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A NOTE ON CONLEY INDEX AND SOME PARABOLIC PROBLEMS WITH LOCALLY LARGE DIFFUSION

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Abstract. We prove singular Conley index continuation results for a class of scalar parabolic equations with locally large diffusion considered by Fusco [7] and Carvalho and Pereira [5].

1. Introduction

Evolution equations with large diffusion were studied in numerous papers, starting with the work [8] by Hale, cf. also [4], [9], [7], [5], [10], [11]. In those papers results like global bounds of solutions, asymptotic spatial homogenization, existence of invariant manifolds and existence of global attractors and their upper or lower semicontinuity, as the diffusion goes to infinity, are obtained.

In a pioneering work [7] Fusco considered the scalar reaction diffusion problem

$$(E_\varepsilon) \qquad u_t = (a_\varepsilon u_x)_x + f(x,u), \quad 0 < x < 1, \quad t > 0,$$

subject to the following separated boundary conditions:

$$(S_\varepsilon) \qquad \begin{cases} \rho u - (1-\rho) a_\varepsilon u_x = 0, & x = 0,\ t > 0 \\ \sigma u + (1-\sigma) a_\varepsilon u_x = 0, & x = 1,\ t > 0. \end{cases}$$

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Here, $0 \le \rho, \sigma < 1$ and f is a C^3 dissipative map. Moreover, the diffusion coefficient a_ε is large except in a neighbourhood of a finite number of points where it becomes small as $\varepsilon \to 0$ and there is some transitory behavior between such neighbourhoods. More precisely, let $\varepsilon_0 \in]0, \infty]$, let $(e_j)_{j\in[1..n]}$, $(l_j)_{j\in[0..n]}$ and $(b_j)_{j\in[0..n]}$ be sequences of positive constants and let $(l'_j)_{j\in[0..n]}$, $(b'_j)_{j\in[0..n]}$ be two sequences of positive functions of $\varepsilon \in]0, \varepsilon_0[$ such that

$$\lim_{\varepsilon\to 0} l'_j(\varepsilon) = l_j \quad \text{and} \quad \lim_{\varepsilon\to 0} b'_j(\varepsilon) = b_j \quad \text{for } j \in [0\,..\,n].$$

Let $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$ be a family of positive C^2 functions defined on $[0,1]$, $n \in \mathbb{N}$, $(x_j)_{j\in[0..n]}$ be a strictly increasing sequence in $[0,1]$ with $x_0 = 0$ with $x_n = 1$ and such that for each $j \in [1\,..\,n]$

$$\begin{aligned} a_\varepsilon(x) &\ge \frac{e_j}{\varepsilon}, && \text{for } x_{j-1} + \varepsilon l'_{j-1} \le x \le x_j - \varepsilon l'_j, \\ a_\varepsilon(x) &\ge \varepsilon b_j, && \text{for } x_j - \varepsilon l'_j \le x \le x_j + \varepsilon l'_j, \\ a_\varepsilon(x) &\le \varepsilon b'_j, && \text{for } x_j - \varepsilon l_j \le x \le x_j + \varepsilon l_j. \end{aligned}$$

Here $x_0 - \varepsilon l'_0 = x_0 - \varepsilon l_0 = 0$ and $x_n + \varepsilon l'_n = x_n + \varepsilon l_n = 1$.

(Note that Fusco writes ν instead of ε, and $(a_j)_{j\in[0..n]}$ and $(a'_j)_{j\in[0..n]}$ instead of $(b_j)_{j\in[0..n]}$ and $(b'_j)_{j\in[0..n]}$, respectively.)

Under some additional technical hypothesis (H), Fusco constructed a system

$$(E_0) \qquad \dot z + A_0 z = g(z)$$

of ordinary differential equations, essentially by discretizing $(E_\varepsilon, S_\varepsilon)$. He proved that the (semi)flow π_0 associated to this system is equivalent (in the sense of Definition 1 in [7]) to the semiflow π_ε defined by $(E_\varepsilon, S_\varepsilon)$ (cf. Theorems 3 and 4 in [7]).

The phase space of (E_0) is the n-dimensional subspace of $L^2(0,1)$ formed by all step functions with respect to the partition $(x_j)_{j\in[0..n]}$, so it can naturally be identified with $\mathbb{R}^n$.

By introducing a nonlinear change of coordinates $u \mapsto (z, v)$ (see equation (9) in [7]) the author proved, under some dissipativeness condition on the nonlinearity, that on the global attractor $\mathcal{A}_\varepsilon$ of the semiflow associated to equation $(E_\varepsilon, S_\varepsilon)$, the variable v becomes very small as $\varepsilon \to 0$ and $\mathcal{A}_\varepsilon$ is contained in an invariant manifold on which $(E_\varepsilon, S_\varepsilon)$ is actually given by a system of ordinary differential equations close to (E_0), cf. Theorems 1 and 2 in [7]. He also obtained some generic structural stability results for $(E_\varepsilon, S_\varepsilon)$.

In the important follow-up paper [5], Carvalho and Pereira approached the above problem from the point of view of spectral convergence: the map

$$u \mapsto -(a_\varepsilon u_x)_x$$

with boundary conditions (S_ε) generates a linear operator A_ε in $L^2(0,1)$ which has a simple spectrum $(\lambda_{l,\varepsilon})_l$ and corresponding appropriately normalized eigenfunctions $(\varphi_{l,\varepsilon})_l$. The authors proved that the above linear operator A_0 has a simple spectrum $(\mu_l)_{l\in[1..n]}$ and corresponding appropriately normalized eigenvectors $(z_l)_{l\in[1..n]}$ such that, for $\varepsilon\to 0$, $\lambda_{l,\varepsilon}\to\mu_l$ for $l\in[1\,..\,n]$ and $\lambda_{l,\varepsilon}\to\infty$ for $l>n$. Moreover, for $l\in[1\,..\,n]$, in some sense, $\varphi_{l,\varepsilon}\to z_l$ as $\varepsilon\to 0$, cf. Section 3 below for the precise statement of these results.

Now, using the linear map $J_\varepsilon\colon\mathbb{R}^n\to H^1(0,1)$, $\sum_{l=1}^n v_l z_l\mapsto\sum_{l=1}^n v_l\varphi_{l,\varepsilon}$ the authors 'embedded' the (semi)flow π_0 into the semiflow π_ε, $\varepsilon>0$. This approach allowed them to prove some of the results from [7] without Fusco's technical hypothesis (H). In addition, they proved an upper semicontinuity result for global attractors and obtained a structural stability result (cf. Theorem 4.1 and Corollary 4.3 in [5]).

It is the purpose of this note to prove Conley index and homology index braids continuation results for the above family $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0[}$ (without any dissipativeness assumption on the nonlinearity), showing in particular that isolated invariant sets K_0 of π_0 continue, for small $\varepsilon>0$, to isolated invariant sets K_ε of π_ε with K_ε 'close' to $J_\varepsilon(K_0)$, and K_0 and K_ε have the same Conley index. In particular, some aspects of the dynamics of the simpler flow π_0 can be found in the more complicated semiflow π_ε. The precise statements of the continuation results and our assumptions on the nonlinearities involved are given in Section 4 below.

To prove our results we use slight extensions of some abstract continuation results established in our previous papers [2], [3]. These results are summarized in Section 2 below.

In this paper, all linear spaces are defined over the field of real numbers.

2. Conley index and homology index braid continuation results

In this section, we will collect, without proof, some abstract singular Conley index continuation results. These results were established in the paper [3] (with some proofs contained in the work [2]) in the special case in which the embedding J_ε considered here is just set inclusion. A careful inspection of the arguments contained in those papers leads to proofs of the more general results presented here.

We start by introducing a basic abstract spectral convergence condition.

Definition 2.1. Given $\widehat{\varepsilon}\in\,]0,\infty[$, we say that the family

$$(H^\varepsilon,\langle\,\cdot\,,\cdot\,\rangle_{H^\varepsilon},A_\varepsilon,J_\varepsilon)_{\varepsilon\in[0,\widehat{\varepsilon}]}$$

satisfies condition (FSpec) if the following properties hold:

(1) for every $\varepsilon\in[0,\widehat{\varepsilon}]$, $(H^\varepsilon,\langle\,\cdot\,,\cdot\,\rangle_{H^\varepsilon})$ is a Hilbert space and $A_\varepsilon\colon D(A_\varepsilon)\subset H^\varepsilon\to H^\varepsilon$ is a densely defined nonnegative self-adjoint linear operator on

$(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon})$ with $(A_\varepsilon+I_\varepsilon)^{-1}\colon H^\varepsilon \to H^\varepsilon$ compact. Here, $I_\varepsilon\colon H^\varepsilon \to H^\varepsilon$ is the identity operator. For $\alpha \in [0,\infty[$ write $H^\varepsilon_\alpha := D(A_\varepsilon + I_\varepsilon)^{\alpha/2}$. In particular, $H^\varepsilon_0 = H^\varepsilon$.

(2) H^0 is n-dimensional with $n \in \mathbb{N}$ while H^ε is infinite dimensional for $\varepsilon \in]0,\widehat{\varepsilon}]$.

(3) For each $\varepsilon \in]0,\widehat{\varepsilon}]$, J_ε is a linear continuous injection from H^0_1 to H^ε_1 and J_0 is the identity operator on H^0_1.

(4) There exists a constant $C \in]1,\infty[$ such that

$$|J_\varepsilon(u)|_{H^\varepsilon_1} \leq C|u|_{H^0_1} \quad \text{and} \quad |u|_{H^0_1} \leq C|J_\varepsilon(u)|_{H^\varepsilon_1}$$

for all $u \in H^0_1$ and all $\varepsilon \in]0,\widehat{\varepsilon}]$.

(5) For every $\varepsilon \in]0,\widehat{\varepsilon}]$ let $(\lambda_{l,\varepsilon})_l$ be the repeated sequence of eigenvalues of A_ε and $(\varphi_{l,\varepsilon})_l$ be the corresponding H^ε-orthonormal sequence of eigenfunctions. Furthermore, let $(\mu_l)_{l\in[1..n]}$ be the repeated sequence of eigenvalues of A_0.

Whenever $(\varepsilon_m)_m$ is a null sequence in $]0,\widehat{\varepsilon}]$, then

(a) $\lambda_{l,\varepsilon_m} \to \mu_l$ as $m \to \infty$, for all $l \in [1\,.\,.\,n]$.

(b) $\lambda_{l,\varepsilon_m} \to \infty$ as $m \to \infty$, for all $l > n$.

Moreover, there is a subsequence $(\varepsilon^1_m)_m$ of $(\varepsilon_m)_m$ and there is an H^0-orthonormal sequence of eigenfunctions $(z_l)_{l\in[1..n]}$ of A_0 corresponding to $(\mu_l)_{l\in[1..n]}$ such that

(c) $|\varphi_{l,\varepsilon^1_m} - J_{\varepsilon^1_m} z_l|_{H^{\varepsilon^1_m}_1} \to 0$ as $m \to \infty$, for all $l \in [1\,.\,.\,n]$.

(d) $\langle J_{\varepsilon^1_m} u, \varphi_{l,\varepsilon^1_m}\rangle_{H^{\varepsilon^1_m}} \to \langle u, z_l\rangle_{H^0}$ as $m \to \infty$, for all $u \in H^0_1$ and all $l \in [1\,.\,.\,n]$.

Such a sequence $(z_l)_{l\in[1..n]}$ is called *adapted* to the sequence $(\varepsilon^1_m)_m$.

The following technical result will be used in the proof of Theorem 4.4 below.

Theorem 2.2 (cf. the proof of Theorem 2.7 in [3]). *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)$, $\varepsilon \in [0,\widehat{\varepsilon}]$ satisfy condition* (FSpec). *Suppose $(\varepsilon_m)_m$ is a null sequence in $]0,\widehat{\varepsilon}]$. Let $u_0 \in H^0$ be arbitrary and let $(u_m)_m$ be a sequence such that $u_m \in H^{\varepsilon_m}$ for $m \in \mathbb{N}$. Suppose that*

(a) *whenever $(m_k)_k$ is a strictly increasing sequence in $\mathbb{N}$ and $(\varepsilon^1_m)_m$ is a subsequence of $(\varepsilon_m)_m$ defined by $\varepsilon^1_k = \varepsilon_{m_k}$, for each $k \in \mathbb{N}$, and whenever $(z_l)_{l\in[1..n]}$ is adapted to $(\varepsilon^1_m)_m$, then $\langle u_{m_k}, \varphi_{l,\varepsilon^1_k}\rangle_{H^{\varepsilon^1_k}} \to \langle u_0, z_l\rangle_{H^0}$ as $k \to \infty$ for all $l \in [1\,.\,.\,n]$.*

(b) $\sup_{m\in\mathbb{N}} |u_m|_{H^{\varepsilon_m}} < \infty$.

Then, for every $\beta \in]0,\infty[$,

$$\sup_{t\in[\beta,\infty[} \big|e^{-tA_{\varepsilon_m}} u_m - J_{\varepsilon_m}(e^{-tA_0} u_0)\big|_{H^{\varepsilon_m}_1} \to 0 \quad \textit{as } m \to \infty.$$

We now introduce an abstract nonlinear convergence condition.

Definition 2.3. Let $\widehat{\varepsilon} \in \,]0,\infty[$ be arbitrary and $(H^\varepsilon, \langle \cdot\,,\,\cdot \rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\widehat{\varepsilon}]}$ be a family satisfying condition (FSpec). We say that the family $(f_\varepsilon)_{\varepsilon\in[0,\widehat{\varepsilon}]}$ of maps *satisfies condition* (Conv) if the following properties hold:

(a) $f_\varepsilon \colon H_1^\varepsilon \to H^\varepsilon$ for every $\varepsilon \in [0,\widehat{\varepsilon}]$.
(b) $\lim_{\varepsilon\to 0^+} |e^{-tA_\varepsilon} f_\varepsilon(J_\varepsilon u) - J_\varepsilon(e^{-tA_0} f_0(u))|_{H_1^\varepsilon} = 0$ for every $u \in H_1^0$ and every $t \in \,]0,\infty[$.
(c) For every $M \in [0,\infty[$ there is an $L = L_M \in [0,\infty[$ such that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon} \le L|u - v|_{H_1^\varepsilon}$$

for all $\varepsilon \in [0,\widehat{\varepsilon}]$ and u, $v \in H_1^\varepsilon$ satisfying $|u|_{H_1^\varepsilon}$, $|v|_{H_1^\varepsilon} \le M$.
(d) For every $u \in H_1^0$ there is an $\widehat{\varepsilon}' \in \,]0,\widehat{\varepsilon}]$ such that

$$\sup_{\varepsilon\in[0,\widehat{\varepsilon}']} |f_\varepsilon(J_\varepsilon u)|_{H^\varepsilon} < \infty.$$

For the rest of this section we assume that the families $(H^\varepsilon, \langle \cdot\,,\,\cdot \rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)$ and f_ε, $\varepsilon \in [0,\widehat{\varepsilon}]$, are as in Definition 2.3. For every $\varepsilon \in [0,\widehat{\varepsilon}]$, let $\pi_\varepsilon := \pi_{A_\varepsilon, f_\varepsilon}$ be the local semiflow on H_1^ε generated by the abstract parabolic equation

$$\dot{u} = -A_\varepsilon u + f_\varepsilon(u). \tag{2.1}$$

For $\varepsilon \in \,]0,\widehat{\varepsilon}]$ let $Q_\varepsilon \colon H_1^\varepsilon \to H_1^\varepsilon$ be the H_1^ε-orthogonal projection of H_1^ε onto (its closed subspace) $J_\varepsilon(H_1^0)$. Moreover, let $R_\varepsilon \colon J_\varepsilon(H_1^0) \to H_1^0$ be the inverse of $J_\varepsilon \colon H_1^0 \to J_\varepsilon(H_1^0)$.

We can now state the following Conley index continuation principle:

Theorem 2.4 (cf. Theorem 4.8 in [3]). *Let N be a closed and bounded isolating neighbourhood of an invariant set K_0 relative to π_0. For $\varepsilon \in \,]0,\widehat{\varepsilon}]$ and for every $\eta \in \,]0,\infty[$ set*

$$N_{\varepsilon,\eta} := \{\, u \in H_1^\varepsilon \mid R_\varepsilon Q_\varepsilon u \in N \text{ and } |(I - Q_\varepsilon)u|_{H_1^\varepsilon} \le \eta \,\}$$

and $K_{\varepsilon,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_{\varepsilon,\eta})$, i.e. $K_{\varepsilon,\eta}$ is the largest π_ε-invariant set in $N_{\varepsilon,\eta}$. Then for every $\eta \in \,]0,\infty[$ there exists an $\varepsilon^{\mathrm{c}} = \varepsilon^{\mathrm{c}}(\eta) \in \,]0,\widehat{\varepsilon}]$ such that for every $\varepsilon \in \,]0,\varepsilon^{\mathrm{c}}]$ the set $N_{\varepsilon,\eta}$ is a strongly admissible isolating neighbourhood of $K_{\varepsilon,\eta}$ relative to π_ε and

$$h(\pi_\varepsilon, K_{\varepsilon,\eta}) = h(\pi_0, K_0).$$

Here, as usual, $h(\pi, K)$ denotes the Conley index of an isolated invariant set K relative to a local semiflow π. Furthermore, for every $\eta \in \,]0,\infty[$, the family $(K_{\varepsilon,\eta})_{\varepsilon\in[0,\varepsilon^{\mathrm{c}}(\eta)]}$ of invariant sets, where $K_{0,\eta} = K_0$, is upper semicontinuous at $\varepsilon = 0$ with respect to the family $|\cdot|_{H_1^\varepsilon}$ of norms, i.e.

$$\lim_{\varepsilon\to 0^+} \sup_{w\in K_{\varepsilon,\eta}} \inf_{u\in K_0} |w - J_\varepsilon u|_{H_1^\varepsilon} = 0.$$

The family $(K_{\varepsilon,\eta})_{\varepsilon\in]0,\varepsilon^{\rm c}(\eta)]}$ *is asymptotically independent of* η*, i.e whenever* η_1 *and* $\eta_2\in\,]0,\infty[$ *then there is an* $\varepsilon'\in\,]0,\min(\varepsilon^{\rm c}(\eta_1),\varepsilon^{\rm c}(\eta_2))]$ *such that* $K_{\varepsilon,\eta_1}=K_{\varepsilon,\eta_2}$ *for* $\varepsilon\in\,]0,\varepsilon']$.

Finally, we have the following homology index braids continuation principle:

Theorem 2.5 (cf. Theorem 4.10 in [3]). *Assume the hypotheses of Theorem 2.4 and for every* $\eta\in\,]0,\infty[$ *let* $\varepsilon^{\rm c}(\eta)\in\,]0,\widehat{\varepsilon}]$ *be as in that theorem. Let* $(P,\prec)$ *be a finite poset. Let* $(M_{p,0})_{p\in P}$ *be a* $\prec$*-ordered Morse decomposition of* K_0 *relative to* π_0*. For each* $p\in P$*, let* $V_p\subset N$ *be closed in* X_0 *and such that* $M_{p,0}=\mathrm{Inv}_{\pi_0}(V_p)\subset\mathrm{Int}_{H^0_1}(V_p)$*. (Such sets* V_p*,* $p\in P$*, exist.) For* $\varepsilon\in\,]0,\widehat{\varepsilon}]$*, for every* $\eta\in\,]0,\infty[$ *and* $p\in P$ *set* $M_{p,\varepsilon,\eta}:=\mathrm{Inv}_{\pi_\varepsilon}(V_{p,\varepsilon,\eta})$*, where*

$$V_{p,\varepsilon,\eta}:=\{\,u\in H^\varepsilon_1\mid R_\varepsilon Q_\varepsilon u\in V_p\ \text{and}\ |(I-Q_\varepsilon)u|_{H^\varepsilon_1}\le\eta\,\}.$$

Then for every $\eta\in\,]0,\infty[$ *there is an* $\widetilde{\varepsilon}=\widetilde{\varepsilon}(\eta)\in\,]0,\varepsilon^{\rm c}(\eta)]$ *such that for every* $\varepsilon\in\,]0,\widetilde{\varepsilon}]$ *and* $p\in P$*,* $M_{p,\varepsilon,\eta}\subset\mathrm{Int}_{H^\varepsilon_1}(V_{p,\varepsilon,\eta})$ *and the family* $(M_{p,\varepsilon,\eta})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition of* $K_{\varepsilon,\eta}$ *relative to* π_ε *and the homology index braids of* $(\pi_0,K_0,(M_{p,0})_{p\in P})$ *and* $(\pi_\varepsilon,K_{\varepsilon,\eta},(M_{p,\varepsilon,\eta})_{p\in P})$*,* $\varepsilon\in\,]0,\widetilde{\varepsilon}]$*, are isomorphic and so they determine the same collection of* C*-connection matrices. For each* $p\in P$*, the family* $(M_{p,\varepsilon,\eta})_{\varepsilon\in[0,\widetilde{\varepsilon}(\eta)]}$*, where* $M_{p,0,\eta}=M_{p,0}$*, is upper semicontinuous at* $\varepsilon=0$ *with respect to the family* $|\cdot|_{H^\varepsilon_1}$ *of norms and the family* $(M_{p,\varepsilon,\eta})_{\varepsilon\in]0,\widetilde{\varepsilon}(\eta)]}$ *is asymptotically independent of* η*.*

3. The spectral convergence result

In this section we will state the spectral convergence result proved in [5]. In what follows let $\varepsilon_0\in\,]0,\infty]$, $(a_\varepsilon)_{\varepsilon\in]0,\varepsilon_0[}$, $(x_j)_{j\in[0..n]}$, $(l_j)_{j\in[0..n]}$ and $(b_j)_{j\in[0..n]}$ be as in Section 1. Set $K_j=[x_j,x_{j+1}]$ and $L_j=x_{j+1}-x_j$ for $j\in[0..n-1]$.

For each $\varepsilon\in\,]0,\varepsilon_0[$ there is a linear operator $A_\varepsilon\colon D_\varepsilon\subset H^1(0,1)\to L^2(0,1)$ associated to problem $(E_\varepsilon,S_\varepsilon)$ defined as follows: D_ε is the set of all $u\in H^2(0,1)$ with $\rho u(0)-(1-\rho)a_\varepsilon(0)u_x(0)=\sigma u(0)+(1-\sigma)a_\varepsilon(1)u_x(1)=0$ and

$$A_\varepsilon u=-(a_\varepsilon\cdot u')'\quad\text{for } u\in D_\varepsilon. \tag{3.1}$$

As a matter of fact, the operator $A_\varepsilon\colon D_\varepsilon\subset H^1(0,1)\to L^2(0,1)$ is generated by the pair $(\tau_\varepsilon,\langle\cdot\,,\cdot\rangle_{L^2})$, where $\tau_\varepsilon\colon H^1(0,1)\times H^1(0,1)\to\mathbb{R}$ is the bilinear form given by

$$\tau_\varepsilon(u,v)=\int_0^1 a_\varepsilon\cdot u'\cdot v'\,dx+\frac{\rho}{1-\rho}\,u(0)v(0)+\frac{\sigma}{1-\sigma}\,u(1)v(1),\quad u,v\in H^1(0,1)$$

and $\langle\cdot\,,\cdot\rangle_{L^2}=\langle\cdot\,,\cdot\rangle_{L^2(0,1)}$ is the standard scalar product on $L^2=L^2(0,1)$.

Let $(\lambda_{l,\varepsilon})_l$ be the increasing sequence of eigenvalues of A_ε (which are all simple). Let $(\varphi_{l,\varepsilon})_l$ be an (appropriately normalized) L^2-orthogonal sequence such that $\varphi_{l,\varepsilon}$ is an eigenfunction of A_ε corresponding to $\lambda_{l,\varepsilon}$, $l\in\mathbb{N}$.

Now define the 'limit' bilinear form $\tau_0\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned}\tau_0(y,z) = {} & \frac{b_0\rho}{l_0\rho + b_0(1-\rho)}\, y_0 z_0 \\ & + \sum_{j=1}^{n-1} \frac{b_j}{2l_j}\,(y_j - y_{j-1})(z_j - z_{j-1}) + \frac{b_n\sigma}{l_n\sigma + b_n(1-\sigma)}\, y_{n-1} z_{n-1}\end{aligned}$$

and the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$ on $\mathbb{R}^n$ by

$$\langle y, z\rangle_{\mathbb{L}} = \sum_{j=0}^{n-1} L_j y_j z_j, \quad y = (y_0, \ldots, y_{n-1}),\ z = (z_0, \ldots, z_{n-1}) \in \mathbb{R}^n.$$

(3.2) Let $A_0\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by the pair $(\tau_0, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}})$.

The matrix representation of A_0 in terms of the standard basis on $\mathbb{R}^n$ is given as $M^{-1}B$, where $M = \operatorname{diag}(L_0, \ldots, L_{n-1})$ and B is the matrix

$$\begin{pmatrix} m_1 & r_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & m_2 & r_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & m_3 & r_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & r_3 & m_4 & r_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & r_{n-3} & m_{n-2} & r_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & r_{n-2} & m_{n-1} & r_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & r_{n-1} & m_n \end{pmatrix},$$

with

$$\begin{aligned} m_1 &= \frac{b_0\rho}{\rho l_0 + b_0(1-\rho)} + \frac{b_1}{2l_1}, \qquad m_n = \frac{b_{n-1}}{2l_{n-1}} + \frac{b_n\sigma}{\sigma l_n + b_n(1-\sigma)}, \\ m_k &= \frac{b_{k-1}}{2l_{k-1}} + \frac{b_k}{2l_k}, \quad k \in [2\,.\,.\, n-1], \end{aligned}$$

and $r_k = b_k/2l_k$, $k \in [1\,.\,.\, n-1]$. It follows that the map A_0 is $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-symmetric and all of its eigenvalues are simple. Denote by $(\mu_l)_{l\in[1..n]}$ the increasing sequence of eigenvalues of A_0 and by $(z_l)_{l\in[1..n]}$ a corresponding appropriately normalized $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{L}}$-orthogonal sequence of eigenvectors.

Lemma 3.1 and Theorem 3.5 from [5] imply the following spectral convergence result:

THEOREM 3.1. *With the above notation and hypotheses the following assertions hold:*

(a) $\lambda_{n+1,\varepsilon} \to \infty$ *as* $\varepsilon \to 0$.
(b) *For each* $l \in [1\,.\,.\, n]$, $\lambda_{l,\varepsilon} \to \mu_l$ *as* $\varepsilon \to 0$.
(c) *If the families* $(\varphi_{l,\varepsilon})_l$ *and* $(z_l)_{l\in[1..n]}$ *are properly chosen, then*

$$\sup_{x\in K_{j,\varepsilon}} |\varphi_{l,\varepsilon}(x) - z_{l,j}| \to 0, \quad \text{as } \varepsilon \to 0,$$

where $z_{l,j}$ is the j-th component of the vector z_l and

$$K_{j,\varepsilon} := [x_j + \varepsilon l_j, x_{j+1} - \varepsilon l_{j+1}], \qquad j \in [0\,.\,.\, n-1].$$

4. Conley index continuation and scalar reaction diffusion equations with large diffusion

In this section we use the notation of Sections 1 and 3. For each $\varepsilon \in \,]0, \varepsilon_0[$ define $H^\varepsilon = L^2$, $\langle \cdot\,,\cdot\rangle_{H^\varepsilon} = \langle \cdot\,,\cdot\rangle_{L^2}$ and A_ε as in (3.1). Define also $H^0 = \mathbb{R}^n$, $\langle \cdot\,,\cdot\rangle_{H^0} = \langle \cdot\,,\cdot\rangle_{\mathbb{L}}$ and A_0 as in (3.2).

In this note we will consider the following norms:

$$\begin{aligned} \|u\|_\varepsilon^2 &:= \tau_\varepsilon(u,u) + \|u\|_{L^2}^2, \quad \varepsilon \in \,]0,\varepsilon_0[\,,\ u \in H^1(0,1), \\ \|u\|_0^2 &:= \tau_0(u,u) + \|u\|_{\mathbb{L}}^2, \qquad u \in \mathbb{R}^n. \end{aligned} \tag{4.1}$$

Notice that for each $\varepsilon \in \,]0, \varepsilon_0[$, $H_1^\varepsilon = H^1(0,1)$ and $|\cdot|_{H_1^\varepsilon} = \|\cdot\|_\varepsilon$. Moreover, $H_1^0 = \mathbb{R}^n$ and $|\cdot|_{H_1^0} = \|\cdot\|_0$.

Theorem 4.1. *There exists an $\varepsilon_1' \in \,]0,\varepsilon_0[$ and a family $(J_\varepsilon)_{\varepsilon\in]0,\varepsilon_1']}$ such that the family $(H^\varepsilon, \langle \cdot\,,\cdot\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon_1']}$ satisfies condition* (FSpec).

To prove the existence of an embedding family $(J_\varepsilon)_{\varepsilon\in]0,\varepsilon_1']}$ let us establish some preliminary estimates. We have

$$\|v\|_\varepsilon^2 = \sum_{l=1}^{\infty} (\lambda_{l,\varepsilon}+1)|\langle v, \varphi_{l,\varepsilon}\rangle_{L^2}|^2, \quad v \in H^1(0,1) \text{ and } \varepsilon \in \,]0,\varepsilon_0[.$$

Moreover,

$$\|u\|_0^2 = \sum_{l=1}^{n} (\mu_l + 1)|\langle u, z_l\rangle_{\mathbb{L}}|^2, \quad u \in \mathbb{R}^n.$$

Proof of Theorem 4.1. It is clear that (1) and (2) of condition (FSpec) hold.

Define the embedding $J_\varepsilon \colon \mathbb{R}^n \to H^1(0,1)$ by

$$J_\varepsilon(u) = \sum_{l=1}^{n} \langle u, z_l\rangle_{\mathbb{L}} \varphi_{l,\varepsilon}, \quad u \in \mathbb{R}^n.$$

It follows that J_ε is $\mathbb{R}$-linear. Suppose that $J_\varepsilon(u) = 0$. Since $\varphi_{l,\varepsilon}$, $l \in [1\,.\,.\,n]$, are linearly independent, we have $\langle u, z_l\rangle_{\mathbb{L}} = 0$ for all $l \in [1\,.\,.\,n]$. Recall that $(z_l)_{l\in[1..n]}$ is an $\langle \cdot\,,\cdot\rangle_{\mathbb{L}}$-orthonormal basis of $\mathbb{R}^n$. Therefore, $u = \sum_{l=1}^{n} \langle u, z_l\rangle_{\mathbb{L}} z_l = 0$. Thus J_ε is injective.

Let $u \in \mathbb{R}^n$ and $v = J_\varepsilon(u) \in H_1^\varepsilon$. A quick calculation shows that

$$\|v\|_\varepsilon^2 = \sum_{l=1}^{n} (\lambda_{l,\varepsilon}+1)|\langle u, z_l\rangle_{\mathbb{L}}|^2.$$

It follows from Theorem 3.1 that there exist a constant $C \in]1,\infty[$ and an $\varepsilon_1' \in]0,\varepsilon_0[$ such that $0 \le \lambda_{l,\varepsilon}+1 \le C^2$ and $0 \le \mu_l+1 \le C^2$, and so $\lambda_{l,\varepsilon}+1 \le C^2(\mu_l+1)$ and $\mu_l+1 \le C^2(\lambda_{l,\varepsilon}+1)$ for $l \in [1..n]$ and $\varepsilon \in]0,\varepsilon_1']$. It follows that

$$\|u\|_0^2 \le C^2\|J_\varepsilon(u)\|_\varepsilon^2 \quad \text{and} \quad \|J_\varepsilon(u)\|_\varepsilon^2 \le C^2\|u\|_0^2 \tag{4.2}$$

for $u \in \mathbb{R}^n$ and $\varepsilon \in]0,\varepsilon_1']$. Now inequalities (4.2) imply (3) and (4) of condition (FSpec).

Let $(\varepsilon_m)_m$ be an arbitrary null sequence in $]0,\varepsilon_1']$. It follows from Theorem 3.1 that (5) (a) and (5) (b) of condition (FSpec) hold. To complete the proof we need to show that (5) (c) and (5) (d) of condition (FSpec) also hold.

Let $l \in [1..n]$ be arbitrary and define $\varepsilon_m^1 = \varepsilon_m$ for $m \in \mathbb{N}$. We have

$$\varphi_{l,\varepsilon_m} - J_{\varepsilon_m}(z_l) = \varphi_{l,\varepsilon_m} - \sum_{p=1}^{n}\langle z_l, z_p\rangle_{\mathbb{L}}\varphi_{p,\varepsilon_m} = \varphi_{l,\varepsilon_m} - \varphi_{l,\varepsilon_m} = 0.$$

Thus, we see that (5) (c) of condition (FSpec) holds. For $u \in \mathbb{R}^n = H_1^0$ and $m \in \mathbb{N}$ we have

$$\langle J_{\varepsilon_m}u, \varphi_{l,\varepsilon_m}\rangle_{H^{\varepsilon_m}} = \langle J_{\varepsilon_m}u, \varphi_{l,\varepsilon_m}\rangle_{L^2} = \sum_{p=1}^{n}\langle u, z_p\rangle_{\mathbb{L}}\langle\varphi_{p,\varepsilon_m}, \varphi_{l,\varepsilon_m}\rangle_{L^2} = \langle u, z_l\rangle_{\mathbb{L}}.$$

Therefore (5) (d) of condition (FSpec) holds. □

Now consider the following nonlinear convergence hypothesis:

Assumption 4.2. (a) For each $\varepsilon \in [0,\varepsilon_0[$ the function $g_\varepsilon\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous and such that for each $M \in]0,\infty[$ there exists an $L_M \in]0,\infty[$ such that for $|s| \le M$ and $|s'| \le M$

$$|g_\varepsilon(x,s) - g_\varepsilon(x,s')| \le L_M|s-s'|, \quad \text{for all } x \in [0,1],\ \varepsilon \in [0,\varepsilon_0[.$$

(b) There is an $\varepsilon_2' \in]0,\varepsilon_0[$ such that

$$\sup_{\varepsilon\in[0,\varepsilon_2']}\ \sup_{x\in[0,1]} |g_\varepsilon(x,0)| < \infty.$$

(c) For each $x \in [0,1]$ and $s \in \mathbb{R}$, $g_\varepsilon(x,s) \to g_0(x,s)$ as $\varepsilon \to 0$.

Let $\varepsilon \in]0,\varepsilon_0[$. Note that each $u \in H^1(0,1)$ is (uniquely represented by) a continuous function. Hence the map $\widehat{g}_\varepsilon(u)\colon [0,1]\to\mathbb{R}$ defined by

$$\widehat{g}_\varepsilon(u)(x) = g_\varepsilon(x,u(x)), \quad x \in [0,1],$$

is continuous and bounded. Moreover, $\widehat{g}_\varepsilon(u)$ is Lebesgue measurable and so it lies in $L^2(0,1)$. Therefore for each $\varepsilon \in]0,\varepsilon_0[$ we obtain a well-defined map $f_\varepsilon\colon H^1(0,1)\to L^2$ given by $f_\varepsilon(u) = \widehat{g}_\varepsilon(u)$, $u \in H^1(0,1)$. Moreover, define $f_0\colon \mathbb{R}^n\to\mathbb{R}^n$ by $f_0(u) = (f_0(u)_1,\dots,f_0(u)_n)$, where

$$f_0(u)_j = \frac{1}{L_j}\int_{K_j} g_0(x,u_j)\,dx,$$

$u = (u_1, \ldots, u_n)$, for $j \in [1\,..\,n]$.

Lemma 4.3. *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon'_1]}$ be as Theorem* 4.1. *Then there exist an $\varepsilon'_3 \in\ ]0, \varepsilon_0[$ and a $C'_1 \in\ ]0, \infty[$ such that for every $v \in H^1(0,1)$ and every $\varepsilon \in\ ]0, \varepsilon'_3]$,*

$$\sup_{x\in[0,1]} |v(x)| \le C'_1 \|v\|_\varepsilon.$$

Proof. Consider first the case $(\rho, \sigma) \neq (0, 0)$. It follows from Lemma 4.2 from [5] that there exist an $\varepsilon' \in\ ]0, \varepsilon_0[$ and a $C' \in\ ]0, \infty[$ such that for every $v \in H^1(0,1)$ and every $\varepsilon \in\ ]0, \varepsilon']$

$$\sup_{x\in[0,1]} |v(x)| \le C' \bigg(\int_0^1 a_\varepsilon \cdot (v')^2\, dx \bigg)^{1/2}.$$

Thus

$$\sup_{x\in[0,1]} |v(x)| \le C' \tau_\varepsilon(v,v)^{1/2} \le C' \|v\|_\varepsilon.$$

Define $\varepsilon'_3 = \varepsilon'$ and $C'_1 = C'$ for this case.

Now consider the case $(\rho, \sigma) = (0, 0)$. It follows from Lemma 4.2 from [5] that there exist an $\varepsilon'' \in\ ]0, \varepsilon_0[$ and a $C'' \in\ ]0, \infty[$ such that for every $\phi \in H^1(0,1)$ with $\phi \perp 1$ and every $\varepsilon \in\ ]0, \varepsilon'']$

$$\sup_{x\in[0,1]} |\phi(x)| \le C'' \bigg(\int_0^1 a_\varepsilon \cdot (\phi')^2\, dx \bigg)^{1/2}.$$

Let $v \in H^1(0,1)$. Hence there are a constant $\alpha \in \mathbb{R}$ and $\phi \in H^1(0,1)$ with $\phi \perp 1$ such that $u = \alpha + \phi$. Let $x \in [0,1]$. Then

$$\begin{aligned} |u(x)|^2 &\le 2\alpha^2 + 2|\phi(x)|^2 \le 2\alpha^2 + 2(C'')^2 \int_0^1 a_\varepsilon \cdot (\phi')^2\, dx \\ &= 2\alpha^2 + 2(C'')^2 \int_0^1 a_\varepsilon \cdot (u')^2\, dx \le \widehat{C} \bigg(\alpha^2 + \int_0^1 a_\varepsilon \cdot (u')^2\, dx \bigg), \end{aligned}$$

where $\widehat{C} = \max\{2, 2(C'')^2\}$. Recall that

$$\begin{aligned} \|u\|_\varepsilon^2 = \tau_\varepsilon(u,u) + \|u\|_{L^2}^2 &= \int_0^1 a_\varepsilon \cdot (u')^2\, dx + \int_0^1 (\alpha + \phi)^2\, dx \\ &= \int_0^1 a_\varepsilon \cdot (u')^2\, dx + \int_0^1 \alpha^2\, dx + 2\alpha \int_0^1 \phi\, dx + \int_0^1 \phi^2\, dx \\ &= \int_0^1 a_\varepsilon \cdot (u')^2\, dx + \alpha^2 + \int_0^1 \phi^2\, dx. \end{aligned}$$

Therefore, $|u(x)|^2 \le \widehat{C} \|u\|_\varepsilon^2$, for all $x \in [0,1]$. Define $\varepsilon'_3 = \varepsilon''$ and $C'_1 = (\widehat{C})^{1/2}$ for this case. □

Theorem 4.4. *Let $(H^\varepsilon, \langle\,\cdot\,,\,\cdot\,\rangle_{H^\varepsilon}, A_\varepsilon, J_\varepsilon)_{\varepsilon\in[0,\varepsilon'_1]}$ be as Theorem* 4.1. *There exists an $\varepsilon'_4 \in\ ]0, \varepsilon'_1]$ such that the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon'_4]}$ satisfies condition* (Conv).

Proof. Let $\varepsilon_4' = \min\{\varepsilon_3', \varepsilon_1', \varepsilon_2'\}$. Part (a) of condition (Conv) has just been proved. Let $M \in]0,\infty[$ be arbitrary. Let $\varepsilon \in]0,\varepsilon_3']$ and u, $v \in H_1^\varepsilon$ be such that $|u|_{H_1^\varepsilon}, |v|_{H_1^\varepsilon} \leq M$. It follows from Lemma 4.3 that

$$\sup_{x\in[0,1]} |u(x)| \leq C_1' M \quad \text{and} \quad \sup_{x\in[0,1]} |v(x)| \leq C_1' M.$$

Hence

$$\int_0^1 |g_\varepsilon(x,u(x)) - g_\varepsilon(x,v(x))|^2\,dx \leq L_{\widetilde{M}}^2 \int_0^1 |u(x)-v(x)|^2\,dx \leq L_{\widetilde{M}}^2 \|u-v\|_\varepsilon^2,$$

where $\widetilde{M} = C_1' M$. This implies that

$$|f_\varepsilon(u) - f_\varepsilon(v)|_{H^\varepsilon} \leq L_{\widetilde{M}} |u-v|_{H_1^\varepsilon}, \quad \text{for all } \varepsilon \in]0,\varepsilon_3'].$$

Moreover, let u, $v \in H_1^0$ satisfy $|u|_{H_1^0}, |v|_{H_1^0} \leq M$.

$$\begin{aligned}
\|f_0(u) - f_0(v)\|_{\mathbb{L}}^2 &= \sum_{j=1}^n L_j (f_0(u)_j - f_0(v)_j)^2 \\
&= \sum_{j=1}^n L_j \frac{1}{L_j^2} \bigg(\int_{K_j} (g_0(x,u_j) - g_0(x,v_j))\,dx \bigg)^2 \\
&= \sum_{j=1}^n \frac{1}{L_j} \bigg(\int_{K_j} |g_0(x,u_j) - g_0(x,v_j)|\,dx \bigg)^2 \\
&\leq \sum_{j=1}^n \frac{L_{M'}^2}{L_j} \bigg(\int_{K_j} |u_j - v_j|\,dx \bigg)^2 \\
&= L_{M'}^2 \sum_{j=1}^n L_j |u_j - v_j|^2 = L_{M'}^2 \|u-v\|_{\mathbb{L}}^2 \leq L_{M'}^2 \|u-v\|_0^2,
\end{aligned}$$

where $M' = M\Big(\min_{j\in[1..n]} L_j\Big)^{-1/2}$. This implies that

$$|f_0(u) - f_0(v)|_{H^0} \leq L_{M'} |u-v|_{H_1^0}.$$

It follows that part (c) of condition (Conv) holds.

Let C be as in formula (4.2). Let $\varepsilon \in]0,\varepsilon_4']$ be arbitrary. Then

$$\begin{aligned}
\|f_\varepsilon(J_\varepsilon(u))\|_{L^2} &\leq \|f_\varepsilon(J_\varepsilon(u)) - f_\varepsilon(0)\|_{L^2} + \|f_\varepsilon(0)\|_{L^2} \leq L_M \|J_\varepsilon(u)\|_\varepsilon + \|f_\varepsilon(0)\|_{L^2} \\
&\leq L_M C \|u\|_0 + \|f_\varepsilon(0)\|_{L^2} \leq L_M C\|u\|_0 + K,
\end{aligned}$$

where $M = C\|u\|_0$ and $K = \sup_{\varepsilon\in[0,\varepsilon_4']} \sup_{x\in[0,1]} |g_\varepsilon(x,0)|$. This implies that statement (d) of condition (Conv) holds.

To complete the proof we need to show that (b) of condition (Conv) holds. To this end we will use Theorem 2.2, which holds in the present case in view of

Theorem 4.1. We claim that:

(4.3) Let $u \in H_1^0 = \mathbb{R}^n$ and $t \in]0, \infty[$. Then

$$\lim_{\varepsilon \to 0^+} |e^{-tA_\varepsilon} f_\varepsilon(J_\varepsilon(u)) - J_\varepsilon(e^{-tA_0} f_0(u))|_{H_1^\varepsilon} = 0.$$

Let $(\varepsilon_m)_m$ be a null sequence in $]0, \varepsilon_4']$. Notice that $f_{\varepsilon_m}(J_{\varepsilon_m} u) \in H^{\varepsilon_m}$ for all $m \in \mathbb{N}$. It follows from (d) of condition (Conv) that

$$\sup_{m \in \mathbb{N}} |f_{\varepsilon_m}(J_{\varepsilon_m}(u))|_{H^{\varepsilon_m}} < \infty. \tag{4.4}$$

Theorem 3.1 implies that for each $l \in [1..n]$ and $j \in [1..n]$,

$$\sup_{x \in K_{j,\varepsilon_m}} |\varphi_{l,\varepsilon_m}(x) - z_{l,j}| \to 0, \quad \text{as } m \to \infty.$$

Let $l \in [1..n]$. We will show that

$$\langle f_{\varepsilon_m}(J_{\varepsilon_m} u), \varphi_{l,\varepsilon_m} \rangle_{L^2} \to \langle f_0(u), z_l \rangle_{\mathbb{L}} \quad \text{as } m \to \infty.$$

For each $m \in \mathbb{N}$ we have

$$\langle f_{\varepsilon_m}(J_{\varepsilon_m} u), \varphi_{l,\varepsilon_m} \rangle_{L^2} = \int_0^1 g_{\varepsilon_m}(x, (J_{\varepsilon_m} u)(x)) \varphi_{l,\varepsilon_m}(x)\, dx =: \sum_{j=1}^n \int_{K_j} T_j(x)\, dx,$$

where $T_j(x) = g_{\varepsilon_m}(x, (J_{\varepsilon_m} u)(x)) \varphi_{l,\varepsilon_m}(x)$, $x \in K_j$, $j \in [1..n]$. For $m \in \mathbb{N}$, $x \in K_j$ and $j \in [1..n]$ we have

$$\begin{aligned} T_j(x) &= \big(g_{\varepsilon_m}(x, (J_{\varepsilon_m} u)(x)) - g_{\varepsilon_m}(x, u_j)\big) \varphi_{l,\varepsilon_m}(x) + g_{\varepsilon_m}(x, u_j)(\varphi_{l,\varepsilon_m}(x) - z_{l,j}) \\ &\quad + (g_{\varepsilon_m}(x, u_j) - g_0(x, u_j)) z_{l,j} + g_0(x, u_j) z_{l,j} \\ &=: S^j_{1,m}(x) + S^j_{2,m}(x) + S^j_{3,m}(x) + S^j_{4,m}(x). \end{aligned}$$

Let $M \in]0, \infty[$ be a positive constant such that for all $\varepsilon \in]0, \varepsilon_4']$, $j \in [1..n]$, $x \in [0,1]$ and $m \in \mathbb{N}$,

$$\begin{aligned} |(J_\varepsilon u)(x)| \leq M, &\qquad |\varphi_{l,\varepsilon}(x)| \leq M, \\ |u_j| \leq M, &\quad |g_\varepsilon(x, u_j)| \leq M. \end{aligned}$$

Therefore,

$$|S^j_{1,m}(x)| \leq L_M |(J_{\varepsilon_m} u)(x) - u_j| M, \quad \text{for all } j \in [1..n],\ x \in [0,1] \text{ and } m \in \mathbb{N},$$

and

$$|S^j_{2,m}(x)| \leq M |\varphi_{l,\varepsilon_m}(x) - z_{l,j}|, \quad \text{for all } j \in [1..n],\ x \in [0,1] \text{ and } m \in \mathbb{N}.$$

Recall that

$$(J_{\varepsilon_m} u)(x) = \sum_{p=1}^n \langle u, z_p \rangle_{\mathbb{L}} \varphi_{p,\varepsilon_m}(x), \quad \text{for } x \in [0,1] \text{ and } m \in \mathbb{N}.$$

Let $j \in [1..n]$. Since $u_j = \sum_{p=1}^{n} \langle u, z_p \rangle_{\mathbb{L}} z_{p,j}$ we obtain

$$(J_{\varepsilon_m} u)(x) - u_j = \sum_{p=1}^{n} \langle u, z_p \rangle_{\mathbb{L}} (\varphi_{p,\varepsilon_m}(x) - z_{p,j}).$$

It follows from Theorem 3.1 and our choice of M that

$$\sup_{x \in K_{j,\varepsilon_m}} |J_{\varepsilon_m} u(x) - u_j| \to 0, \quad \text{as } m \to \infty,$$

and

$$\sup_{m \in \mathbb{N}} \sup_{x \in K_j} |J_{\varepsilon_m} u(x) - u_j| < \infty.$$

Since the Lebesgue measure of $K_j \setminus K_{j,\varepsilon_m}$ goes to zero as $m \to \infty$ it follows that

$$\int_{K_j} S^j_{1,m}(x)\, dx \to 0, \quad \text{as } m \to \infty.$$

Similarly we show that

$$\sup_{x \in K_{j,\varepsilon_m}} |S^j_{2,m}(x)| \to 0 \quad \text{as } m \to \infty \quad \text{and} \quad \sup_{m \in \mathbb{N}} \sup_{x \in K_j} |S^j_{2,m}(x)| < \infty.$$

Hence

$$\int_{K_j} S^j_{2,m}(x)\, dx \to 0 \quad \text{as } m \to \infty.$$

Since $g_\varepsilon(x, s) \to g_0(x, s)$ as $\varepsilon \to 0$ and $\sup_{m \in \mathbb{N}} \sup_{x \in K_j} |g_{\varepsilon_m}(x, u_j)| < \infty$, the Lebesgue Dominated Convergence Theorem implies that

$$\int_{K_j} S^j_{3,m}(x)\, dx \to 0 \quad \text{as } m \to \infty.$$

Finally

$$\int_{K_j} S^j_{4,m}(x)\, dx = \int_{K_j} g_0(x, u_j) z_{l,j}\, dx = L_j (f_0(u))_j z_{l,j}.$$

Thus

$$\sum_{j=1}^{n} \int_{K_j} S^j_{4,m}(x)\, dx = \langle f_0(u), z_l \rangle_{\mathbb{L}}$$

and so

$$\langle f_{\varepsilon_m}(J_{\varepsilon_m} u), \varphi_{l,\varepsilon_m} \rangle_{L^2} \to \langle f_0(u), z_l \rangle_{\mathbb{L}} \quad \text{as } m \to \infty.$$

This together with (4.4) and Theorem 2.2 implies that

$$\left| e^{-tA_{\varepsilon_m}} f_{\varepsilon_m}(J_{\varepsilon_m} u) - J_{\varepsilon_m}(e^{-tA_0} f_0(u)) \right|_{H_1^{\varepsilon_m}} \to 0 \quad \text{as } m \to \infty.$$

This proves claim (4.3). □

By the results of Section 2 we may now consider, for each $\varepsilon \in\ ]0, \varepsilon'_4]$, the abstract parabolic equation

$$\dot{u} = -A_\varepsilon u + f_\varepsilon(u) \tag{4.5}$$

on $H^1(0,1)$. This equation generates a local semiflow π_ε on $H^1(0,1)$. Equation (4.5) is an abstract formulation of the boundary value problem

$$(E_\varepsilon, S_\varepsilon) \qquad \begin{cases} u_t = (a_\varepsilon u_x)_x + g_\varepsilon(x,u), & 0 < x < 1,\ t > 0, \\ \rho u - (1-\rho) a_\varepsilon u_x = 0, & x = 0,\ t > 0, \\ \sigma u + (1-\sigma) a_\varepsilon u_x = 0, & x = 1,\ t > 0. \end{cases}$$

Moreover, we may also consider the system of ordinary diferential equations

$$\dot{z} = -A_0 z + f_0(z) \tag{4.6}$$

on $\mathbb{R}^n$. This system generates a local (semi)flow π_0 on $\mathbb{R}^n$. We now conclude that

Theorem 4.5. *The Conley index and homology index braid continuation results, Theorems* 2.4 *and* 2.5*, hold for the family* $(\pi_\varepsilon)_{\varepsilon \in [0,\varepsilon'_4]}$.

Remark 4.6. If f is *sufficiently smooth and dissipative*, then by results in [5] the semiflows π_ε generated by $(E_\varepsilon, S_\varepsilon)$ have a global attractor $\mathcal{A}_\varepsilon$ and the limit flow π_0 of the limit equation (4.6) has a global attractor $\mathcal{A}_0$. If, in addition, π_0 is *structurally stable on* $\mathcal{A}_0$, then by Corollary 4.3 in [5], for small ε, the flow π_ε on $\mathcal{A}_\varepsilon$ is equivalent to the flow π_0 on $\mathcal{A}_0$, and this in particular implies the assertions of Theorem 4.5. Note, however, that we do not make any dissipativeness or structural stability assumptions here.

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References

[1] M.C. Carbinatto and K.P. Rybakowski, *Continuation of the connection matrix in singular perturbation problems*, Ergodic Theory Dynam. Systems **26** (2006), 1021–1059.

[2] M.C. Carbinatto and K.P. Rybakowski, *Localized singularities and Conley index*, Topol. Methods Nonlinear Anal. **37** (2011), 1–36.

[3] M.C. Carbinatto and K.P. Rybakowski, *On convergence and compactness in parabolic problems with globally large diffusion and nonlinear boundary conditions*, Topol. Methods Nonlinear Anal. **40** (2012), 1–28.

[4] A.N. Carvalho and J. Hale, *Large diffusion with dispersion*, Nonlinear Anal. **17** (1991), 1139–1151.

[5] A.N. Carvalho and A.L. Pereira, *A scalar parabolic equation whose asymptotic behavior is dictated by a system of ordinary differential equations*, J. Differential Equations **112** (1994), 81–130.

[6] C.C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Amer. Math. Soc., vol. 38, Providence, 1978.

[7] G. Fusco, *On the explicit construction of an* ODE *which has the same dynamics as scalar parabolic* PDE, J. Differential Equations **69** (1987), 85–110.

[8] J. Hale, *Large diffusivity and asymptotic behavior in parabolic systems*, J. Math. Anal. Appl. **118** (1986), 455–466.

[9] J. Hale and C. Rocha, *Varying boundary conditions with large diffusivity*, J. Math. Pures Appl. **66** (1987), 139–158.

[10] A. Rodríguez-Bernal, *Localized spatial homogenization and large diffusion*, SIAM J. Math. Anal. **29** (1998), 1361–1380.

[11] A. Rodríguez-Bernal and R. Willie, *Singular large diffusivity and spatial homogenization in a non homogeneous linear parabolic equation*, Discrete Contin. Dyn. Syst. Ser. B **5** (2005), 385–409.

[12] K.P. Rybakowski, *On the homotopy index for infinite-dimensional semiflows*, Trans. Amer. Math. Soc. **269** (1982), 351–382.

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