MAYER–VIETORIS PROPERTY
OF THE FIXED POINT INDEX

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Abstract. We study a Mayer–Vietoris kind formula for the fixed point index of maps of ENR triplets $f\colon (X; X_1, X_2) \to (X; X_1, X_2)$ having compact fixed point set. We prove it under some suitable conditions. For instance when $(X; X_1, X_2) = (E^n; E^n_+, E^n_-)$.

We use these results to generalize the Poincaré–Bendixson index formula for vector fields to continuous maps having a *sectorial decomposition*, to study the fixed point index $i(f, 0)$ of orientation preserving homeomorphisms of E^2_+ and $(E^3; E^3_+, E^3_-)$ and the fixed point index in the invariant subspace.

1. Introduction

In this paper we will deal with triplets $(X; X_1, X_2)$ where X is an ENR and X_1, X_2 are ENR's closed in X such that $X_1 \cap X_2$ is also an ENR and $X = X_1 \cup X_2$. We will call such triplets $(X; X_1, X_2)$ *ENR triplets* and we will denote $X_0 := X_1 \cap X_2$. A continuous map $f\colon (X; X_1, X_2) \to (X; X_1, X_2)$ of a triplet is a continuous map $f\colon X \to X$ satisfying that

$$f(X_i) \subset X_i, \quad i = 1, 2.$$

Notice that $f(X_0) \subset X_0$.

We denote

$$f_i := f|_{X_i} \colon X_i \to X_i, \quad i = 0, 1, 2.$$

Let $\mathrm{Fix}(f) = \{x \in X : f(x) = x\}$ be the set of fixed points of f. Assuming that $\mathrm{Fix}(f)$ is compact it follows that the fixed point indices

$$i(f, X) \quad \text{and} \quad i(f_i, X_i), \quad i = 0, 1, 2,$$

are well defined. Our goal is to study the following Mayer–Vietoris index formula:

$$(1.1) \qquad\qquad i(f, X) + i(f_0, X_0) = i(f_1, X_1) + i(f_2, X_2).$$

If X is compact then (1.1) holds by the Lefschetz–Hopf fixed point theorem and the Mayer–Vietoris sequence of the triad $(X; X_1, X_2)$ (see [1, Corollary 2.4]).

The main motivation to study the non-compact case is when

$$X = E^n, \qquad X_+ = E^n_+, \qquad X_- = E^n_-,$$

where E^n denotes the n-dimensional Euclidean space and

$$E^n_+ := \{(x_1, \ldots, x_n) \in E^n : x_n \geq 0\} \quad \text{and} \quad E^n_- := \{(x_1, \ldots, x_n) \in E^n : x_n \leq 0\}.$$

We will denote by E^n_0 the intersection $E^n_+ \cap E^n_-$.

The interest of this case comes from applications to ecological systems ([6], [11], [16], [26]) where E^n_+ is a natural phase space.

Let M be an n-dimensional manifold with non-empty boundary ∂M. Assume that $f \colon (M, \partial M) \to (M, \partial M)$ is a continuous map and let $f_0 := f|_{\partial M} \colon \partial M \to \partial M$. If $p \in \partial M$ is an isolated fixed point of f, that is, $f(q) \neq q$ for all $q \neq p$ in a neighbourhood of p in M, then p is an isolated fixed point of f_0 as well. A natural problem is to study the relationship between the fixed point indices $i(f, p)$ and $i(f_0, p)$. It follows from [5] that, without further assumptions, the values $i(f, p)$ and $i(f_0, p)$ can be arbitrary integers, with the values unrelated to each other. It can happen even in the smooth category. For instance, if $M = E^n_+$ then, given integers $r, s \in \mathbb{Z}$, there exists a smooth map $f \colon (E^n_+, E^n_0) \to (E^n_+, E^n_0)$ such that the origin 0 is an isolated fixed point of f and

$$i(f, 0) = r \quad \text{and} \quad i(f_0, 0) = s.$$

However, if $f \colon (M, \partial M) \to (M, \partial M)$ is smooth, $p \in \partial M$ is an isolated fixed point of f and $f'(p) - I \colon T_p(\partial M) \to T_p(\partial M)$ is a non-singular linear transformation then either $i(f, p) = 0$ or $i(f, p) = i(f_0, p) = \pm 1$ (see [5], [20], [21]).

The study of the important case $M = E^n_+$ is mostly motivated by ecological systems (compare [6], [11], [16], [26], [33], [34]) and is closely related to permanence theory, where, roughly speaking, conditions for E^n_0 to be a repeller are studied. From the point of view of biological applications the fact that $\emptyset \neq \omega(x) \subset E^n_0$ for some $x \in E^n_+ \setminus E^n_0$, is understood as a strong violation of permanence (the ultimate survival of all species).

In this context, a special kind of compact invariant sets $S \subset E^n_0$, the so-called sets of *repelling type* (see [11], [33], [34]), are interesting. It turns out that it is common in biological applications that S is an isolated invariant set in the sense

of the Conley index theory (i.e. isolated with respect to the entire flow or map on E_+^n). For instance, in [16], Hofbauer initiated the study of S being a finite collection of equilibria by means of the standard degree theory. Capietto and Garay [6] used the fixed point index theory and the Conley index theory. Their approach, however, only worked for flows induced by vector fields and an special kind of isolated invariant sets called saturated invariant sets. Both restrictions were removed and many more Conley-type results were proved in [33], [34]. Generalizations for discrete-time semidynamical systems were presented in [31].

For instance, if $S \subset E_0^n$ is an isolated invariant set of a continuous map $f \colon (E_+^n, E_0^n) \to (E_+^n, E_0^n)$ it follows from the results in [31] that

$$i(f, S) = i(f_0, S),$$

provided $W^u(S) \subset E_0^n$ and $i(f, S) = 0$, if $W^s(S) \subset E_0^n$. In particular, if $\{0\}$ is an isolated invariant set of f, then

$$i(f, 0) = \begin{cases} i(f_0, 0) & \text{if } W^u(0) \subset E_0^n, \\ 0 & \text{if } W^s(0) \subset E_0^n. \end{cases}$$

Let us mention, that it follows from the construction in [29] that for $n \leq 1$, $k \in \mathbb{Z}$ there exists a homeomorphism $f \colon E_+^3 \to E_+^3$ such that $\{0\}$ is an isolated invariant set and

$$i(f, 0) = k, \qquad i(f_0, 0) = n.$$

The paper is structured as follows: In Section 3, the notion of a *trapping set* is introduced and it is proven that if A is a trapping set and $K = \text{Fix}(f|_A)$ is compact, then $i(f, K)$ is well defined and agrees with $i(f|_A, K)$. This result plays a key role in the proof of Proposition 3.4, which is the main result of this section and establishes that if $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ is a continuous map whose fixed point set is compact and X_0 is a trapping set, then (1.1) holds.

In Section 4, we recall the definition and properties of *proper pairs* introduced by Srzednicki in [30]. In addition, we introduce the concept of *admissible* proper pair with respect to a map $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ and we prove Theorem 4.4, which establishes that if $\text{Fix}(f) \subset \text{int}(C \setminus E)$ for some admissible proper pair (C, E) with respect to f, then (1.1) holds.

In Section 5, we study the Brouwer degree of maps

$$f \colon (S^n, S^{n-1}) \to (S^n, S^{n-1}), \quad n \geq 1.$$

The main result of this section is Theorem 5.3 which establishes that if $f_+ := f|_{S_+^n}$ and $f_- := f|_{S_-^n}$ admit continuous extensions F_1 and F_2, respectively, satisfying

$$F_1(S_-^n \setminus S^{n-1}) \subset S_-^n \setminus S^{n-1} \quad \text{and} \quad F_2(S_+^n \setminus S^{n-1}) \subset S_+^n \setminus S^{n-1},$$

then

$$\deg(f) + \deg(f_0) = \deg(F_1) + \deg(F_2),$$

where $f_0 := f|_{S^{n-1}} \colon S^{n-1} \to S^{n-1}$.

As a consequence of this result we obtain Proposition 5.4 which establishes that if $f \colon S^n \to S^n$ is symmetric with respect to S^{n-1}, then $f(S^{n-1}) \subset S^{n-1}$ and $\deg(f_0)$ is congruent to $\deg(f)$ modulo 2.

In Section 6, we prove, using the results of Section 5, Theorem 6.1, which establishes that if $f \colon (E^n; E_+^n, E_-^n) \to (E^n; E_+^n, E_-^n)$ is a continuous map whose fixed point set is compact, then (1.1) holds. In addition, we introduce the concept of *good Euclidean embedding* and we see that if $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ is a continuous map of a triplet which admits a good Euclidean embedding and $\mathrm{Fix}(f)$ is compact, then (1.1) holds.

Section 7 is devoted to applications of the results of Section 6. In particular we prove Lemma 7.4 and Corollary 7.6 which generalize to the context of continuous maps $f \colon E^2 \to E^2$ the classical Poincaré–Bendixson index formula for vector fields. In addition, we obtain some consequences of these results. For instance, we see that if $f \colon E_+^2 \to E_+^2$ is an orientation preserving homeomorphism having 0 as its only fixed point, the sequence $(i(f^n, 0))_{n \geq 0}$ is well defined and constant and we obtain some upper-bounds for $i(f, 0)$ if f is also area preserving.

In Section 8, we recall the concept of an *isolated invariant set* of a map and we prove Proposition 8.1, which is an application of Theorem 6.1 to isolated invariant sets in the invariant subspace.

In Section 9, we study some applications of Theorem 6.1 to homeomorphisms $f \colon (E^3; E_+^3, E_-^3) \to (E^3; E_+^3, E_-^3)$ which are symmetric with respect to to E_0^3 and have the origin as an isolated fixed point.

Finally, in Section 10, we present some open problems.

2. Preliminaries

Throughout the paper we will use some algebraic and differential topology techniques. We recommend to the reader the book of Spanier [32] for questions about algebraic topology and [15], [23], [24] as references for differential topology. Regarding the fixed point index theory we recommend [18]. For the sake of completeness we will recall the basic definitions of the degree of maps between smooth manifolds following [24] and for the fixed point index theory we will follow the exposition presented in [19].

2.1. Degree of maps between manifolds. Let M and N be smooth orientable manifolds and assume that M is compact and N is connected. Let $f \colon M \to N$ be a smooth map. Given $x \in M$, we will denote by $f'(x)$ the differential of the map f in x. A point $x \in M$ is called *regular* if $f'(x)$ is non-singular. Let $y \in N$, y is said to be a *regular value* if $f^{-1}(y)$ only contains regular

points. Notice that from Sard's Theorem, the set of regular values is dense in N. Besides, as a consequence of the inverse function theorem, if y is a regular value, $f^{-1}(y)$ consists of a finite number of points. The *Brouwer degree* of the map f is defined to be

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(\det f'(x)),$$

where y is a regular value and $\operatorname{sgn}(\det f'(x))$ denotes the sign of the determinant of the Jacobian matrix of $f'(x)$. Notice that the definition does not depend on the choice of the regular value. Moreover, $\deg(f)$ is an invariant of the smooth homotopy class of f.

The definition of the degree of a smooth map can be extended to continuous maps in the following way. If $f \colon M \to N$ is a continuous map there exists a smooth map $g \colon M \to N$ homotopic to f and the Brouwer degree of f is defined as $\deg(f) := \deg(g)$.

Notice that the degree of continuous maps is a well-defined integer number and it is an invariant of the homotopy class of the map. This follows from the fact that if two smooth maps between smooth manifolds are homotopic they must be smoothly homotopic.

Let $S^0 = \{-1, 1\}$, i.e. S^0 is the 0-dimensional sphere. For technical reasons we also have to define the degree of a map $f \colon S^0 \to S^0$. In this case any point in S^0 is going to be considered as a regular value and the degree is defined as follows:

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(\det f'(x)),$$

where y is any point in S^0 and

$$\operatorname{sgn}(\det f'(x)) := \begin{cases} -1 & \text{if } f(x) \neq x, \\ 1 & \text{if } f(x) = x. \end{cases}$$

A direct consequence of the definition is that any map $f \colon S^0 \to S^0$ has degree either -1, 0 or 1. Notice that here we use the notation $\operatorname{sgn}(\det f'(x))$ to be in accordance with the higher-dimensional case previously described but it does not have any meaning in connection with differential topology.

More generally, if $J = [a, b]$ is an interval of the real line and $f \colon \partial J \to S^0$ is a map, the degree of f is defined as

$$\deg(f) = \deg(f \circ h),$$

where $h \colon S^0 \to \partial J$ is the map which identifies S^0 with ∂J preserving the natural order induced by the real line in S^0 and ∂J, i.e. such that $h(-1) = a$ and $h(1) = b$.

2.2. Fixed point index on the Euclidean space. Let U be an open subset of E^n and $f: U \to E^n$ a continuous map such that $\mathrm{Fix}(f)$ is compact. Choose a compact neighbourhood $V \subset U$ of $\mathrm{Fix}(f)$ which is a compact smooth n-manifold with boundary. Define $\Phi: \partial V \to S^{n-1}$ as

$$\Phi(x) = \frac{x - f(x)}{\|x - f(x)\|}.$$

The fixed point index of f in U is defined as

$$i(f, U) = \deg(\Phi).$$

Notice that, $i(f, U)$ is an integer number, that it does not depend on the choice of V and that it is an invariant of the homotopy class of f. Notice also that in the case $n = 1$, V could be a disjoint union of closed intervals $I_1, \dots, I_k$. In this case we would define,

$$\deg(\Phi) := \sum_{i=1}^{k} \deg(\Phi|_{\partial I_i}).$$

2.3. Fixed point index on ENR's. Let X be an ENR and $f: X \to X$ be a continuous map satisfying that $\mathrm{Fix}(f)$ is compact. Since X is an ENR, it can be embedded in some E^n in such a way that if $e: X \to E^n$ is an embedding, there exists an open set $W \subset E^n$ such that $e(X) \subset W$ and a retraction $r: W \to e(X)$. Consider the map $F = e \circ f \circ r$. The fixed point index of f in X is defined as

$$i(f, X) := i(F, W).$$

The definition does not depend on the choice of e, W and r. Following this construction, if f is only defined in an open subset U of X, the index is defined as

$$i(f, U) := i(F|_{r^{-1}(U)}, r^{-1}(U)).$$

Suppose that X is a compact ENR and H_* is the singular homology functor with rational coefficients. Then $H_*(X)$ is of finite type, i.e. $H_n(X)$ is finite dimensional for all n, and zero if n is greater than a certain n_0. If $f: X \to X$ is a continuous map, then the *Lefschetz number* of f is defined as

$$L(f) = \sum_{n=1}^{\infty} (-1)^n \operatorname{tr}(f_{*n}),$$

where $f_{*n}: H_n(X) \to H_n(X)$ is the linear map induced from f.

It can be proved that $L(f)$ is an integer and it is obviously an invariant of the homotopy class of f.

The Lefschetz number and the fixed point index of a continuous self-map $f: X \to X$ of a compact ENR are related through the celebrated Lefschetz–Hopf Theorem (see [8]).

THEOREM 2.1. *Let $f: X \to X$ be a continuous self-map of a compact* ENR. *Then, $L(f) = i(f, X)$.*

2.4. Isolated sets of fixed points. Consider a continuous map $f \colon U \to X$, where X is a compact ENR and assume that $\mathrm{Fix}(f)$ is compact. We will say that a closed subset $K \subset \mathrm{Fix}(f)$ is an *isolated set of fixed points* if there exists an open subset U' of U such that $U' \cap \mathrm{Fix}(f) = K$. In particular, if K is a singleton $\{x_0\}$ we will say that x_0 is an *isolated fixed point*.

If K is an isolated set of fixed points we will denote $i(f, K) := i(f|_{U'}, U')$, where U' is any open subset such that $U' \cap \mathrm{Fix}(f) = K$. As a consequence of the localization property of the fixed point index this is well defined. Notice that in general, if U' is an open subset of the domain of f such that $U' \cap \mathrm{Fix}(f)$ is compact we will denote $i(f, U') := i(f|_{U'}, U')$.

3. Trapping sets and the Mayer–Vietoris index formula

In this section we will introduce the concept of a trapping set of a map and we will see that if $(X; X_1, X_2)$ is an ENR triplet, $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ is a continuous map having compact fixed point set and X_0 is a trapping set, then (1.1) holds.

DEFINITION 3.1. Let $f \colon X \to X$ be a continuous map and A a closed subset of X. We say that A is a *trapping set* of f if there exists an open neighbourhood U of A in X such that $f(U) \subset A$.

REMARK 3.2. Notice that if a closed subset A of X is a trapping set of a continuous map $f \colon X \to X$, then $f(A) \subset A$.

LEMMA 3.3. *Assume that (X, A) is a pair of ENR's and $A \subset X$ is closed. Let $f \colon X \to X$ be a continuous map such that $\mathrm{Fix}(f)$ is compact. Suppose that A is a trapping set of f, then $K = \mathrm{Fix}(f|_A)$ is an isolated set of fixed points and*

$$i(f, K) = i(f|_A, A).$$

PROOF. Since K agrees with $\mathrm{Fix}(f) \cap A$, it is compact. Moreover, if U is an open neighbourhood of A trapped by A (i.e. $f(U) \subset A$), then $K = \mathrm{Fix}(f) \cap U$. As a consequence, K is an isolated set of fixed points of f. It follows from the commutativity of the fixed point index that $i(f, K) = i(f|_A, K) = i(f|_A, A)$. $\square$

PROPOSITION 3.4. *Let $(X; X_1, X_2)$ be an ENR triplet and $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ a continuous map having compact fixed point set. If X_0 is a trapping set of f, then (1.1) holds.*

PROOF. Let U be an open neighbourhood of X_0 trapped by X_0. Then, Lemma 3.3 ensures that $\mathrm{Fix}(f)$ is the disjoint union of the compact sets

$$K_0 = \mathrm{Fix}(f) \cap X_0 \quad \text{and} \quad K_i = \mathrm{Fix}(f) \cap (X_i \setminus X_0), \quad i = 1, 2.$$

As a consequence, the additivity property of the fixed point index ensures that

$$(3.1) \qquad i(f, X) = \sum_{i=0}^{3} i(f, K_i).$$

Reasoning in an analogous way with X_i, $i = 1, 2$, we get

$$(3.2) \qquad i(f_i, X_i) = i(f_i, K_i) + i(f_i, K_0).$$

On the other hand, from the localization property of the fixed point index it follows that

$$(3.3) \qquad i(f, K_i) = i(f_i, K_i), \quad i = 1, 2.$$

Finally, from Lemma 3.3 we obtain that for $i = 1, 2$

$$(3.4) \qquad i(f, K_0) = i(f_i, K_0) = i(f_0, K_0) = i(f_0, X_0).$$

Now, the result is straightforward from (3.1)–(3.4). $\square$

4. Proper pairs and the Mayer–Vietoris index formula

In this section we will study (1.1) using the concept of a *proper pair* introduced by Srzednicki in [30]. In particular we will see that if $(X; X_1, X_2)$ is an ENR triplet and $f\colon (X; X_1, X_2) \to (X; X_1, X_2)$ is a continuous map such that $\mathrm{Fix}(f) \subset \mathrm{int}(C \setminus E)$, where (C, E) is what we call an *admissible proper pair* with respect to f, then (1.1) holds.

Let C and E be compact subsets of X, $E \subset C$. Following [30], we say that the pair (C, E) is proper with respect to a continuous map $f\colon X \to X$ if it satisfies that

$$C \cap f(E) \subset E, \qquad C \cap \overline{f(C) \setminus C} \subset E.$$

REMARK 4.1. Notice that if $A \subset X$ is a closed subset such that $f(A) \subset A$, then the pair $(C \cap A, E \cap A)$ is proper with respect to $f|_A\colon A \to A$.

For a compact pair (C, E) contained in X, the following conditions are equivalent:

(a) (C, E) is proper with respect to f.
(b) There exists a compact pair (C', E') in X such that:

$$(C, E) \subset (C', E'), \quad f(C) \subset C', \quad f(E) \subset E', \quad C \setminus E = C' \setminus E'.$$

(c) (C, E) satisfies $C \cap f(E) \subset E$ and the map $f^\dagger\colon C/E \to C/E$, given by

$$f^\dagger([x]) = \begin{cases} [f(x)] & \text{if } f(x) \in C \setminus E, \\ \star & \text{otherwise} \end{cases} \qquad \text{with } \star = [E],$$

is continuous.

Let H^* be the Alexander–Spanier cohomology functor with rational coefficients. For (C', E') we define the transfer homomorphism $f^\sharp := f^\sharp_{(C,E)} \colon H^*(C,E) \to H^*(C,E)$ as the composition

$$f^\sharp \colon H^*(C,E) \xrightarrow{\cong} H^*(C',E') \xrightarrow{f^*} H^*(C,E),$$

where the first arrow is the inverse of the isomorphism induced by the inclusion $(C,E) \hookrightarrow (C',E')$ (we use the strong excision property of the Alexander–Spanier cohomology). The definition of the transfer homomorphism does not depend on the choice of (C', E') in b.

THEOREM 4.2 (Srzednicki). *Assume that X is an ENR. If (C,E) is a pair of compact ENR's in X, proper with respect to f and*

$$\mathrm{Fix}(f) \cap \overline{C \setminus E} \subset \mathrm{int}(C \setminus E),$$

then

$$i(f, \mathrm{int}(C \setminus E)) = L(f^\sharp).$$

Let $(X; X_1, X_2)$ be an ENR triplet and $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ a continuous map. Let (C, E) be a proper pair with respect to f. We denote

$$(C_i, E_i) = (C \cap X_i, E \cap X_i), \quad i = 0, 1, 2.$$

DEFINITION 4.3. We say that a proper pair (C, E) is *admissible* if (C, E), and (C_i, E_i) are pairs of compact ENR's for $i = 0, 1, 2$.

THEOREM 4.4. *Let $(X; X_1, X_2)$ be an ENR triplet and $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ a continuous map. Suppose that there exists an admissible proper pair (C, E) with respect to f such that $\mathrm{Fix}(f) \subset \mathrm{int}(C \setminus E)$. Then (1.1) holds.*

PROOF. Notice that, since $\mathrm{Fix}(f)$ is a closed subset of the compact set C, it must be compact. It follows from Remark 4.1 that (C_i, E_i) is a proper pair with respect to f_i, $i = 0, 1, 2$. Moreover,

$$\mathrm{Fix}(f_i) \subset \mathrm{int}(C_i \setminus E_i), \quad i = 0, 1, 2.$$

The following diagram, whose rows are the Mayer–Vietoris sequences of the pairs of excessive triads $(E; E_1, E_2) \subset (C; C_1, C_2)$ is commutative and, since (C, E) is an admissible proper pair, all its entries are finite-dimensional vector spaces and only a finite number of them are non-trivial.

$$\cdots \longrightarrow H^i(C,E) \longrightarrow H^i(C_1,E_1) \oplus H^i(C_2,E_2) \longrightarrow H^i(C_0,E_0) \longrightarrow \cdots$$
$$ \downarrow f^\sharp_{(C,E)} \downarrow f^\sharp_{(C_1,E_1)} \oplus f^\sharp_{(C_2,E_2)} \downarrow f^\sharp_{(C_0,E_0)}$$
$$\cdots \longrightarrow H^i(C,E) \longrightarrow H^i(C_1,E_1) \oplus H^i(C_2,E_2) \longrightarrow H^i(C_0,E_0) \longrightarrow \cdots$$

As a consequence, from [1, Theorem 2.2] we get that

$$L(f^\sharp_{(C,E)}) + L(f^\sharp_{(C_0,E_0)}) = L(f^\sharp_{(C_1,E_1)}) + L(f^\sharp_{(C_2,E_2)}),$$

and, hence, Theorem 4.2 ensures that

$$(4.1) \quad i(f, \text{int}(C \setminus E)) + (f_0, \text{int}(C_0 \setminus E_0))$$
$$= (f_1, \text{int}(C_1 \setminus E_1)) + (f_2, \text{int}(C_2 \setminus E_2)).$$

The result follows from (4.1) and the localization property of the index.

5. A degree formula for sphere maps

This section is devoted to proving a formula for the degree of maps

$$f\colon (S^n, S^{n-1}) \to (S^n, S^{n-1}), \quad n \geq 1,$$

which will play a fundamental role in the proof of (1.1) in the Euclidean space.

By S^n we mean the unit sphere centered at the origin of the Euclidean space E^{n+1}. We will denote

$$S_+^n := S^n \cap E_+^{n+1} \quad \text{and} \quad S_-^n := S^n \cap E_-^{n+1}.$$

The intersection $S_+^n \cap S_-^n$ is exactly the unit sphere S^{n-1} of the Euclidean space $E_0^{n+1} = E^n \times \{0\}$.

Consider $S^{n-1} \subset E^n \times \{0\}$ and $J \subset [-1, 1]$. We use the notation

$$S_J := S^{n-1} \times J \subset E^{n+1}.$$

Consider the projection $\pi\colon S_{[-1,1]} \to S^n$ from the cylinder $S_{[-1,1]}$ onto the sphere S^n which sends each $(p, t) \in S_{[-1,1]}$ to the point in S^n lying on the straight line joining (p, t) with the point $(0, \ldots, 0, t) \in E^{n+1}$. In particular, we have that

$$S_+^n = \pi(S_{[0,1]}), \qquad S_-^n = \pi(S_{[-1,0]}),$$

and, if $J \subset (-1, 1)$ is a subinterval, the map $\pi|_{S_J}\colon S_J \to \pi(S_J)$ is a diffeomorphism.

Let $f\colon (S^n, S^{n-1}) \to (S^n, S^{n-1})$ be a continuous map. We denote

$$f_+ := f|_{S_+^n}, \ f_- := f|_{S_-^n} \quad \text{and} \quad f_0 := f|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}.$$

To prove the formula about the degree we will make use of the following lemma.

LEMMA 5.1. *Let $f\colon (S_+^n, S^{n-1}) \to (S^n, S^{n-1})$ be a continuous map. Then, there exists a homotopy $H_t\colon (S_+^n, S^{n-1}) \to (S^n, S^{n-1})$ such that $H_0 = f$ and H_1 is smooth. Moreover, if $f(S_+^n) \subset S_+^n$ then H can be chosen such that $H_t(S_+^n) \subset S_+^n$.*

PROOF. It follows from [15, Theorem 3.5] that f can be arbitrarily close approximated by a smooth map $g\colon (S_+^n, S^{n-1}) \to (S^n, S^{n-1})$. Moreover, if

$f(S^n_+) \subset S^n_+$ then g can be chosen such that $g(S^n_+) \subset S^n_+$. Let W be an ε-neighbourhood of S^n in E^{n+1} and let $r \colon W \to S^n$ be a radial retraction. We define $h_t \colon E^{n+1} \to E^{n+1}$ by

$$h_t(x) = (1-t)f(x) + tg(x).$$

We can choose g in such a way that it approximates f so closely that $h_t(x) \in W$ for each $x \in S^n$ and $t \in [0,1]$. The desired homotopy is $H_t := r \circ h_t$. Notice that $H_t(S^{n-1}) \subset S^{n-1}$ since $h_t(x) \in E^n \times \{0\}$ for every $x \in S^{n-1}$ and $t \in [0,1]$. $\quad\square$

The following lemma states a smooth version of the formula we are looking for.

LEMMA 5.2. *Let $f \colon (S^n, S^{n-1}) \to (S^n, S^{n-1})$ be a smooth map. Suppose that f_+ and f_- admit smooth extensions F_1 and F_2, respectively, satisfying*

$$(5.1) \qquad F_1(S^n_- \setminus S^{n-1}) \subset S^n_- \setminus S^{n-1} \quad and \quad F_2(S^n_+ \setminus S^{n-1}) \subset S^n_+ \setminus S^{n-1},$$

then

$$(5.2) \qquad\qquad \deg(f) + \deg(f_0) = \deg(F_1) + \deg(F_2).$$

PROOF. Let $g \colon (S^n, S^{n-1}) \to (S^n, S^{n-1})$ be a continuous map satisfying that if $x \in \pi(S_{[-1/2,1/2]})$, then

$$(5.3) \qquad\qquad g(x) = \pi(g_0(p), t),$$

where (p,t) is the only point in $S_{[-1/2,1/2]}$ such that $x = \pi(p,t)$. Then, by Sard's Theorem we have that there exists $x_0 = \pi(p_0, t_0) \in \pi(S_{[-1/2,1/2]})$ a regular value of g. Then, $g^{-1}(x_0)$ consists of a finite number of points and

$$\deg(g) = \sum_{x \in g^{-1}(x_0)} \operatorname{sgn}(\det g'(x)).$$

On the other hand, let

$$U = g^{-1}(x_0) \cap \pi(S_{[-1/2,1/2]}), \qquad V = g^{-1}(x_0) \cap \pi(S_{(1/2,1]}),$$
$$W = g^{-1}(x_0) \cap \pi(S_{[-1,-1/2)}).$$

Hence,

$$\deg(g) = \sum_{x \in U} \operatorname{sgn}(\det g'(x)) + \sum_{x \in V} \operatorname{sgn}(\det g'(x)) + \sum_{x \in W} \operatorname{sgn}(\det g'(x)).$$

In addition, (5.3) ensures that p_0 is also a regular value for g_0, that $\pi(g_0^{-1}(p_0) \times \{t_0\})$ agrees with U and that, for $x = \pi(p, t_0) \in U$,

$$\operatorname{sgn}(\det g'(x)) = \operatorname{sgn}(\det g_0'(p)).$$

Therefore,

$$\deg(g_0) = \sum_{x \in U} \operatorname{sgn}(\det g'(x)),$$

and we get that

$$(5.4) \qquad \deg(g) = \deg(g_0) + \sum_{x \in V} \text{sgn}(\det g'(x)) + \sum_{x \in W} \text{sgn}(\det g'(x)).$$

Notice that we can find smooth maps $\widetilde{f}, \widetilde{F}_1$ and $\widetilde{F}_2$ homotopic to f, F_1, and F_2 relative to S^{n-1} respectively and such that $\widetilde{f}$ satisfies (5.3),

$$(5.5) \qquad \widetilde{F}_1|_{\pi(S_{[-1/2,1]})} = \widetilde{f}|_{\pi(S_{[-1/2,1]})}, \qquad \widetilde{F}_2|_{\pi(S_{[-1,1/2]})} = \widetilde{f}|_{\pi(S_{[-1,1/2]})}$$

and

$$(5.6) \qquad \widetilde{F}_1(\pi(S_{[-1,-1/2])})) \subset \pi(S_{[-1,-1/2)}), \qquad \widetilde{F}_2(\pi(S_{(1/2,1]})) \subset \pi(S_{(1/2,1]}).$$

Now, $\widetilde{f}, \widetilde{F}_1$ and $\widetilde{F}_2$ satisfy (5.3) and, hence, if $x_0 \in \pi(S_{[-1/2,1/2]})$ is a common regular value of the three maps, which exists as a consequence of Sard's Theorem, then from (5.4), (5.5) and (5.6) it follows that

$$(5.7) \qquad \deg(\widetilde{f}) = \deg(\widetilde{f}_0) + \sum_{x \in V} \text{sgn}(\det \widetilde{f}'(x)) + \sum_{x \in W} \text{sgn}(\det \widetilde{f}'(x)),$$

$$(5.8) \qquad \deg(\widetilde{F}_1) = \deg(\widetilde{f}_0) + \sum_{x \in V} \text{sgn}(\det \widetilde{f}'(x)),$$

$$(5.9) \qquad \deg(\widetilde{F}_2) = \deg(\widetilde{f}_0) + \sum_{x \in W} \text{sgn}(\det \widetilde{f}'(x)).$$

The result follows from (5.7), (5.8) and (5.9) and the homotopy invariance of the degree. $\qquad \square$

THEOREM 5.3. *Let $f \colon (S^n, S^{n-1}) \to (S^n, S^{n-1})$ be a continuous map. If F_1 and F_2 are continuous extensions of f_+ and f_- respectively, satisfying (5.1), then (5.2) holds.*

PROOF. As a consequence of Lemma 5.2 it will be sufficient to find a smooth map $\widetilde{f} \colon (S^n, S^{n-1}) \to (S^n, S^{n-1})$ homotopic to f as maps of pairs, and such that the corresponding maps $\widetilde{f}_+$ and $\widetilde{f}_-$ have smooth extensions $\widetilde{F}_1$ and $\widetilde{F}_2$, homotopic F_1 and F_2 and satisfying the assumption (5.1) of Lemma 5.2.

Let $\widetilde{f}_0 \colon S^{n-1} \to S^{n-1}$ be a smooth map homotopic to f_0. By Lemma 5.1 there exists a smooth map $g \colon (S^n_+, S^{n-1}) \to (S^n, S^{n-1})$ homotopic to $f_+ \colon (S^n_+, S^{n-1}) \to (S^n, S^{n-1})$ as maps of pairs. Hence, $\widetilde{f}_0$ and g_0 are homotopic and, since they are smooth, there exists a smooth homotopy $H_t \colon S^{n-1} \to S^{n-1}$ such that $H_0 = \widetilde{f}_0$ and $H_1 = g_0$.

We extend $\widetilde{f}_0$ to a map $\widetilde{f}_+ \colon S^n_+ \to S^n$ letting

$$\widetilde{f}_+(x) = \begin{cases} \pi(\widetilde{f}_0(p), t) & \text{if } x = \pi(p, t) \in \pi(S_{[0,1/3]}), \\ \pi(H_{6t-2}(p), t) & \text{if } x = \pi(p, t) \in \pi(S_{(1/3,1/2]}), \\ \widetilde{g}(x) & \text{if } x \in \pi(S_{(1/2,1]}), \end{cases}$$

where $\widetilde{g}$ is obtained by rescaling g in the obvious way.

In a symmetric way, it is possible to extend $\widetilde{f}_+$ to S_-^n ending up with a smooth function $\widetilde{f}\colon S^n \to S^n$ homotopic to f. To construct a smooth extension $\widetilde{F}_1$ of $\widetilde{f}_+$ such that (5.1) holds we define $\widetilde{F}_1\colon \pi(S_{[-1/2,1]}) \to S^n$ as follows:

$$\widetilde{F}_1 = \begin{cases} \widetilde{f}_+(x) & \text{if } x \in S_+^n, \\ \pi(\widetilde{f}_0(p), t) & \text{if } x = \pi(p, t) \in \pi(S_{[-1/2,0)}). \end{cases}$$

By proceeding in an analogous way as in the construction of $\widetilde{f}$ it is possible to extend $\widetilde{F}_1$ to a smooth map $\widetilde{F}_1\colon S^n \to S^n$ homotopic to F_1 and such that

$$\widetilde{F}_1(\pi(S_{[-1,-1/2)})) \subset \pi(S_{[-1,-1/2)}).$$

The construction $\widetilde{F}_2$ is analogous. □

The next result is a nice consequence of Theorem 5.3.

PROPOSITION 5.4. *Assume that $f\colon S^n \to S^n$ is a continuous map such that $f = J \circ f \circ J$, where $J\colon S^n \to S^n$ is given by $J(\pi(p,t)) = \pi(p,-t)$. Then $f(S^{n-1}) \subset S^{n-1}$ and*

$$\deg(f) \equiv \deg(f_0) \pmod 2.$$

PROOF. We define the extension $F_1\colon S^n \to S^n$ of f_+ by

$$F_1(x) = \begin{cases} f_+(x) & \text{if } x \in S_+^n, \\ \pi(f_0(p), t) & \text{if } x = \pi(p, t) \in \pi(S_{(0,-1)}), \\ p_S & \text{if } x = p_S, \end{cases}$$

where p_S denotes the south pole of S^n.

The extension F_2 of f_- is defined in an analogous way. On the other hand, since $f = J \circ f \circ J$ it follows that $\deg(F_1) = \deg(F_2)$ and, as a consequence, Theorem 5.3 ensures that

$$\deg(f) + \deg(f_0) = 2\deg(F_1).$$

6. Mayer–Vietoris index formula for Euclidean spaces

In this section we prove the main results of this note. In particular we will see that if $f\colon (X; X_1, X_2) \to (X; X_1, X_2)$ is a continuous map with compact fixed point set and either $(X; X_1, X_2)$ is $(E^n; E_+^n, E_-^n)$ or admits a good Euclidean embedding, then (1.1) holds.

Let $f\colon (E^n; E_+^n, E_-^n) \to (E^n; E_+^n, E_-^n)$ be a continuous map. We will use the notation

$$f_+ := f|_{E_+^n}\colon E_+^n \to E_+^n, \qquad f_- := f|_{E_-^n}\colon E_-^n \to E_-^n \qquad f_0 := f|_{E_0^n}\colon E_0^n \to E_0^n.$$

THEOREM 6.1. *Suppose $f\colon (E^n; E_+^n, E_-^n) \to (E^n; E_+^n, E_-^n)$ is a continuous map satisfying that $\mathrm{Fix}(f)$ is compact. Then (1.1) holds.*

Proof. The case $n = 1$ is easier and is left to the reader, so we will assume that $n \geq 2$. Since $\mathrm{Fix}(f)$ is compact, there exists a metric closed ball B in E^n centered at the origin containing $\mathrm{Fix}(f)$ in its interior. Consider

$$r_1 \colon E^n \to E^n_+ \quad \text{and} \quad r_2 \colon E^n \to E^n_-,$$

the natural retractions, and

$$j_1 \colon E^n_+ \hookrightarrow E^n \quad \text{and} \quad j_2 \colon E^n_- \hookrightarrow E^n,$$

the inclusions. We consider the maps $F_i \colon E^n \to E^n$, $i = 1, 2$, given by

$$F_1 := j_1 \circ f_+ \circ r_1, \qquad F_2 := j_2 \circ f_- \circ r_2.$$

The indices of the maps f, F_1, F_2 and f_0 agree with the degree of the maps $\Phi, \Phi_i \colon \partial B \to S^{n-1}$, $i = 1, 2$, and $\Phi_0 \colon \partial B \cap E^n_0 \to S^{n-2}$ is given by

$$\Phi(x) = \frac{x - f(x)}{\|x - f(x)\|}, \qquad \Phi_i(x) = \frac{x - F_i(x)}{\|x - F_i(x)\|}, \qquad \Phi_0(x) = \frac{x - f_0(x)}{\|x - f_0(x)\|}.$$

Notice that, if we denote by ∂B_+ and ∂B_- the intersections of ∂B with E^n_+ and E^n_-, respectively, then $\Phi_1|_{\partial B_+} = \Phi|_{\partial B_+}$, $\Phi_2|_{\partial B_-} = \Phi|_{\partial B_-}$ and $\Phi_0 = \Phi|_{\partial B \cap E^n_0}$.

On the other hand, it is clear that the natural radial diffeomorphism from S^{n-1} onto ∂B is a diffeomorphism of triplets

$$h \colon (S^{n-1}; S^{n-1}_+, S^{n-1}_-) \to (\partial B; \partial B_+, \partial B_-).$$

It is straightforward to see that the maps $\Phi \circ h$ and $\Phi_i \circ h$, $i = 1, 2$, satisfy the assumptions of Theorem 5.3. The result follows by combining these remarks with the fact that the degree of a composition is the product of the degrees and that $\deg(h) = \deg(h|_{S^{n-2}}) = 1$.   □

Remark 6.2. Notice that, from the proof of Theorem 6.1, it follows that if $f \colon (E^n; E^n_+, E^n_-) \to (E^n; E^n_+, E^n_+)$ is a continuous map and U is a metric open ball with no fixed points on its boundary, then

$$i(f, U) + i(f_0, U \cap E^n_0) = i(f_+, U \cap E^n_+) + i(f_-, U \cap E^n_-).$$

An immediate consequence of this fact is that, if 0 is an isolated fixed point of f then

$$i(f, 0) + i(f_0, 0) = i(f_+, 0) + i(f_-, 0).$$

Corollary 6.3. *Assume that $f \colon (E^n; E^n_+, E^n_-) \to (E^n; E^n_+, E^n_-)$ is smooth and 0 is an isolated fixed point. If $f_0'(0) - I \colon E^n_0 \to E^n_0$ is a non-singular linear transformation then $i(f, 0) \in \{-1, 0, 1\}$.*

Proof. It follows from [5, Theorem 5.1] that $i(f_\pm, 0) \in \{0, i(f_0, 0)\}$. Then, it follows from Remark 6.2 that $i(f, 0) \in \{0, \pm i(f_0, 0)\}$. The result follows since $i(f_0, 0) = \pm 1$.   □

The remaining part of the section deals with the concept of good Euclidean embedding.

DEFINITION 6.4. We say that the triplet $(X; X_1, X_2)$ admits a *good Euclidean embedding* if:

(a) there exists $n \geq 1$ and an embedding $e \colon X \to E^n$ such that $e(X_1) \subset E_+^n$ and $e(X_2) \subset E_-^n$,

(b) there exists a retraction $r \colon E^n \to e(X)$ such that
$$r(E_+^n) = e(X_1), \quad \text{and} \quad r(E_-^n) = e(X_2).$$

REMARK 6.5. If the triplet $(X; X_1, X_2)$ admits a good Euclidean embedding, it is straightforward from Definition 6.4 that $r|_{E_0^n}$ is also a retraction onto X_0. As a consequence, X and X_i, $i = 0, 1, 2$, are ENR's being homeomorphic to retracts of ENR's.
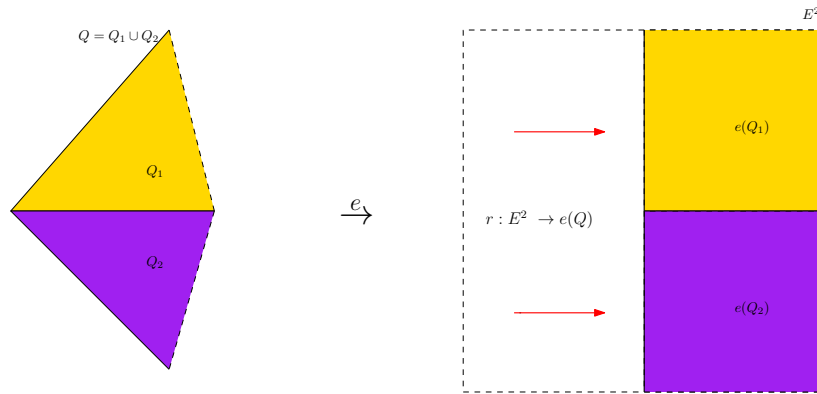

FIGURE 1. Example of a good Euclidean embedding of a triplet $(Q; Q_1, Q_2)$ in E^2.

THEOREM 6.6. *Let $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ be a continuous map satisfying that $\mathrm{Fix}(f)$ is compact and suppose that the triplet $(X; X_1, X_2)$ admits a good Euclidean embedding. Then* (1.1) *holds.*

PROOF. Since it should not cause any confusion we will identify X and X_i, $i = 0, 1, 2$ with $e(X)$ and $e(X_i)$, $i = 0, 1, 2$, respectively.

Let $e_0 \colon X_0 \to E_0^n$, $e_1 \colon X_1 \to E_+^n$, $e_2 \colon X_2 \to E_-^n$ be the respective restrictions of the embedding e to X_i, $i = 0, 1, 2$, and $k_1 \colon E_+^n \hookrightarrow E^n$, $k_2 \colon E_-^n \hookrightarrow E^n$ be the inclusions. We define
$$F = e \circ f \circ r \colon E^n \to E^n.$$
Notice that $\mathrm{Fix}(F) = \mathrm{Fix}(f)$ and, as a consequence, it is compact. It follows from Theorem 6.1 that
$$i(F, E^n) + i(F_0, E_0^n) = i(F_+, E_+^n) + i(F_-, E_-^n).$$

On the other hand, from the definition of the fixed point index we have $i(F, E^n) = i(F, X)$ and, since $F_0 = e_0 \circ f_0 \circ r|_{E_0^n}$ and $E_0^n = E^{n-1} \times \{0\}$, it follows that $i(F_0, E_0^n) = i(f_0, X_0)$.

The remaining part of the proof is to see that $i(F_+, E_+^n) = i(f_1, X_1)$ and $i(F_-, E_-^n) = i(f_2, X_2)$. Let $r_1 \colon E^n \to E_+^n$ be the natural retraction. From the definition of the fixed point index we have

$$i(F_+, E_+^n) = i(k_1 \circ F_+ \circ r_1, E^n) = i(k_1 \circ (e_1 \circ f_+ \circ r|_{E_+^n}) \circ r_1, E^n).$$

Since $r|_{E_+^n} \circ r_1 \colon E^n \to X_1$ is a retraction and $k_1 \circ e_1 \colon X_1 \to E^n$ is an embedding, it follows that $i(F_+, E_+^n) = i(f_1, X_1)$. The equality $i(F_-, E_-^n) = i(f_1, X_1)$ is proven in an analogous way. $\qquad\square$

7. Sectorial decomposition — generalization of the Poincaré–Bendixson index formula

In this section we generalize the Poincaré–Bendixson index formula for vector fields to the context of continuous maps $f \colon E^2 \to E^2$ whose fixed point set is compact. In addition we present some applications of this formula and the previous results to orientation preserving homeomorphisms of E_+^2 satisfying that $\mathrm{Fix}(f) = \{0\}$.

We will say that $Q \subset E^2$ is a *sector* if it is a region bounded by two half-lines with common starting point (including the half-lines).

REMARK 7.1. Notice that the union of two sectors whose intersection is one of their boundary half-lines is also a sector.

DEFINITION 7.2. A continuous map $f \colon E^2 \to E^2$ has a *sectorial decomposition* $\{Q_1, \ldots, Q_p\}$, $p \geq 2$, if there exist different points $z_1, \ldots, z_p \in S^1$, counter clockwise ordered on the circle such that for each $i = 1, \ldots, p - 1$, Q_i is the sector bounded by the half-lines L_i and L_{i+1} starting at the origin and passing through z_i and z_{i+1} respectively, Q_p is the sector bounded by the half-lines z_p and z_1 and $f(Q_i) \subset Q_i$ for each $i = 1, \ldots, p$.

REMARK 7.3. From Definition 7.2 it readily follows that $f(L_i) \subset L_i$ for all $i = 1, \ldots, p$.

LEMMA 7.4. *Let $f \colon E^2 \to E^2$ be a continuous map satisfying that $\mathrm{Fix}(f)$ is compact. If f has a sectorial decomposition $\{Q_1, \ldots, Q_p\}$, then*

$$i(f, E^2) = 1 + \sum_{i=1}^{p} i(f|_{Q_i}, Q_i) - \sum_{i=1}^{p} i(f|_{L_i}, L_i).$$

PROOF. It is easy to check that in general, if Q_1 and Q_2 are two sectors whose intersection is one of their boundary half-lines L, the triplet $(Q_1 \cup Q_2; Q_1, Q_2)$

admits a good Euclidean embedding (see Figure 1). Then, from Theorem 6.6 it follows that

$$(7.1) \qquad i(f|_{Q_1 \cup Q_2}, Q_1 \cup Q_2) + i(f|_L, L) = i(f|_{Q_1}, Q_1) + i(f|_{Q_2}, Q_2).$$

Now, suppose that f has sectorial decomposition $\{Q_1, \ldots, Q_p\}$. Then, if we consider $Q = Q_1 \cup \ldots \cup Q_{p-1}$, Remark 7.1 and (7.1) together with the induction argument ensure that

$$(7.2) \qquad i(f|_Q, Q) = -\sum_{i=2}^{p-1} i(f|_{L_i}, L_i) + \sum_{i=1}^{p-1} i(f|_{Q_i}, Q_i).$$

On the other hand, again from Theorem 6.6 we get

$$(7.3) \qquad i(f, E^2) + i(f|_{L_1 \cup L_p}, L_1 \cup L_p) = i(f|_Q, Q) + i(f|_{Q_p}, Q_p),$$

and

$$(7.4) \qquad i(f|_{L_1 \cup L_p}, L_1 \cup L_p) + 1 = i(f|_{L_1}, L_1) + i(f|_{L_p}, L_p).$$

Hence, the result follows from (7.2), (7.3) and (7.4).

REMARK 7.5. Assume that a continuous map $f \colon E^2 \to E^2$ admits a sectorial decomposition $\{Q_1, \ldots, Q_p\}$ and $\mathrm{Fix}(f) = \{0\}$. Since for each $i = 1, \ldots, p$, L_i may be embedded in $\mathbb{R}$ as $[0, +\infty)$ and easy computation shows that

$$i(f|_{L_i}, 0) \in \{0, 1\}.$$

COROLLARY 7.6. *Assume that $f \colon E^2 \to E^2$ admits a sectorial decomposition $\{Q_1, \ldots, Q_p\}$ and $\mathrm{Fix}(f) = \{0\}$. If we denote by $l = \mathrm{card}\{i \in \{1, \ldots, p\} : i(f|_{L_i}, 0) = 1\}$, then*

$$i(f, 0) = 1 - l + \sum_{i=1}^{p} i(f|_{Q_i}, 0).$$

If, in addition $i(f|_{Q_i}, 0) \in \{0, 1\}$ for each $i = 1, \ldots, p$ and $e = \mathrm{card}\{i \in \{1, \ldots, p\} : i(f|_{Q_i}, 0) = 1\}$, then

$$(7.5) \qquad i(f, 0) = 1 - l + e.$$

REMARK 7.7. One can think about (7.5) as a kind of Euler characteristic formula. In the classical Poincaré–Bendixson index formula (see [10, Proposition 6.32, p. 179] and [17]) the number of sectors would be $p = 2n$ and

$$i(f|_{L_i}, 0) = \begin{cases} 1 & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

hence $l = n$. The sectors with index 1 are called *elliptic sectors* and the sectors with index 0 are called *hyperbolic sectors*. Hence, the number of hyperbolic sectors is $h = 2n - e$. It follows that $l = (e + h)/2$, so

$$i(f, 0) = 1 - \frac{e + h}{2} + e = 1 + \frac{e - h}{2}.$$

Let $U \subset E^2$ be an open set. A homeomorphism $f \colon U \to U$ is called *free* provided whenever $D \subset U$ is a topological disk such that $f(D) \cap D = \emptyset$, then $f^i(D) \cap f^j(D) = \emptyset$ for all $i \neq j$. Obviously, the only periodic points of a free homeomorphism are fixed points.

PROPOSITION 7.8. *Let* $f \colon E_+^2 \to E_+^2$ *be an orientation preserving homeomorphism such that* $\mathrm{Fix}(f) = \{0\}$. *Then* $f|_{E_+^2 \setminus E_0^2}$ *is a free homeomorphism, and, hence,* f *has no other periodic points than the fixed point* 0 *and*

$$i(f^n, 0) = i(f, 0), \quad n > 0.$$

PROOF. We extend $f = (f_1, f_2)$ to the homeomorphism $F \colon E^2 \to E^2$ given by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in E_+^2, \\ (f_1(x, -y), -f_2(x, -y)) & \text{if } (x, y) \in E_-^2. \end{cases}$$

From Theorem 6.1 combined with the fact that $i(F_+, 0) = i(F_-, 0) = i(f, 0)$ it follows that

(7.6) $$i(F, 0) + i(f_0, 0) = 2i(f, 0).$$

Case 1. Assume that $i(f_0, 0) = 0$. Then, from (7.6) it follows that

$$i(F, 0) = 2i(f, 0),$$

so $i(F, 0)$ is even. Since F is an orientation preserving homeomorphism with the unique fixed point at the origin and $i(F, 0) \neq 1$, it follows from Theorem 3 in [4] that F is a free homeomorphism and $i(F^n, 0) = i(F, 0)$ for $n > 0$, so the result follows.

Case 2. Either ($i(f_0, 0) = 1$ and $i(f, 0) \neq 1$) or ($i(f_0, 0) = -1$ and $i(f, 0) \neq 0$). It follows from (7.6) that $i(F, 0) \neq 1$, so we can use the same arguments as in the proof of Case 1.

Case 3. $i(f_0, 0) = 1$ and $i(f, 0) = 1$. We extend f to a homeomorphism $H \colon E^2 \to E^2$ such that $\mathrm{Fix}(H^n) = \{0\}$ for $n > 0$ and H admits a sectorial decomposition $\{Q_1, Q_2, Q_3\}$ where

$$\begin{aligned} Q_1 &= E_+^2, \\ Q_2 &= \{(x, y) \in E^2 : x \leq 0, \ y \leq 0\}, \\ Q_3 &= \{(x, y) \in E^2 : x \geq 0, y \leq 0\} \end{aligned}$$

with rays

$$L_1 = \{(x, 0) : x \geq 0\}, \qquad L_2 = \{(x, 0) : x \leq 0\}, \qquad L_3 = \{(0, y) : y \leq 0\},$$

and such that $i((H|_{L_3})^n, 0) = 0$ ($n > 0$). Observe that such an H must satisfy $i((H|_{L_i})^n, 0) = 1$ for $i = 1, 2$ and $n > 0$, since $i(f_0, 0) = 1$. To define H,

consider f_1 to be the first component of f and let $h\colon (-\infty, 0] \to (-\infty, 0]$ be a homeomorphism such that $\lim_{n\to+\infty} h^n(x) = -\infty$ for $x \neq 0$. We define

$$H(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in E_+^2, \\ (f_1(x,0), h(y)) & \text{if } (x,y) \in E_-^2. \end{cases}$$

Since $H|_{Q_i}$ $(i = 2, 3)$ are in Case 1, then $i((H|_{Q_i})^n, 0) = i(H|_{Q_i}, 0)$ for $n > 0$ and $i = 2, 3$. It follows from Lemma 7.4 that

$$i(H, 0) = 1 + i(H_+, 0) + i(H|_{Q_2}, 0) + i(H|_{Q_3}, 0) - 2 = 2i(H|_{Q_2}, 0) \neq 1,$$

since $i(H|_{E_+^2}, 0) = 1$ and $i(H|_{Q_2}, 0) = i(H|_{Q_3}, 0)$. We can use the arguments similar to the proof of Case 1.

Case 4. If $i(f_0, 0) = -1$ and $i(f, 0) = 0$, we are in a dual situation to Case 3. □

COROLLARY 7.9. *Assume that $f\colon E^2 \to E^2$ is an orientation preserving homeomorphism of the plane with the origin as its unique fixed point. If f has an invariant sector, then the sequence $\{i(f^n, 0)\}_{n\geq 1}$ is well defined and constant.*

If we consider orientation and area preserving homeomorphisms $f\colon E_+^2 \to E_+^2$ which have the origin as their unique fixed point, we can obtain upper bounds for $i(f, 0)$ by combining Theorem 6.1 with some results of [25].

PROPOSITION 7.10. *Let $f\colon E_+^2 \to E_+^2$ be an orientation and area preserving homeomorphism and $\mathrm{Fix}(f) = \{0\}$. Then*

  (a) *if $i(f_0, 0) \in \{0, -1\}$ then $i(f, 0) \leq 0$,*
  (b) *if $i(f_0, 0) = 1$ then $i(f, 0) \leq 1$.*

PROOF. Let $F\colon E^2 \to E^2$ be a homeomorphism defined in the proof of Proposition 7.8. Since F is area preserving it follows from [25] that $i(F, 0) \leq 1$. Hence the result follows using this fact in (7.6). □

8. Isolated invariant sets in the invariant subspace

The aim of this section is to give an application of Theorem 6.1 to the index of a continuous map $f\colon (E^n; E_+^n, E_-^n) \to (E^n; E_+^n, E_-^n)$ in an isolated invariant set. For this purpose we start by recalling some definitions about these sets.

Let (X, d) be a locally compact metric space and $f\colon X \to X$ a continuous map. The sequence $\sigma\colon \mathbb{Z}_- \to X$ is said to be a *left solution to f through x* if $\sigma(0) = x$ and $f(\sigma(i-1)) = \sigma(i)$ for each $i \in \mathbb{Z}_-$.

Given $N \subset X$, by $\mathrm{Inv}_f^+ N, \mathrm{Inv}_f^- N$, and $\mathrm{Inv}_f N$ we shall denote, respectively, the *positively invariant, negatively invariant* and *invariant parts* of N relative

to f, defined by

$$\mathrm{Inv}_f^+ N = \{x \in X : f^i(x) \in X \text{ for all } i \in \mathbb{Z}_+\},$$

$$\mathrm{Inv}_f^- N = \{x \in X : \exists \sigma : \mathbb{Z}_- \to N \text{ a left solution to } f \text{ through } x\},$$

$$\mathrm{Inv}_f N = \mathrm{Inv}_f^+ N \cap \mathrm{Inv}_f^- N.$$

A set S is said to be *invariant* if $f(S) = S$. This is easily seen to be equivalent to $S = \mathrm{Inv}_f S$. We say that S is an *isolated invariant set* if it admits a compact neighbourhood N such that $S = \mathrm{Inv}_f N$. Such a neighbourhood N is said to be an *isolating neighbourhood* of S. If X is an ENR then $i(f, S)$ is defined as the fixed point index of f on the interior of any isolating neighbourhood of S.

We define the *stable* and *unstable* sets of the compact invariant set S by

$$W^s(S, f) = \big\{x \in X : \lim_{i \to \infty} d(f^i(x), S) = 0\big\},$$

$$W^u(S, f) = \big\{x \in X : \exists \sigma \text{ a left solution of } f \text{ through } x$$

$$\text{such that } \lim_{i \to \infty} d(\sigma(-i), S) = 0\big\}.$$

PROPOSITION 8.1. *Let* $f \colon (E^n; E_+^n, E_-^n) \to (E^n; E_+^n, E_-^n)$ *be a continuous map and* $S \subset E_0^n$ *an isolated invariant set for* f *such that* $\mathrm{Fix}(f) \subset S$.

(a) *If* $W^u(S, f) \subset E_+^n$ *then* $i(f, S) = i(f_+, S)$. *Moreover, if* $W^u(S, f) \subset E_0^n$ *then*

$$i(f, S) = i(f_+, S) = i(f_-, S) = i(f_0, S).$$

(b) *If* $W^s(S, f) \subset E_-^n$ *then* $i(f_+, S) = 0$. *Moreover, if* $W^s(S, f) \subset E_0^n$ *then*

$$i(f, S) = -i(f_0, S), \qquad i(f_\pm, S) = 0.$$

(c) *If* $W^s(S, f) \subset E_-^n$ *and* $W^u(S, f) \subset E_+^n$ *then* $i(f, S) = 0$.

PROOF. Suppose that $W^u(S, f) \subset E_+^n$. As a consequence, $W^u(S, f_-) \subset E_0^n$ and it follows from [31, Corollary 6] that $i(f_-, S) = i(f_0, S)$. Thus, the first part of (a) follows from this together with Theorem 6.1. If additionally, $W^u(S, f) \subset E_0^n$ then $i(f_-, S) = i(f, S)$ by symmetry and, hence, (a) holds.

If $W^s(S, f) \subset E_-^n$ then $W^s(S, f_+) \subset E_0^n$ and, hence, [31, Corollary 6] ensures that $i(f_+, S) = 0$. In order to finish the proof of (a), suppose that $W^s(S, f) \subset E_0^n$. Reasoning in an analogous way we get that

$$i(f, S) = -i(f_0, S), \qquad i(f_\pm, S) = 0.$$

One can check that (c) follows by combining (a) and (b). □

REMARK 8.2. Let $f \colon (E^3; E_+^3, E_-^3) \to (E^3, E_+^3, E_-^3)$ be a continuous map with the origin being an isolated invariant set. Proposition 8.1 together with [22], [28] sheds some light on the behaviour of the sequence $(i(f^n, 0))_{n \geq 1}$ in dimension 3. Suppose that $W^u(0, f^n) \subset E_0^3$ for every $n \geq 1$. Hence, item (a) in Proposition 8.1 ensures that $i(f^n, 0) = i(f_0^n, 0)$ for each $n \geq 1$.

Following [14] we say that the isolated invariant set $\{0\}$ is a *sink* provided there exists an isolating neighbourhood N of 0 such that $f(N) \subset N$. On the other hand, $\{0\}$ is called a *source* if there exists an isolating neighbourhood N such that $f(\partial N) \cap \text{int}(N) = \emptyset$ and the forward orbit of every point $x \in N \setminus \{0\}$ eventually exits N.

If $\{0\}$ is neither a sink nor a source with respect to f_0 and f_0 is an orientation preserving homeomorphism, then there exist positive integers r, q, such that for each $n \geq 1$ (compare [22])

$$i(f^n, 0) = i(f_0^n, 0) = \begin{cases} 1 - rq & \text{if } q|n, \\ 1 & \text{otherwise.} \end{cases}$$

If f_0 is an orientation reversing homeomorphism instead of orientation preserving one can use [28] to get that there are integers $\delta \in \{0, 1, 2\}$ and q such that

$$i(f^n, 0) = i(f_0^n, 0) = \begin{cases} 1 - \delta & \text{if } n \text{ is odd,} \\ 1 - \delta - 2q & \text{if } n \text{ is even.} \end{cases}$$

If the origin 0 is a source for f_0 then by [14] there exists an integer d such that $i(f^n, 0) = i(f_0^n, 0) = d^n$ for any $n \geq 1$.

9. Maps of E^3 symmetric with respect E_0^3

In this section we consider continuous maps $f \colon (E^3; E_+^3, E_-^3) \to (E^3; E_+^3, E_-^3)$ which are symmetric with respect to E_0^3, i.e. $f = J \circ f \circ J$, where $J(x, y, z) = (x, y, -z)$.

The following result deals with the fixed point index of isolated fixed points. Its first part is an application of Theorem 6.1 to the index of an isolated stable fixed point. The index of stable fixed points has been extensively studied. For instance, Bonatti and Villadelprat (see [2]) have shown that, in dimension greater or equal to 3, the index of a stable critical point of a vector field can be any integer. Dancer and Ortega [7] proved that for any orientation preserving homeomorphism of the plane such that 0 is an isolated and stable fixed point, the fixed point index at 0 is equal to 1. The same was proved in [27] for orientation reversing homeomorphisms. Let us mention that if 0 is a sink, its index is equal to 1 in any dimension.

PROPOSITION 9.1. *Let $f \colon (E^3; E_+^3, E_-^3) \to (E^3; E_+^3, E_-^3)$ be a continuous map symmetric with respect to E_0^3 satisfying that the origin is an isolated fixed point. Then*

  (a) *If f_0 is a homeomorphism and 0 is stable with respect to f_0 then $i(f, 0)$ is odd.*

(b) *If 0 is an isolated fixed point of f^n for all $n \geq 1$ and f_0 is an orientation preserving homeomorphism then there is an integer $p \in \mathbb{Z} \setminus \{0, 2\}$ such that*

$$i(f^n, 0) \in \begin{cases} i(f_0, 0) + 2\mathbb{Z} & \text{if } i(f_0, 0) \neq 1, \\ 1 + 2\mathbb{Z} & \text{if } i(f_0^n, 0) = 1, \\ p + 2\mathbb{Z} & \text{if } i(f_0^n, 0) = p. \end{cases}$$

(c) *If 0 is an isolated fixed point of f^n for all $n \geq 1$ and f_0 is an orientation reversing homeomorphism then for every odd $n \geq 1$*

$$i(f^n, 0) \in \begin{cases} 2\mathbb{Z} & \text{if } i(f_0, 0) = 0, \\ 1 + 2\mathbb{Z} & \text{if } i(f_0, 0) \neq 0. \end{cases}$$

PROOF. Observe that if 0 is an isolated fixed point of f^n for some $n \geq 1$ then it follows from the proof of Theorem 6.1 that

$$i(f^n, 0) + i(f_0^n, 0) = 2i(f_+^n, 0),$$

and, as a consequence, $i(f^n, 0) \equiv i(f_0^n, 0) \pmod 2$. The case (a) follows since $i(f_0, 0) = 1$ by [7, 27].

We prove (b) and (c). Assume that 0 is an isolated fixed point of f^n for every $n \geq 1$ and let f_0 be orientation preserving. If $i(f, 0) \neq 1$ then $i(f_0, 0)$ is also different from 1 and, hence, $i(f_0^n, 0) = i(f_0, 0)$ for every $n \geq 1$ by [4]. As a consequence, $i(f^n, 0) \equiv i(f_0, 0) \pmod 2$ for $n \geq 1$.

If $i(f_0, 0) = 1$ then there exists $p \in \mathbb{Z}$ such that $i(f_0^n, 0) \in \{1, p\}$ for $n \geq 1$ ([4]). It follows from [13] that $p = 0$ and $p = 2$ cannot occur as indices of any iteration if $i(f_0, 0) = 1$, hence (2) holds.

If f_0 is orientation reversing then [13, Theorem 4.2] (compare with [3]) ensures that for every n odd $i(f_0^n, 0) \in \{-1, 0, 1\}$ and

$$i(f_0^n, 0) = \begin{cases} i(f_0, 0) & \text{if } n > 0, \\ -i(f_0, 0) & \text{if } n < 0, \end{cases}$$

which proves (c). $\qquad\square$

10. Final remarks and open problems

Throughout the paper we have studied continuous maps $f \colon (X; X_1, X_2) \to (X; X_1, X_2)$ of ENR triplets having compact fixed point set. We saw that the Mayer–Vietoris formula of the fixed point holds in the following situations:

(1) if X is compact,
(2) if X_0 is a trapping set,
(3) if $\mathrm{Fix}(f) \subset \mathrm{int}(C \setminus E)$ for an admissible proper pair,
(4) if $(X; X_1, X_2) = (E^n; E_+^n, E_-^n)$, and
(5) if $(X; X_1, X_2)$ admits a good Euclidean embedding.

However, the general case remains open.

PROBLEM 10.1. Does the Mayer–Vietoris formula (1.1) hold for general self-maps $f\colon (X; X_1, X_2) \to (X; X_1, X_2)$ of ENR triplets whose fixed point set is compact?

Let X be an ENR and $f\colon X \to X$ be a continuous map having x_0 as an isolated fixed point for all its iterates. The sequence of fixed point indices $(i(f^n, x_0))_{n \geq 1}$ must satisfy the following congruences, known as *Dold relations* [9]:

$$\sum_{k \mid n} \mu(n/k) i(f^k, x_0) \equiv 0 \pmod{n},$$

where μ denotes the arithmetic Möbius function which assigns to each natural number n a value $-1, 0$, or 1 depending on its prime decomposition. In particular, $\mu(n) = 0$ if n has a repeated factor in its prime decomposition and $\mu(n) = (-1)^s$ otherwise, where s is the number of prime factors in its prime decomposition.

PROBLEM 10.2. Determine which integer sequences $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$ satisfying Dold relations can be realized in such a way that there exists a homeomorphism $f\colon E_+^n \to E_+^n$ such that 0 is an isolated fixed point for all its iterates and for each $k \geq 1$

$$a_k = i(f^k, 0) \quad \text{and} \quad b_k = i(f_0^k, 0).$$

Consider also the analogous question in the smooth category.

Problem 10.2 for odd iterates was recently solved by Graff and Jezierski in the smooth category [12].

The solution of the following problem could find some applications in mathematical ecology.

PROBLEM 10.3. Suppose that $f\colon (E_+^n, E_0^n) \to (E_+^n, E_0^n)$ is a continuous map. It is known (see [31]) that if $S \subset E_0^n$ is an isolated invariant set with respect to f (in the entire space) then

$$i(f, S) = \begin{cases} i(f_0, S) & \text{if } W^u(S, f) \subset E_0^n, \\ 0 & \text{if } W^s(S, f) \subset E_0^n. \end{cases}$$

Does it remain true if $S \subset E_0^n$ is an isolated compact set of fixed points of f but not necessarily isolated as an invariant set?

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