}$. Hence

$$\deg(F_0, B(\widetilde{y}, \|A^{-1}\|M, 0) = 1.$$

Therefore $F_\lambda(x) = 0$ has a solution for any $\widetilde{y} \in \mathbb{R}^n$. □

2.2. g_λ are homeomorphisms under assumptions of Theorem 2.1. The following lemma can be found for example in [17, Lemma 1] and [24, Proposition II.2].

LEMMA 2.5. *Let A and h be as in Theorem* 2.1. *Let* $\varepsilon_1(A) = 1/\|A^{-1}\| > 0$. *If $\varepsilon < \varepsilon_1(A)$, then g_λ is a homeomorphism and g_λ^{-1} is Lipschitz.*

PROOF. The surjectivity follows from Lemma 2.4. The injectivity is obtained as follows:

$$\begin{aligned}\|g_\lambda(z_1) - g_\lambda(z_2)\| &= \|Az_1 + \lambda h(z_1) - (Az_2 + \lambda h(z_2))\| \\ &\geq \|A(z_1) - A(z_2)\| - \lambda\|h(z_1) - h(z_2)\| \\ &\geq \frac{1}{\|A^{-1}\|}\|z_1 - z_2\| - \varepsilon\|z_1 - z_2\| = \left(\frac{1}{\|A^{-1}\|} - \varepsilon\right)\|z_1 - z_2\|.\end{aligned}$$

From the above formula it follows also that

$$\|z_1 - z_2\| \geq \left(\frac{1}{\|A^{-1}\|} - \varepsilon\right)\|g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)\|.$$

Therefore

$$\|g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)\| \leq \left(\frac{1}{\|A^{-1}\|} - \varepsilon\right)^{-1}\|z_1 - z_2\|.$$ □

2.3. Cone condition for g_λ under assumptions of Theorem 2.1. Throughout this subsection we work under assumptions of Theorem 2.1.

We will establish the cone condition for g_λ using the approach from [25], where the cones are defined in terms of a quadratic form. Let Q be a quadratic form in $\mathbb{R}^n = \mathbb{R}^u \times \mathbb{R}^s$ given by $Q(x,y) = (x,x) - (y,y)$. Our goal is to show the following *cone condition*: for sufficiently small $\eta > 0$ it holds

$$Q(Az_1 - Az_2) > (1 \pm \eta) Q(z_1 - z_2), \quad z_1, z_2 \in \mathbb{R}^n,\ z_1 \neq z_2. \tag{2.5}$$

This will be established in Lemma 2.7.

By Q we will also denote a matrix such that $Q(z) = z^t Q z$. In our case $Q = \left[\begin{smallmatrix} I_u & 0 \\ 0 & -I_s \end{smallmatrix}\right]$, where $I_u \in \mathbb{R}^{u\times u}$ and $I_s \in \mathbb{R}^{s \times s}$ are the identity matrices.

Lemma 2.6. *For $0 \leq \eta \leq \min(c_u^2 - 1, 1 - c_s^2)$ the matrix $A^t Q A - (1 \pm \eta) Q$ is positive definite.*

Proof. Easy computations show that

$$A^t Q A = \begin{pmatrix} A_u^t A_u & 0 \\ 0 & A_s^t A_s \end{pmatrix}.$$

Hence for any $z = (x,y) \in \mathbb{R}^u \times \mathbb{R}^s \setminus \{0\}$ we have

$$\begin{aligned} z^t(A^t Q A - (1 \pm \eta) Q) z &= x^t A_u^t A_u x - (1 \pm \eta) x^2 + (1 \pm \eta) y^2 - y^t A_s^t A_s y \\ &= (A_u x, A_u x) - (1 \pm \eta) x^2 + (1 \pm \eta) y^2 - (A_s y, A_s y) \\ &\geq (c_u^2 - 1 - \eta) x^2 + (1 - \eta - c_s^2) y^2 > 0, \end{aligned}$$

if $c_u^2 - 1 > \eta$ and $1 - c_s^2 > \eta$. □

Lemma 2.7. *There exists $\varepsilon_0(A) > 0$ such that if $0 \leq \varepsilon < \varepsilon_0(A)$, then there exists $\eta \in (0,1)$ such that for any $\lambda \in [0,1]$ the following* cone condition *holds:*

$$Q(g_\lambda(z_1) - g_\lambda(z_2)) > (1 \pm \eta) Q(z_1 - z_2), \quad \text{for all } z_1, z_2 \in \mathbb{R}^n,\ z_1 \neq z_2. \tag{2.6}$$

Proof. We have

$$Q(g_\lambda(z_1) - g_\lambda(z_2)) = (z_1 - z_2)^t (D(z_1,z_2)^t Q D(z_1,z_2))(z_1 - z_2),$$
$$D(z_1, z_2) = \int_0^1 Dg_\lambda(t(z_1 - z_2) + z_2)\, dt.$$

Let

$$C(z_1,z_2) = \int_0^1 Dh(t(z_1 - z_2) + z_2)\, dt,$$

then $D(z_1,z_2) = A + \lambda C(z_1,z_2)$. Observe that $\|C(z_1,z_2)\| \leq \varepsilon$.

From Lemma 2.6 it follows that $A^t Q A - (1 \pm \eta) Q$ is positive definite for sufficiently small $\eta > 0$. Let us fix such η. Since being a positively defined symmetric matrix is an open condition, there exists $\varepsilon_0(A) > 0$ such that the matrix

$$(A + \lambda C)^t Q(A + \lambda C) - (1 \pm \eta)Q \tag{2.7}$$

is positive definite for any $\lambda \in [0,1]$ and $C \in \mathbb{R}^{n\times n}$ satisfying $\|C\| \le \varepsilon_0$. □

From Lemma 2.5 it follows that for any $\lambda \in [0,1]$ and any point z we can define a full orbit for g_λ through this point, i.e. $g_\lambda^k(z)$ makes sense for any $k \in \mathbb{Z}$.

LEMMA 2.8. *Assume that* $\varepsilon < \min(\varepsilon_0(A), \varepsilon_1(A))$ *is from Lemmas* 2.7 *and* 2.5. *Let* $\lambda \in [0,1]$. *If* $z_1, z_2 \in \mathbb{R}^n$ *and* β *are such that*

$$\|g_\lambda^k(z_1) - g_\lambda^k(z_2)\| \le \beta, \quad \textit{for all } k \in \mathbb{Z}, \tag{2.8}$$

then $z_1 = z_2$.

PROOF. The proof is by contradiction. Let $z_1 \ne z_2$. Either $Q(z_1 - z_2) \ge 0$ or $Q(z_1 - z_2) < 0$.

Let us consider first the case $Q(z_1 - z_2) \ge 0$. By the cone condition (Lemma 2.7) we obtain, for any $k > 0$,

$$Q(g_\lambda(z_1) - g_\lambda(z_2)) > Q(z_1 - z_2) \ge 0,$$
$$\|\pi_x(g_\lambda^k(z_1) - g_\lambda^k(z_2))\| \ge Q(g_\lambda^k(z_1) - g_\lambda^k(z_2)) > (1+\eta)^{k-1} Q(g_\lambda(z_1) - g_\lambda(z_2)).$$

Therefore $g_\lambda^k(z_1) - g_\lambda^k(z_2)$ is unbounded. This contradicts (2.8).

Now we consider the case $Q(z_1 - z_2) < 0$. The cone condition (Lemma 2.7) applied to the inverse map gives, for any $k > 0$,

$$Q(z_1 - z_2) > (1-\eta) Q(g_\lambda^{-1}(z_1) - g_\lambda^{-1}(z_2)) > (1-\eta)^k Q(g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)).$$

Therefore we obtain

$$-Q(g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)) > \frac{1}{(1-\eta)^k}\,(-Q(z_1 - z_2)).$$

Therefore $g_\lambda^{-k}(z_1) - g_\lambda^{-k}(z_2)$ is unbounded. This contradicts (2.8). □

2.4. Covering relations. We assume that the reader is familiar with the notion of an h-set and covering relation [26]. For the convenience of the reader we recall these notions in Appendix A.

DEFINITION 2.9. For any $z \in \mathbb{R}^n$, $\alpha > 0$ we define an h-set (with a natural structure) $N(z, \alpha) = z + \overline{B}_u(0, \alpha) \times \overline{B}_s(0, \alpha)$.

The following theorem follows immediately from Theorem A.5 in Appendix A.

THEOREM 2.10. *Assume that we have a bi-infinite chain of covering relations*

$$N_i \overset{f}{\Longrightarrow} N_{i+1}, \quad i \in \mathbb{Z}.$$

Then there exists a sequence $\{z_i\}_{i\in\mathbb{Z}}$ *such that* $z_i \in N_i$ *and* $f(z_i) = z_{i+1}$.

The following lemma plays the crucial role in the construction of ρ from Theorem 2.1.

Lemma 2.11. *Assume the standard conditions. Let*

$$\widehat{\alpha} = \widehat{\alpha}(A, M) = \max\left(\frac{2M}{c_u - 1}, \frac{2M}{1 - c_s}\right).$$

Then, for any $\alpha > \widehat{\alpha}$, $\lambda_1, \lambda_2 \in [0,1]$ *and* $z \in \mathbb{R}^n$ *it holds that*

$$N(z, \alpha) \overset{A+\lambda_1 h}{\Longrightarrow} N((A + \lambda_2 h)(z), \alpha). \tag{2.9}$$

Proof. Let us fix $z \in \mathbb{R}^n$ and let us define the homotopy $H\colon [0,1] \times \overline{B}_u(0,\alpha) \times \overline{B}_u(0,\alpha) \to \mathbb{R}^n$ as follows:

$$H_t((x,y)) = (A_u x, (1-t)A_s y) + (1-t)\lambda_1 h(z + (r,y)) + (A + t\lambda_2 h)(z).$$

We have

$$\begin{aligned} H_0(x,y) &= A(z + (x,y)) + \lambda_1 h(z + (x,y)) = (A + \lambda_1 h)(z + (x,y)),\\ H_1(x,y) &= (A + \lambda_2 h)(z) + (A_u x, 0). \end{aligned}$$

For the proof of Lemma 2.11 it is enough to show the following conditions, for all $t, \lambda_1, \lambda_2 \in [0,1]$:

$$\|\pi_x(H_t(x,y) - (A + \lambda_2 h)(z))\| > \alpha, \quad (x,y) \in (\partial B_u(0,\alpha)) \times \overline{B}_s(0,\alpha), \tag{2.10}$$

$$\|\pi_y(H_t(x,y) - (A + \lambda_2 h)(z))\| < \alpha, \quad (x,y) \in \overline{B}_u(0,\alpha) \times \overline{B}_s(0,\alpha). \tag{2.11}$$

First we establish (2.10). We have

$$\begin{aligned} \|\pi_x(H_t((x,y)) &- (A + \lambda_2 h)(z))\| \\ &= \|A_u x + (1-t)\lambda_1 \pi_x h(z + (x,y)) + (t-1)\lambda_2 \pi_x h(z)\| \\ &\geq \|A_u x\| - \|h(z + (x,y))\| - \|h(z)\| \geq c_u \alpha - 2M. \end{aligned}$$

Hence (2.10) holds if the following inequality is satisfied:

$$(c_u - 1)\alpha > 2M. \tag{2.12}$$

Now we deal with (2.11). We have

$$\begin{aligned} \|\pi_y(H_t(x,y) &- (A + \lambda_2 h)(z))\| \\ &= \|(1-t)A_s y + (1-t)\lambda_1 \pi_y h(z + (x,y)) + (t-1)\lambda_2 \pi_y h(z)\| \\ &\leq \|A_s y\| + \|h(z + (x,y))\| + \|h(z)\| \leq c_s \alpha + 2M. \end{aligned}$$

Hence (2.11) holds if the following inequality is satisfied:

$$(1 - c_s)\alpha > 2M. \tag{2.13}$$

Hence it is enough to take $\widehat{\alpha} = \max(2M/(c_u - 1), 2M/(1 - c_s))$. $\square$

2.5. Proofs of Theorems 2.1 and 2.2. Under assumptions of Theorem 2.1 it follows from Lemma 2.5 that g is a homeomorphism. Under assumptions of Theorem 2.2 it follows from Lemma 2.4 that g is a homeomorphism. Therefore we can talk of the full orbit of g passing through an arbitrary point $z \in \mathbb{R}^n$.

We define $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ and a multivalued map ρ from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$. In the case of the proof of Theorem 2.1 we will show that ρ is single-valued, i.e. $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$.

1. Let us fix $\alpha > \widehat{\alpha}$, where $\widehat{\alpha}$ is obtained in Lemma 2.11.
2. For $z \in \mathbb{R}^n$, from Lemma 2.11 with $\lambda_1 = 1$ and $\lambda_2 = 0$ we have a bi-infinite chain of covering relations

$$\cdots \overset{g}{\Longrightarrow} N(A^{-2}z,\alpha) \overset{g}{\Longrightarrow} N(A^{-1}z,\alpha) \overset{g}{\Longrightarrow} N(z,\alpha) \overset{g}{\Longrightarrow} N(Az,\alpha) \overset{g}{\Longrightarrow} N(A^2z,\alpha) \overset{g}{\Longrightarrow} N(A^3z,\alpha) \overset{g}{\Longrightarrow} \cdots \tag{2.14}$$

3.1. In the context of the proof of Theorem 2.1: from Theorem 2.10 and Lemma 2.8 it follows that the chain of covering relations (2.14) defines a unique point, which we will denote by $\rho(z)$, such that

$$g^k(\rho(z)) \in N(A^k(z),\alpha), \quad k \in \mathbb{Z}. \tag{2.15}$$

3.2. In the context of the proof of Theorem 2.2: from Theorem 2.10 it follows that (2.14) defines for each $z \in \mathbb{R}^n$ a non-empty set $\rho(z)$ such that for each $z_1 \in \rho(z)$ it holds

$$g^k(z_1) \in N(A^k(z),\alpha), \quad k \in \mathbb{Z}. \tag{2.16}$$

4. For $z \in \mathbb{R}^n$, from Lemma 2.11 with $\lambda_1 = 0$ and $\lambda_2 = 1$ we have a bi-infinite chain of covering relations

$$\cdots \overset{A}{\Longrightarrow} N(g^{-2}(z),\alpha) \overset{A}{\Longrightarrow} N(g^{-1}(z),\alpha) \overset{A}{\Longrightarrow} N(z,\alpha) \overset{A}{\Longrightarrow} N(g(z),\alpha) \overset{A}{\Longrightarrow} N(g^2(z),\alpha) \overset{A}{\Longrightarrow} N(g^3(z),\alpha) \overset{A}{\Longrightarrow} \cdots \tag{2.17}$$

5. From Theorem 2.10 and the hyperbolicity of A it follows that the chain of covering relations (2.17) defines a unique point, which we will denote by $\sigma(z)$, such that

$$A^k(\sigma(z)) \in N(g^k(z),\alpha), \quad k \in \mathbb{Z}. \tag{2.18}$$

The following lemma shows that in the context of Theorem 2.1 the map ρ in fact does not depend on α.

LEMMA 2.12. *Under assumptions of Theorem* 2.1 *assume that*

$$\varepsilon < \min(\varepsilon_0(A),\varepsilon_1(A)).$$

Assume $\widehat{\alpha} < \beta$. *Let* $z \in \mathbb{R}^n$. *If* z_1 *is such that*

$$g^k(z_1) \in N(A^k z, \beta), \quad k \in \mathbb{Z}, \tag{2.19}$$

then $z_1 = \rho(z)$.

Proof. Observe that (2.15) and (2.19) imply

$$\|g^k(z_1) - g^k(\rho(z))\| \leq \alpha + \beta.$$

The assertion follows from Lemma 2.8. □

The following lemma a consequence of the hyperbolicity of A.

Lemma 2.13. *Under assumptions of Theorem* 2.2*, let* $\widehat{\alpha} < \beta$ *and* $z \in \mathbb{R}^n$. *If* z_1 *is such that*

$$A^k(z_1) \in N(g^k(z), \beta), \quad k \in \mathbb{Z}, \tag{2.20}$$

then $z_1 = \sigma(z)$.

Lemma 2.14. *Under assumptions of Theorem* 2.2*,* σ *is continuous.*

Proof. Assume that $z_j \to \overline{z}$, we will show that the sequence $\{\sigma(z_j)\}_{j\in\mathbb{N}}$ is bounded and each converging subsequence converges to $\sigma(\overline{z})$.

We can assume that $\|z_j - \overline{z}\| < \alpha$. Then, since $\|\sigma(z_j) - z_j\| < \alpha$, we obtain

$$\|\sigma(z_j) - \overline{z}\| < 2\alpha.$$

Hence $\{\sigma(z_j)\}_{j\in\mathbb{N}}$ is bounded.

Now let us take a convergent subsequence, which we will again index by j, hence $z_j \to \overline{z}$ and $\sigma(z_j) \to w$ for $j \to \infty$, where $w \in \mathbb{R}^n$. We will show that $w = \sigma(\overline{z})$. This implies that $\sigma(z_i) \to \sigma(\overline{z})$.

Let us fix $k \in \mathbb{Z}$. From the continuity of $z \mapsto g^k(z)$ it follows, that there exists j_0 such for $j \geq j_0$,

$$\|g^k(z_j) - g^k(\overline{z})\| < \alpha. \tag{2.21}$$

Since by the definition of σ we have $A^k(\sigma(z_j)) \in N(g^k(z_j), \alpha)$, (2.21) implies that $\|A^k(\sigma(z_j)) - g^k(\overline{z})\| \leq 2\alpha$. By passing to the limit with j we obtain

$$\|A^k(w) - g^k(\overline{z})\| \leq 2\alpha. \tag{2.22}$$

Since (2.22) holds for all $k \in \mathbb{Z}$, by Lemma 2.13, $w = \sigma(\overline{z})$. □

We continue with the proofs of Theorems 2.1 and 2.2. From the definition of ρ and σ we immediately conclude that $\sigma \circ g = A \circ \sigma$ and in the context of Theorem 2.2 we also have $\rho \circ A = g \circ \rho$.

We will show that $\sigma(\rho(z)) = \{z\}$. Let us fix $z \in \mathbb{R}^n$ and $z_1 \in \rho(z)$, then for any $k \in \mathbb{Z}$ it holds that

$$\|g^k(z_1) - A^k(z)\| \leq \alpha, \qquad \|A^k(\sigma(z_1)) - g^k(z_1)\| \leq \alpha.$$

Hence

$$\|A^k(\sigma(z_1)) - A^k(z)\| \le 2\alpha, \quad k \in \mathbb{Z}.$$

From the hyperbolicity of A (see also Lemma 2.8) it follows that $z = \sigma(z_1)$. Therefore,

$$\sigma(\rho(z)) = \{z\}. \tag{2.23}$$

Observe that (2.23) implies that σ is a surjection. This finishes the proof of Theorem 2.2.

From now on we work under assumptions of Theorem 2.1 and with

$$\varepsilon < \min(\varepsilon_0(A), \varepsilon_1(A)).$$

We will prove that $\rho \circ \sigma = \mathrm{Id}$. Let us fix $z \in \mathbb{R}^n$. For all $k \in \mathbb{Z}$ we have

$$\|A^k\sigma(z) - g^k(z)\| \le \alpha, \qquad \|g^k(\rho(\sigma(z))) - A^k\sigma(z)\| \le \alpha,$$

hence

$$\|g^k(\rho(\sigma(z))) - g^k(z)\| \le 2\alpha.$$

From Lemma 2.8 we obtain that $\rho(\sigma(z)) = z$.

It remains to show that $\sigma^{-1} = \rho$ is continuous. The proof is virtually the same as the proof of continuity of σ. The only difference is the use of Lemma 2.12 in place of Lemma 2.13. □

2.6. Proof of Theorem 2.3. This time we can only consider forward orbits. To define a map σ_u we proceed as follows. For any $z \in \mathbb{R}^n$, from Lemma 2.11 with $\lambda_1 = 0$ and $\lambda_2 = 1$ we have the following chain of covering relations:

$$N(z,\alpha) \overset{A}{\Longrightarrow} N(g(z),\alpha) \overset{A}{\Longrightarrow} N(g^2(z),\alpha) \overset{A}{\Longrightarrow} N(g^3(z),\alpha) \overset{A}{\Longrightarrow} \cdots \tag{2.24}$$

From Theorem A.5 applied to (2.24) it is easy to show that there exists $z_1 = (x_1, y_1) \in \mathbb{R}^u \times \mathbb{R}^s$ such that $A^k(z_1) \in N(g^k(z),\alpha)$, for $k \in \mathbb{N}$.

We set $\sigma_u(z) = x_1$. We need to show first that $\sigma_u(z)$ is well defined. Let $z_2 = (x_2, y_2)$ be another point such that $A^k(z_2) \in N(g^k(z),\alpha)$, for $k \in \mathbb{N}$. Then

$$\|A_u^k(x_1) - A_u^k(x_2)\| \le 2\alpha, \quad k \in \mathbb{N}. \tag{2.25}$$

On the other side, from our assumptions on A its follows that

$$\|A_u^k(x_1) - A_u^k(x_2)\| \ge c_u^k \|x_1 - x_2\|, \quad k \in \mathbb{N}. \tag{2.26}$$

Since $c_u > 1$, we conclude that $x_1 = x_2$.

From the above reasoning it follows immediately that $\sigma_u(z)$ is defined by the following condition:

$$\exists\, \sigma_s(z) \in \mathbb{R}^s \ \forall\, k \in \mathbb{N} \quad A^k(\sigma_u(z), \sigma_s(z)) \in N(g^k(z), \beta), \tag{2.27}$$

where $\beta \ge \alpha$.

Let us stress that $\sigma_s(z)$ is not a well-defined map, there exist many possibilities for $\sigma_s(z)$. However using the functional notation $\sigma_s(z)$ will facilitate further discussions.

To establish the semiconjugacy (2.3) observe that from (2.27) we obtain, for all $k \in \mathbb{N} \setminus \{0\}$,

$$A^{k-1}(A(\sigma_u(z), \sigma_s(z))) = A^{k-1}(A_u\sigma_u(z), A_s\sigma_s(z)) \in N(g^{k-1}(g(z)), \alpha).$$

This implies that $A_u\sigma_u(z) = \sigma_u(g(z))$.

The next step is the continuity of σ_u.

Lemma 2.15. *σ_u is continuous.*

Proof. Assume that $z_j \to \overline{z}$, we will show that the sequence $\{\sigma_u(z_j)\}_{j\in\mathbb{N}}$ is bounded and each converging subsequence converges to $\sigma_u(\overline{z})$. We can assume that $\|z_j - \overline{z}\| < \alpha$. Then, since $\|(\sigma_u(z_j), \sigma_s(z_j)) - z_j\| < \alpha$, we obtain

$$\|\sigma_u(z_j) - \pi_x\overline{z}\| < 2\alpha, \qquad \|\sigma_s(z_j) - \pi_y\overline{z}\| < 2\alpha.$$

Hence $\{\sigma_u(z_j), \sigma_s(z_j)\}_{j\in\mathbb{N}}$ is bounded.

Now, let us take a convergent subsequence, which we will again index by j, hence $z_j \to \overline{z}$, $\sigma_u(z_j) \to w$ and $\sigma_s(z_j) \to v$ for $j \to \infty$, where $w \in \mathbb{R}^u$. We will show that $w = \sigma_u(\overline{z})$. This implies that $\sigma_u(z_i) \to \sigma_u(\overline{z})$.

Let us fix $k \in \mathbb{N}$. From the continuity of $z \mapsto g^k(z)$ it follows that there exists j_0 such, for $j \geq j_0$,

$$\|g^k(z_j) - g^k(\overline{z})\| < \alpha. \tag{2.28}$$

By the definition of σ_u, we have

$$A^k(\sigma_u(z_j), \sigma_s(z_j)) \in N(g^k(z_j), \alpha).$$

Inequality (2.28) implies that

$$\|A^k(\sigma_u(z_j), \sigma_s(z_j)) - g^k(\overline{z})\| \leq 2\alpha.$$

By passing to the limit with j we obtain

$$\|A^k(w, v) - g^k(\overline{z})\| \leq 2\alpha. \tag{2.29}$$

Since (2.29) holds for all $k \in \mathbb{N}$, by (2.27), $w = \sigma_u(\overline{z})$. □

It remains to show the surjectivity of σ_u. For this let us set $z = (x_0, 0)$ and consider the following chain of covering relations:

$$N(z,\alpha) \overset{g}{\Longrightarrow} N(Az,\alpha) \overset{g}{\Longrightarrow} N(A^2z,\alpha) \overset{g}{\Longrightarrow} N(A^3z,\alpha) \overset{g}{\Longrightarrow} \cdots \tag{2.30}$$

From Theorem A.5 applied to (2.30) it follows that there exists $\overline{z}$ such that

$$g^k(\overline{z}) \in N(A^k(z), \alpha), \quad k \in \mathbb{N}.$$

Hence

$$A^k((x_0,0)) \in N(g^k(\overline{z}), 2\alpha), \quad \text{for all } k \in \mathbb{N}. \tag{2.31}$$

From (2.27) it follows that $x_0 = \sigma_u(\overline{z})$. Since x_0 was arbitrary, σ_u is onto. □

2.7. From global to local Grobman–Hartman theorem. The transition from the global to the local version of the Grobman–Hartman theorem is very standard, see for example [17], [24]. We include it here for the sake of completeness.

Assume that $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism satisfying

$$\varphi(z) = Az + h(z),$$

where $A \in \mathbb{R}^{n\times n}$ is a linear hyperbolic isomorphism and $h(0) = 0$ and $Dh(0) = 0$.

Let us fix $\varepsilon > 0$. There exists $\delta > 0$ such that $\|Dh(z)\| < \varepsilon$, $\|z\| \le \delta$.

Let $t\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function such that

$$\begin{aligned} t(r) &= r && \text{if } r \le \delta/2,\\ t(r) &= w < \delta && \text{if } r \ge \delta,\\ t(r_1) &\le t(r_2) && \text{if } r_1 < r_2,\\ 0 < t'(r) &< 1 && \text{if } r \in [\delta/2, \delta]. \end{aligned}$$

Consider now the function $R\colon \mathbb{R}^n \to \mathbb{R}^n$ given by

$$R(0) = 0, \quad R(z) = \frac{t(\|z\|)z}{\|z\|}, \quad z \ne 0. \tag{2.32}$$

It is easy to see that $R(z) = z$, for $z \in \overline{B}(0, \delta/2)$, $R(\mathbb{R}^n) \subset \overline{B}(0, w)$, $\|DR\| \le 1$.

Consider now a modification of φ given by $\widehat{\varphi}(z) = Az + h(R(z))$. It is easy to see that

$$\begin{aligned} \widehat{\varphi}(z) &= \varphi(z), && z \in \overline{B}(0, \delta_2),\\ \|h(R(z))\| &\le \varepsilon\delta, && z \in \mathbb{R}^n,\\ \|D(h\circ R)(z)\| &\le \varepsilon, && z \in \mathbb{R}^n. \end{aligned}$$

It is clear that, by taking ε and δ small enough, $h \circ R$ will satisfy the smallness assumption in Theorem 2.1 hence we will obtain the local conjugacy, which is the Grobman–Hartman theorem.

3. Hölder regularity of ρ

It is known that the conjugating homeomorphism from Theorem 2.1 is Hölder. The geometric proof of this fact is given in the Katok and Hasselblatt book [13]. In fact this is a particular case of a more general result about the Hölder regularity of the conjugacy between hyperbolic invariant sets. In [13] no effort was made to estimate the Hölder exponent in the context of the global Grobman–Hartman theorem.

Using the functional analysis type approach, the Hölder continuity of the conjugating homeomorphism was established by Barreira and Valls [2], Belitskiĭ [3], Belitskiĭ and Rayskin [4] (see [2] and references given there for other related papers) and apparently the best value of the Hölder exponent was obtained.

Our goal is to show the Hölder property for $\rho = \sigma^{-1}$, the map from the conjugacy established in Theorem 2.1. The main result in this section is Theorem 3.4. The same arguments apply also to σ. We show that we can obtain the same estimate as in [2]–[4].

LEMMA 3.1. *Let Q, A, g be as in the proof of Theorem 2.1. If $Q(z_1 - z_2) \geq 0$, $z_1 \neq z_2$, then $Q(g(z_1) - g(z_2)) > 0$ and*

$$\|\pi_x g(z_1) - \pi_x g(z_2)\| > \theta_u \|\pi_x z_1 - \pi_x z_2\|,$$

where $\theta_u = c_u - 2\varepsilon_0 > 1$.

PROOF. From the cone condition (Lemma 2.7) it follows that $Q(g(z_1) - g(z_2)) > 0$. Since $Q(z_1 - z_2) \geq 0$,

$$\|\pi_x z_1 - \pi_x z_2\| \geq \|\pi_y z_1 - \pi_y z_2\|.$$

We have

$$\begin{aligned}
\pi_x g(z_1) - \pi_x g(z_2) &= \int_0^1 D\pi_x g(t(z_1 - z_2) + z_2)\, dt \cdot (z_1 - z_2) \\
&= A_u \pi_x (z_1 - z_2) + \int_0^1 \frac{\partial \pi_x h}{\partial x} (t(z_1 - z_2) + z_2)\, dt \cdot \pi_x (z_1 - z_2) \\
&\quad + \int_0^1 \frac{\partial \pi_x h}{\partial y} (t(z_1 - z_2) + z_2)\, dt \cdot \pi_y (z_1 - z_2).
\end{aligned}$$

Hence, if $Q(z_1 - z_2) \geq 0$, we obtain

$$\|\pi_x g(z_1) - \pi_x g(z_2)\| \geq c_u \|\pi_x (z_1 - z_2)\| - 2\varepsilon \|\pi_x (z_1 - z_2)\|. \qquad \square$$

An analogous lemma holds for the inverse map.

LEMMA 3.2. *Let Q, A, g, ρ be as in the proof of Theorem 2.1. If $Q(z_1 - z_2) \leq 0$, $z_1 \neq z_2$, then $Q(g^{-1}(z_1) - g^{-1}(z_2)) < 0$ and*

$$\|\pi_y g^{-1}(z_1) - \pi_y g^{-1}(z_2)\| > \theta_s \|\pi_y z_1 - \pi_y z_2\|,$$

where $\theta_s = 1/(c_s + 2\varepsilon) > 1$.

PROOF. From the cone condition (Lemma 2.7) it follows that $Q(g^{-1}(z_1) - g^{-1}(z_2)) < 0$. Since $Q(z_1 - z_2) \leq 0$,

$$\|\pi_y z_1 - \pi_y z_2\| \geq \|\pi_x z_1 - \pi_x z_2\|.$$

We have, for any z_1, z_2,

$$\begin{aligned}\pi_y g(z_1) - \pi_y g(z_2) &= \int_0^1 D\pi_y g(t(z_1 - z_2) + z_2)\, dt \cdot (z_1 - z_2) \\ &= A_s \pi_y (z_1 - z_2) + \int_0^1 \frac{\partial \pi_y h}{\partial x}(t(z_1 - z_2) + z_2)\, dt \cdot \pi_x (z_1 - z_2) \\ &\quad + \int_0^1 \frac{\partial \pi_y h}{\partial y}(t(z_1 - z_2) + z_2)\, dt \cdot \pi_y (z_1 - z_2).\end{aligned}$$

Hence, if $Q(g(z_1) - g(z_2)) \leq 0$, then $Q(z_1 - z_2) < 0$ and we have

$$\begin{aligned}\|\pi_y g(z_1) - \pi_y g(z_2)\| &\leq c_s \|\pi_y (z_1 - z_2)\| + 2\varepsilon \|\pi_y (z_1 - z_2)\| \\ &= (c_s + 2\varepsilon)\|\pi_y (z_1 - z_2)\|,\end{aligned}$$

which, after the substitution $z_i \to g^{-1} z_i$, gives for $Q(z_1 - z_2) \leq 0$,

$$\|\pi_y z_1 - \pi_y z_2\| \leq (c_s + 2\varepsilon)\|\pi_y (g^{-1}(z_1) - g^{-1}(z_2))\|. \qquad \square$$

Lemma 3.3. *Let Q, A, g, ρ be as in the proof of Theorem* 2.1. *Then, for any $k \in \mathbb{Z}_+$, it holds*

$$\|\rho(z_1) - \rho(z_2)\| \leq \frac{2\alpha}{\theta_u^k} + \left(\frac{\|A_u\|}{\theta_u}\right)^k \|z_1 - z_2\|, \quad \text{if } Q(\rho(z_1) - \rho(z_2)) \geq 0, \tag{3.1}$$

$$\|\rho(z_1) - \rho(z_2)\| \leq \frac{2\alpha}{\theta_s^k} + \left(\frac{\|A_s^{-1}\|}{\theta_s}\right)^k \|z_1 - z_2\|, \quad \text{if } Q(\rho(z_1) - \rho(z_2)) \leq 0. \tag{3.2}$$

Proof. We will consider the case $Q(\rho(z_1) - \rho(z_2)) \geq 0$, the case $Q(\rho(z_1) - \rho(z_2)) \leq 0$ is analogous, one just need to consider the inverse maps.

From Lemma 3.1 (or Lemma 3.2 in the second case) applied to $\rho(z_1)$ and $\rho(z_2)$ it follows that, for any $k > 0$,

$$\begin{aligned}\|g^k(\rho(z_1)) - g^k(\rho(z_2))\| &= \|\pi_x g^k(\rho(z_1)) - \pi_x g^k(\rho(z_2))\| \\ &\geq \theta_u^k \|\pi_x \rho(z_1) - \pi_x \rho(z_2)\| = \theta_u^k \|\rho(z_1) - \rho(z_2)\|.\end{aligned}$$

Now we derive an upper bound on $\|g^k(\rho(z_1)) - g^k(\rho(z_2))\|$. Since $g^k(\rho(z_i)) \in N(A^k z_i, \alpha)$, for $i = 1, 2$, we obtain

$$\begin{aligned}\|g^k(\rho(z_1)) - g^k(\rho(z_2))\| &\leq \|g^k(\rho(z_1)) - A^k z_1\| \\ &\quad + \|A^k z_1 - A^k z_2\| + \|A^k z_2 - g^k(\rho(z_2))\| \\ &\leq \alpha + \|A\|^k \|z_1 - z_2\| + \alpha = 2\alpha + \|A_u\|^k \|z_1 - z_2\|.\end{aligned}$$

By combining the above inequalities, we obtain

$$\|\rho(z_1) - \rho(z_2)\| \leq \frac{2\alpha}{\theta_u^k} + \left(\frac{\|A_u\|}{\theta_u}\right)^k \|z_1 - z_2\|. \qquad \square$$

We are now ready to prove the Hölder regularity of ρ.

Theorem 3.4. *Let* $\gamma = \min(\ln\theta_u/\ln\|A_u\|, \ln\theta_s/\ln\|A_s^{-1}\|)$. *There exists* $C > 0$ *such that for any* $z_1, z_2 \in \mathbb{R}^n$, $z_1 \neq z_2$ *and* $\|z_1 - z_2\| < 1$, *it holds*

$$\frac{\|\rho(z_1) - \rho(z_2)\|}{\|z_1 - z_2\|^\gamma} \leq C. \tag{3.3}$$

Proof. Observe first that $\|A_u\| \geq \theta_u > 1$ and $\|A_s^{-1}\| \geq \theta_s > 1$. Let us set $\delta_0 = 1$. Let us denote $\delta = \|z_1 - z_2\|$. For any $\gamma > 0$ and $k \in \mathbb{Z}_+$ from Lemma 3.3 we have

$$\frac{\|\rho(z_1) - \rho(z_2)\|}{\|z_1 - z_2\|^\gamma} \leq \frac{2\alpha}{\theta^k}\,\delta^{-\gamma} + \left(\frac{L}{\theta}\right)^k \delta^{1-\gamma},$$

where $(\theta, L) = (\theta_u, \|A_u\|)$ or $(\theta, L) = (\theta_s, \|A_s^{-1}\|)$.

In the sequel we will find C which is good for each case separately, and then we choose the larger C. Observe that (3.3) holds if there exist constants C_1 and C_2 such that for each $0 < \delta < \delta_0$ there exists $k \in \mathbb{Z}_+$ such that the following inequalities are satisfied:

$$\frac{2\alpha}{\theta^k}\,\delta^{-\gamma} \leq C_1, \tag{3.4}$$

$$\left(\frac{L}{\theta}\right)^k \delta^{1-\gamma} \leq C_2. \tag{3.5}$$

We show that we can take

$$C_1 = 2\alpha, \tag{3.6}$$

$$C_2 = \frac{L}{\theta}. \tag{3.7}$$

The strategy is as follows: first from (3.4) we compute k and then we insert it into (3.5), which will give an inequality which should hold for any $0 < \delta < \delta_0$, this will produce a bound for γ, C_1 and C_2. From (3.4) we obtain

$$\theta^k \geq \frac{2\alpha\delta^{-\gamma}}{C_1}, \qquad k\ln\theta \geq \ln\frac{2\alpha}{C_1} - \gamma\ln\delta.$$

Taking into account (3.6), we have

$$k\ln\theta \geq -\gamma\ln\delta. \tag{3.8}$$

We set $k_0 = k_0(\delta) = -\gamma\ln\delta/\ln\theta$. k_0 might not belong to $\mathbb{Z}$, but $k_0 > 0$. We set $k = k(\delta) = \lfloor k_0 + 1\rfloor$, where $\lfloor z\rfloor$ is the integer part of z. With this choice of k equation (3.8) is satisfied. Hence also (3.4) holds.

Now we work on (3.5). Since

$$\left(\frac{L}{\theta}\right)^k \leq \left(\frac{L}{\theta}\right)^{k_0+1},$$

(3.5) is satisfied if the following inequality holds:

$$\left(\frac{L}{\theta}\right)^{1-\gamma\ln\delta/\ln\theta} \delta^{1-\gamma} \leq C_2.$$

By taking the logarithm of both sides of the above inequality we obtain

$$\left(1-\frac{\gamma}{\ln\theta}\ln\delta\right)\ln\left(\frac{L}{\theta}\right)+(1-\gamma)\ln\delta\le\ln C_2.$$

Finally, after an rearrangement of terms we arrive at

$$\left(1-\gamma\left(1+\frac{\ln(L/\theta)}{\ln\theta}\right)\right)\ln\delta\le\ln C_2-\ln\frac{L}{\theta}.$$

The last inequality should be satisfied for all $\delta\le\delta_0=1$. Therefore, we need the coefficient on the lhs by $\ln\delta$ to be nonnegative and the rhs to be nonnegative. It is easy to see that the rhs is nonnegative with C_2 given by (3.7). For the lhs observe that

$$1+\frac{\ln(L/\theta)}{\ln\theta}=1+\frac{\ln L-\ln\theta}{\ln\theta}=\frac{\ln L}{\ln\theta}.$$

Hence we obtain $1-\gamma\ln L/\ln\theta\ge 0$ and finally $\gamma\le\ln\theta/\ln L$. □

3.1. Comparison with known estimates. In [2, Theorem 1] (see also [3], [4]) the following estimate has been given for the Hölder exponent for ρ and ρ^{-1} if the size of the perturbation goes to 0 (we use our notation)

$$\alpha<\alpha_0=\min\left\{-\frac{\ln r(A_s)}{\ln r(A_s^{-1})},-\frac{\ln r(A_u^{-1})}{\ln r(A_u)}\right\},\tag{3.9}$$

where $r(A)$ denotes the spectral radius of the matrix A.

Let us consider our estimate of the Hölder exponent from Theorem 3.4. In the limit of vanishing perturbation we obtain (see Lemmas 3.1 and 3.2)

$$\theta_u=c_u,\qquad\theta_s=\frac{1}{c_s}.$$

Since from assumptions of Theorem 3.4 it follows that we can assume that

$$\frac{1}{c_u}=\|A_u^{-1}\|,\qquad c_s=\|A_s\|,\tag{3.10}$$

we obtain

$$\frac{\ln\theta_u}{\ln\|A_u\|}=\frac{\ln\frac{1}{\|A_u^{-1}\|}}{\ln\|A_u\|}=-\frac{\ln\|A_u^{-1}\|}{\ln\|A_u\|},$$
$$\frac{\ln\theta_s}{\ln\|A_s^{-1}\|}=\frac{\ln\frac{1}{\|A_s\|}}{\ln\|A_s^{-1}\|}=-\frac{\ln\|A_s\|}{\ln\|A_s^{-1}\|}.$$

Therefore our estimate for the Hölder exponent is

$$\alpha_1<\min\left\{-\frac{\ln\|A_u^{-1}\|}{\ln\|A_u\|},-\frac{\ln\|A_s\|}{\ln\|A_s^{-1}\|}\right\}.$$

It differs from (3.9) by the exchange of the spectral radius of matrices in (3.9) by the norms of matrices. It is quite obvious that by using the adapted norm we can get arbitrary close to the bound given by (3.9). For example, if A_u and A_s are diagonalizable over $\mathbb{R}$, if we define the scalar product so that the eigenvectors are orthogonal, then we obtain $\|A_{u,s}^{\pm1}\|=r(A_{u,s}^{\pm1})$.

To conclude, we claim that we were able to reproduce the Hölder exponent from [2]–[4].

4. Grobman–Hartman theorem for ODEs

Consider an ODE

$$z' = f(z), \quad z \in \mathbb{R}^n, \tag{4.1}$$

such that $f \in C^1$ and 0 is a hyperbolic fixed point. It is well known that the Grobman–Hartman theorem is also valid for (4.1). It can be obtained from Theorem 2.1 for time one map. In this section we would like to give a geometric proof, which will not reduce the proof to the map case, but rather we prefer a clean ODE version.

In such approach, the chain of covering relations along the full orbit will be replaced by an isolating segment along the orbit of a fixed diameter in the extended phase space (i.e. $(t, z) \in \mathbb{R} \times \mathbb{R}^n$). The cone conditions for maps have also its natural analog, we will demand that

$$\frac{d}{dt} Q(\varphi(t, z_1) - \varphi(t, z_2)) > 0.$$

We will consider an ODE

$$z' = Az + h(z), \quad z \in \mathbb{R}^n.$$

We will have the following set of assumptions on A and h, which we will refer to as the *ODE-standard conditions*:

- Assume that $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear map of the form

  $$A(x, y) = (A_u x, A_s y),$$

  where $n = u + s$, $A_u\colon \mathbb{R}^u \to \mathbb{R}^u$ and $A_s\colon \mathbb{R}^s \to \mathbb{R}^s$ are linear maps such that

  $$(x, A_u x) \geq c_u \|x\|^2, \quad c_u > 0, \text{ for all } x \in \mathbb{R}^u,$$
  $$(y, A_s y) \leq -c_s \|y\|^2, \quad c_s > 0, \text{ for all } y \in \mathbb{R}^s.$$

- Assume that $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and there exists $M > 0$ such that

  $$\|h(x)\| \leq M, \quad \text{for all } x \in \mathbb{R}^n.$$

Let φ be the (local) dynamical system induced by $z' = Az + h(z)$. Here is a global version of the Grobman–Hartman theorem for ODEs, which is similar in spirit to Theorem 2.1.

Theorem 4.1. *Assume* ODE-*standard conditions. Assume additionally that*

$$\|Dh(x)\| \leq \varepsilon, \quad \textit{for all } x \in \mathbb{R}^n.$$

Under the above assumptions there exists $\varepsilon_0 = \varepsilon_0(A) > 0$ such that if $\varepsilon < \varepsilon_0(A)$, then there exists a homeomorphism $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ such that for any $t \in \mathbb{R}$,

$$\rho(\exp(At)z) = \varphi(t, \rho(z)). \tag{4.2}$$

Theorem 4.2. *Assume* ODE-*standard conditions. Then there exists a continuous surjective map $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ such that for any $t \in \mathbb{R}$,*

$$(\exp(At)\sigma(z)) = \sigma(\varphi(t, z)). \tag{4.3}$$

In the sequel for $\lambda \in [0, 1]$ by $\varphi^\lambda\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ we will denote the dynamical system induced by

$$z' = f^\lambda(z) := Az + \lambda h(z).$$

Before the proofs of Theorems 4.1 and 4.2 we need first to develop some technical tools. The basic steps and constructions used in the proofs are given in Section 4.4. We invite the reader to jump first to this section to see the overall picture of the proof and then consult other more technical sections when necessary.

4.1. φ^λ is a global dynamical system.

Lemma 4.3. *Assume ODE-standard conditions. Then, for every $(t, z) \in \mathbb{R} \times \mathbb{R}^n$, $\varphi^\lambda(t, z)$ is defined.*

Proof. Observe that $\|f_\lambda(z)\| \le \|A\|\|z\|+M$. From this, using the Gronwall inequality, we obtain the following estimate:

$$\|z(t)\| \le \|z(0)\|e^{\|A\|\cdot|t|} + \frac{M}{\|A\|}\big(e^{\|A\|\cdot|t|} - 1\big). \tag{4.4}$$

This implies that $\varphi^\lambda(t, z)$ is defined. □

4.2. Isolating segment. We assume that the reader is familiar with the notion of an isolating segment for an ODE. It has its origin in the Conley index theory [7] and was developed in papers by Roman Srzednicki and his coworkers [18]–[21], [23].

Roughly speaking, an isolating segment for a (non-autonomous) ODE is the set in the extended phasespace (i.e. $(t, z) \in \mathbb{R} \times \mathbb{R}^n$), whose boundaries are sections of the vector field. The precise definition can be found in Appendix B.

Lemma 4.4. *Assume* ODE-*standard conditions. There exists*

$$\widehat{\alpha} = \max\left(\frac{2M}{c_u}, \frac{2M}{c_s}\right)$$

such that for $\alpha > \widehat{\alpha}$ and for any $\lambda_1, \lambda_2 \in [0, 1]$ and $z_0 \in \mathbb{R}^n$ the set

$$N_{\lambda_1}(z_0, \alpha) = \{(t,(x, y)) \mid (x - \pi_x\varphi^{\lambda_1}(t, z_0))^2 \le \alpha^2,\ (y - \pi_y\varphi^{\lambda_1}(t, z_0))^2 \le \alpha^2\},$$

with

$$N^-_{\lambda_1}(z_0,\alpha) = \{(t,(x,y)) \in N_{\lambda_1}(z_0,\alpha) \mid (x - \pi_x\varphi^{\lambda_1}(t,z_0))^2 = \alpha^2\},$$
$$N^+_{\lambda_1}(z_0,\alpha) = \{(t,(x,y)) \in N_{\lambda_1}(z_0,\alpha) \mid (y - \pi_y\varphi^{\lambda_1}(t,z_0))^2 = \alpha^2\},$$

is an isolating segment for φ^{λ_2}.

PROOF. Let us introduce the following notation:

$$L^-(t,x,y) = (x - \pi_x\varphi^{\lambda_1}(t,z_0))^2 - \alpha^2,$$
$$L^+(t,x,y) = (y - \pi_y\varphi^{\lambda_1}(t,z_0))^2 - \alpha^2.$$

The outside normal vector field to $N^-_{\lambda_1}(z_0,\alpha)$ is given by ∇L^-. We have

$$\begin{aligned}
\frac{\partial L^-}{\partial t}(t,x,y) &= -2(x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot \pi_x f^{\lambda_1}(\varphi^{\lambda_1}(t,z_0))) \\
&= -2(x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot (A_u\varphi^{\lambda_1}(t,z_0) + \lambda_1\pi_x h(\varphi^{\lambda_1}(t,z_0))), \\
\frac{\partial L^-}{\partial x}(t,x,y) &= 2(x - \pi_x\varphi^{\lambda_1}(t,z_0)), \\
\frac{\partial L^-}{\partial y}(t,x,y) &= 0.
\end{aligned}$$

We verify the exit condition by checking that for $(t,z) \in N^-_{\lambda_1}(z_0,\alpha)$, $\nabla L^-(t,z) \cdot (1, f^{\lambda_2}(t,z)) > 0$. We have, for $(t,(x,y)) \in N^-_{\lambda_1}(z_0,\alpha)$,

$$\begin{aligned}
&\frac{1}{2}\nabla L^-(t,z) \cdot (1, f^{\lambda_2}(t,z)) \\
&\qquad = -(x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot (A_u\varphi^{\lambda_1}(t,z_0) + \lambda_1\pi_x h(\varphi^{\lambda_1}(t,z_0))) \\
&\qquad\quad + (x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot (A_u x + \lambda_2\pi_x h(x,y)) \\
&\qquad = (x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot (A_u(x - \pi_x\varphi^{\lambda_1}(t,z_0))) \\
&\qquad\quad + (x - \pi_x\varphi^{\lambda_1}(t,z_0)) \cdot (-\lambda_1\pi_x h(\varphi^{\lambda_1}(t,z_0)) + \lambda_2\pi_x h(x,y)) \\
&\qquad \geq c_u\alpha^2 - 2\alpha M = \alpha(c_u\alpha - 2M).
\end{aligned}$$

We see that it is enough to take $\widehat{\alpha} > 2M/c_u$.

For the verification of the entry condition we will show that for $(t,z) \in N^+_{\lambda_1}(z_0,\alpha)$, $\nabla L^+(t,z) \cdot (1, f^{\lambda_2}(t,z)) < 0$.

The outside normal vector field to $N^+_{\lambda_1}(z_0,\alpha)$ is given by ∇L^+. We have

$$\begin{aligned}
\frac{\partial L^+}{\partial t}(t,x,y) &= -2(y - \pi_y\varphi^{\lambda_1}(t,z_0)) \cdot \pi_y f^{\lambda_1}(\varphi^{\lambda_1}(t,z_0))) \\
&= -2(y - \pi_y\varphi^{\lambda_1}(t,z_0)) \cdot (A_s\varphi^{\lambda_1}(t,z_0) + \lambda_1\pi_y h(\varphi^{\lambda_1}(t,z_0))), \\
\frac{\partial L^+}{\partial x}(t,x,y) &= 0, \\
\frac{\partial L^+}{\partial y}(t,x,y) &= 2(y - \pi_y\varphi^{\lambda_1}(t,z_0)).
\end{aligned}$$

We have, for $(t,(x,y)) \in N^{+}_{\lambda_1}(z_0,\alpha)$,

$$\begin{aligned}\frac{1}{2}\nabla L^{+}(t,z) &\cdot (1, f^{\lambda_2}(t,z)) \\ &= -(y-\pi_y\varphi^{\lambda_1}(t,z_0))\cdot(A_s\varphi^{\lambda_1}(t,z_0)+\lambda_1\pi_y h(\varphi^{\lambda_1}(t,z_0))) \\ &\quad + (y-\pi_y\varphi^{\lambda_1}(t,z_0))\cdot(A_y y+\lambda_2\pi_y h(x,y)) \\ &= (y-\pi_y\varphi^{\lambda_1}(t,z_0))\cdot(A_s(y-\pi_y\varphi^{\lambda_1}(t,z_0))) \\ &\quad + (y-\pi_y\varphi^{\lambda_1}(t,z_0))\cdot(-\lambda_1\pi_y h(\varphi^{\lambda_1}(t,z_0))+\lambda_2\pi_y h(x,y)) \\ &\le -c_s\alpha^2+2\alpha M = \alpha(-c_s\alpha+2M).\end{aligned}$$

We see that it is enough to take $\widehat{\alpha} > 2M/c_s$. □

The following theorem will be obtained using the ideas from the proof of the Ważewski Rectract Theorem [22] (see also [7]). We will present the details.

Theorem 4.5. *Assume* ODE-*standard conditions. Let $\alpha > \widehat{\alpha}$, where $\widehat{\alpha}$ is defined in Lemma* 4.4. *Then, for any $\lambda_1,\lambda_2 \in [0,1]$ and $z_0 \in \mathbb{R}^n$, there exists $z_1 \in \mathbb{R}^n$ such that, for all $t \in \mathbb{R}$, it holds*

$$\varphi^{\lambda_2}(t,z_1) \in \varphi^{\lambda_1}(t,z_0) + B_u(0,\alpha)\times B_s(0,\alpha). \tag{4.5}$$

Proof. We will show that for any $T>0$ there exists $z_T \in z_0 + \overline{B}_u(0,\alpha)\times \overline{B}_s(0,\alpha)$ such that

$$\varphi^{\lambda_2}(t,z_T) \in \varphi^{\lambda_1}(t,z_0) + \overline{B}_u(0,\alpha)\times \overline{B}_s(0,\alpha), \quad t\in[-T,T]. \tag{4.6}$$

Observe that once (4.6) is established, by choosing a convergent subsequence from $z_n \to \overline{z}$ for $n\in\mathbb{Z}_+$ we obtain an orbit for φ^{λ_2} satisfying

$$\varphi^{\lambda_2}(t,z_1) \in \varphi^{\lambda_1}(t,z_0) + \overline{B}_u(0,\alpha)\times \overline{B}_s(0,\alpha). \tag{4.7}$$

From Lemma 4.4 it follows that $N_{\lambda_1}(z,\alpha)$ is an isolating segment for φ^{λ_2} for any λ_2.

Let us fix $T>0$. We define a map $h\colon [0,2T]\times \overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha) \to \overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha)$ as follows. Let $\tau\colon N_{\lambda_1}(z_0,\alpha)\to\mathbb{R}\cup\{\infty\}$ be the exit time function from the isolating segment $N_{\lambda_1}(z_0,\alpha)$ for the process φ^{λ_2}. From the properties of the isolating segments (see Appendix B) it follows that this function is continuous.

The map $h(s,\cdot\,)$ does the following: in the coordinate frame with moving origin given by $\varphi^{\lambda_1}(s-T,z_0)$ to a point z we assign $\varphi^{\lambda_2}(s,z)$ if s is smaller than the exit time, or the exit point (all in the moving coordinate frame).

The precise definition of h is as follows: let

$$i(z) = z+\varphi^{\lambda_1}(-T,z_0), \quad \tau_i(z) = \tau(-T,i(z)),$$

then

$$h(s,z)=\begin{cases}\varphi^{\lambda_2}(s,i(z))-\varphi^{\lambda_1}(s-T,z_0), & \text{if } s\geq\tau_i(z),\\ \varphi^{\lambda_2}(\tau_i(z),i(z))-\varphi^{\lambda_1}(\tau_i(z)-T,z_0) & \text{otherwise.}\end{cases}$$

To prove (4.6) it is enough to show that there exists $z\in z_0+\overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha)$ such that

$$\tau(-T,z+\varphi^{\lambda_1}(-T,z_0))<2T. \tag{4.8}$$

We will reason by contradiction and assume that no such z exists.

Since $N_{\lambda_1}(z_0,\alpha)$ is an isolating segment, we see that h satisfies the following conditions:

$$\begin{aligned} h(2T,z)&\in(\partial B_u(0,\alpha))\times\overline{B}_s(0,\alpha) && \text{for all } z\in\overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha),\\ h(0,z)&=z, && \text{for all } z\in\overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha),\\ h(s,z)&=z, && \text{for all } s\in[0,2T],\\ & && \text{for all } z\in(\partial B_u(0,\alpha))\times\overline{B}_s(0,\alpha).\end{aligned}$$

This implies that h is a deformation retraction of $\overline{B}_u(0,\alpha)\times\overline{B}_s(0,\alpha)$ onto $(\partial B_u(0,\alpha))\times\overline{B}_s(0,\alpha)$. This is not possible because the homology groups of both spaces are different, hence (4.8) is true for some z.

Hence we obtain (4.7). To have (4.5) for z_1, observe that from Lemma 4.4 it follows that $(t,\varphi^{\lambda_2}(t,z_1))\in\operatorname{int}N_{\lambda_1}(z,\alpha)$ for all $t\in\mathbb{R}$, otherwise it will leave $N_{\lambda_1}(z,\alpha)$ forward or backward in time. Therefore (4.5) is satisfied. This finishes the proof. □

4.3. Cone condition. The cone condition for ODEs is treated using the methods from [25] and the cones are defined in terms of a quadratic form. In this subsection we work under assumptions of Theorem 4.1.

Let $Q(x,y)=(x,x)-(y,y)$ be a quadratic form on $\mathbb{R}^n$. By Q we will also denote a matrix such that $Q(z)=z^tQz$. In our case $Q=\left[\begin{smallmatrix} I_u & 0\\ 0 & -I_s\end{smallmatrix}\right]$, where $I_u\in\mathbb{R}^{u\times u}$ and $I_s\in\mathbb{R}^{s\times s}$ are the identity matrices.

Lemma 4.6. *There exists $\varepsilon_0=\varepsilon_0(A)>0$ such that, if $\varepsilon<\varepsilon_0$, then there exists $\eta>0$ such that for $\lambda\in[0,1]$ the following cone condition holds:*

$$\frac{d}{dt}Q(\varphi^\lambda(t,z_1)-\varphi^\lambda(t,z_2))\geq\pm\eta Q(\varphi^\lambda(t,z_1)-\varphi^\lambda(t,z_2)), \tag{4.9}$$

for all $z_1,z_2\in\mathbb{R}^n$.

Proof. It is enough to consider (4.9) for $t=0$. We have

$$\begin{aligned}\frac{d}{dt}Q(\varphi^\lambda(t,z_1)&-\varphi^\lambda(t,z_2))_{t=0}\\ &=(f^\lambda(z_1)-f^\lambda(z_2))^tQ(z_1-z_2)+(z_1-z_2)^tQ(f^\lambda(z_1)-f^\lambda(z_2))\\ &=(z_1-z_2)^t(D(z_1,z_2)^tQ+QD(z_1,z_2))(z_1-z_2),\end{aligned}$$

where

$$D(z_1, z_2) = \int_0^1 Df^\lambda(z_2 + t(z_1 - z_2))\,dt = A + \lambda \int_0^1 Dh(z_2 + t(z_1 - z_2))\,dt.$$

We set

$$C(z_1, z_2) = \int_0^1 Dh(z_2 + t(z_1 - z_2))\,dt,$$

hence

$$D(z_1, z_2) = A + \lambda C(z_1, z_2), \qquad \|C(z_1, z_2)\| \le \varepsilon.$$

It is enough to prove that $D^tQ + QD$ is positive definite. Observe first that $A^tQ + QA$ is positive definite. Indeed, we have for any $z = (x, y) \in \mathbb{R}^n$,

$$\begin{aligned} v^t(A^tQ + QA)v &= v^t \cdot \begin{pmatrix} A_u^t + A_u & 0 \\ 0 & -(A_s^t + A_s) \end{pmatrix} \cdot v \\ &= x^t(A_u^t + A_u)x - y^t(A_s^t + A_s)y \\ &= 2(x, A_u x) - 2(y, A_s y) \ge 2c_u x^2 + 2c_s y^2 \ge 2\min(c_u, c_s)\|v\|^2. \end{aligned}$$

Since being positive definite is an open property we see that the desired $\eta > 0$ and $\varepsilon_0 > 0$ exist. □

Lemma 4.7. *Assume that $\varepsilon < \varepsilon_0$ is as in Lemma 4.6. Let $\lambda \in [0,1]$. Assume that for some $z_1, z_2 \in \mathbb{R}^n$ there exists β such that for all $t \in \mathbb{R}$,*

$$\|\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)\| \le \beta.$$

Then $z_1 = z_2$.

Proof. Observe that from our assumption it follows that there exists β_1 such that

$$|Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2))| \le \beta_1, \quad \text{for all } t \in \mathbb{R}. \tag{4.10}$$

We consider two cases: $Q(z_1 - z_2) \ge 0$ and $Q(z_1 - z_2) < 0$.

Consider first $Q(z_1 - z_2) \ge 0$. From Lemma 4.6 it follows that for all $t > 0$, $Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) > 0$, and for any $t_0, t > 0$,

$$Q(\varphi^\lambda(t + t_0, z_1) - \varphi^\lambda(t + t_0, z_2)) \ge \exp(\eta t)\, Q(\varphi^\lambda(t_0, z_1) - \varphi^\lambda(t_0, z_2)).$$

This is in contradiction with (4.10).

Now we consider the case $Q(z_1 - z_2) < 0$. It is easy to see that $Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) < 0$ for $t < 0$. From the cone condition (Lemma 4.6) it follows that

$$Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2)) < \exp(-\eta t)\, Q(z_1 - z_2), \quad t < 0.$$

Hence

$$|Q(\varphi^\lambda(t, z_1) - \varphi^\lambda(t, z_2))| > \exp(\eta |t|)\, |Q(z_1 - z_2)|, \quad t < 0.$$

This is in contradiction with (4.10). □

4.4. Proofs of Theorems 4.1 and 4.2. The proofs follow the pattern of the proofs of Theorems 2.1 and 2.2. Below we will just list the basic steps.

We define $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ and a multivalued map ρ from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$. In the case of the proof of Theorem 2.1 we will show that ρ is single-valued, i.e. $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$.

1. Let us fix $\alpha > \widehat{\alpha}$, where $\widehat{\alpha}$ is obtained in Lemma 4.4.
2. For $z \in \mathbb{R}^n$, from Lemma 4.4 with $\lambda_1 = 1$ and $\lambda_2 = 0$ we have an isolating segment $N_0(z,\alpha)$ for φ^1.

3.1. In the context of the proof of Theorem 4.1: from Theorem 4.5 and Lemma 4.7 it follows that $N_0(z,\alpha)$ defines a unique point, which we will denote by $\rho(z)$, such that

$$\varphi^1(t,\rho(z)) \in B(\varphi^0(t,z),\alpha) \quad t \in \mathbb{R}. \tag{4.11}$$

3.2. In the context of the proof of Theorem 4.2: from Theorem 4.5 it follows that $N_0(z,\alpha)$ defines for each $z \in \mathbb{R}^n$ a non-empty set $\rho(z)$, such that for each $z_1 \in \rho(z)$,

$$\varphi^1(t,z_1) \in B(\varphi^0(t,z),\alpha), \quad t \in \mathbb{R}. \tag{4.12}$$

4. For $z \in \mathbb{R}^n$, from Lemma 4.4 with $\lambda_1 = 0$ and $\lambda_2 = 1$ we have an isolating segment $N_1(z,\alpha)$ for φ^0.
5. From Theorem 4.5 and the hyperbolicity of A it follows that the isolating segment $N_1(z,\alpha)$ defines a unique point, which we will denote by $\sigma(z)$, such that

$$\varphi^0(t,\sigma(z)) \in B(\varphi^1(t,z),\alpha), \quad t \in \mathbb{R}. \tag{4.13}$$

The details of the proofs are basically the same as in the proofs of the map case and are left to the reader.

Appendix A. h-set and covering relations

The goal of this section is to present the notions of an h-set and covering relation, and to state the theorem about the existence of a point realizing the chain of covering relations.

A.1. h-set and covering relations.

Definition A.1 ([26, Definition 1]). An h-set N is a quadruple $(|N|, u(N), s(N), c_N)$ such that:

(a) $|N|$ is a compact subset of $\mathbb{R}^n$,
(b) $u(N), s(N) \in \{0,1,2,\ldots\}$ are such that $u(N)+s(N)=n$,
(c) $c_N\colon \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ is a homeomorphism such that

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned} \dim(N) &:= n, & N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\ N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, & N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\ N^- &:= c_N^{-1}(N_c^-), & N^+ &= c_N^{-1}(N_c^+). \end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by the homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$, and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both an h-set and its support. Sometimes we will call N^- the *exit set of* N and N^+ the *entry set of* N.

Definition A.2 ([26, Definition 6]). Assume that N, M are h-sets such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f\colon N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1}\colon N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) if and only if the following conditions are satisfied:

(a) There exists a continuous homotopy $h\colon [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ such that the following conditions hold true:

$$h_0 = f_c, \tag{A.1}$$

$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{A.2}$$

$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{A.3}$$

(b) If $u > 0$, then there exists a map $A\colon \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h_1(p,q) = (A(p), 0), \quad \text{for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \tag{A.4}$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \setminus \overline{B_u}(0,1). \tag{A.5}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w, \tag{A.6}$$

We will call condition (A.2) the *exit condition* and condition (A.3) will be called the *entry condition*.

Note that in the case $u = 0$, if $N \overset{f,w}{\Longrightarrow} M$, then $f(N) \subset \operatorname{int} M$ and $w = 1$.

Remark A.3. If the map A in condition (b) of Definition A.2 is a linear map, then condition (A.5) implies that $\deg(A, \overline{B_u}(0,1), 0) = \pm 1$. Hence condition (A.6) is in this situation automatically fulfilled with $w = \pm 1$. In fact, this is the most common situation in applications of covering relations.

Most of the time we will not be interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$ instead. Sometimes we may even drop the symbol f and write $N \Longrightarrow M$.

A.2. Main theorem about chains of covering relations.

THEOREM A.4 ([26, Theorem 9]). *Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i. \tag{A.7}$$

Then there exists a point $x \in \operatorname{int} N_0$ such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \ldots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \ldots, k, \\ f_k \circ f_{k-1} \circ \ldots \circ f_1(x) &= x. \end{aligned}$$

We point the reader to [26] for the proof.

The following result follows from Theorem A.4.

THEOREM A.5. *Assume that $I = \mathbb{Z}$ or $I = \mathbb{N}$. Let N_i, $i \in I$, be h-sets. Assume that, for each $i \in I$, we have*

$$N_i \overset{f_{i+1}, w_{i+1}}{\Longrightarrow} N_{i+1}.$$

Then there exists a sequence $\{x_i\}_{i \in I}$ such that $x_i \in \operatorname{int} N_i$ and $f_{i+1}(x_i) = x_{i+1}$, for all $i \in I$.

PROOF. We will consider the case $I = \mathbb{Z}$, the proof for the other case is almost the same. For any $k \in \mathbb{Z}_+$ let us consider a closed loop of covering relations

$$N_{-k} \overset{f_{-k+1}}{\Longrightarrow} N_{-k+1} \overset{f_{-k+2}}{\Longrightarrow} N_{-k+2} \Longrightarrow \cdots \overset{f_{k-1}}{\Longrightarrow} N_{k-1} \overset{f_k}{\Longrightarrow} N_k \overset{A_k}{\Longrightarrow} N_{-k},$$

where A_k is some artificial map such that $N_k \overset{A_k}{\Longrightarrow} N_{-k}$. From Theorem A.4 it follows that there exists a finite sequence $\{x_i^k\}_{i=-k,\ldots,k}$ such that

$$x_i^k \in \operatorname{int} N_i \quad \text{and} \quad f_i(x_{i-1}^k) = x_i^k, \quad i = -k+1, \ldots, k.$$

Since N_i are compact, it is easy to construct a desired sequence, by taking suitable subsequences. □

A.3. Natural structure of h-set. Observe that all the conditions appearing in the definition of the covering relation are expressed in 'internal' coordinates c_N and c_M. Also the homotopy is defined in terms of these coordinates. This sometimes makes the matter and the notation look a bit cumbersome. With this in mind we introduce the notion of a 'natural' structure on h-set.

Definition A.6. We will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_1) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an *h-set* with a natural structure given by

$$u(N) = u, \qquad s(N) = s, \qquad c_N(x, y) = \left(\frac{x - x_0}{r_1}, \frac{y - y_0}{r_2}\right).$$

Appendix B. Isolating segments for ODEs

Let us consider the differential equation

$$\dot{x} = f(t, x) \tag{B.1}$$

where $x \in \mathbb{R}^n$ and $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Let $x(t_0, x_0; \cdot)$ be the solution of (B.1) such that $x(t_0, x_0; t_0) = x_0$, we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0, x_0; t_0 + \tau).$$

The range of τ for which $\varphi_{(t_0,\tau)}(x_0)$ is defined might depend on (t_0, x_0). φ defines a local flow Φ on $\mathbb{R} \times \mathbb{R}^n$ by the formula

$$\Phi_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)}(x)). \tag{B.2}$$

In the sequel we will often call the first coordinate in the extended phase space $\mathbb{R} \times \mathbb{R}^n$ the *time*.

We use the following notation: by $\pi_1\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\pi_2\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the projections and for a subset $Z \subset \mathbb{R} \times \mathbb{R}^n$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in \mathbb{R}^n : (t, x) \in Z\}.$$

Now we are going to state the definition of an isolating segment for (B.1), which is a modification of the notion of a periodic isolating segment over $[0, T]$ or T-periodic isolating segment in [18]–[21], [23].

Definition B.1. Let $(W, W^-) \subset \mathbb{R} \times \mathbb{R}^n$ be a pair of subsets. We call W an *isolating segment* for (B.1) (or φ) if:

(a) $(W, W^-) \cap ([a, b] \times \mathbb{R}^n)$ is a pair of compact sets,
(b) for every $\sigma \in \mathbb{R}$, $x \in \partial W_\sigma$ there exists $\delta > 0$ such that for all $t \in (0, \delta)$, $\varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x) \in \operatorname{int} W_{\sigma+t}$,
(c) $W^- = \{(\sigma, x) \in W : \exists\, \delta > 0\ \forall\, t \in (0, \delta)\ \varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}\}$, $W^+ := \operatorname{cl}(\partial W \setminus W^-)$,
(d) for all $(\sigma, x) \in W^+$ there exists $\delta > 0$ such that for all $t \in (0, \delta)$, $\varphi_{(\sigma,-t)}(x) \notin W_{\sigma-t}$,
(e) there exists $\eta > 0$ such that for all $x \in W^-$ there exists $t > 0$ such that for all $\tau \in (0, t]$, $\Phi_\tau(x) \notin W$ and $\rho(\Phi_t(x), W) > \eta$.

Roughly speaking, W^- and W^+ are sections for (B.1), through which trajectories leave and enter the segment W, respectively.

Definition B.2. For the isolating segment W we define the *exit time function* $\tau_{W,\varphi}$ as

$$\tau_{W,\varphi}\colon W \ni (t_0, x_0) \mapsto \sup\{t \geq 0 : \forall\, s \in [0,t]\ (t_0 + s, \varphi_{(t_0,s)}(x_0)) \in W\} \in [0,\infty].$$

By the Ważewski Retract Theorem [22] the map $\tau_{W,\varphi}$ is continuous (compare with [7]).

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Piotr Zgliczyński
Jagiellonian University
Institute of Computer Science
and Computational Mathematics
Łojasiewicza 6
30–348 Kraków, POLAND

E-mail address: piotr.zgliczynski@ii.uj.edu.pl