Topological Methods in Nonlinear Analysis
Volume 50, No. 1, 2017, 217–229
DOI: 10.12775/TMNA.2017.027

GENERALIZED RECURRENCE IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

Boyang Ding — Changming Ding

Abstract. We aim to introduce the generalized recurrence into the theory of impulsive semidynamical systems. Similarly to Auslander's construction in [J. Auslander, *Generalized recurrence in dynamical systems*, Contrib. Differential Equations **3** (1964), 65–74], we present two different characterizations, respectively, by Lyapunov functions and higher prolongations. In fact, we show that if the phase space is a locally compact separable metric space, then the generalized recurrent set is the same as the quasi prolongational recurrent set. Also, we see that many new phenomena appear for the impulse effects in the semidynamical system.

1. Introduction

Since at least the time of Poisson, mathematicians have pondered the notion of recurrence for differential equations. Solutions that exhibit recurrent behavior provide insight into the behavior of general solutions. In the theory of dynamical systems, the different notions of recurrence all express the idea of a point returning to itself, in some sense, for arbitrarily large time. Using continuous real valued functions on the phase space, Auslander [1] introduced the concept of generalized recurrence in dynamical systems. In the literature, the generalized recurrence is also called to be the Auslander recurrence or prolongational recurrence. The generalized recurrent set contains periodic points, recurrent

2010 *Mathematics Subject Classification.* 37B20, 37C10, 54H20.
Key words and phrases. Impulsive semidynamical system; generalized recurrence; Lyapunov function; higher prolongation.

(or Poisson stable) points and non-wandering points. It is known that the generalized recurrence is a very important concept in the theory of stabilities, for example, Nitecki [15] showed the role of generalized recurrence in a completely unstable flow, and Peixoto [16] perturbed a vector field with a non-periodic prolongational recurrent point to get a periodic orbit. Recently, in [9] we have used prolongational recurrence to generalize Birkhoff center and its depth.

An impulsive semidynamical system is a discontinuous semidynamical system, which is a natural generalization of a classical dynamical system. Systems with impulses may present many interesting and unexpected phenomena such as 'beating', 'merging' and 'noncontinuation of solutions'. Since an impulsive system admits abrupt perturbations, its dynamical behavior is much richer than that of the corresponding system. Kaul [11] began to investigate limit sets and the periodicity of impulsive orbits. Later, using a discrete dynamical system associated to the given impulsive semidynamical system, he studied recursive properties in [12]. Ciesielski applied his section theory to obtain the continuity of impulsive time functions and stabilities in [4], [5]. The second author of this paper presented some results on the structure of limit sets in [7], [8]. Now, the theory of impulsive systems is an important and flourishing area of investigation.

The aim of this paper is to introduce the notion of generalized recurrence for impulsive dynamical systems. Since there exist impulse effects in the impulsive systems, analogous results to those established by Auslander in [1] in the impulsive case are not true, our examples also show that many new phenomena occur. In this paper, we will define two different prolongational recurrent sets, and show that if the phase space is a locally compact separable metric space, the generalized recurrent set is the same as the quasi prolongational recurrent set (for definition, see Section 4).

This paper is organized as follows. In Section 2, we recall the definition of an impulsive dynamical system, and fix some notations that will be used in the sequel. In Section 3, following Auslander, we use Lyapunov functions of an impulsive system to define the generalized recurrence. Finally, in Section 4, we introduce the high prolongations and present their fundamental properties, which lead to a variant characterization of generalized recurrence.

2. Impulsive dynamics

Throughout the paper, $X = (X, d)$ denotes a metric space with metric d. For a subset $A \subseteq X$, $\overline{A}$ denotes the closure of A. Let $B(x, r) = \{y \in X : d(x, y) < r\}$ be the open ball with center x and radius $r > 0$. Let $\mathbb{R}$ be the real line, and $\mathbb{R}^+$ be the subset of $\mathbb{R}$ consisting of nonnegative real numbers.

A *semidynamical system* (or *semiflow*) on X is a triple $(X, \pi, \mathbb{R}^+)$, where π is a continuous mapping from $X \times \mathbb{R}^+$ onto X satisfying the following axioms:

(1) $\pi(x,0)=x$ for each $x\in X$,
(2) $\pi(\pi(x,t),s)=\pi(x,t+s)$ for each $x\in X$ and $t,\, s\in\mathbb{R}^+$.

We often denote a semidynamical system $(X,\pi,\mathbb{R}^+)$ by (X,π). Also, for brevity, we write $x\cdot t=\pi(x,t)$, and let $A\cdot T=\{x\cdot t : x\in A, t\in T\}$ for $A\subseteq X$ and $T\subseteq\mathbb{R}^+$. If either A or T is a singleton, i.e. $A=\{x\}$ or $T=\{t\}$, then we simply write $x\cdot T$ and $A\cdot t$ in place of $\{x\}\cdot T$ and $A\cdot\{t\}$, respectively. For any $x\in X$, the function $\pi_x\colon\mathbb{R}^+\to X$ defined by $\pi_x(t)=\pi(x,t)$ is clearly continuous, which is called the *trajectory* of x. The set $x\cdot\mathbb{R}^+$ is said to be the (*positive*) *orbit* of x. In the above definition, replacing $\mathbb{R}^+$ by $\mathbb{R}$, we get the notion of a dynamical system (or flow). For elementary properties of dynamical systems and semidynamical systems, the reader is referred to [2], [3].

Let M be a nonempty closed subset in X and $\Omega=X\setminus M$. Let $I\colon M\to\Omega$ be a continuous function and $I(M)=N$. If $x\in M$, we shall denote $I(x)$ by x^+ and say that x jumps to x^+. Meanwhile, I and M are said to be an *impulsive function* and an *impulsive set*, respectively. For each $x\in\Omega$, by $M^+(x)$ we mean the set $x\cdot\mathbb{R}^+\cap M$. We can define a function $\phi\colon\Omega\to\mathbb{R}^+\cup\{+\infty\}$ by

$$\phi(x)=\begin{cases} s & \text{if } x\cdot s\in M \text{ and } x\cdot t\notin M \text{ for } t\in[0,s),\\ +\infty & \text{if } M^+(x)=\emptyset.\end{cases}$$

In general, $\phi\colon\Omega\to\mathbb{R}^+\cup\{+\infty\}$ is not continuous. Ciesielski [4] has established some easy conditions to guarantee the continuity of ϕ. In this paper, we always assume that ϕ is a continuous function on Ω.

Now, we recall the notion of an impulsive semidynamical system $(\Omega,\pi,\mathbb{R}^+;M,I)$, which is defined by portraying the trajectory of each point in Ω. The impulsive trajectory of $x\in\Omega$ is an Ω-valued function $\widetilde{\pi}_x$ defined on a subset of $\mathbb{R}^+$. If $M^+(x)=\emptyset$, then $\phi(x)=+\infty$, and we set $\widetilde{\pi}_x(t)=x\cdot t$ for all $t\in\mathbb{R}^+$. If $M^+(x)\neq\emptyset$, it is easy to see that there is a positive number t_0 such that $x\cdot t_0=x_1\in M$ and $x\cdot t\notin M$ for $0\le t<t_0$. Thus, we define $\widetilde{\pi}_x$ on $[0,t_0]$ by

$$\widetilde{\pi}_x(t)=\begin{cases} x\cdot t & \text{for } 0\le t<t_0,\\ x_1^+ & \text{for } t=t_0,\end{cases}$$

where $\phi(x)=t_0$ and $x_1^+=I(x_1)\in\Omega$.

Since $t_0<+\infty$, we continue the process by starting with x_1^+. Similarly, if $M^+(x_1^+)=\emptyset$, i.e. $\phi(x_1^+)=+\infty$, we define $\widetilde{\pi}_x(t)=x_1^+\cdot(t-t_0)$ for $t_0<t<+\infty$. Otherwise, let $\phi(x_1^+)=t_1$ and $x_1^+\cdot t_1=x_2\in M$, then we define $\widetilde{\pi}_x(t)$ on $[t_0,t_0+t_1]$ by

$$\widetilde{\pi}_x(t)=\begin{cases} x_1^+\cdot(t-t_0) & \text{for } t_0\le t<t_0+t_1,\\ x_2^+ & \text{for } t=t_0+t_1,\end{cases}$$

where $x_2^+=I(x_2)$.

Thus, continuing inductively, the process above either ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n, or it continues infinitely, if $M^+(x_n^+) \neq \emptyset$ for $n = 1, 2, \ldots$, and $\widetilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} t_i$. We call $\{t_i\}$ the *impulsive intervals* of $\widetilde{\pi}_x$, and call

$$\bigg\{t^n = \sum_{i=0}^{n} t_i \;\bigg|\; n = 0, 1, \ldots\bigg\}$$

the *impulsive times* of $\widetilde{\pi}_x$. Obviously, this gives rise to either a finite or infinite number of jumps at points $\{x_n\}$ for the trajectory $\widetilde{\pi}_x$. Having the trajectory $\widetilde{\pi}_x$ for each point x in Ω, we let $\widetilde{\pi}(x, t) = \widetilde{\pi}_x(t)$ for $x \in \Omega$ and $t \in [0, T(x))$, and obtain a discontinuous system $(\Omega, \pi, \mathbb{R}^+; M, I)$, or $(\Omega, \widetilde{\pi})$, with the following properties:

(i) $\widetilde{\pi}(x, 0) = x$ for $x \in \Omega$,
(ii) $\widetilde{\pi}(\widetilde{\pi}(x, t), s) = \widetilde{\pi}(x, t + s)$ for $x \in \Omega$ and $t, s \in [0, T(x))$, such that $t + s \in [0, T(x))$.

We call $(\Omega, \pi, \mathbb{R}^+; M, I)$, or $(\Omega, \widetilde{\pi})$ with $\widetilde{\pi}$ as defined above, an *impulsive semidynamical system* associated with (X, π). For simplicity of exposition, in the remainder of this paper we denote the trajectory $\widetilde{\pi}(x, t)$ by $x * t$. Thus, (ii) reads $(x * t) * s = x * (t + s)$. The set $x * \mathbb{R}^+$ is said to be the *orbit* of x, and sometimes denoted by $\gamma(x)$. Given $x \in \Omega$, if $M^+(x) = \emptyset$, the trajectory $\widetilde{\pi}_x$ is continuous; otherwise, it has discontinuities at a finite or infinite number of its *impulsive points* $\{x_n^+\}$. At any such point, however, $\widetilde{\pi}_x$ is continuous from the right.

From the point of view of an impulsive semidynamical system, the trajectories that are of interest are those with an infinite number of discontinuities and with $[0, +\infty)$ as the interval of definition. Following Kaul [11], we call them *infinite trajectories*. For an impulsive system, Ciesielski [6] uses the time reparametrization to get an isomorphic system whose impulsive trajectories are global, i.e. the resulting dynamics is defined for all positive times. In this paper, we do not deal with the Zeno orbits, i.e. orbits that involve infinitely many resetting in finite time. Hence, from now on we assume $T(x) = +\infty$ for each $x \in \Omega$.

Now, we recall some basic concepts in the impulsive systems, which will be used in the sequel. A point x in Ω is a *rest point* if $x * t = x$ for every $t \in \mathbb{R}^+$. Clearly, $x \in \Omega$ is a rest point of $(\Omega, \widetilde{\pi})$ if and only if it is a rest point of (X, π). An orbit $\gamma(x)$ is said to be *periodic* of *period* $\tau > 0$ and *order* k if $x * \mathbb{R}^+$ has k components and τ is the least positive number such that $x * \tau = x$. A point x in Ω is a *non-wandering point*, if for every neighborhood U of x and every $T > 0$, $U \cap U * t \neq \emptyset$ for some $t > T$. The set of all non-wandering points is called the *non-wandering set* of $(\Omega, \widetilde{\pi})$. A subset S of Ω is said to be *positively invariant* if for any $x \in S$, $\gamma(x) \subseteq S$. Further, it is said to be *invariant* if it is positively invariant and for any $x \in S$, $t \in \mathbb{R}^+$ there exists a $y \in S$ such that $y * t = x$.

Clearly, a periodic orbit of $(\Omega, \widetilde{\pi})$ is an invariant closed set in Ω, and it is not connected as long as $k \neq 1$.

We end this section with a lemma, which is proved in [8].

Lemma 2.1. *Assume that $\{x_n\}$ is a sequence in Ω, convergent to a point $y \in \Omega$. Then for any $t \geq 0$, there exists a sequence of real positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, such that $x_n * (t + \varepsilon_n) \to y * t$.*

3. Generalized recurrence

In an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, the notion of a Lyapunov function was introduced by Kaul in [12]. A function $V\colon X \to \mathbb{R}$ is said to be a Lyapunov function, if

(a) V is continuous in X,
(b) $V(I(x)) \leq V(x)$ for $x \in M$,
(c) $\dot{V}(x) \leq 0$ for $x \in X$, where $\dot{V}(x) = \lim_{t \to 0^+} (V(x * t) - V(x))/t$.

Clearly, item (c) implies the monotonicity on orbits, i.e. $V(x*t) \leq V(x)$ for every $t \geq 0$ and $x \in \Omega$. In order to introduce the concept of generalized recurrence for $(\Omega, \widetilde{\pi})$, we just use the monotonicity on orbits to define the Lyapunov function as follows.

Definition 3.1. A continuous function $f\colon \Omega \to \mathbb{R}$ is said to be a *Lyapunov function* of $(\Omega, \widetilde{\pi})$, if it is decreasing along each orbit of $(\Omega, \widetilde{\pi})$, i.e. $f(x*t) \leq f(x)$ for all $x \in \Omega$ and $t \in \mathbb{R}^+$.

Let $\mathscr{V}$ denote the class of all Lyapunov functions of $(\Omega, \widetilde{\pi})$, and call it the *Lyapunov function family* of $\widetilde{\pi}$. Observe that if $f \in \mathscr{V}$, then f is constant on $\gamma(x)$ for a non-wandering point x. To show this, let $\tau > 0$. Since x is non-wandering, there exist sequences $p_n \to x$ and $t_n \to +\infty$ such that $p_n * t_n \to x$. By Lemma 2.1, let a sequence $\varepsilon_n \to 0^+$ be such that $p_n * (\tau + \varepsilon_n) \to x * \tau$. For n sufficiently large, we have $t_n > \tau + 1$, and then $f(p_n * t_n) \leq f(p_n * (\tau + \varepsilon_n))$. By the continuity of f, it follows that $f(x) \leq f(x * \tau)$. On the other hand, $f(x * \tau) \leq f(x)$ is always true for $f \in \mathscr{V}$, and therefore $f(x * \tau) = f(x)$.

Definition 3.2. Let $\mathcal{R} = \{x \in \Omega : f(x * t) = f(x)$ for all $f \in \mathscr{V}$ and $t \geq 0\}$. The set $\mathcal{R}$ is called *generalized recurrent set*, and an element of $\mathcal{R}$ is said to be a *generalized recurrent point*.

Lemma 3.3. *In Ω, the generalized recurrent set $\mathcal{R}$ is a closed and positively invariant set.*

Proof. By the continuity of a Lyapunov function, $\mathcal{R}$ is closed. Let $x \in \mathcal{R}$ and $\tau > 0$. If $f \in \mathscr{V}$ and $t > 0$, then $f((x*\tau)*t) = f(x*(\tau+t)) = f(x) = f(x*\tau)$, which implies that $x * \tau \in \mathcal{R}$, and therefore $\mathcal{R}$ is positively invariant. □

Note that $\mathcal{R}$ may not be invariant, as we can see in the following.

Example 3.4. Consider a dynamical system π on $\mathbb{R}^2$ defined by the differential equations: $\dot{x} = 0$, $\dot{y} = -(x^2 + y^2)$. Let $X = \{(x,y) \in \mathbb{R}^2 : x = 0 \text{ or } 1/k,\ k = 1,2,\ldots\} \setminus \{(0,0)\}$, $M = \{(x,0) \in \mathbb{R}^2 : x = 1/k,\ k = 1,2,\ldots\}$ is a closed subset of X. Let $\Omega = X \setminus M$, define the impulsive function $I\colon M \to \Omega$ by $I(1/k,0) = (1/(k+1),1)$ for all $k = 1,2,\ldots$ Then, we get an impulsive dynamical system $(\Omega,\widetilde{\pi})$. Now we define a continuous function from X to $\mathbb{R}$ as follows:

$$v(x,y) = \begin{cases} y - 1 + x & \text{for } 1 \le y, \\ \dfrac{x^2}{x+1}(y-1) + x & \text{for } 0 < y < 1, \\ y + \dfrac{x}{x+1} & \text{for } y \le 0. \end{cases}$$

It is easy to see that in Ω, $v(x,y)$ is a Lyapunov function of $(\Omega,\widetilde{\pi})$. Thus, $\mathcal{R} = \{(0,y) \in \mathbb{R}^2 : 0 < y \le 1\}$ (see Theorem 3.6), it is positively invariant but not invariant.

Lemma 3.5. *Let a and b be real numbers with $a < b$. Let $\mathscr{V}_{a,b} = \{f \in \mathscr{V} : a \le f(x) \le b$ for all $x \in \Omega\}$. Then $x \in \mathcal{R}$ if and only if $f(x * t) = f(x)$ for all $f \in \mathscr{V}_{a,b}$ and all $t \ge 0$.*

Proof. If $x \in \mathcal{R}$, then $f(x*t) = f(x)$ for all $f \in \mathscr{V}$ and $t \ge 0$. Since $\mathscr{V}_{a,b} \subset \mathscr{V}$, it is obviously true that $f(x*t) = f(x)$ for all $f \in \mathscr{V}_{a,b}$ and $t \ge 0$. Next, given an $x \in \Omega$, assume that $f(x*t) = f(x)$ for all $f \in \mathscr{V}_{a,b}$ and $t \ge 0$. Let $f \in \mathscr{V}$ and $t \ge 0$. Then, we have $-\pi/2 \le \arctan f(y) \le \pi/2$ for each $y \in \Omega$. Clearly, there exist real numbers d and $c > 0$ such that $c\arctan f + d \in \mathscr{V}_{a,b}$, which implies that $c\arctan f(x*t) + d = c\arctan f(x) + d$. Therefore, $f(x*t) = f(x)$ is true for $f \in \mathscr{V}$ and $t \ge 0$, it is $x \in \mathcal{R}$. □

Theorem 3.6. *Let X be a locally compact separable metric space. Then, there is an f in $\mathscr{V}$ with the following properties*:

(a) *$x \in \mathcal{R}$ if and only if f is constant on $\gamma(x)$;*
(b) *if $x \notin \mathcal{R}$ and $t > 0$, then $f(x*t) < f(x)$.*

Proof. Since Ω is an open subset of the metric space X, Ω is also locally compact and separable. Let $C(\Omega)$ denote the continuous real valued functions on Ω, provided with the topology of uniform convergence on compact sets (i.e. the compact-open topology). Then, $C(\Omega)$ is a separable metric space (see [17, p. 271, Theorem 7]), and so is $\mathscr{V}_{-1,1}$. Let $\{f_n : n = 1,2,\ldots\}$ be a countable dense set in $\mathscr{V}_{-1,1}$. Clearly, $x \in \mathcal{R}$ if and only if $f_n(x*t) = f_n(x)$ for all $n = 1,2,\ldots$ and all $t \ge 0$. Then, let $f_0 = \sum_{n=1}^{+\infty} 2^{-n} f_n$, since $|f_n(x)| \le 1$, it follows that $f_0 \in \mathscr{V}_{-1,1}$ is continuous.

Now, if $x \in \mathcal{R}$, then $f_0(x * t) = f_0(x)$ for all $t \geq 0$. Conversely, let $f_0(x * t) = f_0(x)$ for all $t \geq 0$, so we have

$$0 = \sum_{n=1}^{\infty} 2^{-n}[f_n(x * t) - f_n(x)].$$

Since $f_n(x*t) - f_n(x) \leq 0$ for each $n = 1, 2, \ldots$, it follows that $f_n(x*t) = f_n(x)$ for each $n = 1, 2, \ldots$ and $t \geq 0$. Thus, we have $x \in \mathcal{R}$. Finally, we consider the case: $x \notin \mathcal{R}$ but f_0 is constant on some segments of $\gamma(x)$. In this case, there is at least a $t_1 > 0$ such that $f_0(x) > f_0(x * t_1)$. Define

$$f(x) = \int_0^{+\infty} e^{-s} f_0(x * s)\, ds.$$

It is easy to verify that $f \in \mathscr{V}_{-1,1}$ and has the required properties. $\square$

4. Higher prolongations

In the impulsive system $(\Omega, \widetilde{\pi})$, let $x \in \Omega$, the omega limit set of x is defined by $\widetilde{\omega}(x) = \{y \in \Omega : x * t_n \to y \text{ for some } t_n \to +\infty\}$. The first prolongational set and first prolongational limit set of x are defined, respectively, by $\widetilde{D}_1(x) = \{y \in \Omega :$ there are two sequences $\{x_n\} \subseteq \Omega$, $\{t_n\} \subseteq \mathbb{R}^+$ such that $x_n \to x$ and $x_n * t_n \to y\}$ and $\widetilde{J}_1(x) = \{y \in \Omega :$ there are two sequences $\{x_n\} \subseteq \Omega$, $\{t_n\} \subseteq \mathbb{R}^+$ such that $x_n \to x$, $t_n \to +\infty$ and $x_n * t_n \to y\}$. Clearly, $\widetilde{\omega}(x) \subseteq \widetilde{J}_1(x)$ for each $x \in \Omega$, and also $\widetilde{D}_1(x) = x * \mathbb{R}^+ \cup \widetilde{J}_1(x)$ holds. Note that $\widetilde{\omega}(x)$, $\widetilde{J}_1(x)$ and $\widetilde{D}_1(x)$ are closed and positively invariant, see [13].

Let $\mathscr{X}$ be the collection of all subsets of Ω, and $\mathcal{M} = \{\Gamma : \Gamma \text{ is a map from } \Omega \text{ to } \mathscr{X}\}$. For $\Gamma \in \mathcal{M}$ and $A \in \mathscr{X}$, we define $\Gamma(A) = \bigcup\{\Gamma(x) : x \in A\}$. If n is a positive integer, the map $\Gamma^n \colon \Omega \to \mathscr{X}$ is defined inductively by $\Gamma^1 = \Gamma$ and $\Gamma^n = \Gamma \circ \Gamma^{n-1}$, i.e., $\Gamma^1(x) = \Gamma(x)$ and $\Gamma^n(x) = \Gamma(\Gamma^{n-1}(x))$ for $x \in \Omega$.

Now, we introduce two operators $\mathscr{D}$ and $\mathscr{S}$ on the collection $\mathcal{M}$. If $\Gamma \in \mathcal{M}$, $\mathscr{D}\Gamma$ and $\mathscr{S}\Gamma$ are defined, respectively, by

$$\mathscr{D}\Gamma(x) = \bigcap_{r>0} \overline{\Gamma(B(x,r))} \quad \text{and} \quad \mathscr{S}\Gamma(x) = \bigcup_{n=1}^{+\infty} \Gamma^n(x) \quad \text{for } x \in \Omega.$$

It is easy to see that $y \in \mathscr{D}\Gamma(x)$ if and only if there are sequences $\{x_n\}$ and $\{y_n\}$ with $y_n \in \Gamma(x_n)$ such that $x_n \to x$ and $y_n \to y$. Also, $y \in \mathscr{S}\Gamma(x)$ if and only if there are points $x = x_0, x_1, \ldots, x_m = y$ with $x_i \in \Gamma(x_{i-1})$ $(i = 1, 2, \ldots, m)$, i.e. $y \in \Gamma^m(x)$. Obviously, $\mathscr{D}$ and $\mathscr{S}$ may be regarded as a 'closure' operator and a 'transitizing' operator on $\mathcal{M}$, respectively. Let $\Gamma_1, \Gamma_2 \in \mathcal{M}$, if $\Gamma_1(x) \subseteq \Gamma_2(x)$ for all $x \in \Omega$, then we write $\Gamma_1 \subseteq \Gamma_2$. Thus, if $\Gamma_1 \subseteq \Gamma_2$, it follows that $\mathscr{D}\mathscr{S}\Gamma_1 \subseteq \mathscr{D}\mathscr{S}\Gamma_2$. Also, for every $\Gamma \in \mathcal{M}$, we have $\Gamma \subseteq \mathscr{D}\mathscr{S}\Gamma$.

In the study of dynamics, the set valued maps $\widetilde{\omega}, \widetilde{D}_1$ and $\widetilde{J}_1$ are the most important ones in $\mathcal{M}$. Starting with $\widetilde{D}_1$, we use the operators $\mathscr{D}$ and $\mathscr{S}$ to define higher prolongational maps $\{\widetilde{D}_\alpha\}$ (α is an ordinal) as follows. By means of transfinite induction, if α is a successor ordinal, then having defined $\widetilde{D}_{\alpha-1}$, we set $\widetilde{D}_\alpha = \mathscr{D}\mathscr{S}\widetilde{D}_{\alpha-1}$; if α is a limit ordinal, then having defined $\widetilde{D}_\beta$ for every $\beta < \alpha$, we set $\widetilde{D}_\alpha = \mathscr{D}\Big(\bigcup\limits_{\beta<\alpha} \mathscr{S}\widetilde{D}_\beta\Big)$. Similarly, we can consider the higher prolongational limit maps, which lead to the concept of prolongational recurrence. Let $\Gamma = \widetilde{J}_1$, define $\widetilde{J}_2 = \mathscr{D}\mathscr{S}\widetilde{J}_1$. Inductively, if α is a successor ordinal, having defined $\widetilde{J}_{\alpha-1}$ we set $\widetilde{J}_\alpha = \mathscr{D}\mathscr{S}\widetilde{J}_{\alpha-1}$; if α is a limit ordinal, having defined $\widetilde{J}_\beta$ for each $\beta < \alpha$ we set $\widetilde{J}_\alpha = \mathscr{D}\Big(\bigcup\limits_{\beta<\alpha} \mathscr{S}\widetilde{J}_\beta\Big)$. Clearly, for each ordinal α and $x \in \Omega$, $\widetilde{D}_\alpha(x)$ and $\widetilde{J}_\alpha(x)$ are closed sets. If $\beta < \alpha$, then $\widetilde{J}_\beta(x) \subseteq \widetilde{J}_\alpha(x)$, i.e. $\widetilde{J}_\beta \subseteq \widetilde{J}_\alpha$, also $\widetilde{D}_\beta \subseteq \widetilde{D}_\alpha$.

Lemma 4.1. *Suppose that Ω is a separable metric space. Then, there exists a countable ordinal η such that $\widetilde{J}_\eta(x) = \widetilde{J}_\alpha(x)$ for all ordinals $\alpha > \eta$. Also, $\widetilde{D}_\varsigma(x) = \widetilde{D}_\alpha(x)$ for all ordinals $\alpha > \varsigma$, where ς is a countable ordinal.*

Proof. For $x \in \Omega$, $\{\widetilde{J}_\alpha(x) : \alpha \geq 1\}$ is a class of nested closed sets, i.e. $\widetilde{J}_1(x) \subseteq \widetilde{J}_2(x) \subseteq \ldots \subseteq \widetilde{J}_n(x) \subseteq \ldots \subseteq \widetilde{J}_\sigma(x) \subseteq \widetilde{J}_{\sigma+1}(x) \subseteq \ldots$, where σ is the first infinite ordinal number. From the definition of $\widetilde{J}_\alpha$, it is easy to see that if $\widetilde{J}_\rho(x) = \widetilde{J}_{\rho+1}(x)$ for some ordinal ρ, then $\widetilde{J}_\rho(x) = \widetilde{J}_\mu(x)$ for all $\mu \geq \rho$. Thus, by the Baire Category Theorem (see [14, p. 312] and [10, p. 249]), there exists a countable ordinal η such that $\widetilde{J}_\eta(x) = \widetilde{J}_\alpha(x)$ for all ordinals $\alpha > \eta$. The same argument works for the case of higher prolongations $\widetilde{D}_\alpha(x)$. $\square$

Clearly, if Ω is a separable metric space, there exists a sufficiently large ordinal ς (e.g. the first uncountable ordinal) such that for all $x \in \Omega$, $\widetilde{J}_\varsigma(x) = \widetilde{J}_\alpha(x)$ and $\widetilde{D}_\varsigma(x) = \widetilde{D}_\alpha(x)$ for all $\alpha > \varsigma$. We write $\widetilde{J}^* = \widetilde{J}_\varsigma$ and $\widetilde{D}^* = \widetilde{D}_\varsigma$. In the following, our discussion is focused on $\widetilde{J}_\alpha$, similar results hold for $\widetilde{D}_\alpha$.

Lemma 4.2. *Let α be an ordinal and $x \in \Omega$. $\widetilde{J}_\alpha(x)$ is a positively invariant set, also is $\widetilde{J}^n_\alpha(x)$, where n is a positive integer.*

Proof. We prove this lemma by transfinite induction. Let $y \in \widetilde{J}_1(x)$, and $x_n * t_n \to y$ for $x_n \to x$ and $t_n \to +\infty$. For any $t \geq 0$, it follows from Lemma 2.1 that there is a sequence of real positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, such that $x_n * (t_n + t + \varepsilon_n) \to y * t$. Hence, $y * t \in \widetilde{J}_1(x)$, and $\widetilde{J}_1(x)$ is positively invariant. Since $\widetilde{J}^n_1(x) = \bigcup\{\widetilde{J}_1(y) : y \in \widetilde{J}^{n-1}_1(x)\}$, which is the union of positively invariant sets, then $\widetilde{J}^n_1(x)$ is also positively invariant. Now let $\alpha > 1$ be an ordinal, and suppose that the lemma is true for all $\beta < \alpha$. Let $y \in \widetilde{J}_\alpha(x)$ and $t \geq 0$. Let $x_n \to x$, $y_n \to y$, where $y_n \in \widetilde{J}^{k_n}_{\beta_n}(x_n)$ ($\beta_n < \alpha$, k_n a positive integer). By the induction hypothesis, $y_n * (t + \varepsilon_n) \in \widetilde{J}^{k_n}_{\beta_n}(x_n)$, where $\varepsilon_n \to 0^+$ such that $y_n * (t + \varepsilon_n) \to y * t$. Thus, $y * t \in \widetilde{J}_\alpha(x)$, and $\widetilde{J}_\alpha(x)$ is positively invariant. So is $\widetilde{J}^n_\alpha(x)$. $\square$

Example 3.4 shows that $\widetilde{J}_\alpha(x)$ may not be an invariant set.

Lemma 4.3. *For any ordinal* $\alpha \geq 1$, $\widetilde{J}_\alpha(x) \subseteq \widetilde{J}_\alpha(x * t)$ *for all* $t \geq 0$.

Proof. We prove this lemma by transfinite induction. Let $y \in \widetilde{J}_1(x)$ and $t \geq 0$. Let $x_n \to x$, $x_n * t_n \to y$ for $t_n \to +\infty$. Then, by Lemma 2.1, there is a sequence of real positive numbers $\{\varepsilon_n\}$, $\varepsilon_n \to 0^+$, such that $x_n*(t+\varepsilon_n) \to x*t$. Since $t_n > t+\varepsilon_n$ for all $n \geq n_0$ for some n_0, we have $(x_n*(t+\varepsilon_n))*(t_n-t-\varepsilon_n) = x_n * t_n \to y$. Note that $t_n - t - \varepsilon_n \to +\infty$, it follows that $y \in \widetilde{J}_1(x * t)$. Now, suppose that the lemma is true for all $\beta < \alpha$. Let $y \in \widetilde{J}_\alpha(x)$, and $x_n \to x$, $y_n \to y$, where $y_n \in \widetilde{J}^{k_n}_{\beta_n}(x_n)$ ($\beta_n < \alpha$, k_n a positive integer). Then, let $\varepsilon_n \to 0^+$ so that $x_n*(t+\varepsilon_n) \to x*t$. By the induction hypothesis, $y_n \in \widetilde{J}^{k_n}_{\beta_n}(x_n*(t+\varepsilon_n))$, it implies $y \in \widetilde{J}_\alpha(x * t)$. □

Note that $\widetilde{J}_\alpha(x * t)$ may not be contained in $\widetilde{J}_\alpha(x)$.

Example 4.4. Consider a dynamical system π on $\mathbb{R}^2$ defined by the differential equations: $\dot{x} = x(x-1)$, $\dot{y} = -y$. Clearly, π has two rest points, a sink $(0,0)$ and a saddle $(1,0)$. Let $M = \{(3,y) : y \in \mathbb{R}\}$, and $\Omega = \mathbb{R}^2 \setminus M$.

Define the impulsive function $I\colon M \to \Omega$ by $I(3,y) = (1,2)$. Thus, we get an impulsive dynamical system $(\Omega, \widetilde{\pi})$.

Let $p = (2,0)$ and $q = (1,1)$, then $q = p * t_0$ for some $t_0 > 0$. It is easy to see that $\widetilde{J}_1(p) = \{(1,0)\}$ and $\widetilde{J}_1(q) = \{(x,0) : 0 \leq x < 3\}$. Hence, $\widetilde{J}_1(p * t_0)$ is not contained in $\widetilde{J}_1(p)$.

Lemma 4.5. *If* $y \in \widetilde{J}_\alpha(x)$ *and* $z \in \widetilde{J}_\beta(y)$, *then* $z \in \widetilde{J}_{\lambda+1}(x)$, *where* $\lambda = \max\{\alpha, \beta\}$.

Proof. For two ordinals ρ and ϱ, if $\rho < \varrho$, then we have $\widetilde{J}_\rho \subseteq \widetilde{J}_\varrho$, i.e. $\widetilde{J}_\rho(x) \subseteq \widetilde{J}_\varrho(x)$ for $x \in \Omega$. It follows that $\widetilde{J}_\rho(A) \subseteq \widetilde{J}_\varrho(A)$ for any subset $A \subseteq \Omega$. Thus,

$$\widetilde{J}_\beta(y) \subseteq \widetilde{J}_\beta(\widetilde{J}_\alpha(x)) \subseteq \widetilde{J}_\lambda(\widetilde{J}_\lambda(x)) \subseteq \mathscr{S}\widetilde{J}_\lambda(x) \subseteq \widetilde{J}_{\lambda+1}(x).$$

So, we have $z \in \widetilde{J}_{\lambda+1}(x)$. □

Observe that $\widetilde{D}_1(x) = \gamma(x) \cup \widetilde{J}_1(x)$, its proof is straightforward. For the case of dynamical systems, Auslander [1] established the formula $D_\alpha(x) = \gamma(x) \cup J_\alpha(x)$ for any ordinal α, however it is not true for the impulsive system $(\Omega, \widetilde{\pi})$.

Example 4.6. Consider a dynamical system π on $\mathbb{R}^2$ defined by the differential equations: $\dot{x} = x$, $\dot{y} = -y$. Clearly, π has a saddle $O = (0,0)$.

Let $X = \mathbb{R}^2 \setminus \{O\}$, and let the line $M = \{(2,y) : y \in \mathbb{R}\}$ be the impulsive set. For $\Omega = X \setminus M$, define the impulsive function $I\colon M \to \Omega$ by $I(2,y) = (0,1)$. Thus, we get an impulsive dynamical system $(\Omega, \widetilde{\pi})$.

Let $p = (1,0)$, we have $\widetilde{J}_1(p) = \emptyset$ and $\widetilde{D}_1(p) = \gamma(p) = [1,2)\times\{0\}\cup\{0\}\times(0,1]$. Further, it is easy to see that $\widetilde{J}^2_1(p) = \emptyset$ and $\widetilde{D}^2_1(p) = (-\infty,0)\times\{0\}\cup(0,2)\times\{0\}\cup\{0\} \times (0,1]$. Hence, $\widetilde{D}^2_1(p) \neq \gamma(x) \cup \widetilde{J}^2_1(p)$, it follows that $\widetilde{D}_2(p) \neq \gamma(x) \cup \widetilde{J}_2(p)$.

However, we have the following new formula.

Lemma 4.7. *Let $\alpha \geq 2$ be an ordinal, then $\widetilde{D}_\alpha(x) = \gamma(x) \cup \widetilde{J}_\alpha(\gamma(x))$ for $x \in \Omega$.*

Proof. First, we prove $\widetilde{D}_1^{n+1}(x) = \gamma(x) \cup \widetilde{J}_1^n(\gamma(x)) \cup \widetilde{J}_1^{n+1}(x)$ for $n \geq 1$ by induction on n. Since $\widetilde{J}_1(x)$ is positively invariant, $\gamma(\widetilde{J}_1(x)) = \widetilde{J}_1(x)$. Also, it is clear that $\widetilde{J}_1(x) \subseteq \widetilde{J}_1(\gamma(x))$ and $\widetilde{J}_1(x) \subseteq \widetilde{J}_1^2(x)$. Hence, we have

$$\begin{aligned}\widetilde{D}_1^2(x) &= \widetilde{D}_1(\gamma(x) \cup \widetilde{J}_1(x)) = \gamma(\gamma(x) \cup \widetilde{J}_1(x)) \cup \widetilde{J}_1(\gamma(x) \cup \widetilde{J}_1(x)) \\ &= \gamma(x) \cup \gamma(\widetilde{J}_1(x)) \cup \widetilde{J}_1(\gamma(x)) \cup \widetilde{J}_1(\widetilde{J}_1(x)) = \gamma(x) \cup \widetilde{J}_1(\gamma(x)) \cup \widetilde{J}_1^2(x).\end{aligned}$$

Suppose that $\widetilde{D}_1^n(x) = \gamma(x) \cup \widetilde{J}_1^{n-1}(\gamma(x)) \cup \widetilde{J}_1^n(x)$ for $n > 2$. Then, by the induction assumption, we have

$$\begin{aligned}\widetilde{D}_1^{n+1}(x) &= \widetilde{D}_1(\gamma(x) \cup \widetilde{J}_1^{n-1}(\gamma(x)) \cup \widetilde{J}_1^n(x)) \\ &= \gamma(x) \cup \widetilde{J}_1^{n-1}(\gamma(x)) \cup \widetilde{J}_1^n(x) \cup \widetilde{J}_1(\gamma(x)) \cup \widetilde{J}_1^n(\gamma(x)) \cup \widetilde{J}_1(\widetilde{J}_1^n(x)) \\ &= \gamma(x) \cup \widetilde{J}_1^n(\gamma(x)) \cup \widetilde{J}_1^{n+1}(x).\end{aligned}$$

So, the above formula is true.

Next, we prove this lemma by transfinite induction. Note that $\widetilde{J}_1^n(x) \subseteq \widetilde{J}_1^n(\gamma(x))$ for $n \geq 1$, it is clear to see that $\mathscr{S}\widetilde{D}_1(x) = \gamma(x) \cup \mathscr{S}\widetilde{J}_1(\gamma(x))$. Also, from $\overline{\gamma(x)} \subseteq \gamma(x) \cup \widetilde{J}_1(\gamma(x)) \subseteq \gamma(x) \cup \widetilde{J}_1^n(\gamma(x))$, it follows that $\widetilde{D}_2(x) = \gamma(x) \cup \widetilde{J}_2(\gamma(x))$. By a similar argument as above, we have $\widetilde{D}_2^k(x) = \gamma(x) \cup \widetilde{J}_2^k(\gamma(x))$ for $k \geq 1$.

Now, using the transfinite induction, we suppose that the equality $\widetilde{D}_\beta(x) = \gamma(x) \cup \widetilde{J}_\beta(\gamma(x))$ is true for all $\beta < \alpha$. Then,

$$\begin{aligned}\widetilde{D}_\beta^2(x) &= \widetilde{D}_\beta(\gamma(x) \cup \widetilde{J}_\beta(\gamma(x))) \\ &= \gamma(x) \cup \gamma(\widetilde{J}_\beta(\gamma(x))) \cup \widetilde{J}_\beta(\gamma(x)) \cup \widetilde{J}_\beta^2(\gamma(x)) = \gamma(x) \cup \widetilde{J}_\beta^2(\gamma(x)).\end{aligned}$$

Similarly, we have $\widetilde{D}_\beta^k(x) = \gamma(x) \cup \widetilde{J}_\beta^k(\gamma(x))$ for $k \geq 2$. By the definition, it is easy to see that $\mathscr{S}\widetilde{D}_\beta(x) = \gamma(x) \cup \mathscr{S}\widetilde{J}_\beta(\gamma(x))$. Thus, from $\overline{\gamma(x)} \subseteq \gamma(x) \cup \widetilde{J}_\alpha(\gamma(x))$, it follows that $\widetilde{D}_\alpha(x) = \gamma(x) \cup \widetilde{J}_\alpha(\gamma(x))$. □

It is easy to see that x is a non-wandering point if and only if $x \in \widetilde{J}_1(x)$. Similarly, $x \in \widetilde{J}_\alpha(x)$ should imply some recurrent property. Hence, following Auslander, we introduce the prolongational recurrence in the impulsive system $(\Omega, \widetilde{\pi})$.

Definition 4.8. For each ordinal α, let $\mathcal{R}_\alpha$ be the set $\{x \in \Omega : x \in \widetilde{J}_\alpha(x)\}$, which is called the *$\alpha$-order prolongational recurrent set*. Then, define $\mathcal{R}^p = \bigcup \mathcal{R}_\alpha$, and $\mathcal{R}^p$ is said to be the *prolongational recurrent set*. An element of $\mathcal{R}^p$ is called a *prolongational recurrent point*.

A point $x \in \Omega$ is *recurrent* or *Poisson stable* if $x \in \widetilde{\omega}(x)$. It is not difficult to see that a recurrent point is non-wandering, in turn a non-wandering point is prolongational recurrent. However, easy examples show the converse may not be true.

Definition 4.9. For each ordinal α, let $\mathcal{R}^*_\alpha$ be the set $\{x \in \Omega : x \in \widetilde{J}_\alpha(\gamma(x))\}$, which is called the *quasi α-order prolongational recurrent set*. Then, define $\mathcal{R}^* = \bigcup \mathcal{R}^*_\alpha$, and $\mathcal{R}^*$ is said to be the *quasi prolongational recurrent set*. An element of $\mathcal{R}^*$ is called a *quasi prolongational recurrent point*.

If x is a quasi prolongational recurrent point, it means that x goes along the orbit and arrives at a point $x * t_0$, then goes back to itself in the sense of prolongations. Clearly, an (α-order) prolongational recurrent point is a quasi (α-order) prolongational recurrent point, but the converse is not true. In fact, the point $p = (1, 0)$ in Example 4.6 is a quasi prolongational recurrent point, but not a prolongational recurrent point. It is easy to see that all rest points, periodic points and non-wandering points are contained in $\mathcal{R}^p$ and $\mathcal{R}^*$.

Clearly, $\widetilde{D}_1(x) = \widetilde{J}_1(x)$ if and only if $x \in \widetilde{J}_1(x)$, i.e. x is non-wandering. We assert that for $\alpha \geq 2$, $\widetilde{D}_\alpha(x) = \widetilde{J}_\alpha(\gamma(x))$ if and only if $x \in \mathcal{R}^*_\alpha$. Actually, since $\widetilde{J}_\alpha(\gamma(x))$ is positively invariant, $x \in \mathcal{R}^*_\alpha$ if and only if $\gamma(x) \subseteq \widetilde{J}_\alpha(\gamma(x))$, by Lemma 4.7 which is true if and only if $\widetilde{D}_\alpha(x) = \widetilde{J}_\alpha(\gamma(x))$.

Lemma 4.10. *If $f \in \mathscr{V}$ and $y \in \widetilde{D}_\alpha(x)$, then $f(y) \leq f(x)$.*

Proof. Let $f \in \mathscr{V}$ and $y \in \widetilde{D}_1(x) = \gamma(x) \cup \widetilde{J}_1(x)$. We only need consider the case where $y \in \widetilde{J}_1(x)$. Let $x_n \to x$, $x_n * t_n \to y$ for $t_n \to +\infty$. Since $f(x_n * t_n) \leq f(x_n)$ for $t_n > 0$, by the continuity of f we have $f(y) \leq f(x)$.

Next, if $y \in \widetilde{D}_2(x) = \gamma(x) \cup \widetilde{J}_2(\gamma(x))$, we also need to consider the case where $y \in \widetilde{J}_2(\gamma(x))$. Let $y \in \widetilde{J}_2(x * t)$ for some $t \geq 0$. Then, there exist sequences $x_n \to x * t$ and $y_n \to y$, where $y_n \in \widetilde{J}_1^{k_n}(x_n)$ (k_n is a positive integer). Let $y_n \in \widetilde{J}_1(z_1)$, $z_1 \in \widetilde{J}_1(z_2)$, $\ldots$, $z_{i-1} \in \widetilde{J}_1(z_i)$, $\ldots$, $z_{k_n-1} \in \widetilde{J}_1(x_n)$. Clearly, it follows that $f(y_n) \leq f(z_1) \leq \ldots \leq f(z_{k_n-1}) \leq f(x_n)$. Hence, we obtain $f(y) \leq f(x * t)$ from the continuity of f. Since $f(x * t) \leq f(x)$ for all $t \geq 0$, we have $f(y) \leq f(x)$.

Now, suppose that the lemma is true for all $\beta < \alpha$, equivalently, if $y \in \widetilde{J}_\beta(\gamma(x))$ then $f(y) \leq f(x)$. Let $y \in \widetilde{D}_\alpha(x) = \gamma(x) \cup \widetilde{J}_\alpha(\gamma(x))$, we just consider the case $y \in \widetilde{J}_\alpha(\gamma(x))$, or $y \in \widetilde{J}_\alpha(x * t)$ for some $t \geq 0$. Let $x_n \to x * t$, $y_n \to y$, where $y_n \in \widetilde{J}^{k_n}_{\beta_n}(x_n)$ ($\beta_n < \alpha$, k_n a positive integer). Then, using a similar argument as above and the induction hypothesis, we have $f(y_n) \leq f(x_n)$. So, it follows that $f(y) \leq f(x * t) \leq f(x)$. $\square$

In the following, let X be a locally compact separable metric space and so is Ω. By a quasi order on Ω we mean a reflexive, transitive, but not necessarily anti-symmetric relation. We define the relation $\preccurlyeq$ on Ω by $y \preccurlyeq x$ if and only if $y \in \widetilde{D}^*(x)$. Clearly, $x \in \widetilde{D}^*(x)$, so $x \preccurlyeq x$. Next, if $y \preccurlyeq x$ and $z \preccurlyeq y$, then $y \in \widetilde{D}^*(x)$ and $z \in \widetilde{D}^*(y)$, it follows that $z \in \widetilde{D}^*(y) \subset \widetilde{D}^*(\widetilde{D}^*(x)) = \widetilde{D}^*(x)$, or $z \in \widetilde{D}^*(x)$. Thus, $\preccurlyeq$ is a closed quasi order on Ω. If $x \preccurlyeq y$ but not $y \preccurlyeq x$, we write $x \prec y$. To show the relationship between $\mathcal{R}$ and $\mathcal{R}^*$, we recall a topological theorem established by Auslander in [1].

Theorem 4.11 (Auslander). *Let Ω be a locally compact separable metric space and let $\preccurlyeq$ be a closed quasi-order on Ω. Let x and y in Ω be such that $x \preccurlyeq y$ does not hold. Then there is an $f\colon \Omega \to \mathbb{R}$ such that*

(a) *f is continuous,*
(b) *if $z \preccurlyeq z'$, then $f(z) \leq f(z')$,*
(c) *$f(y) < f(x)$.*

Theorem 4.12. *If X is a locally compact separable metric space, then $\mathcal{R} = \mathcal{R}^*$. That is $x \in \mathcal{R}$ if and only if $x \in \mathcal{R}^*_\alpha(x)$ for some ordinal α.*

Proof. Let $x \in \mathcal{R}^*_\alpha$, and let $f \in \mathscr{V}$. We have $x \in \widetilde{J}_\alpha(x * \tau)$ for some $\tau \geq 0$. Since $\widetilde{J}_\alpha(x * t) \subseteq \widetilde{J}_\alpha(\gamma(x * t)) \subseteq \widetilde{D}_\alpha(x * t)$ for all $t \geq 0$, we have $x \in \widetilde{D}_\alpha(x * t)$ for $t \geq \tau$. Thus, Lemma 4.10 implies $f(x) \leq f(x * t)$ for $t \geq \tau$.

Note that $f(x * t) \leq f(x)$ is always true for all $t \geq 0$, so $f(x) = f(x * t)$ for $t \geq \tau$. If $\theta \in [0, \tau]$, then $f(x * \tau) \leq f(x * \theta) \leq f(x)$. Thus, we obtain $f(x) = f(x * t)$ for $t \geq 0$, i.e. $x \in \mathcal{R}$. We have shown that $\mathcal{R}^* \subseteq \mathcal{R}$. Conversely, observe that $x * t \preccurlyeq x$ holds whenever $x \in \Omega$ and $t > 0$.

Now, we assert that if $x \notin \mathcal{R}^*$ and $t > 0$ then $x * t \prec x$, i.e. if $x \notin \mathcal{R}^*$ then $x \preccurlyeq x * t$ does not hold. Otherwise, $x \preccurlyeq x * t$, that is $x \in \widetilde{D}^*(x * t) = \gamma(x * t) \cup \widetilde{J}^*(\gamma(x * t))$.

Since $x \notin \mathcal{R}^*$, it follows that $x \notin \gamma(x*t)$. If $x \in \widetilde{J}^*(\gamma(x*t))$, then $x \in \widetilde{J}^*(x*t_0)$ for some $t_0 \geq 0$, it also means $x \in \mathcal{R}^*$, which is a contradiction. Thus, by Theorem 4.11, if $x \notin \mathcal{R}^*$, then there exists an $f \in \mathscr{V}$ such that $f(x * t) < f(x)$ for $t > 0$. This implies that $x \notin \mathcal{R}$, so we have $\mathcal{R} \subseteq \mathcal{R}^*$. □

Acknowledgements. The authors sincerely thank the referees for many valuable suggestions.

References

[1] J. Auslander, *Generalized recurrence in dynamical systems*, Contrib. Differential Equations **3** (1964), 65–74.

[2] N.P. Bhatia and O. Hajek, *Local Semi-Dynamical Systems*, Lecture Notes in Mathematics, vol. 90, Springer, Berlin, 1970.

[3] N.P. Bhatia and G.P. Szegő, *Stability Theory of Dynamical Systems*, Springer, Berlin, 2002.

[4] K. Ciesielski, *On semicontinuity in impulsive systems*, Bull. Polish Acad. Sci. Math. **52** (2004), 71–80.

[5] K. Ciesielski, *On stability in impulsive dynamical systems*, Bull. Polish Acad. Sci. Math. **52** (2004), 81–91.

[6] K. Ciesielski, *On time reparametrizations and isomorphisms of impulsive dynamical systems*, Ann. Polon. Math. **84** (2004), 1–25.

[7] C. Ding, *Lyapunov quasi-stable trajectories*, Fund. Math. **220** (2013), 139–154.

[8] C. Ding, *Limit sets in impulsive semidynamical systems*, Topol. Methods Nonlinear Anal. **43** (2014), 97–115.

[9] B. Ding and C. Ding, *Prolongational centers and their depths*, Fund. Math. **234** (2016), 287–296.

[10] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[11] S.K. Kaul, *On impulsive semidynamical systems*, J. Math. Anal. Appl. **150** (1990), 120–128.

[12] S.K. Kaul, *On impulsive semidynamical systems* II. *Recursive properties*, Nonlinear Anal. **16** (1991), 635–645.

[13] S.K. Kaul, *Stability and asymptotic stability in impulsive semidynamical systems*, J. Appl. Math. Stochastic Anal. **7** (1991), 509–523.

[14] V.V. Nemytskiĭ and V.V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, 1966.

[15] Z. Nitecki, *Recurrent structure of completely unstable flows on surfaces of finite Euler characteristic*, Amer. J. Math. **103** (1981), 143–180.

[16] M.L.A. Peixoto, *Characterizing Ω-stability for flows in the plane*, Proc. Amer. Math. Soc. **104** (1988), 981–984.

[17] S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. **25** (1958), 265–282.

Manuscript received August 29, 2016

accepted December 27, 2016

Boyang Ding
Department of Economics
University of Copenhagen
Øster Farimagsgade 5, building 26
DK-1353 Copenhagen, DENMARK

E-mail address: byding@foxmail.com

Changming Ding (corresponding author)
School of Mathematical Science
Xiamen University
Xiamen, Fujian 361005, P.R. CHINA

E-mail address: cding@xmu.edu.cn