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GLOBAL AND LOCAL STRUCTURES OF OSCILLATORY BIFURCATION CURVES WITH APPLICATION TO INVERSE BIFURCATION PROBLEM

Tetsutaro Shibata

Abstract. We consider the bifurcation problem

$$-u''(t) = \lambda(u(t) + g(u(t))), \quad u(t) > 0, \quad t \in I := (-1, 1), \quad u(\pm 1) = 0,$$

where $g(u) = g_1(u) := \sin\sqrt{u}$ and $g_2(u) := \sin u^2 (= \sin(u^2))$, and $\lambda > 0$ is a bifurcation parameter. It is known that λ is parameterized by the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ and is written as $\lambda = \lambda(g, \alpha)$. When $g(u) = g_1(u)$, this problem has been proposed in Cheng [4] as an example which has arbitrary many solutions near $\lambda = \pi^2/4$. We show that the bifurcation diagram of $\lambda(g_1, \alpha)$ intersects the line $\lambda = \pi^2/4$ infinitely many times by establishing the precise asymptotic formula for $\lambda(g_1, \alpha)$ as $\alpha \to \infty$. We also establish the precise asymptotic formulas for $\lambda(g_i, \alpha)$ $(i = 1, 2)$ as $\alpha \to \infty$ and $\alpha \to 0$. We apply these results to the new concept of inverse bifurcation problems.

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1. Introduction

This paper is concerned with the following nonlinear eigenvalue problems:

$$-u''(t) = \lambda(u(t) + g(u(t))), \quad t \in I := (-1, 1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1) = u(1) = 0, \tag{1.3}$$

where $g(u)$ is an oscillatory nonlinear term and $\lambda > 0$ is a parameter. If $u + g(u) > 0$ for $u > 0$, it is known from [13] that, for any given $\alpha > 0$, there exists a unique solution pair (λ, u_α) of (1.1)–(1.3) with $\alpha = \|u_\alpha\|_\infty$ and λ is parameterized by α as $\lambda = \lambda(\alpha)$. Furthermore, $\lambda(\alpha)$ is continuous in $\alpha > 0$. Since λ also depends on g, we write $\lambda = \lambda(g, \alpha)$.

The study of the global and local structures of bifurcation diagrams is one of the main interest in the field of nonlinear eigenvalue problems, and many topics arising from mathematical biology, engineering, etc. have been investigated by many authors. We refer to [2], [3], [5], [6] and the references therein.

In particular, when the equations contain oscillatory nonlinear terms, sometimes the bifurcation curves have the oscillatory structures, which reflect the oscillatory properties of the nonlinear terms. We refer to [7], [9], [11], [14]–[16] and the references therein.

Relevant to the viewpoint above, the equation (1.1)–(1.3) with $g(u) = \sin\sqrt{u}$ has been proposed in Cheng [4] as a model problem which has arbitrary many solutions near $\lambda = \pi^2/4$.

THEOREM 1.1 ([4, Theorem 6]). *Let $g(u) = g_1(u) = \sin\sqrt{u}$ $(u \geq 0)$. Then, for any integer $r \geq 1$, there is $\delta > 0$ such that if $\lambda \in (\pi^2/4 - \delta, \pi^2/4 + \delta)$, then* (1.1)–(1.3) *has at least r distinct solutions.*

Certainly, Theorem 1.1 gives us the information about the structure of the solution set of (1.1)–(1.3), and it is quite natural for us to expect that $\lambda(g_1, \alpha)$ oscillates and intersects the line $\lambda = \pi^2/4$ infinitely many times for $\alpha \gg 1$.

In this paper, we first prove that the expectation above is valid. Precisely, we establish the asymptotic formula for $\lambda(g_1, \alpha)$ as $\alpha \to \infty$, which gives us the well understanding why $\lambda(g_1, \alpha)$ intersect the line $\lambda = \pi^2/4$ infinitely many times. We also obtain the asymptotic formula for $\lambda(g_1, \alpha)$ as $\alpha \to 0$. These two formulas clarify the whole structure of $\lambda(g_1, \alpha)$.

We next calculate the asymptotic length $L(g_1, \alpha)$ of $\lambda(g_1, \alpha)$ as $\alpha \to \infty$, where

$$L(g, \alpha) := \int_\alpha^{2\alpha} \sqrt{1 + (\lambda'(g, s))^2}\, ds. \tag{1.4}$$

This concept was introduced in [15] as a new idea to distinguish two unknown nonlinear terms g and $\widetilde{g}$ by the difference between $L(g, \alpha)$ and $L(\widetilde{g}, \alpha)$ for $\alpha \gg 1$.

Now we state our main results.

THEOREM 1.2. *Let $g(u) = g_1(u) = \sin\sqrt{u}$. Then as $\alpha \to \infty$,*

$$\lambda(g_1, \alpha) = \frac{\pi^2}{4} - \pi^{3/2}\alpha^{-5/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-5/4}), \tag{1.5}$$

$$\lambda'(g_1, \alpha) = \frac{1}{2}\pi^{3/2}\alpha^{-7/4}\sin\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-7/4}), \tag{1.6}$$

$$L(g_1, \alpha) = \alpha + \frac{1}{40}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + o(\alpha^{-5/2}). \tag{1.7}$$

THEOREM 1.3. *Let $g(u) = g_1(u) = \sin\sqrt{u}$.*

(a) *As $\alpha \to 0$, the following asymptotic formula for $\lambda(g_1, \alpha)$ holds:*

$$\lambda(g_1, \alpha) = \frac{3}{4}C_1^2\sqrt{\alpha} + \frac{3}{2}C_1C_2\alpha + O(\alpha^{3/2}), \tag{1.8}$$

where

$$C_1 := \int_0^1 \frac{1}{\sqrt{1 - s^{3/2}}}\,ds, \qquad C_2 := -\frac{3}{8}\int_0^1 \frac{1 - s^2}{(1 - s^{3/2})^{3/2}}\,ds. \tag{1.9}$$

(b) *Let v_0 be a unique classical solution of the following equation:*

$$-v_0''(t) = \frac{3}{4}C_1^2\sqrt{v_0(t)}, \quad t \in I, \tag{1.10}$$

$$v_0(t) > 0, \qquad t \in I, \tag{1.11}$$

$$v_0(-1) = v_0(1) = 0. \tag{1.12}$$

Furthermore, let $v_\alpha(t) := u_\alpha(t)/\alpha$. Then $v_\alpha \to v_0$ in $C^2(\overline{I})$ as $\alpha \to 0$.

For the uniqueness of the positive solution of (1.10)–(1.12), we refer to [1]. By Theorems 1.2 and 1.3, we see that the shape of $\lambda(g_1, \alpha)$ is like in Figure 1 below.

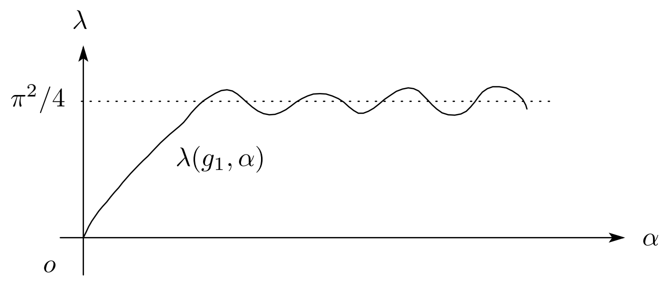

FIGURE 1. Bifurcation curve for $\lambda(g_1, \alpha)$ with $g_1(u)$.

When we consider an oscillatory nonlinear term $g(u)$, the most natural one is $g(u) = \sin u$, which has been already considered in [15]. In general, it seems quite difficult to treat the case $g_n(u) = \sin u^n$, where $n > 2$ is an integer. Therefore, the second purpose of this paper is to consider the case where $g(u) = \sin u^2$.

THEOREM 1.4. *Let $g(u) = g_2(u) = \sin u^2$. Then, as $\alpha \to \infty$,*

$$\lambda(g_2, \alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{2}\,\alpha^{-2}\cos\left(\alpha^2 - \frac{3}{4}\,\pi\right) + o(\alpha^{-2}), \tag{1.13}$$

$$\lambda'(g_2, \alpha) = \frac{\pi^{3/2}}{\alpha}\sin\left(\alpha^2 - \frac{3}{4}\,\pi\right) + o(\alpha^{-1}), \tag{1.14}$$

$$L(g_2, \alpha) = \alpha + \frac{\pi^3}{8\alpha} + o(\alpha^{-1}). \tag{1.15}$$

THEOREM 1.5. *Let $g(u) = g_2(u) = \sin u^2$. Then, as $\alpha \to 0$,*

$$\lambda(g_2, \alpha) = \frac{\pi^2}{4} - \frac{1}{3}\,\pi A_1 \alpha + \left(\frac{1}{9}\,A_1^2 + \frac{1}{6}\,\pi A_2\right)\alpha^2 + o(\alpha^2), \tag{1.16}$$

where

$$A_1 = \int_0^1 \frac{1 - s^3}{(1 - s^2)^{3/2}}\,ds, \qquad A_2 = \int_0^1 \frac{(1 - s^3)^2}{(1 - s^2)^{5/2}}\,ds. \tag{1.17}$$

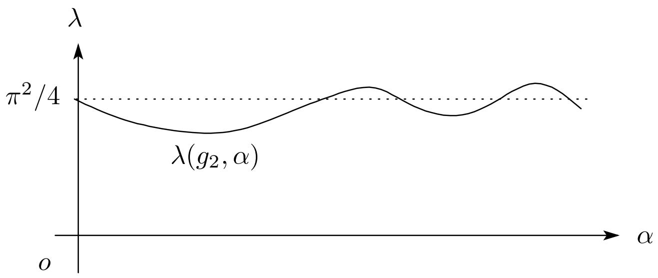

FIGURE 2. Bifurcation curve for $\lambda(g_2, \alpha)$.

We now consider an application of the asymptotic length obtained above to the inverse bifurcation problem, which has been proposed in [15]. Assume that there is an unknown nonlinear term $\widetilde{g}(u)$. Then is it possible to distinguish g_i $(i = 1, 2)$ and $\widetilde{g}$ by using $L(g_i, \alpha)$ and $L(\widetilde{g}, \alpha)$?

The advantage to consider $L(g, \alpha)$ in the inverse problem is as follows. On the theoretical side, we encounter the difficulty to obtain the precise shape of bifurcation curves. However, on the practical side, it sometimes happens that it is rather easy to measure the length of these curves. Therefore, if we can distinguish two unknown nonlinear terms from the information to get easily, then this approach may give us the new light to inverse bifurcation problems.

However, without any conditions on $\widetilde{g}$, it is quite difficult to treat the problem above. Therefore, we assume that $\widetilde{g}(u) \in C^1([0, \infty))$ satisfies the following assumption (A.1), which was introduced in [15].

(A.1) $\widetilde{g}(0) = \widetilde{g}'(0) = 0$, $\widetilde{g}'(u) \geq 0$ for $u > 0$ and $\widetilde{g}(u) = Cu^m$ for $u \geq 1$, where $C > 0$ and $0 < m < 1$ are constants.

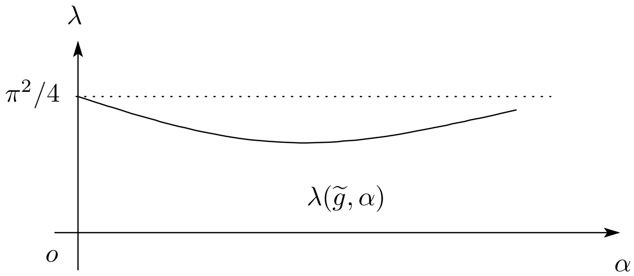

FIGURE 3. Bifurcation curve for $\lambda(\widetilde{g}, \alpha)$ with $\widetilde{g}(u) \simeq Cu^m$.

Then can we distinguish $\widetilde{g}(u)$ from $g_i(u)$ $(i = 1, 2)$ by $L(g, \alpha)$?

THEOREM 1.6 ([15]). *Let* $\widetilde{g}(u)$ *satisfy* (A.1). *Then, as* $\alpha \to \infty$,

$$L(\widetilde{g}, \alpha) = \alpha + \frac{2^{2m-3} - 1}{2(2m-3)} A(m)^2 \alpha^{2m-3} + o(\alpha^{2m-3}), \tag{1.18}$$

where

$$A(m) := \frac{(1-m)\pi C C(m)}{1+m}, \qquad C(m) = \int_0^1 \frac{1 - s^{m+1}}{(1-s^2)^{3/2}}\, ds. \tag{1.19}$$

By Theorems 1.4 and 1.6, we can distinguish g_2 and $\widetilde{g}$. On the contrary, if we put $m = 1/4$ and $C = 5/(6\sqrt{2}\pi C(1/4))$, then we see from Theorems 1.2 and 1.6 that the second terms of $L(g_1, \alpha)$ and $L(\widetilde{g}, \alpha)$ coincide. From this point, we might go to a more precise consideration of the concept of the asymptotic length of the bifurcation curves.

The proofs of Theorems 1.2 and 1.5 basically depend on the time-map argument and the asymptotic formulas for Bessel functions. In particular, the key tool of the proof of Theorem 1.2 is the asymptotic formula for the Bessel functions obtained in [12]. From this point of view, the proofs of Theorems 1.2 and 1.5 are different from those used in [14]–[16].

2. Proof of (1.5) in Theorem 1.2

In what follows, we eliminate g from $\lambda(g, \alpha)$ and write $\lambda = \lambda(\alpha)$ for simplicity. In this section, let $\alpha \gg 1$. Furthermore, we denote by C the various positive constants independent of α. For $u \ge 0$, let $g(u) = g_1(u) = \sin\sqrt{u}$ and

$$G(u) = \int_0^u g(s)\, ds = 2\sin\sqrt{u} - 2\sqrt{u}\cos\sqrt{u}. \tag{2.1}$$

It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\overline{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\alpha(t) = u_\alpha(-t), \qquad 0 \le t \le 1, \tag{2.2}$$

$$u_\alpha(0) = \max_{-1 \le t \le 1} u_\alpha(t) = \alpha, \tag{2.3}$$

$$u_\alpha'(t) > 0, \qquad -1 < t < 0. \tag{2.4}$$

By (1.1), we have

$$\big(u_\alpha''(t) + \lambda\big(u_\alpha(t) + \sin\sqrt{u_\alpha(t)}\big)\big)u_\alpha'(t) = 0.$$

By this and putting $t = 0$, we obtain

$$\frac{1}{2}\,u_\alpha'(t)^2 + \lambda\left(\frac{1}{2}\,u_\alpha(t)^2 + G(u_\alpha(t))\right) = \text{constant} = \lambda\left(\frac{1}{2}\,\alpha^2 + G(\alpha)\right).$$

This along with (2.4) implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{\lambda}\,\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \tag{2.5}$$

For $0 \le s \le 1$, we have

$$\left|\frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)}\right| = \left|\frac{\int_{\alpha s}^{\alpha} g(t)dt}{\alpha^2(1-s^2)}\right| \le \frac{\alpha(1-s)}{\alpha^2(1-s^2)} \le \alpha^{-1}. \tag{2.6}$$

By (2.5), (2.6), putting $s := u_\alpha(t)/\alpha$ and by Taylor expansion, we obtain

$$\begin{aligned}
\sqrt{\lambda} &= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}}\,dt \\
&= \int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}}\,ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}}\,\frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}}\,ds \\
&= \int_0^1 \frac{1}{\sqrt{1-s^2}}\left\{1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)}(1 + o(1))\right\} ds \\
&= \frac{\pi}{2} - \frac{1}{\alpha^2}\,(1 + o(1)) \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}}\,ds.
\end{aligned} \tag{2.7}$$

We put

$$K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}}\,ds. \tag{2.8}$$

Lemma 2.1. *As $\alpha \to \infty$,*

$$K(\alpha) = \sqrt{\pi}\alpha^{3/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\,\pi\right) + o(\alpha^{3/4}). \tag{2.9}$$

It is clear that (1.5) in Theorem 1.2 follows immediately from (2.7) and Lemma 2.1. We prove Lemma 2.1 by the series of several lemmas.

LEMMA 2.2. $K(\alpha) = \sqrt{2}\alpha^{3/2}R(\alpha)$ *for* $\alpha \gg 1$, *where*

$$R(\alpha) := \int_0^{\pi/2} \sqrt{1 - \frac{\cos^2\theta}{2}} \cos^2\theta \cos(\sqrt{\alpha}\sin\theta)\, d\theta. \tag{2.10}$$

PROOF. We put $s = \sin\theta$ in (2.8). Then, by integration by parts, we obtain

$$\begin{aligned} K(\alpha) &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} (G(\alpha) - G(\alpha\sin\theta))\, d\theta \\ &= \int_0^{\pi/2} (\tan\theta)'(G(\alpha) - G(\alpha\sin\theta))\, d\theta \\ &= [\tan\theta(G(\alpha) - G(\alpha\sin\theta))]_0^{\pi/2} \\ &\quad + \alpha \int_0^{\pi/2} \tan\theta(\cos\theta \sin\sqrt{\alpha\sin\theta})\, d\theta. \end{aligned} \tag{2.11}$$

By l'Hôpital's rule, we obtain

$$\begin{aligned} &\lim_{\theta\to\pi/2} \frac{\sin\sqrt{\alpha} - \sin\sqrt{\alpha\sin\theta} - \sqrt{\alpha}\cos\sqrt{\alpha} + \sqrt{\alpha\sin\theta}\cos\sqrt{\alpha\sin\theta}}{\cos\theta} \\ &\qquad = \lim_{\theta\to\pi/2} \frac{\alpha\cos\theta\sin\sqrt{\alpha\sin\theta}}{2\sin\theta} = 0. \end{aligned} \tag{2.12}$$

By this and (2.11), we obtain

$$K(\alpha) = \alpha L(\alpha) := \alpha \int_0^{\pi/2} \sin\theta \sin\sqrt{\alpha\sin\theta}\, d\theta. \tag{2.13}$$

We put $t = \sqrt{\sin\theta}$. Then by (2.13) and integration by parts, we obtain

$$\begin{aligned} L(\alpha) &= \int_0^1 \frac{2t^3}{\sqrt{1-t^4}} \sin\sqrt{\alpha}t\, dt = -\int_0^1 (\sqrt{1-t^4})' \sin\sqrt{\alpha}t\, dt \\ &= \sqrt{\alpha}\int_0^1 \sqrt{1-t^4}\cos(\sqrt{\alpha}t)\, dt \quad (\text{put } t = \sin\theta \text{ again}) \\ &= \sqrt{\alpha}\int_0^{\pi/2} \sqrt{1-\sin^4\theta}\cos(\sqrt{\alpha}\sin\theta)\cos\theta\, d\theta \\ &= \sqrt{\alpha}\int_0^{\pi/2} \sqrt{1+\sin^2\theta}\cos^2\theta\cos(\sqrt{\alpha}\sin\theta)\, d\theta \\ &= \sqrt{2\alpha}\int_0^{\pi/2} \sqrt{1-\frac{\cos^2\theta}{2}}\cos^2\theta\cos(\sqrt{\alpha}\sin\theta)\, d\theta. = \sqrt{2\alpha}\, R(\alpha). \end{aligned} \tag{2.14}$$

Thus the proof is complete. □

Let ν be a positive integer, and $J_\nu(x)$ be the Bessel function. Then we see from [12, Theorem 4] that for $x > 0$,

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos(x - w_\nu) + \theta c\mu x^{-3/2}, \tag{2.15}$$

where $w_\nu = (2\nu+1)\pi/4$, θ is a number with the absolute value not exceeding one, $\mu = |\nu^2 - (1/4)|$ and

$$c = 4/5 \quad (0 < x < \sqrt{\mu},\ \nu > 1/2),$$
$$c = 2/\pi \quad (x \geq \sqrt{\mu},\ \nu > 1/2).$$

Lemma 2.3. *For $\alpha \gg 1$,*

$$R(\alpha) = \sqrt{\frac{\pi}{2}}\,\alpha^{-3/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-3/4}). \tag{2.16}$$

Proof. We know from [8, p. 424] that for $n = 0, 1, \ldots$,

$$\int_0^{\pi/2} \cos^{2(n+1)}\theta \cos(x\sin\theta)\,d\theta = \frac{\pi}{2}\,(2n+1)!!\,x^{-(n+1)}J_{n+1}(x). \tag{2.17}$$

We know that for $|x| < 1$,

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!\,2^n}\,x^n. \tag{2.18}$$

Let $N > 0$ be an integer specified later. By (2.10), (2.15), (2.17) and (2.18), Taylor expansion and Lebesgue's convergence theorem, we have

$$\begin{aligned} R(\alpha) &= \int_0^{\pi/2} \left\{1 - \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!\,2^n}\,\frac{\cos^{2n}\theta}{2^n}\right\} \cos^2\theta\cos(\sqrt{\alpha}\sin\theta)\,d\theta \\ &= \frac{\pi}{2\sqrt{\alpha}}\,J_1(\sqrt{\alpha}) - \sum_{n=1}^{N} \frac{(2n-3)!!}{(2n)!!}\,\frac{\pi}{2}\,\frac{(2n+1)!!}{2^n\alpha^{(n+1)/2}}\,J_{n+1}(\sqrt{\alpha}) \\ &\quad - \sum_{n=N+1}^{\infty} \int_0^{\pi/2} \frac{(2n-3)!!}{(2n)!!}\,\frac{\cos^{2(n+1)}\theta}{2^n}\cos(\sqrt{\alpha}\sin\theta)\,d\theta \\ &:= \sqrt{\frac{\pi}{2}}\,\alpha^{-3/4}\cos\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + O(\alpha^{-5/4}) - Q_1 - Q_2. \end{aligned} \tag{2.19}$$

Here

$$\begin{aligned} (2n+1)!! &= (2n+1)(2n-1)\cdots 3\cdot 1, && (n = 1, 2, \ldots), \\ (2n)!! &= (2n)\cdot(2n-2)\cdots 4\cdot 2, && (n = 1, 2, \ldots), \\ (2n-3)!! &= (2n-3)(2n-5)\cdots 3\cdot 1, && (n = 2, 3, \ldots), \\ (2n-3)!! &= 1, && (n = 1). \end{aligned}$$

We show that Q_1 and Q_2 are remainder terms. We first calculate Q_2. Clearly, for $n \in \mathbb{N}$,

$$\int_0^{\pi/2} \cos^{2(n+1)}\theta|\cos(\sqrt{\alpha}\sin\theta)|\,d\theta < \frac{\pi}{2}. \tag{2.20}$$

By this, we obtain

$$|Q_2| \le \sum_{n=N+1}^{\infty} \frac{(2n-3)!!}{(2n)!!}\,\frac{1}{2^n}\int_0^{\pi/2} \cos^{2(n+1)}\theta |\cos(\sqrt{\alpha}\sin\theta)|\,d\theta \tag{2.21}$$

$$\le \sum_{n=N+1}^{\infty} \frac{\pi}{2}\,\frac{1}{2^n} = \frac{\pi}{2}\,\frac{1}{2^{N+1}}.$$

We next calculate Q_1. By (2.15), (2.17) and (2.19), we have

$$\begin{aligned} Q_1 =& \sum_{n=1}^{N} \frac{(2n-3)!!}{(2n)!!}\,\frac{\pi}{2}\,\frac{(2n+1)!!}{2^n\alpha^{(n+1)/2}}\sqrt{\frac{2}{\pi\alpha^{1/2}}} \\ &\times\left\{\cos(\sqrt{\alpha}-w_{n+1}) + \theta c\left(n^2+2n+\frac{3}{4}\right)\alpha^{-1/2}\right\} \\ =& \sqrt{\frac{\pi}{2}}\,\frac{1}{\alpha^{5/4}}\sum_{n=1}^{N}\frac{(2n-3)!!}{(2n)!!}\,\frac{(2n+1)!!}{2^n\alpha^{(n-1)/2}} \\ &\times\left\{\cos(\sqrt{\alpha}-w_{n+1}) + \theta c\left(n^2+2n+\frac{3}{4}\right)\alpha^{-1/2}\right\}. \end{aligned} \tag{2.22}$$

We choose N satisfying $N \le \alpha^{1/6} < N+1$. Then, for $1 \le n \le N$, we have

$$\frac{(2n+1)!!}{2^n\alpha^{(n-1)/2}} = \frac{n+(1/2)}{\alpha^{1/2}}\cdot\frac{n-(1/2)}{\alpha^{1/2}}\cdot\dots\cdot\frac{(5/2)}{\alpha^{1/2}}\cdot\frac{3}{2} < 1. \tag{2.23}$$

Moreover,

$$\sum_{n=1}^{N}\left(n^2+2n+\frac{3}{4}\right)\alpha^{-1/2} \le C. \tag{2.24}$$

By this, we obtain

$$|Q_1| \le C\alpha^{-5/4}N \le C\alpha^{-13/(12)}. \tag{2.25}$$

Furthermore, for $\alpha \gg 1$, we have $2^{-(N+1)} = o(\alpha^{-3/4})$. Then by this, (2.19), (2.21) and (2.25), we obtain

$$R(\alpha) = \sqrt{\frac{\pi}{2}}\,\alpha^{-3/4}\cos\left(\sqrt{\alpha}-\frac{3}{4}\pi\right) + o(\alpha^{-3/4}). \tag{2.26}$$

This implies (2.16). □

Now Lemma 2.1 follows from Lemmas 2.2 and 2.3. Then we obtain (1.5) of Theorem 1.1 from (2.7) and Lemma 2.1. Thus the proof is complete. □

3. Proofs of (1.6) and (1.7) in Theorem 1.2

In this section, let $\alpha \gg 1$. By direct calculation, we obtain

$$\lambda'(\alpha) = 2\sqrt{\lambda(\alpha)}\,\frac{d}{d\alpha}\left(\sqrt{\lambda(\alpha)}\right). \tag{3.1}$$

We see from (3.1) that (1.6) in Theorem 1.2 follows from (1.5), (3.1) and the following Lemma 3.1.

LEMMA 3.1. *As* $\alpha \to \infty$,

$$\frac{d}{d\alpha}\left(\sqrt{\lambda(\alpha)}\right) = \frac{1}{2}\sqrt{\pi}\alpha^{-7/4}\sin\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{-7/4}). \tag{3.2}$$

By (2.6), (2.7), Lemma 2.1 and Lebesgue's convergence theorem, we have

(3.3)

$$\begin{aligned}\frac{d}{d\alpha}\left(\sqrt{\lambda(\alpha)}\right) &= \frac{d}{d\alpha}\int_0^1 \frac{1}{\sqrt{1 - s^2 + 2(G(\alpha) - G(\alpha s))/\alpha^2}}\,ds \\ &= -\frac{1}{\alpha^2}(1 + o(1))\int_0^1 \frac{g(\alpha) - sg(\alpha s)}{(1 - s^2)^{3/2}}\,ds \\ &\quad + \frac{2}{\alpha^3}(1 + o(1))\int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}}\,ds \\ &= -\frac{1}{\alpha^2}(1 + o(1))\int_0^1 \frac{g(\alpha) - sg(\alpha s)}{(1 - s^2)^{3/2}}\,ds + \frac{2}{\alpha^3}(1 + o(1))K(\alpha) \\ &:= -\frac{1}{\alpha^2}(1 + o(1))M(\alpha) + O(\alpha^{-9/4}).\end{aligned}$$

Therefore, Lemma 3.1 follows from (3.3) and the following Lemma 3.2.

LEMMA 3.2. *As* $\alpha \to \infty$,

$$M(\alpha) = -\frac{\sqrt{\pi}}{2}\alpha^{1/4}\sin\left(\sqrt{\alpha} - \frac{3}{4}\pi\right) + o(\alpha^{1/4}). \tag{3.4}$$

Lemma 3.2 is proved by a series of lemmas.

LEMMA 3.3. *As* $\alpha \to \infty$,

$$M(\alpha) = \frac{1}{2}\alpha^{1/2}M_1(\alpha) + O(\alpha^{-1/4}), \tag{3.5}$$

where

$$M_1(\alpha) := \int_0^{\pi/2} \sin^{3/2}\theta\cos\sqrt{\alpha\sin\theta}\,d\theta. \tag{3.6}$$

PROOF. By putting $s = \sin\theta$ in (3.3), (2.12), (2.14), (2.16) and Lemma 2.3, we obtain

$$\begin{aligned}M(\alpha) &= \int_0^1 \frac{\sin\sqrt{\alpha} - s\sin\sqrt{\alpha s}}{(1 - s^2)^{3/2}}\,ds \\ &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}\left\{\sin\sqrt{\alpha} - \sin\theta\sin\sqrt{\alpha\sin\theta}\right\}d\theta \\ &= \left[\tan\theta\left\{\sin\sqrt{\alpha} - \sin\theta\sin\sqrt{\alpha\sin\theta}\right\}\right]_0^{\pi/2} \\ &\quad + \int_0^{\pi/2} \tan\theta\left\{\cos\theta\sin\sqrt{\alpha\sin\theta}\right.\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
&+ \frac{1}{2}\,\alpha^{1/2} \sin^{1/2}\theta \cos\theta \cos\sqrt{\alpha\sin\theta}\Big\}\, d\theta \\
&= \frac{1}{2}\,\alpha^{1/2} \int_0^{\pi/2} \sin^{3/2}\theta \cos\sqrt{\alpha\sin\theta}\, d\theta + \int_0^{\pi/2} \sin\theta \sin\sqrt{\alpha\sin\theta}\, d\theta \\
&:= \frac{1}{2}\,\alpha^{1/2} M_1(\alpha) + L(\alpha) = \frac{1}{2}\,\alpha^{1/2} M_1(\alpha) + O(\alpha^{-1/4}).
\end{aligned}
$$

Thus the proof is complete. □

Lemma 3.4. *As $\alpha \to \infty$,*

$$
M_1(\alpha) = -\sqrt{\pi}\alpha^{-1/4} \sin\left(\sqrt{\alpha} - \frac{3}{4}\,\pi\right) + o(\alpha^{-1/4}). \tag{3.8}
$$

Proof. The proof is divided into several steps.

Step 1. By putting $t = \sqrt{\sin\theta}$ and integration by parts, (2.14) and (2.16), we obtain

$$
\begin{aligned}
M_1(\alpha) &= 2 \int_0^1 \frac{t^4}{\sqrt{1-t^4}} \cos\sqrt{\alpha}t\, dt = \int_0^1 \frac{2t^3}{\sqrt{1-t^4}}\,(t\cos\sqrt{\alpha}t)\, dt \\
&= -\int_0^1 \{(1-t^4)^{1/2}\}'(t\cos\sqrt{\alpha}t)\, dt \\
&= -\left[(1-t^4)^{1/2} t\cos\sqrt{\alpha}t\right]_0^1 \\
&\quad + \int_0^1 (1-t^4)^{1/2}\{\cos\sqrt{\alpha}t - \sqrt{\alpha}t\sin\sqrt{\alpha}t\}\, dt \\
&= \int_0^1 (1-t^4)^{1/2}\{\cos\sqrt{\alpha}t - \sqrt{\alpha}t\sin\sqrt{\alpha}t\}\, dt \\
&= -\sqrt{\alpha}\int_0^1 (1-t^4)^{1/2} t\sin\sqrt{\alpha}t\, dt + \sqrt{2}R(\alpha) \\
&= -\sqrt{\alpha}\int_0^1 (1-t^4)^{1/2} t\sin\sqrt{\alpha}t\, dt + O(\alpha^{-3/4}).
\end{aligned} \tag{3.9}
$$

By this, (2.18) and putting $t = \sin\theta$, we obtain

$$
\begin{aligned}
M_1(\alpha) &= -\sqrt{\alpha}\int_0^{\pi/2} (1+\sin^2\theta)^{1/2}(1-\sin^2\theta)^{1/2} \\
&\qquad \cdot \sin\theta\sin(\sqrt{\alpha}\sin\theta)\cos\theta\, d\theta + O(\alpha^{-3/4}) \\
&= -\sqrt{\alpha}\int_0^{\pi/2} (1+\sin^2\theta)^{1/2} \\
&\qquad \cdot \cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\, d\theta + O(\alpha^{-3/4}) \\
&= -\sqrt{2\alpha}\int_0^{\pi/2} \sqrt{1-\frac{\cos^2\theta}{2}} \\
&\qquad \cdot \cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\, d\theta + O(\alpha^{-3/4})
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&= -\sqrt{2\alpha}\int_0^{\pi/2}\left\{1-\sum_{n=1}^{\infty}\frac{(2n-3)!!}{n!\,2^n}\,\frac{\cos^{2n}\theta}{2^n}\right\}\\
&\qquad \cdot\cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\,d\theta + O(\alpha^{-3/4})\\
&= -\sqrt{2\alpha}\int_0^{\pi/2}\cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)d\theta\\
&\qquad + \sqrt{2\alpha}\int_0^{\pi/2}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{n!\,2^n}\,\frac{\cos^{2n}\theta}{2^n}\\
&\qquad \cdot\cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\,d\theta + O(\alpha^{-3/4})\\
&:= -\sqrt{2\alpha}M_2(\alpha) + \sqrt{2\alpha}\sum_{n=1}^{N}\frac{(2n-3)!!}{n!\,2^n}\,\frac{1}{2^n}\,Q_n\\
&\qquad + \sqrt{2\alpha}\sum_{n=N+1}^{\infty}\frac{(2n-3)!!}{n!\,2^n}\,\frac{1}{2^n}\,Q_n + O(\alpha^{-3/4})\\
&:= -\sqrt{2\alpha}\,M_2(\alpha) + \sqrt{2\alpha}\,M_3(\alpha) + \sqrt{2\alpha}\,M_4(\alpha) + O(\alpha^{-3/4}),
\end{aligned}
$$

where $N \gg 1$ will be an integer specified later, and

$$
Q_n := \int_0^{\pi/2}\cos^{2(n+1)}\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\,d\theta. \tag{3.11}
$$

Step 2. We show that $-\sqrt{2\alpha}\,M_2(\alpha)$ is the leading term of $M_1(\alpha)$. By (2.15) and (2.17), we have

$$
\begin{aligned}
M_2(\alpha) &= \int_0^{\pi/2}\cos^2\theta\sin\theta\sin(\sqrt{\alpha}\sin\theta)\,d\theta \qquad (3.12)\\
&= \int_0^{\pi/2}\left(-\frac{1}{3}\cos^3\theta\right)'\sin(\sqrt{\alpha}\sin\theta)\,d\theta\\
&= \left[-\frac{1}{3}\cos^3\theta\sin(\sqrt{\alpha}\sin\theta)\right]_0^{\pi/2}\\
&\qquad + \frac{1}{3}\sqrt{\alpha}\int_0^{\pi/2}\cos^4\theta\cos(\sqrt{\alpha}\sin\theta)\,d\theta\\
&= \frac{1}{3}\sqrt{\alpha}\int_0^{\pi/2}\cos^4\theta\cos(\sqrt{\alpha}\sin\theta)\,d\theta = \frac{1}{3}\sqrt{\alpha}\,\frac{\pi}{2}\,\frac{3!!}{(\sqrt{\alpha})^2}J_2(\sqrt{\alpha})\\
&= \frac{1}{\sqrt{\alpha}}\,\frac{\pi}{2}\left\{\sqrt{\frac{2}{\pi\sqrt{\alpha}}}\cos\left(\sqrt{\alpha}-\frac{5}{4}\pi\right) + O(\alpha^{-3/4})\right\}\\
&= \sqrt{\frac{\pi}{2}}\,\alpha^{-3/4}\sin\left(\sqrt{\alpha}-\frac{3}{4}\pi\right) + o(\alpha^{-3/4}).
\end{aligned}
$$

Step 3. We show that $\sqrt{2\alpha}\, M_3(\alpha)$ and $\sqrt{2\alpha}\, M_4(\alpha)$ in (3.10) are negligible. By using integration by parts, (2.16) and (2.17), we obtain

$$
\begin{aligned}
Q_n = & \int_0^{\pi/2} \left(- \frac{1}{2n+3} \cos^{2n+3}\theta \right)' \sin(\sqrt{\alpha}\sin\theta)\, d\theta \\
= & \left[\left(- \frac{1}{2n+3} \cos^{2n+3}\theta \right) \sin(\sqrt{\alpha}\sin\theta) \right]_0^{\pi/2} \\
& + \frac{1}{2n+3} \sqrt{\alpha} \int_0^{\pi/2} \cos^{2(n+2)}\theta \cos(\sqrt{\alpha}\sin\theta)\, d\theta \\
= & \frac{1}{2n+3} \sqrt{\alpha}\, \frac{(2n+3)!!}{\alpha^{(n+2)/2}} J_{n+2}(\sqrt{\alpha}) \\
= & \frac{1}{2n+3} \sqrt{\alpha}\, \frac{(2n+3)!!}{\alpha^{(n+2)/2}} \\
& \times \left[\sqrt{\frac{2}{\pi\sqrt{\alpha}}} \cos\left(\sqrt{\alpha} - \frac{\pi}{2}(n+2) - \frac{1}{4}\pi \right) + \theta c \mu \alpha^{-3/4} \right].
\end{aligned}
\tag{3.13}
$$

We choose N satisfying $N + 1 \le \alpha^{1/6} < N + 2$. Recall that c and θ are the constants defined in Lemma 2.1. Then by (3.13), we obtain

$$
\begin{aligned}
Q_n \le & \sqrt{\alpha}\, \frac{(2n+1)!!}{\alpha^{(n+2)/2}} \left\{ C\alpha^{-1/4} + \theta c \left(n^2 + 4n + \frac{15}{4} \right) \alpha^{-3/4} \right\} \\
\le & C\sqrt{\alpha}\, \frac{(2n+1)!!}{\alpha^{(n+2)/2}} \left\{ \alpha^{-1/4} + \left(n^2 + 4n + \frac{15}{4} \right) \alpha^{-3/4} \right\}.
\end{aligned}
\tag{3.14}
$$

By this and (3.10), we obtain

$$
\begin{aligned}
|M_3(\alpha)| \le & C \sum_{n=1}^{N} \sqrt{\alpha}\, \frac{(2n-3)!!}{(2n)!!} \frac{(2n+1)!!}{2^n \alpha^{n/6}} \frac{1}{\alpha^{(n+3)/3}} \\
& \times \left(\alpha^{-1/4} + \left(n^2 + 4n + \frac{15}{4} \right) \alpha^{-3/4} \right) \\
\le & C\alpha^{-13/12} N \le C\alpha^{-11/12}.
\end{aligned}
\tag{3.15}
$$

Since $|Q_n| \le \pi/2$ for $n \in \mathbb{N}$, we have

$$
|M_4(\alpha)| \le C 2^{-N} \le C\alpha^{-1}. \tag{3.16}
$$

By (3.10), (3.12), (3.15) and (3.16), we obtain (3.8). Thus the proof is complete. □

Lemma 3.1 follows from Lemmas 3.2–3.4. Then we obtain (1.6) in Theorem 1.2 by (1.5), (3.1) and Lemma 3.1. Thus the proof is complete. □

PROOF OF (1.7). By (1.4), (1.6) and Taylor expansion, for $\alpha \gg 1$, we obtain

$$L(g_1,\alpha) = \int_\alpha^{2\alpha} \sqrt{1 + \frac{1}{4}\,\pi^3 s^{-7/2}(1+o(1))\sin^2\left(\sqrt{s} - \frac{3}{4}\,\pi\right)}\,ds \tag{3.17}$$
$$= \int_\alpha^{2\alpha} \left\{1 + \frac{1}{8}\,\pi^3(1+o(1))s^{-7/2}\sin^2\left(\sqrt{s} - \frac{3}{4}\,\pi\right)\right\} ds$$
$$= \alpha + \frac{1}{4}\,\pi^3(1+o(1))\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}\sin^2\left(t - \frac{3}{4}\,\pi\right) dt.$$

By integration by parts, we obtain

$$\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}\sin^2\left(t - \frac{3}{4}\,\pi\right) dt = \frac{1}{2}\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}(\sin t + \cos t)^2\,dt \tag{3.18}$$
$$= \frac{1}{2}\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}\,dt + \frac{1}{2}\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}\sin 2t\,dt$$
$$= \frac{1}{10}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + \frac{1}{2}\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-6}\left(-\frac{1}{2}\cos 2t\right)' dt$$
$$= \frac{1}{10}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + \left[-\frac{1}{4}\,t^{-6}\cos 2t\right]_{\sqrt{\alpha}}^{\sqrt{2\alpha}}$$
$$- \frac{3}{2}\int_{\sqrt{\alpha}}^{\sqrt{2\alpha}} t^{-7}\cos 2t\,dt$$
$$= \frac{1}{10}\left(1 - \frac{1}{4\sqrt{2}}\right)\alpha^{-5/2} + O(\alpha^{-3}).$$

By this and (3.17), we obtain (1.7). □

4. Proof of Theorem 1.3

In this section, let $0 < \alpha \ll 1$.

PROOF OF THEOREM 1.3 (a). By (2.7),

$$\sqrt{\lambda} = \int_0^1 \frac{\alpha}{\sqrt{\alpha^2(1-s^2) + 2\alpha^{3/2}(G(\alpha) - G(\alpha s))/\alpha^{3/2}}}\,ds \tag{4.1}$$
$$= \alpha^{1/4}\int_0^1 \frac{1}{\sqrt{\alpha^{1/2}(1-s^2) + 2(G(\alpha) - G(\alpha s))/\alpha^{3/2}}}\,ds.$$

By Taylor expansion, for $0 \le s \le 1$, we have

$$G(\alpha) - G(\alpha s) = \int_{\alpha s}^{\alpha} \sin\sqrt{t}\,dt = \int_{\alpha s}^{\alpha}\left(\sqrt{t} - \frac{1}{6}\,t^{3/2} + O(\alpha^{5/2})\right) dt$$
$$= \frac{2}{3}\,\alpha^{3/2}(1 - s^{3/2}) - \frac{1}{15}\,\alpha^{5/2}(1 - s^{5/2}) + O(\alpha^{7/2})(1-s)$$
$$= \frac{2}{3}\,\alpha^{3/2}(1 - s^{3/2}) - \frac{1}{15}\,\alpha^{5/2}(1+o(1))(1 - s^{5/2}).$$

By this and (4.1), we obtain

$$
\begin{aligned}
(4.2)\qquad \sqrt{\lambda} &= \alpha^{1/4}\int_0^1 \frac{1}{\sqrt{\alpha^{1/2}(1-s^2)+\frac{4(1-s^{3/2})}{3}-\frac{2\alpha(1+o(1))(1-s^{5/2})}{15}}}\,ds \\
&= \frac{\sqrt{3}}{2}\,\alpha^{1/4} \\
&\quad\times\int_0^1 \frac{1}{\sqrt{1-s^{3/2}}\sqrt{1+\alpha^{1/2}\frac{3(1-s^2)}{4(1-s^{3/2})}-\alpha(1+o(1))\,\frac{1-s^{5/2}}{10(1-s^{3/2})}}}\,ds \\
&= \frac{\sqrt{3}}{2}\,\alpha^{1/4}\int_0^1 \frac{1}{\sqrt{1-s^{3/2}}}\left(1-\alpha^{1/2}\,\frac{3(1-s^2)}{8(1-s^{3/2})}+O(\alpha)\right)ds \\
&= \frac{\sqrt{3}}{2}\,\alpha^{1/4}(C_1+C_2\alpha^{1/2}+O(\alpha)),
\end{aligned}
$$

where C_1 and C_2 are constants defined in (1.9). □

Proof of Theorem 1.3 (b). By (1.1) and Theorem 1.3 (a), we see that v_α satisfies

$$(4.3)\qquad -v_\alpha''(t) = \frac{3}{4}\,C_1^2(1+o(1))\left(\alpha^{1/2}v_\alpha(t)+\frac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha v_\alpha(t)}\right).$$

By this, we see that $\|v_\alpha''\|_\infty \le C, \|v_\alpha'\|_\infty \le C, \|v_\alpha\|_\infty = 1$. We choose an arbitrary subsequence of $\{v_\alpha\}$, which is denoted by $\{v_\alpha\}$ again, for simplicity. Let $\alpha \to 0$. By these inequalities, (4.4) and Ascoli–Arzelà theorem, we can choose a subsequence of $\{v_\alpha\}$, which is denoted by $\{v_\alpha\}$ again, such that $v_\alpha \to v_0$ in $C^2(\overline{I})$. This implies that v_0 is a classical solution of (1.10)–(1.12). Then, by a standard compactness argument, we see that $v_\alpha \to v_0$ in $C^2(\overline{I})$ as $\alpha \to 0$. □

5. Proof of Theorem 1.4

In this section, let $g(u) = g_2(u) = \sin u^2$ and $\alpha \gg 1$. We know that

$$(5.1)\qquad G(u) = \int_0^u \sin t^2\,dt = \sqrt{\frac{\pi}{2}}\,S(u),$$

where $S(u)$ is the Fresnel sine integral defined by

$$(5.2)\qquad S(u) = \sqrt{\frac{2}{\pi}}\int_0^u \sin x^2\,dx.$$

Further, let $C(\alpha)$ be the Fresnel cosine integral defined by

$$(5.3)\qquad C(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\alpha \cos x^2\,dx.$$

Then we know (cf. [8, pp. 898–899]) that as $\alpha \to \infty$,

$$(5.4)\qquad \begin{aligned} S(\alpha) &= \frac{1}{2} - \frac{1}{\sqrt{2\pi\alpha}} \cos^2 \alpha + O(\alpha^{-2}), \\ C(\alpha) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi\alpha}} \sin^2 \alpha + O(\alpha^{-2}). \end{aligned}$$

Since (2.6) also holds in this case, for $\alpha \gg 1$, we have (2.7). We calculate (2.7) by (2.8).

LEMMA 5.1. *As* $\alpha \to \infty$,

$$(5.5)\qquad K(\alpha) = \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}} \, ds = \frac{\sqrt{\pi}}{2} \cos\left(\alpha^2 - \frac{3}{4}\pi\right) + o(1).$$

PROOF. For $0 \le \theta \le \pi/2$, we put

$$(5.6)\qquad P(\theta) := \int_{\alpha \sin\theta}^{\alpha} \sin t^2 \, dt.$$

We put $s = \sin\theta$ in (5.5). Then by (5.5) and integration by parts, we obtain

$$(5.7)\qquad \begin{aligned} K(\alpha) &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} P(\theta) \, d\theta \\ &= [\tan\theta \, P(\theta)]_0^{\pi/2} + \alpha \int_0^{\pi/2} \tan\theta \sin(\alpha\sin\theta)^2 \cos\theta \, d\theta \\ &:= K_1(\alpha) + \alpha K_2(\alpha). \end{aligned}$$

By l'Hôpital's rule, we have

$$(5.8)\qquad \lim_{\theta \to \pi/2} \frac{P(\theta)}{\cos\theta} = \lim_{\theta \to \pi/2} \frac{\alpha \cos\theta \sin(\alpha \sin\theta)^2}{\sin\theta} = 0.$$

So we see that $K_1(\alpha) = 0$. Now we calculate K_2.

$$(5.9)\qquad \begin{aligned} K_2(\alpha) &= \int_0^{\pi/2} \sin\theta \sin(\alpha\sin\theta)^2 \, d\theta \\ &= \int_0^{\pi/2} \sin\theta \sin(\alpha^2 - \alpha^2\cos^2\theta) \, d\theta \\ &= \sin\alpha^2 \int_0^{\pi/2} \sin\theta \cos(\alpha^2\cos^2\theta) \, d\theta \\ &\quad - \cos\alpha^2 \int_0^{\pi/2} \sin\theta \sin(\alpha^2\cos^2\theta) \, d\theta \\ &:= K_{21}(\alpha)\sin\alpha^2 - K_{22}(\alpha)\cos\alpha^2. \end{aligned}$$

Putting $t = \cos\theta$, we obtain by (5.4) that as $\alpha \to \infty$,

$$(5.10)\qquad K_{21}(\alpha) = \int_0^1 \cos(\alpha^2 t^2) \, dt = \frac{1}{\alpha} \int_0^{\alpha} \cos x^2 \, dx = \sqrt{\frac{\pi}{2}} \frac{1}{2\alpha}(1 + o(1)).$$

By the same calculation as that to obtain (5.10), we obtain

$$K_{22}(\alpha) = \int_0^1 \sin(\alpha^2 t^2)\,dt = \frac{1}{\alpha}\sqrt{\frac{\pi}{2}}\,S(\alpha) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)). \tag{5.11}$$

By (5.9)–(5.11), we obtain

$$\begin{aligned} K(\alpha) &= \alpha K_2 = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,(1+o(1))(\sin\alpha^2 - \cos\alpha^2) \\ &= \frac{\sqrt{\pi}}{2}\cos\left(\alpha^2 - \frac{3}{4}\pi\right) + o(1). \end{aligned} \tag{5.12}$$

This implies (5.5). □

By Lemma 5.1 and (2.7), we obtain (1.13) in Theorem 1.4. □

We next prove (1.14). We apply (3.1) to the proof. By (2.6), (2.7), (3.3) and Lemma 5.1, we have

$$\begin{aligned} (\sqrt{\lambda})' &= -\frac{1}{\alpha^2}\,(1+o(1))\int_0^1 \frac{g(\alpha) - sg(\alpha s)}{(1-s^2)^{3/2}}\,ds + \frac{2}{\alpha^3}\,(1+o(1))K(\alpha) \\ &:= -\frac{1}{\alpha^2}\,(1+o(1))\,T(\alpha) + O(\alpha^{-3}). \end{aligned} \tag{5.13}$$

Lemma 5.2. *As $\alpha \to \infty$,*

$$T(\alpha) = -\sqrt{\pi}\alpha\sin\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha). \tag{5.14}$$

Proof. By putting $s = \sin\theta$, integration by parts and l'Hôpital's rule, for $\alpha \gg 1$, we obtain (cf. (5.8))

$$\begin{aligned} T(\alpha) &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}\,(\sin\alpha^2 - \sin\theta\sin(\alpha\sin\theta)^2)\,d\theta \\ &= [\tan\theta(\sin\alpha^2 - \sin\theta\sin(\alpha\sin\theta)^2)]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan\theta\big(-\cos\theta\sin(\alpha\sin\theta)^2 \\ &\qquad - 2\alpha^2\sin^2\theta\cos\theta\cos(\alpha\sin\theta)^2\big)\,d\theta \\ &= \int_0^{\pi/2} \sin\theta\sin(\alpha\sin\theta)^2\,d\theta + 2\alpha^2\int_0^{\pi/2} \sin^3\theta\cos(\alpha\sin\theta)^2\,d\theta \\ &:= K_2(\alpha) + 2\alpha^2 T_2(\alpha). \end{aligned} \tag{5.15}$$

Then

$$T_2(\alpha) = \int_0^{\pi/2} \sin^3\theta \cos(\alpha^2 - \alpha^2\cos^2\theta)\,d\theta \tag{5.16}$$
$$= \cos\alpha^2 \int_0^{\pi/2} \sin^3\theta\cos(\alpha^2\cos^2\theta)\,d\theta$$
$$+ \sin\alpha^2 \int_0^{\pi/2} \sin^3\theta\sin(\alpha^2\cos^2\theta)\,d\theta$$
$$:= T_{21}(\alpha)\cos\alpha^2 + T_{22}(\alpha)\sin\alpha^2.$$

By putting $x = \cos\theta$, (5.10), (5.11) and integration by parts, we obtain

$$T_{21}(\alpha) = \int_0^{\pi/2} (1-\cos^2\theta)\sin\theta\cos(\alpha^2\cos^2\theta)\,d\theta \tag{5.17}$$
$$= \int_0^1 (1-x^2)\cos(\alpha^2x^2)\,dx$$
$$= \int_0^1 \cos(\alpha^2x^2)\,dx - \int_0^1 x\cdot(x\cos(\alpha^2x^2))\,dx$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)) - \int_0^1 x\cdot\left(\frac{1}{2\alpha^2}\sin(\alpha^2x^2)\right)'dx$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)) - \left[x\cdot\frac{1}{2\alpha^2}\sin(\alpha^2x^2)\right]_0^1$$
$$+ \frac{1}{2\alpha^2}\int_0^1 \sin(\alpha^2x^2)\,dx$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)) - \frac{1}{2\alpha^2}\sin\alpha^2 + \frac{1}{2\alpha^2}\sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1))$$
$$= \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)).$$

By the same calculation as that above, we also obtain

$$T_{22}(\alpha) = \sqrt{\frac{\pi}{2}}\,\frac{1}{2\alpha}\,(1+o(1)). \tag{5.18}$$

By (5.16)–(5.18), we obtain

$$T_2 = \frac{\sqrt{\pi}}{2\alpha}\sin\left(\alpha^2 + \frac{1}{4}\pi\right) + o(\alpha^{-1}). \tag{5.19}$$

By (5.9), (5.15) and (5.19), we obtain

$$T(\alpha) = \sqrt{\pi}\alpha\sin\left(\alpha^2 + \frac{1}{4}\pi\right) + o(\alpha) = -\sqrt{\pi}\alpha\sin\left(\alpha^2 - \frac{3}{4}\pi\right) + o(\alpha). \qquad \square$$

By (3.1), (5.13) and Lemma 5.2, we obtain (1.14). □

Proof of (1.15). By (1.14) and Taylor expansion, we have

$$L(g_2,\alpha) = \int_\alpha^{2\alpha} \sqrt{1 + \frac{\pi^3}{t^2}\,(1+o(1))\sin^2\left(t^2 - \frac{3\pi}{4}\right)}\,dt \tag{5.20}$$
$$= \int_\alpha^{2\alpha} \left\{1 + \frac{\pi^3}{2t^2}\,(1+o(1))\sin^2\left(t^2 - \frac{3\pi}{4}\right)\right\} dt$$
$$= \alpha + \frac{\pi^3}{4}\,(1+o(1)) \int_\alpha^{2\alpha} \left(\frac{\sin^2 t^2}{t^2} + \frac{\cos^2 t^2}{t^2} + \frac{2\sin t^2 \cos t^2}{t^2}\right) dt.$$

Clearly,

$$\int_\alpha^{2\alpha} \left(\frac{\sin^2 t^2}{t^2} + \frac{\cos^2 t^2}{t^2}\right) dt = \int_\alpha^{2\alpha} \frac{1}{t^2}\,dt = \frac{1}{2\alpha}. \tag{5.21}$$

Furthermore,

$$\int_\alpha^{2\alpha} \frac{2\sin t^2 \cos t^2}{t^2}\,dt = \int_\alpha^{2\alpha} \frac{\sin(2t^2)}{t^2}\,dt = \sqrt{2}\int_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} \frac{\sin x^2}{x^2}\,dx \tag{5.22}$$
$$= \sqrt{2}\left[-\frac{1}{2t^3}\cos t^2\right]_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} - \frac{3\sqrt{2}}{2}\int_{\sqrt{2}\alpha}^{2\sqrt{2}\alpha} \frac{\cos t^2}{t^4}\,dt = O(\alpha^{-3}).$$

By (5.20)–(5.22), we obtain (1.15). □

6. Proof of Theorem 1.5

Let $0 < \alpha \ll 1$ in this section. For $0 < s < 1$, let

$$D_\alpha(s) := \frac{2}{\alpha^2}\,\frac{1}{1-s^2}\int_{\alpha s}^{\alpha} \sin x^2\,dx. \tag{6.1}$$

Then by Taylor expansion, as $\alpha \to 0$,

$$D_\alpha(s) = \frac{2}{\alpha^2}\,\frac{1}{1-s^2}\int_{\alpha s}^{\alpha}\left(x^2 - \frac{1}{6}\,(1+o(1))x^6\right) dx \tag{6.2}$$
$$= \frac{2(1-s^3)}{3(1-s^2)}\,\alpha - \frac{1}{21}\,(1+o(1))\,\frac{1-s^7}{1-s^2}\,\alpha^5.$$

By (2.7), (6.2), Taylor expansion and direct calculation, we have

$$\sqrt{\lambda} = \int_0^1 \frac{1}{\sqrt{1-s^2}}\,\frac{1}{\sqrt{1+D_\alpha(s)}}\,ds \tag{6.3}$$
$$= \int_0^1 \frac{1}{\sqrt{1-s^2}}\left\{1 - \frac{1}{2}\,D_\alpha(s) + \frac{3}{8}\,(1+o(1))D_\alpha(s)^2\right\} ds$$
$$= \frac{\pi}{2} - \frac{1}{3}\,\alpha\int_0^1 \frac{1-s^3}{(1-s^2)^{3/2}}\,ds + \frac{1}{6}\,\alpha^2\int_0^1 \frac{(1-s^3)^2}{(1-s^2)^{5/2}}\,ds + o(\alpha^2).$$

By this, we directly obtain Theorem 1.5. □

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Tetsutaro Shibata
Laboratory of Mathematics
Graduate School of Engineering
Hiroshima University
Higashi-Hiroshima, 739-8527, JAPAN

E-mail address: shibata@amath.hiroshima-u.ac.jp