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ON A CLASS OF COCYCLES HAVING ATTRACTORS WHICH CONSIST OF SINGLETONS

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Abstract. We give a new simple sufficient condition for existence of the global pullback attractor which consists of singletons for general cocycle mappings on an arbitrary complete metric space. In particular, we need not have any structure on a parameter space, so the criterion can be applied in both cases: nonautonomous as well as random dynamical systems. Our considerations lead us also to new large class of iterated function systems with point-fibred attractors.

1. Introduction

In the theory of nonautonomous as well as random dynamical systems the notion of a cocycle mapping is fundamental. A cocycle is, roughly speaking, a mapping which acts on the product of a parameter space and a phase space (usually of different nature), inducing an autonomous skew product (semi)flow. On a parameter space an autonomous (semi)flow is given. In general, it can be interpreted as perturbations or a noise. The reader interested in applications of nonautonomous/random dynamical systems can find a vast literature, for example cited in monographs [3], [9] and [19].

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The notion of attractor is one of the basic concepts in the theory of dynamical systems. Such a set attracts, in some sense, sets from a desired class of subsets of phase space and determines the long-term behavior of a dynamical system. Attractors of classical autonomous dynamical systems are investigated since many years, and different types of attractors of nonautonomous as well as random dynamical systems are intensively studied during last two decades (see for example [10], [19], and also the recent paper [6] for details and the references therein). In particular, so-called pullback (cocycle) attractors are known as the best tool bringing much information on asymptotic behavior of such systems. Cocycle attractors are typically families of (compact) sets contrary to the case of autonomous systems when the unique set is considered as an attractor. On the other hand, attractors obtained in many examples of nonautonomous/random systems consist of singletons, so asking on sufficient conditions when a cocycle has a pullback attractor being a family of singletons seems to be natural. It is interesting that such attractors could be also strange nonchaotic ones (see [17]).

Our study was primary inspired by papers [1] and [2], when some curious particular systems on a real interval satisfying some kind of contractivity on fibers were studied. In the present paper we introduce the notion of a pullback uniform contractivity of cocycles on fibers and we show the existence of a pullback attractor which is a single point on each fiber. It is weaker than condition of uniform contractivity considered in our previous papers. It was fruitfully explored first in [21] by A. Lasota and J. Myjak in the context of iterated function systems and in [13], [14] by the present author in the context of discrete as well as general cocycles, respectively. Our methodology is different then the standard one (cf. [10] and [19]), since we mostly deal with topological limits as the main tool. This approach let us to omit every standard assumptions on a parameter space — it is supposed to be a non-void set only (cf. [3] and [19]).

The organization of the rest of the paper is the following. In the next section we introduce the notion and properties of topological (Kuratowski's) limits of nets of subsets of a metric space. In Section 3 we define a general cocycle mapping in the common way. Section 4 contains the main result of the paper (Theorem 4.4) and in Section 5 we show Proposition 5.3 on existence of a pullback cocycle attractor consisting of singletons. In particular, in the considered realities the standard assumption on the existence of the so-called absorbing compact set can be weakened. In the last section (Section 6) we consider a special kind of discrete cocycles, namely iterated function systems. Usually they are studied by using completely different methods. But by unifying presented here and some our older results we obtain the new comprehensive class of iterated function systems for which there exists a point-fibred attractor (cf. [5, Theorem 6.2], see also [4] and [23]). In particular, the main result from [16] can be obtained as a corollary.

2. Topological limits

Let (X, ϱ) be a metric space. By $B^o(x, \varepsilon)$ we denote the open ball with center x and radius ε, $\operatorname{cl} A$ stays for the closure of A and $\operatorname{int} A$ for its interior.

We need to recall some basic definitions and notions on nets of points and sets (see [8, Chapter 2]). Let $(\Sigma, \leq)$ be a directed set, i.e. a set with a partial order satisfying the following property: for all $\sigma_1, \sigma_2 \in \Sigma$ there is $\sigma \in \Sigma$ such that $\sigma_1 \leq \sigma$ and $\sigma_2 \leq \sigma$.

Any mapping $S \colon \Sigma \ni \sigma \mapsto x_\sigma \in X$ we call a *net* (or a *generalized sequence*) and we denote $S = (x_\sigma)_{\sigma\in\Sigma}$.

We say that x is a *limit* of a net $(x_\sigma)_{\sigma\in\Sigma}$ (or a net $(x_\sigma)_{\sigma\in\Sigma}$ *converges* to x) if for every $\varepsilon > 0$ there is σ_0 such that $x_\sigma \in B^o(x, \varepsilon)$ for every $\sigma \geq \sigma_0$. Therefore, we denote $x = \lim_{\sigma\in\Sigma} x_\sigma$. It is known that $x \in \operatorname{cl} A$ if and only if there exists a net $(x_\sigma)_{\sigma\in\Sigma}$ of elements of A converging to x.

We say that a set $\Sigma' \subset \Sigma$ is *cofinal* with a directed set Σ if for every $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ such that $\sigma \leq \sigma'$.

A net $S' = (x_{\sigma'})_{\sigma'\in\Sigma'}$ is a *subnet* of a net $S = (x_\sigma)_{\sigma\in\Sigma}$ if there exists a non-increasing mapping $\alpha \colon \Sigma' \to \Sigma$ (namely, $\sigma_1 \leq \sigma_2$ implies $\alpha(\sigma_1) \leq \alpha(\sigma_2)$) such that $\alpha(\Sigma')$ is cofinal with Σ and $x_{\alpha(\sigma')} = x_{\sigma'}$ for every $\sigma' \in \Sigma'$.

A limit of a subnet S' of a net S is called a *cluster point*. It can be proved that A is a closed set if and only if it contains all cluster points of each net of its elements.

Any mapping $S \colon \Sigma \to 2^X$, $\sigma \mapsto A_\sigma$ is called a *net of subsets* of X and we denote $S = (A_\sigma)_{\sigma\in\Sigma}$. We say that a set $U \subset X$ intersects *almost all* (or *eventually*) sets A_σ if there is $\sigma_0 \in \Sigma$ such that $A_\sigma \cap U \neq \emptyset$, for every $\sigma \geq \sigma_0$, and we say that U intersects *infinitely many* (or *frequently*) sets A_σ if for every $\sigma_0 \in \Sigma$ there is $\sigma \geq \sigma_0$ such that $A_\sigma \cap U \neq \emptyset$ holds.

We define the *lower limit* (or *interior limit*) $\liminf_\sigma A_\sigma$ and the *upper limit* (or *exterior limit*) $\limsup_\sigma A_\sigma$ as follows: $x \in \liminf_\sigma A_\sigma$ if for every $\varepsilon > 0$ the ball $B^o(x, \varepsilon)$ intersects almost all sets A_σ, and $x \in \limsup_\sigma A_\sigma$ if for every $\varepsilon > 0$ the ball $B^o(x, \varepsilon)$ intersects infinitely many sets A_σ. If both limits are equal we say that the net $(A_\sigma)_{\sigma\in\Sigma}$ is *topologically convergent*. We denote this common limit as $\lim_\sigma A_\sigma$ and call it a *topological limit* of this net.

It is clear that $\liminf_\sigma A_\sigma \subset \limsup_\sigma A_\sigma$ and $\liminf_\sigma A_\sigma = \liminf_\sigma \operatorname{cl} A_\sigma$ (the same is valid for the upper limit). Notice that $\liminf_\sigma A_\sigma$ and $\limsup_\sigma A_\sigma$ are closed sets. Moreover if $A = \operatorname{cl} A$ and $A_\sigma \subset A$ for every $\sigma \in \Sigma$ then $\liminf_\sigma A_\sigma \subset A$ and

$\limsup_{\sigma} A_\sigma \subset A$. Other properties of topological limits can be found in [8] and, in the case of countable sequences, in [20].

Given a net $(A_\sigma)_{\sigma\in\Sigma}$ of subsets of X, define a set

$$\mathcal{L} := \bigcap_{\sigma\in\Sigma} \mathrm{cl}\Big(\bigcup_{\delta\geq\sigma} A_\delta\Big). \tag{2.1}$$

The following useful characterization is known (see, for example, [14, Lemma 2.1]).

Lemma 2.1. *If $(A_\sigma)_{\sigma\in\Sigma}$ is a net of subsets of X, then $\mathcal{L} = \limsup_{\sigma} A_\sigma$.*

3. Cocycles and skew product semiflows

In what follows let $\mathbb{T}$ be a subgroup of the additive group $(\mathbb{R},+)$ of all reals containing as a subsemigroup the set $\mathbb{N}$ of all positive integers. Let $\mathbb{T}^+ := \mathbb{T}\cap(0,\infty)$. We consider the sets $\mathbb{T}$ and $\mathbb{T}^+$ as directed sets with natural order induced from the real line.

First we are going to define a cocycle mapping with some base map and fiber maps, and also an induced skew product semiflow. Let Ω be a nonempty set and (X,ϱ) be an arbitrary metric space. Further Ω is called the *base space* and X the *fiber space* or the *phase space*. Let $\theta = \{\theta_t\colon \Omega\to\Omega : t\in\mathbb{T}\}$ be a group of bijective transformations i.e.

$$\theta_{s+t} = \theta_t\circ\theta_s \quad \text{for } s,t\in\mathbb{T} \quad \text{and} \quad \theta_0 = \mathrm{id}_\Omega. \tag{3.1}$$

The group θ is called a *base flow*. Consider the mapping $\varphi\colon \mathbb{T}^+\times\Omega\to X^X$ satisfying the following equation:

$$\varphi(s+t,\omega) = \varphi(t,\theta_s\omega)\circ\varphi(s,\omega) \quad \text{for } s,t\in\mathbb{T}^+ \quad \text{and} \quad \omega\in\Omega. \tag{3.2}$$

Throughout the paper we will assume that every function $\varphi(t,\omega)\colon X\to X$ is continuous. This assumption will not be repeated. A pair (θ,φ) we call a *cocycle* (over θ).

Observe that the given cocycle (θ,φ) induces a *skew product semigroup* of mapping of $\Omega\times X$ into itself given by

$$\Theta_t(\omega,x) = (\theta_t\omega, \varphi(t,\omega)(x)) \quad \text{for } t\in\mathbb{T}^+,$$

i.e. $\Theta_{s+t} = \Theta_t\circ\Theta_s$ for all $s,t\in\mathbb{T}^+$.

Given a cocycle (θ,φ), we define in the standard way the following *limit set* for an $\omega\in\Omega$ and a subset D of X:

$$\mathcal{L}(\omega,D) := \bigcap_{t\in\mathbb{T}^+} \mathrm{cl}\Big(\bigcup_{s\geq t} \varphi(s,\theta_{-s}\omega)(D)\Big).$$

Observe that, by Lemma 2.1,

$$\mathcal{L}(\omega,D) = \limsup_{t} \varphi(t,\theta_{-t}\omega)(D). \tag{3.3}$$

Then define a family $\mathcal{A} = \{\mathcal{A}_\omega : \omega \in \Omega\}$ by

$$\mathcal{A}_\omega := \mathrm{cl}\Big(\bigcup_D \mathcal{L}(\omega, D)\Big) \quad \text{for } \omega \in \Omega, \tag{3.4}$$

where the sum on the right-hand side is taken over all bounded subsets D of X.

4. Uniform contractivity on fibers

We say that the cocycle (θ, φ) is *pullback uniformly contractive* if for every nonempty bounded subset D of X and every $\varepsilon > 0$ there is a $t_0 = t_0(\varepsilon, D) \in \mathbb{T}^+$ such that

$$\mathrm{diam}(\varphi(t, \theta_{-t}\omega)(D)) < \varepsilon \quad \text{for all } t \geq t_0 \text{ and } \omega \in \Omega. \tag{4.1}$$

The cocycle (θ, φ) is said to be *pullback uniformly contractive on fibers* if for every $\omega \in \Omega$, every nonempty bounded subset D of X and every $\varepsilon > 0$ there is $t_0 = t_0(\varepsilon, \omega, D) \in \mathbb{T}^+$ such that for every $t \geq t_0$ condition (4.1) is satisfied. Clearly a pullback uniform contractive cocycle is also pullback uniformly contractive on fibers.

We prove the following result using the similar techniques as in [13, Theorem 4.1] and [14, Theorem 4.4].

Proposition 4.1. *Assume that the cocycle (θ, φ) is pullback uniformly contractive on fibers. Then for all nonempty and bounded subsets A, B of X*

$$\liminf_t \varphi(t, \theta_{-t}\omega)(A) = \liminf_t \varphi(t, \theta_{-t}\omega)(B), \tag{4.2}$$

$$\limsup_t \varphi(t, \theta_{-t}\omega)(A) = \limsup_t \varphi(t, \theta_{-t}\omega)(B) \tag{4.3}$$

for every $\omega \in \Omega$.

Proof. Let $A, B \subset X$ be nonempty and bounded and fix $\omega \in \Omega$. Owing to the symmetry of conditions (4.2) and (4.3) it is sufficient to show that

$$\liminf_t \varphi(t, \theta_{-t}\omega)(A) \subset \liminf_t \varphi(t, \theta_{-t}\omega)(B),$$
$$\limsup_t \varphi(t, \theta_{-t}\omega)(A) \subset \limsup_t \varphi(t, \theta_{-t}\omega)(B).$$

We prove the first inclusion. The proof of the second one is similar.

Fix $u \in \liminf_t \varphi(t, \theta_{-t}\omega)(A)$ and $\varepsilon > 0$. By the definition of the lower limit, there exists $s_0 \in \mathbb{T}^+$ such that for every $t \geq s_0$,

$$\varphi(t, \theta_{-t}\omega)(A) \cap B^o(u, \varepsilon/2) \neq \emptyset. \tag{4.4}$$

Let now $t_0 = t_0(\varepsilon/2, A \cup B)$ be a number from $\mathbb{T}^+$ corresponding to the sum $A \cup B$ and to $\varepsilon/2$ according to the definition of the uniform contractivity on fibers. Put $\tau_0 := \max\{s_0, t_0\}$ and fix $t \geq \tau_0$. By (4.4) there is a point $w \in \varphi(t, \theta_{-t}\omega)(A)$ such that $\varrho(w, u) < \varepsilon/2$. Therefore there is $x \in A$ such that $w = \varphi(t, \theta_{-t}\omega)(x)$.

Take an arbitrary $y \in B$ and set $v = \varphi(t, \theta_{-t}\omega)(y)$. Since $x, y \in A \cup B$ and $t \geq \tau_0 \geq t_0$, the uniform contractivity implies that $\varrho(w, v) < \varepsilon/2$. Consequently, $\varrho(u, v) < \varepsilon$. Since $v \in \varphi(t, \omega)(B)$, it follows that

$$\varphi(t, \theta_{-t}\omega)(B) \cap B^o(u, \varepsilon/2) \neq \emptyset.$$

It holds for every $t \geq \tau_0$, therefore from the fact that $\varepsilon > 0$ was arbitrary it follows that $u \in \liminf_t \varphi(t, \theta_{-t}\omega)(B)$. □

The proposition above and equality (3.3) give us immediately

Corollary 4.2. *If the cocycle (θ, φ) is pullback uniformly contractive on fibers, then*

$$\mathcal{A}_\omega = \mathcal{L}(\omega, D) \quad \textit{for } \omega \in \Omega$$

for every nonempty bounded subset D of X, where $\mathcal{A}_\omega$ is defined by (3.4).

Remark 4.3. Unfortunately in general the set $\mathcal{L}(\omega, D)$ can be empty for some $\omega \in \Omega$.

The main result of this section shows that under some weak and natural assumptions sets $\mathcal{A}_\omega$ given by (3.4) are non-empty and consist of singletons.

Theorem 4.4. *Let (X, ϱ) be a complete metric space and (θ, φ) be a cocycle pullback uniformly contractive on fibers. Suppose that there exists a nonempty bounded subset A of X such that*

$$\varphi(t, \omega)(A) \subset A \quad \textit{for } t \in \mathbb{T}^+ \textit{ and } \omega \in \Omega. \tag{4.5}$$

Then for every $\omega \in \Omega$ there is a unique point $x_\omega \in X$ such that $\mathcal{A}_\omega = \{x_\omega\}$.

Proof. Let A be a nonempty bounded set satisfying (4.5). Fix $\omega \in \Omega$. Due to Corollary 4.2 it suffices to prove that the limit set $\mathcal{L}(\omega, A)$ is a singleton. To this end we show that

$$\varphi(t_2, \theta_{-t_2}\omega)(A) \subset \varphi(t_1, \theta_{-t_1}\omega)(A) \quad \text{for } t_1 < t_2,\ t_1, t_2 \in \mathbb{T}^+. \tag{4.6}$$

Indeed, if $t_1 < t_2$ then there is $\tau \in \mathbb{T}^+$ such that $t_2 = t_1 + \tau$ and hence, using condition (4.5) and the cocycle equation (3.2), we get

$$\begin{aligned}\varphi(t_2, \theta_{-t_2}\omega)(A) &= \varphi(t_1 + \tau, \theta_{-(t_1+\tau)}\omega)(A) \\ &= \varphi(t_1, \theta_{-t_1}\omega) \circ \varphi(\tau, \theta_{-(t_1+\tau)}\omega)(A) \subset \varphi(t_1, \theta_{-t_1}\omega)(A).\end{aligned}$$

Formula (4.6) implies that for $t \in \mathbb{T}^+$ we have

$$\bigcup_{s \geq t} \varphi(s, \theta_{-s}\omega)(A) = \varphi(t, \theta_{-t}\omega)(A),$$

and, by the definition of a limit set,

$$\mathcal{L}(\omega, A) = \bigcap_{t \in \mathbb{T}^+} \operatorname{cl}\big(\varphi(t, \theta_{-t}\omega)(A)\big).$$

Formula (4.6) implies also that sets $\mathrm{cl}(\varphi(t,\theta_{-t}\omega)(A))$, $t \in \mathbb{T}^+$ form a decreasing family of closed sets with diameters tending to zero as $t \to \infty$, by uniform contractivity on fibers. It means, by completeness of X, that there is exactly one point $x_\omega \in X$ such that $\mathcal{L}(\omega, A) = \{x_\omega\}$, and this completes the proof. □

Remark 4.5. It is clear that if X is compact, then the whole space X satisfies condition (4.5).

A mapping $\Omega \ni \omega \mapsto x_\omega \in X$ is said to be an *equilibrium* of the cocycle (θ, φ) if

$$\varphi(t,\omega)(x_\omega) = x_{\theta_t\omega} \quad \text{for } t \in \mathbb{T}^+ \text{ and } \omega \in \Omega$$

(cf. [9, Definition 1.7.1]).

Theorem 4.4 gives us immediately

Corollary 4.6. *If $\mathcal{A}_\omega = \{x_\omega\}$ for $\omega \in \Omega$ are as in Theorem 4.4, then the mapping $\omega \mapsto x_\omega$ is an equilibrium of the cocycle (θ, φ).*

Proof. Indeed, using standard arguments (see, e.g., [10, Lemma 3.2]) one can see that $\mathcal{A}_\omega$, $\omega \in \Omega$, as limit sets satisfy $\varphi(t,\omega)(\mathcal{A}_\omega) \subset \mathcal{A}_{\theta_t\omega}$ for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$. But since $\mathcal{A}_\omega$ is a singleton for every $\omega \in \Omega$, the last inclusion can be replaced by the equality. □

5. Pullback attractors of cocycles

In the theory of nonautonomous as well as random dynamical systems the following definition arises (see, for example, [19] and [10]). The family $A = \{A_\omega : \omega \in \Omega\}$ of compact subsets of X is called a (*global*) *pullback attractor* of the cocycle (θ, φ) if

(i) it is *strictly invariant*, i.e. $\varphi(t,\omega)(A_\omega) = A_{\theta_t\omega}$ for all $t \in \mathbb{T}^+$ and $\omega \in \Omega$,

(ii) it is *pullback attracting*, i.e.

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta_{-t}\omega)(D), A_\omega) = 0$$

for every $\omega \in \Omega$ and every bounded subset D of X.

Here dist denotes usual Hausdorff semidistance, i.e. $\mathrm{dist}(A,B) = \sup_{x\in A} \varrho(x,B)$.

Remark 5.1. In general the upper limit needs not attract elements of a given net of sets. Consider, for example, a sequence $(\{a_n\})_{n\in\mathbb{N}}$ of singletons such that

$$a_{2k-1} = 0 \quad \text{and} \quad a_{2k} = k \quad \text{for } k \in \mathbb{N}.$$

Hence $\limsup_n \{a_n\} = \{0\}$ and $\lim_{n\to\infty} \mathrm{dist}(\{a_n\},\{0\}) \neq 0$.

Lemma 5.2. *Consider a net $(A_t)_{t\in\mathbb{T}^+}$ of subsets of X and let $\mathcal{L} = \limsup_t A_t$. If there exits a compact set $K \subset X$ and $t_0 \in \mathbb{T}^+$ such that $A_t \subset K$ for every $t \geq t_0$, then $\lim_{t\to\infty} \mathrm{dist}(A_t, \mathcal{L}) = 0$.*

Proof. Suppose *a contrario* that there is an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of numbers from $\mathbb{T}^+$, elements $a_{t_n} \in A_{t_n}$ and $\delta > 0$ such that

$$\varrho(a_{t_n}, \mathcal{L}) \geq \delta \tag{5.1}$$

for sufficiently large $n \in \mathbb{N}$. But for sufficiently large $n \in \mathbb{N}$ we have $A_{t_n} \subset K$, which implies that the sequence $(a_{t_n})_{n\in\mathbb{N}}$ has a subsequence convergent to some element $a \in K$. Finally, by the definition of upper limit, $a \in \mathcal{L} \subset K$, and this contradicts (5.1). □

If for a given net there exists a compact a set K satisfying conditions of Lemma 5.2, it is said to be *absorbing* for this net.

Proposition 5.3. *Assume that (X, ϱ) is a complete metric space and (θ, φ) is a cocycle pullback uniformly contractive on fibers. If there exists a bounded subset A of X satisfying condition* (4.5) *and for all bounded set $D \subset X$ and $\omega \in \Omega$ there is a compact set $K(\omega, D)$ absorbing the net $(\varphi(t, \theta_{-t})(D))_{t\in\mathbb{T}^+}$, then the family $\mathcal{A} = \{\mathcal{A}_\omega : \omega \in \Omega\}$ of singletons $\mathcal{A}_\omega = \{x_\omega\}$ obtained as in Theorem* 4.4 *is a pullback attractor of (θ, φ).*

Proof. In view of Corollary 4.6 it is enough to see that the considered family is attracting. Hence, using Corollary 4.2 and Lemma 5.2 one can see that $\mathcal{A} = \{\mathcal{A}_\omega : \omega \in \Omega\}$ as a family of upper limits is pullback attracting. Precisely,

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega)(D), A_\omega) = \lim_{t\to\infty} \operatorname{dist}\Big(\varphi(t, \theta_{-t}\omega)(D), \limsup_{t} \varphi(t, \theta_{-t}\omega)(D)\Big) = 0. \qquad \square$$

Remark 5.4. (a) Typically in existence results it is supposed that compact absorbing sets depend only on $\omega \in \Omega$ and are universal for all bounded sets (see [10, Theorem 3.11]) or even such set is supposed to be unique (see [19, Theorem 3.20]), so our result is less restrictive.

(b) It is obvious that if X is a compact space, then the whole X is a universal absorbing set.

6. Point fibred attractors of iterated function systems

One of the most prominent examples of (discrete) cocycles are iterated function systems (IFSs, for short; see [13, Example 3.1], also [19, Example 2.10]). Namely, consider an arbitrary nonempty set Σ and a family of continuous mappings $\{S_\sigma \colon X \to X : \sigma \in \Sigma\}$. Such a family is called an *iterated function system*. Let now $\Omega = \Sigma^{\mathbb{Z}}$ be a set of all bi-infinite sequences on Σ and $\theta \colon \Omega \to \Omega$ be a left shift operator, i.e. for $\omega = (\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots)$, $(\theta\omega)(n) = \omega(n+1)$, where $\omega(k)$ denotes the k-th term of the sequence ω.

If now, for $\omega \in \Omega$, $\varphi(1,\omega) := S_{\sigma_1}$, and $\varphi(n,\omega) := S_{\sigma_n} \circ \ldots \circ S_{\sigma_1}$, for every $n \geq 2$, the pair (θ, φ) is a discrete cocycle (over the shift θ). Moreover, observe that

$$\varphi(n, \theta^{-n}\omega) = S_{\sigma_1} \circ \ldots \circ S_{\sigma_n} \quad \text{for } n \in \mathbb{N}.$$

This formula describes the so-called inverse iterations or inverse process considered by many authors (see, for example [11], [18] and the references therein).

Consider an IFS $\{S_\sigma(x) \colon X \to X : \sigma \in \Sigma\}$ and the induced cocycle (θ, φ) as above. Denote by F the *Barnsley–Hutchinson multifunction* given by $F(x) = \{S_\sigma(x) : \sigma \in \Sigma\}$. Denote moreover by F^n the n-th iterate of F. If there exists the common topological limit $A_* = \lim_n F^n(D)$ independent of a bounded nonvoid set $D \subset X$ then it is called the *Lasota–Myjak attractor* (see [15]). Notice that the notion of Lasota–Myjak attractor coincides with the notion of global strict attractor of an IFS (see [5]) in the case of compact metric space.

It was proved in [14, Theorem 4.5] (cf. also [21, Theorem 4.2] and [13, Theorem 4.4]) that, under the assumption of pullback uniform contractivity and the existence of convergent trajectory any cocycle mapping admits the Lasota–Myjak attractor. More precisely, we have obtained the following result in the case of iterated function systems.

Proposition 6.1. *Assume that an iterated function system $\{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ of continuous selfmappings of a metric space X is such that*

(a) *for every $\varepsilon < 0$ there is an integer $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every bounded set $D \subset X$ we have*

$$\operatorname{diam}(S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(D)) < \varepsilon \quad \textit{for all } \sigma_1, \ldots, \sigma_n \in \Sigma,\ n \in \mathbb{N};$$

(b) *there is a convergent trajectory, namely there exists the limit*

$$\lim_{n\to\infty} S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(x_0) \quad \textit{for some } x_0 \in X.$$

Then there exists the Lasota–Myjak attractor A_*.

Remark 6.2. It can be shown that A_* satisfies condition (4.5). Indeed $F(A_*) \subset A_*$ (see [15, Theorem 5.7] and [21, Theorem 3.1]), so also $F^n(A_*) \subset A_*$ for every $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$, $\omega \in \Sigma^{\mathbb{Z}}$ we have

$$\varphi(n,\omega)(A_*) \subset \{\varphi(n,\omega)(A_*) : \omega \in \Omega\} = F^n(A_*) \subset A_*.$$

In view of Proposition 6.1 and Theorem 4.4 we can get the theorem which generalizes [5, Theorem 6.2]. It is remarkable that in that theorem only finite IFSs are considered.

Theorem 6.3. *Assume that an iterated function system $\{S_\sigma \colon X \to X : \sigma \in \Sigma\}$ of continuous selfmappings of a complete metric space X satisfies assumptions* (a) *and* (b) *of Proposition* 6.1. *Let A_* be its Lasota–Myjak attractor.*

If A_* is a bounded set then the sets

$$\mathcal{A}_\omega = \bigcap_{n\in\mathbb{N}} \mathrm{cl}\,(S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(A_*)) \quad \textit{for } \omega = (\ldots \sigma_{-1}, \sigma_0, \sigma_1, \ldots) \in \Sigma^{\mathbb{Z}}$$

are singletons. Moreover,

$$A_* = \bigcup_{\omega\in\Omega} \mathcal{A}_\omega. \tag{6.1}$$

Proof. In view of Theorem 4.4 it suffices to prove (6.1). Clearly $\bigcup_{\omega\in\Omega} \mathcal{A}_\omega \subset A_*$, since $\mathcal{A}_\omega$ are simply limit sets (cf. [14, Proposition 4.3]). To prove the opposite inclusion take an arbitrary element $a \in A_*$. Therefore, by the definitions of topological limit and the Lasota–Myjak attractor, there is a sequence $(a_n)_{n\in\mathbb{N}}$ of elements $a_n \in F^n(a)$ such that $a_n \to a$ as $n \to \infty$. Hence there is $\omega = (\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots) \in \Sigma^{\mathbb{Z}}$ such that $a_n = S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(a)$. Finally, $a \in \liminf_n S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(a) \subset \limsup_n S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(a) = \mathcal{A}_\omega = \{x_\omega\}$, so $a = x_\omega$. □

If the attractor A_* of an IFS has the form (6.1) with $\mathcal{A}_\omega$ being singletons, then it is called *point-fibred* [5, Definition 6.1].

The following class of functions was introduced by F.E. Browder [7]. We say that the function $S\colon X \to X$ is *ϕ-contraction* if there exists an upper semicontinuous and non-decreasing function $\phi\colon [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that

$$\varrho(S(x), S(y)) \le \phi(\varrho(x,y)) \quad \text{for } x, y \in X. \tag{6.2}$$

J. Matkowski [22, Theorem1.2] proved that each ϕ-contraction has a globally attractive fixed point.

The following corollary is related to [16, Theorem 1]. One can see that in [16] only countable IFSs are considered.

Corollary 6.4. *For every* IFS $\{S_\sigma\colon X \to X : \sigma \in \Sigma\}$ *consisting of ϕ-contractions of complete metric space X with ϕ independent on $\sigma \in \Sigma$ conditions* (a) *and* (b) *of Proposition* 6.1 *are fulfilled, so it admits the Lasota–Myjak attractor which is point-fibred, whenever it is a bounded set.*

Proof. Condition (6.2) implies that for every $n \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_n \in I$ we have

$$\varrho(S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(x), S_{\sigma_1} \circ \ldots \circ S_{\sigma_n}(y)) \le \phi^n(\varrho(x,y))$$

which tends to 0 as $n \to \infty$ and it is independent on the choice of a sequence $\sigma_1, \sigma_2, \ldots$, so it means that an IFS is pullback uniformly contractive. Moreover, existence of globally attractive fixed points for every transformation leads us to existence of convergent trajectory, therefore, by Proposition 6.1, the considered

IFS admits the Lasota–Myjak attractor. If it is bounded it suffices to apply Theorem 6.3. □

It is easy to see that every contraction is also a ϕ-contraction with $\phi(t) = Lt$, $t \in [0,\infty)$, where $L \in (0,1)$ is a Lipschitz constant.

Example 6.5. Let $(X, \|\cdot\|)$ be a Banach space and Σ be a non-void set. Consider two maps $L\colon \Sigma \to \mathbb{R}$ and $M\colon \Sigma \to X$ and the following affine difference equation on X:

$$x_{n+1} = L(\sigma_n)x_n + M(\sigma_n) \quad \text{for } n \in \mathbb{N}. \tag{6.3}$$

Observe that mappings $S_\sigma\colon X \to X$ given by

$$S_\sigma(x) = L(\sigma)x + M(\sigma) \tag{6.4}$$

for all $\sigma \in \Sigma$ and $x \in X$ form an IFS.

Assume that $K := \sup\{|L(\sigma)| : \sigma \in \Sigma\} < 1$ and M is bounded, i.e. there is a positive real number $R > 0$ such that $M(\sigma) \in B^o(0,R)$ for $\sigma \in \Sigma$. Suppose moreover, that there is $\sigma_0 \in \Sigma$ such that $M(\sigma_0) = 0$.

By these assumptions transformations $S(\sigma)$ are strict contractions. Indeed,

$$\|S(\sigma)(x) - S(\sigma)(y)\| \le |L(\sigma)|\|x-y\| \le K\|x-y\|$$

for all $x, y \in X$. In addition, since $\|L^n(\sigma_0)(x)\| \to 0$ as $n \to \infty$, there is a convergent trajectory (through $\omega_0 = (\ldots,\sigma_0,\sigma_0,\sigma_0,\ldots)$). Hence, in view of Proposition 6.1, considered IFS admits the Lasota–Myjak attractor and, consequently, by Theorem 4.4, the sets $\mathcal{A}_\omega$ defined by (3.4) are singletons.

Let us calculate the fibers $\mathcal{A}_\omega = \{x_\omega\}$, $\omega \in \Omega$ *explicitly.* Observe that using the induction argument we obtain for all $n \in \mathbb{N}$, $\omega = (\ldots,\sigma_{-1},\sigma_0,\sigma_1,\ldots) \in \Sigma^{\mathbb{Z}}$ and $x \in X$ that

$$\varphi(n,\theta^{-n}\omega)(x) = \Big(\prod_{k=1}^{n} L(\sigma_k)\Big)x + \sum_{k=1}^{n}\Big(\prod_{j=1}^{k-1} L(\sigma_j)\Big)M(\sigma_k).$$

Now

$$\Big(\prod_{k=1}^{n} L(\sigma_k)\Big)x \to 0 \quad \text{as } n \to \infty$$

independently of $x \in X$. Moreover,

$$\bigg\|\Big(\prod_{j=1}^{k-1} L(\sigma_j)\Big)M(\sigma_k)\bigg\| \le K^{k-1}R,$$

which means that the series

$$\sum_{n=1}^{\infty}\Big(\prod_{j=1}^{n-1} L(\sigma_j)\Big)M(\sigma_n) \tag{6.5}$$

is absolutely summable. Consequently, there is a unique element $x_\omega \in X$ which is the sum of the series (6.5). Since $\mathcal{A}_\omega = \mathcal{L}(\omega, D)$ does not depend on a bounded set $D \subset X$, we can put $D = \{x\}$ for any $x \in X$ and we infer that $\mathcal{A}_\omega = \{x_\omega\}$, where x_ω is defined as above.

As we have seen the set $A := \bigcup_{\omega\in\Omega} \{x_\omega\}$ is a subset of the Lasota–Myjak attractor A_*. Now we will show that in fact the equality holds. To this end consider the Barnsley–Hutchinson multifunction given by $F(x) = \{S_\sigma(x) : \sigma \in \Sigma\}$ and observe that $F(A) \subset A$. Indeed, if $\omega = (\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots)$ and $x_\omega \in A$ we have

$$F(x_\omega) = F\bigg(\sum_{n=1}^{\infty}\bigg(\prod_{j=1}^{n-1} L(\sigma_j)\bigg)M(\sigma_n)\bigg) = \bigg\{L(\sigma)\sum_{n=1}^{\infty}\bigg(\prod_{j=1}^{n-1} L(\sigma_j)\bigg)M(\sigma_n) + M(\sigma)\bigg\}$$

for some $\sigma \in \Sigma$. But it is easy to see that it is equal to the singleton $\{x_{\overline{\omega}}\} \subset A$, where $\overline{\omega} = (\ldots, \sigma, \sigma_1, \sigma_2, \ldots)$. Hence, the desired inclusion holds. Farther, it is known (see [21, Proposition 3.1] and [15, Proposition 5.6]) that A_* is a subset of any non-empty set $B \subset X$ such that $F(B) \subset B$, so $A_* \subset A$.

Notice that the difference equation (6.3) plays an important role in the theory of perpetuities (see [12] and the references therein, also [18, Remark 1]). Moreover, the IFS (6.4) is crucial in solving of a large class of refinement equations (see [24], [18] and the references therein). On the other hand, one can treat the mapping M as random interventions in a linear IFS or even, when L is a constant function, equation (6.3) can be seen as an inducing perturbed discrete dynamical system (cf. for example [13, Example 5.2] or [19, Example 2.11]). Under our assumptions perturbations are bounded.

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