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SINGULARLY PERTURBED N-LAPLACIAN PROBLEMS WITH A NONLINEARITY IN THE CRITICAL GROWTH RANGE

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Abstract. We consider the following singularly perturbed problem:

$$-\varepsilon^N \Delta_N u + V(x)|u|^{N-2}u = f(u), \quad u(x) > 0 \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$ and $\Delta_N u$ is the N-Laplacian operator. In this paper, we construct a solution u_ε which concentrates around any given isolated positive local minimum component of V, as $\varepsilon \to 0$, in the Trudinger–Moser type of subcritical or critical case. In the subcritical case, we only impose on f the Berestycki and Lions conditions. In the critical case, a global condition on the nonlinearity f is imposed. However, any *monotonicity* of $f(t)/t^{N-1}$ or *Ambrosetti–Rabinowitz* type conditions are not required.

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1. Introduction

In this paper, we are concerned with the concentration phenomena of positive solutions to the following singularly perturbed elliptic problem:

$$-\varepsilon^N \Delta_N v + V(x)|v|^{N-2}v = f(v), \quad v > 0, \quad v \in W^{1,N}(\mathbb{R}^N), \tag{1.1}$$

where $\Delta_N v = \operatorname{div}(|\nabla v|^{N-2}\nabla v)$ and $N \geq 2$. For $\varepsilon > 0$ sufficiently small, these solutions are referred to as "semi-classical states". In the sequel, we assume that the potential V satisfies the following conditions:

- (V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x)$;
- (V2) there is a bounded domain O such that

$$m \equiv \inf_{x\in O} V(x) < \min_{x\in\partial O} V(x).$$

In 2008, J. Byeon, L. Jeanjean and K. Tanaka in [11] considered the above problem (1.1) in the cases: $N = 2$ and subcritical growth. Precisely, in addition to the hypotheses on V, (V1) and (V2), they assumed that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

- (F1) $\lim_{t\to 0} f(t)/t^{N-1} = 0$;
- (F2) for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \leq C_\alpha \exp(\alpha t^{N/(N-1)})$ for $t \geq 0$;
- (F3) there exists $T > 0$ such that $T^N m < NF(T)$, where $F(s) := \int_0^s f(t)\,dt$.

They proved that problem (1.1), with $N = 2$, possesses a positive solution which concentrates around a local minimum of the V. These hypotheses, (F1)–(F3), are called Berestycki–Lions conditions, which were firstly proposed in the classical paper [6] to guarantee the existence of ground states to problem (1.1) with $N = 2$. Moreover, (F1)–(F3) are almost optimal (see [11]).

To state our results, we start with Lemma 1.1 due to J.M. do Ó [21] (see also [15] for $N = 2$) and Lemma 1.2 due to S. Adachi and K. Tanaka [2].

LEMMA 1.1. *If $N \geq 2$, $\alpha > 0$ and $u \in W^{1,N}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \big(\exp\big(\alpha|u|^{N/(N-1)}\big) - S_{N-2}(\alpha, u)\big)\,dx < \infty,$$

where

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!}\,|u|^{kN/(N-1)}.$$

Moreover, if $\alpha < \alpha_N$, then for any positive constant M, there exists $C = C(\alpha, N, M)$ such that

$$\int_{\mathbb{R}^N} \big(\exp\big(\alpha|u|^{N/(N-1)}\big) - S_{N-2}(\alpha, u)\big)\,dx \leq C,$$

for any $u \in W^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_N \leq 1$ and $\|u\|_N \leq M$.

LEMMA 1.2. *If $N \geq 2$ and $\alpha \in (0, \alpha_N)$, there exists $C_\alpha > 0$ such that*

$$\|\nabla u\|_N^N \int_{\mathbb{R}^N} \Psi_N\left(\frac{u}{\|\nabla u\|_N}\right) dx \leq C_\alpha \|u\|_N^N,$$

for any $u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}$, where $\Psi_N(t) = \exp(\alpha|t|^{N/(N-1)}) - S_{N-2}(\alpha, t)$.

Motivated by the above papers, the purpose of this work is to extend results obtained in [11] to higher dimension $N \geq 3$ and a nonlinearity involving critical growth. First, we start establishing the subcritical case, because in the proof of the critical case we will use some arguments made for the subcritical case.

THEOREM 1.3. *Suppose that* (V1)–(V2) *and* (F1)–(F3) *hold. Then, for sufficiently small $\varepsilon > 0$,* (1.1) *admits a positive solution v_ε, which satisfies*

(a) *there exists a maximum point x_ε of v_ε such that $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$ and for any such x_ε, $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon)$ converges (up to a subsequence) uniformly to a least energy solution of*

$$-\Delta_N u + m u^{N-1} = f(u), \quad u > 0, \quad u \in W^{1,N}(\mathbb{R}^N); \tag{1.2}$$

(b) *$v_\varepsilon(x) \leq C \exp(-c|x - x_\varepsilon|/\varepsilon)$ for some $c, C > 0$.*

Naturally, since we are interested in the critical growth case, we need to assume some additional hypotheses on $f \in C(\mathbb{R}^+, \mathbb{R})$, namely

(F4) $\displaystyle\lim_{s \to +\infty} f(s) \exp(-\alpha s^{N/(N-1)}) = \begin{cases} 0 & \text{for all } \alpha > \alpha_N, \\ +\infty & \text{for all } \alpha < \alpha_N, \end{cases}$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$,

(F5) $\displaystyle\lim_{|t| \to +\infty} t f(t) \exp(-\alpha_N t^{N/(N-1)}) \geq \beta_0$.

The main result of this paper reads as

THEOREM 1.4. *Suppose that* (V1)–(V2), (F1) *and* (F4)–(F5) *hold with*

$$\beta_0 > \frac{e}{w_{N-1}} \cdot \frac{(N-2)!}{N^{N-2}}\, m.$$

Then, for $\varepsilon > 0$ sufficiently small, (1.1) *admits a positive solution v_ε, which satisfies*

(a) *there exists a maximum point x_ε of v_ε such that $\lim_{\varepsilon \to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$ and for any such x_ε, $w_\varepsilon(x) \equiv v_\varepsilon(\varepsilon x + x_\varepsilon)$ converges (up to a subsequence) uniformly to a least energy solution of*

$$-\Delta_N u + m u^{N-1} = f(u), \quad u > 0,\ u \in W^{1,N}(\mathbb{R}^N); \tag{1.3}$$

(b) *$v_\varepsilon(x) \leq C \exp(-c|x - x_\varepsilon|/\varepsilon)$ for some $c, C > 0$.*

Remark 1.5. Without loss of generality, in the present paper we can assume that $V_0 = 1$. The assumptions (F1)–(F3) are called the Berestycki–Lions type conditions, which we believe to be almost optimal.

This paper is also motivated by some works addressing the so-called singularly perturbed problem, when $t = 0$, of the type

$$-\varepsilon^p \Delta_p v + V(x)|v|^{p-2}v = f(v), \quad v > 0,\ v \in W^{1,p}(\mathbb{R}^N), \tag{1.4}$$

where $N \geq 3$, $1 < p \leq N$, and Δ_p is the well-known p-Laplacian operator. When $p = 2$, many authors have studied problem (1.4) investigating, not only the existence question, but also the behavior of some families of solutions, e.g. solutions which develop a spike shape around some point in $\mathbb{R}^N$ as $\varepsilon \to 0$. In the pioneering work [26] (see also [31] for higher dimensions), by a reduction method, A. Floer and A. Weinstein studied the single peak solutions around any given non-degenerate critical point of V for $N = 1$ and $f(s) = s^3$. In [33], Paul Rabinowitz used the variational approach to consider the existence of positive solutions to (1.4) without the uniqueness and non-degeneracy condition, but imposing a global condition on V and by considering a subcritical growth condition on the nonlinearity. Indeed this family of the solutions has a concentration phenomenon, which was proved in [38]. Still in the subcritical case, in [17], using a penalization approach, M. del Pino and P. Felmer obtained a single-peak solution around some minimal point of V, assuming only a local condition on the potential. See also related papers [18]–[20] and [3] for $1 < p < N$. In the works above, more restrictions on f are imposed, such as the monotonicity:

(H) $f(t)/t^{N-1}$ is nondecreasing in $(0, \infty)$,

and the Ambrosetti–Rabinowitz condition:

(AR) there exists $\mu > N$ such that $0 \leq \mu F(x,u) \leq uf(x,u)$, for all $u > 0$, $x \in \mathbb{R}^N$.

Recently, some efforts have been made to weaken or eliminate assumptions (AR) or (H). In this direction, J. Byeon and L. Jeanjean [9] developed a new variational approach and established the concentration phenomenon around any isolated component of the local minimal points of V. For the related results, when $p = 2$, we also refer to [12], [13], [8], [16] and [28] for $1 < p < N$.

With the penalized argument, J.M. do Ó [22] considered the concentration phenomenon of (1.4) with $1 < p < N$ in the critical case and constructed a single peak solution around the local minimal point of V. Here, we also would like to mention [25]. For $p = N$, C. Alves and G. Figueiredo [4], under conditions (H) and (AR), considered the N-Laplacian problem (1.4) with a Trudinger–Moser type critical growth and proved the existence and concentration of solutions. In [41], by a truncation argument, J. Zhang and J. do Ó considered the semiclassical states of (1.4) for $N = 2$ and extended the result in [11] to the

Trudinger–Moser critical case without restrictions of the type (AR) and monotonicity condition on f. To conclude this section, we would like to point out some additional difficulties of the case $N > 2$ in contrast with the case $N = 2$. First, in the present paper, Proposition 2.7 (see Section 2) plays a crucial role. When $N > 2$, the underlying space $W^{1,N}(\mathbb{R}^N)$ is not a Hilbert space any more, which causes that it is more complicated to prove the following splitting property in Proposition 2.7:

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1) \quad \text{as } \varepsilon \to 0.$$

Second, to get the concentration, we need refined $C^{1,\alpha}$-estimates for N-Laplace equations instead of $W^{2,p}$-estimates for Laplace equations. Therefore, the methods in [11], [41] cannot be used directly and some more tricks are given.

The paper is organized as follows. Section 2 is dedicated to the proof of Theorem 1.3. In Section 3, we use a truncation approach to prove Theorem 1.4.

Notations.

- $\|u\|_s := \left(\int_{\mathbb{R}^N} |u|^s\,dx\right)^{1/s}$ for $s \in [N, \infty)$.
- $\|u\|_{L^s(B)} := \left(\int_B |u|^s\,dx\right)^{1/s}$ for $s \in [N, \infty)$, $B \subset \mathbb{R}^N$.
- $\|u\| := \left(\|u\|_N^N + \|\nabla u\|_N^N\right)^{1/N}$ for $u \in W^{1,N}(\mathbb{R}^N)$.
- $W_r^{1,N}(\mathbb{R}^N)$ stands for the subspace of $W^{1,N}(\mathbb{R}^N)$ formed by the radially symmetric functions.
- C, c denote positive constants, which may change from line to line.

2. Proof of Theorem 1.3

In this section, to prove our result, we will use the framework made in [11], when $p = 2$, combined with some arguments made in [28], for polynomial subcritical situation. Since we are concerned with positive solutions to (1.1), from now on, we can assume that $f(t) = 0$ for $t < 0$. By denoting $u(x) = v(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$, (1.1) is equivalent to

$$-\Delta_N u + V_\varepsilon(x)|u|^{N-2}u = f(u), \quad u > 0 \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

To study (1.1), it suffices to study (2.1). Let W_ε be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon |u|^N)\,dx\right)^{1/N}.$$

For any set $B \subset \mathbb{R}^N$ and $\varepsilon > 0$, we define $B_\varepsilon \equiv \{x \in \mathbb{R}^N : \varepsilon x \in B\}$. Now, we modify the nonlinearity f as in [17], [28]. By (F1) there exists $a > 0$ such that

$f(t) \le t^{N-1}/2$ for $t \in (0,a)$. For $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, let

$$g(x,t) = \chi_O(x) f(t) + (1-\chi_O(x))\widetilde{f}(t),$$

where $\chi_O(x) = 1$ if $x \in O$, $\chi_O(x) = 0$ if $x \notin O$ and

$$\widetilde{f}(t) = \begin{cases} f(t) & \text{if } t \le a, \\ \min\left\{ f(t), \dfrac{1}{2}\, t^{N-1} \right\} & \text{if } t > a. \end{cases}$$

It is easy to check that $g(x,t) = f(t)$ for $x \in \mathbb{R}^N$, $t \in [0,a]$ and $g(x,t) \le f(t)$ for any $x \in \mathbb{R}^N$, $t \ge 0$. Now, we consider the modified problem

$$-\Delta_N u + V_\varepsilon(x)|u|^{N-2}u = g(\varepsilon x, u), \quad u > 0, \quad u \in W_\varepsilon, \tag{2.2}$$

where $g(\varepsilon x, t) = \chi_{O_\varepsilon}(x) f(t) + (1-\chi_{O_\varepsilon}(x))\widetilde{f}(t)$. Obviously, if u_ε is a solution to (2.2) satisfying $u_\varepsilon(x) \le a$ for $x \in \mathbb{R}^N \setminus O_\varepsilon$, then u_ε is a solution to the original problem (2.1).

For $u \in W_\varepsilon$, let

$$P_\varepsilon(u) = \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon |u|^N)\, dx - \int_{\mathbb{R}^N} G(\varepsilon x, u)\, dx,$$

where $G(x,t) = \int_0^t g(x,s)\, ds$. Fixing an arbitrary $\mu > 0$, we define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in O_\varepsilon, \\ \varepsilon^{-\mu} & \text{if } x \in \mathbb{R}^N \setminus O_\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon |u|^N\, dx - 1 \right)_+^2.$$

This type of penalization was firstly introduced in [14] (see also [9]), which will act as a penalization to force the concentration phenomena to occur inside O. Finally, let $\Gamma_\varepsilon \colon H_\varepsilon \to \mathbb{R}$ be given by

$$\Gamma_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u).$$

Obviously, $\Gamma_\varepsilon \in C^1(H_\varepsilon)$. In the following, to find solutions to (2.2) which concentrate around O as $\varepsilon \to 0$, we shall search critical points of Γ_ε such that Q_ε is zero.

First, we study the properties of ground state solutions to the limit problem (1.2). We define an energy functional for the limiting problem (1.2) by

$$L_m(u) = \frac{1}{N}\int_{\mathbb{R}^N} \big(|\nabla u|^N + m\,|u|^N\big)\, dx - \int_{\mathbb{R}^N} F(u)\, dx, \quad u \in W^{1,N}(\mathbb{R}^N).$$

By combining some arguments made in [23] with those used in [27], we can prove that, with the same assumptions on f as in Theorem 1.3, there exists a positive radially symmetric ground state solution U to (1.2). Moreover, the least energy E_m gives a mountain pass level. Let S_m be the set of positive ground state solutions U to (1.2) satisfying $U(0) = \max_{x\in\mathbb{R}^N} U(x)$. Then $S_m \neq \phi$ and we have the following result.

PROPOSITION 2.1. *Under the same assumptions as in Theorem* 1.3*, we have*

(a) *for any* $U \in S_m$, $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ $(\alpha \in (0,1))$ *is radially symmetric and* $\partial U/\partial r \leq 0$, $r = |x|$;
(b) S_m *is compact in* $W^{1,N}(\mathbb{R}^N)$;
(c) $0 < \inf\{\|U\|_\infty : U \in S_m\} \leq \sup\{\|U\|_\infty : U \in S_m\} < \infty$;
(d) *there exist* $C, c > 0$, *independent of* $U \in S_m$, *such that* $|D^\alpha U(x)| \leq C\exp(-c|x|)$, $x \in \mathbb{R}^N$ *for* $|\alpha| = 0, 1$.

To prove Proposition 2.1, we recall some results involving regularity of solutions to (1.2), as well as, the following $C^{1,\alpha}$-estimates for N-Laplace equations instead of $W^{2,p}$-estimates for uniform elliptic equations.

LEMMA 2.2 ([34]). *Assume* Ω *is a smooth bounded domain in* $\mathbb{R}^N$ *and* $u \in W^{1,N}(\Omega)$ *is a weak solution to* $-\Delta_N u = f$, *where* $f \in L^q(\Omega)$ *for some* $q > 1$, *then for any* $\Omega' \Subset \Omega$, *there exists a constant* C *depending only on* Ω, Ω', q *and* N *such that*

$$\|u\|_{L^\infty(\Omega')} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{L^N(\Omega)}).$$

LEMMA 2.3 ([36]). *Let* Ω *be a smooth bounded domain in* $\mathbb{R}^N$ *and* $u \in W^{1,N}(\Omega)$ *be a weak solution to* $-\Delta_N u = f$. *If* $\|u\|_{L^\infty(\Omega)} \leq a$ *and* $\|f\|_{L^\infty(\Omega)} \leq b$, *then* $u \in C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$. *Moreover, for any* $\Omega' \Subset \Omega$, *there exists a constant* C *depending only on* $\Omega, \Omega', a, b, \alpha$ *such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C.$$

Similarly to [11], by Lemma 1.2 we can get

LEMMA 2.4. *Assume that* (F1)–(F2) *hold, then for any bounded sequence* $\{u_n\}_n$ *in* $W^{1,N}(\mathbb{R}^N)$ *with*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^N \, dx = 0,$$

it holds that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) \, dx = 0.$$

Now, we will adopt some ideas from [11] and [41] to prove Proposition 2.1. We will give only the sketch of the proof.

PROOF OF PROPOSITION 2.1.

STEP 1. We show that for any $U \in S_m$, $U \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. For any $r > 0$, U is a weak solution to the following problem:

$$-\Delta_N u + mu^{N-1} = f(u) \quad \text{in } B_r, \quad u - U \in W^{1,N}(B_r), \tag{2.3}$$

where $B_r(0) := \{x \in \mathbb{R}^N : |x| < r\}$. By Lemma 1.1, it follows that $f(U) \in L^N(\mathbb{R}^N)$, which implies from Lemma 2.2 that for each open $\Omega \Subset B_r$ with $\partial\Omega \in C^1$,

$$\|U\|_{L^\infty(\Omega)} \leq C\big(\|f(U)\|_{L^N(B_r)} + \|U\|_{L^N(B_r)}\big), \tag{2.4}$$

where C depends only on Ω, r. Meanwhile, by Lemma 2.3, we get that $U \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$ and there exists c (depending only on $\|U\|_{L^\infty(B_r)}$, Ω, r, α) such that

$$\|U\|_{C^{1,\alpha}(\Omega)} \leq c. \tag{2.5}$$

Now, to prove that U vanishes at infinity, it suffices to prove that for any $\delta > 0$, there exists $R > 0$ such that $U(x) \leq \delta$, for all $|x| \geq R$. If not, there exists $\{x_j\} \subset \mathbb{R}^N$ with $|x_j| \to \infty$ as $j \to \infty$ and $\liminf_{j\to\infty} U(x_j) > 0$. Let $v_j(x) = U(x + x_j)$ and assume that $v_j \to v$ weakly in $W^{1,N}(\mathbb{R}^N)$, we claim that $v \not\equiv 0$. In fact, noting that v_j is a weak solution to (2.3), it follows from (2.4) and (2.5) that, up to a subsequence, $v_j \to v$ uniformly in Ω. Hence,

$$v(0) = \liminf_{j\to\infty} v_j(0) = \liminf_{j\to\infty} U(x_j) > 0,$$

which implies that $v \not\equiv 0$. On the other hand, for any fixed $R > 0$ and j large enough, we have

$$\begin{aligned}\int_{\mathbb{R}^N} U^N\,dx &\geq \int_{B_R(0)} U^N\,dx + \int_{B_R(x_j)} U^N\,dx \\ &= \int_{B_R(0)} U^N\,dx + \int_{B_R(0)} v^N\,dx + o_j(1),\end{aligned}$$

where $o_j(1) \to 0$ as $j \to \infty$. Since R is arbitrary, we get that $v \equiv 0$, which is a contradiction. Thus, $U(x) \to 0$ as $|x| \to \infty$, which implies that $U \in L^\infty(\mathbb{R}^N)$.

Step 2. We use a result of [10] to prove that any $U \in S_m$ is radially symmetric. Let

$$T(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N\,dx, \qquad G(u) = \int_{\mathbb{R}^N} F(u) - \frac{m}{N}|u|^N\,dx,$$

we consider the constraint minimization problem

$$T_0 := \inf\big\{T(u) : G(u) = 0,\ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}\big\}. \tag{2.6}$$

Arguing as in [23] together with [27], it follows that $T_0 = E_m > 0$ and it is achieved. On the other hand, for any minimizer u of (2.6), as we can see in [7], there exists $\theta > 0$ such that u is a weak solution to the following problem:

$$-\Delta_N u + \theta m|u|^{N-2}u = \theta f(u), \quad u \in W^{1,N}(\mathbb{R}^N). \tag{2.7}$$

Similarly to Step 1, for any solution u to (2.7), $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u(x) \to 0$ as $|x| \to \infty$. By a classical comparison argument, u decays exponentially at infinity, which implies from the Pohozaev–Pucci and Serrin [32] (see also [27]) that u satisfies $G(u) = 0$. By (F1),

$$F(t) - \frac{m}{N}|t|^N < 0 \quad \text{for small enough } |t| > 0.$$

Therefore, it follows from [10, Proposition 4] that U is radially symmetric and nonincreasing with respect to $r = |x|$.

STEP 3. We show the compactness of S_m. First, by Lemma 2.4, similarly to [11], we know that S_m is bounded in $W^{1,N}(\mathbb{R}^N)$. Recalling that $\partial U/\partial r \leq 0$, by the radial lemma [7],

$$\lim_{|x|\to\infty} U(x) = 0 \quad \text{uniformly for } U \in S_m. \tag{2.8}$$

Second, assume that $\{U_j\} \subset S_m$ with $U_j \to U$ weakly in $W^{1,N}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$. By (F2), without loss of generality, we can assume that $\limsup_{j\to\infty} \|\nabla U_j\|_N \leq 1$. By Lemma 1.1, $\|f(U_j)\|_{L^N(\mathbb{R}^N)}$ is uniformly bounded for j, which implies from Lemmas 2.2–2.3 that $\|U_j\|_{C^{1,\alpha}(B_r)} \leq C$, where C does not depend on j. Due to $E_m > 0$, it is easy to prove that $\liminf_{j\to\infty} \|U_j\|_\infty > 0$ since $\lim_{s\to 0} f(t)/t^{N-1} = 0$. Noting that $U_j(0) = \|U_j\|_\infty$, we know that $U \not\equiv 0$. On the other hand, by Lemmas 1.1 and 2.8, it follows from the compactness lemma of Strauss [35] (see also [7]) that

$$\int_{\mathbb{R}^N} F(U_j) \to \int_{\mathbb{R}^N} F(U) \quad \text{as } j \to \infty,$$

which implies that $G(U) \geq 0$ since $G(U_j) = 0$. Due to

$$\|\nabla U\|_N^N \leq \liminf_{j\to\infty} \|\nabla U_j\|_N^N = NE_m,$$

as we can see in [6] (see also [5]), $G(U) = 0$ and $\|U\|_N^N = NE_m$. Thus, $U_j \to U$ strongly in $W^{1,N}(\mathbb{R}^N)$ and S_m is compact in $W^{1,N}(\mathbb{R}^N)$.

STEP 4. We give the decay estimate of S_m at infinity. Similarly to Step 3, by (2.8) and Lemma 2.2, $0 < \inf\{\|U\|_\infty : U \in S_m\} \leq \sup\{\|U\|_\infty : U \in S_m\} < \infty$. By a classical comparison principle, there exist $c, C > 0$ such that

$$U(x) + |\nabla U(x)| \leq C\exp(-c|x|), \quad x \in \mathbb{R}^N,$$

for any $U \in S_m$. The proof is completed. □

Now, we are ready to prove Theorem 1.3. Without loss of generality, we may assume that $0 \in \mathcal{M}$. For any set $B \subset \mathbb{R}^N$ and $\delta > 0$, we define $B^\delta \equiv \{x \in \mathbb{R}^N : \mathrm{dist}(x, B) \leq \delta\}$. Let $E_m = L_m(U)$ for $U \in S_m$ and $10\delta = \mathrm{dist}(\mathcal{M}, O^c)$. We fix $\beta \in (0, \delta)$ and a cut-off $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $|x| \leq \beta$

and $\varphi(x) = 0$ for $|x| \geq 2\beta$. Let $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$, $y \in \mathbb{R}^N$, and for each $x \in \mathcal{M}^\beta$ and $U \in S_m$, we define

$$U_\varepsilon^x(y) = \varphi_\varepsilon\bigg(y - \frac{x}{\varepsilon}\bigg)U\bigg(y - \frac{x}{\varepsilon}\bigg).$$

As in [9], we can find, for sufficiently small $\varepsilon > 0$, a solution near the set

$$X_\varepsilon = \{U_\varepsilon^x(y) : x \in \mathcal{M}^\beta,\, U \in S_m\}.$$

By Proposition 2.1, similarly to [11], we can construct a family of good mountain pass paths at the level E_m.

Proposition 2.5. *There exists $T > 0$ such that, for any $\delta > 0$, there exists a path $\gamma^\delta \in C([0,T], W^{1,N}(\mathbb{R}^N))$ with the following properties*:

(a) $\gamma^\delta(0) = 0$, $L_m(\gamma^\delta(T)) < -1$ *and* $\max\limits_{t\in[0,T]} L_m(\gamma^\delta(t)) = E_m$*;*

(b) *there exists $T_0 \in (0,T_0)$ such that $\gamma^\delta(T_0) \in S_m$, $L_m(\gamma^\delta(T_0)) = E_m$ and $L_m(\gamma^\delta(t)) < E_m$ for $\|\gamma^\delta(t) - \gamma^\delta(T_0)\| \geq \delta$;*

(c) *there exist $C, c > 0$ such that for any $t \in [0,T]$,*

$$|D_x^\alpha(\gamma^\delta(t))(x)| \leq C\exp(-c|x|), \quad x \in \mathbb{R}^N,\ |\alpha| = 0, 1.$$

We note that $0 \in \mathcal{M}$ and define

$$\gamma_\varepsilon^\delta(t)(y) = \varphi_\varepsilon(y)\gamma^\delta(t)(y),$$

then $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t)) = P_\varepsilon(\gamma_\varepsilon^\delta(t))$ for $t \in [0,T]$. Now, we define a min-max value C_ε:

$$C_\varepsilon = \inf_{\gamma\in\Phi_\varepsilon}\max_{s\in[0,1]}\Gamma_\varepsilon(\gamma(s)),$$

where $\Phi_\varepsilon = \{\gamma \in C([0,1], W_\varepsilon) : \gamma(0) = 0,\, \gamma(1) = \gamma_\varepsilon^\delta(T)\}$. By Proposition 2.5, $\gamma_\varepsilon^\delta(T\,\cdot\,) \in \Phi_\varepsilon$ and $\Gamma_\varepsilon(\gamma_\varepsilon^\delta(T)) < 0$ for small enough $\varepsilon > 0$. Let

$$D_\varepsilon^\delta := \max_{s\in[0,1]}\Gamma_\varepsilon(\gamma_\varepsilon^\delta(Ts)). \tag{2.9}$$

Obviously, $C_\varepsilon \leq D_\varepsilon^\delta$. Then similarly to [11], we have

Proposition 2.6. $\lim\limits_{\varepsilon\to0} C_\varepsilon = \lim\limits_{\varepsilon\to0} D_\varepsilon^\delta = E_m$.

Set $\Gamma_\varepsilon^\alpha := \{u \in W_\varepsilon : \Gamma_\varepsilon(u) \leq \alpha\}$ and for a set $A \subset W_\varepsilon$ and $\alpha > 0$, let

$$A^\alpha := \Big\{u \in W_\varepsilon : \inf_{v\in A}\|u - v\|_\varepsilon \leq \alpha\Big\}.$$

Proposition 2.7. *Let $\{\varepsilon_i\}_{i=1}^\infty$ be such that $\lim\limits_{i\to\infty}\varepsilon_i = 0$ and $\{u_{\varepsilon_i}\} \subset X_{\varepsilon_i}^d$ such that*

$$\lim_{i\to\infty}\Gamma_{\varepsilon_i}(u_{\varepsilon_i}) \leq E_m \quad \text{and} \quad \lim_{i\to\infty}\Gamma'_{\varepsilon_i}(u_{\varepsilon_i}) = 0.$$

Then, for sufficiently small $d > 0$, there exists, up to a subsequence, $\{y_i\}_{i=1}^\infty \subset \mathbb{R}^N$, $x \in \mathcal{M}$, $U \in S_m$, such that

$$\lim_{i\to\infty}|\varepsilon_i y_i - x| = 0 \quad \text{and} \quad \lim_{i\to\infty}\|u_{\varepsilon_i} - \varphi_{\varepsilon_i}(\,\cdot - y_i)U(\,\cdot - y_i)\|_{\varepsilon_i} = 0.$$

PROOF. For convenience, we write ε for ε_i. By the definition of X_ε^d, there exist $\{U_\varepsilon\} \subset S_m$ and $\{x_\varepsilon\} \subset \mathcal{M}^\beta$ with

$$\left\| u_\varepsilon - \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) U_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_\varepsilon \le d.$$

Since S_m and $\mathcal{M}^\beta$ are compact, there exist $Z \in S_m$, $x \in \mathcal{M}^\beta$ such that $U_\varepsilon \to Z$ in $W^{1,N}(\mathbb{R}^N)$ and $x_\varepsilon \to x$. Thus, for small enough $\varepsilon > 0$,

$$\left\| u_\varepsilon - \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) Z\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right)\right\|_\varepsilon \le 2d. \tag{2.10}$$

Moreover, by (F2) we may assume that $\sup \|\nabla u_\varepsilon\|_N \le 1$.

STEP 1. We claim that

$$\liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \, dx = 0,$$

where $A_\varepsilon = B(x_\varepsilon/\varepsilon, 3\beta/\varepsilon) \setminus B(x_\varepsilon/\varepsilon, \beta/2\varepsilon)$. If the claim is true, by Lions' lemma,

$$u_\varepsilon \to 0 \quad \text{strongly in } L^q(B_\varepsilon) \text{ for any } q > N,$$

where $B_\varepsilon = B(x_\varepsilon/\varepsilon, 2\beta/\varepsilon) \setminus B(x_\varepsilon/\varepsilon, \beta/\varepsilon)$. Assume by contradiction that there exists $r > 0$ such that

$$\liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \, dx = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for small enough $\varepsilon > 0$, $\int_{B(y_\varepsilon,1)} |u_\varepsilon|^N \, dx \ge r$. Note that $y_\varepsilon \in A_\varepsilon$ and there exists $x_0 \in \mathcal{M}^{4\beta} \subset O$ such that $\varepsilon y_\varepsilon \to x_0$. Let $v_\varepsilon(y) = u_\varepsilon(y + y_\varepsilon)$, then

$$\begin{aligned} -\Delta_N v_\varepsilon + V_\varepsilon(y+y_\varepsilon)|v_\varepsilon|^{N-2}v_\varepsilon - g(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon) \\ = -2NQ_\varepsilon^{1/2}(u_\varepsilon)\chi_\varepsilon(y+y_\varepsilon)|v_\varepsilon|^{N-2}v_\varepsilon + h_\varepsilon, \end{aligned} \tag{2.11}$$

where $h_\varepsilon \to 0$ strongly in $W^{-1,N'}(\mathbb{R}^N)$ as $\varepsilon \to 0$, and for ε small enough,

$$\int_{B(0,1)} |v_\varepsilon|^N \, dy \ge r, \tag{2.12}$$

and up to a subsequence, $v_\varepsilon \to v$ weakly in $W^{1,N}(\mathbb{R}^N)$, almost everywhere in $\mathbb{R}^N$. Recalling that the embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^N(B(0,1))$ is compact, it follows from (2.12) that $v \not\equiv 0$. Now, we claim that v satisfies

$$-\Delta_N v + V(x_0)|v|^{N-2}v = f(v) \quad \text{in } \mathbb{R}^N. \tag{2.13}$$

In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we use $(v_\varepsilon - v)\varphi$ as a test function in (2.11). By the definition of χ and g, we know that for ε small enough,

$$\begin{aligned} &\chi_\varepsilon(y+y_\varepsilon)|v_\varepsilon|^{N-2}v_\varepsilon(v_\varepsilon - v)\varphi = 0 && \text{for all } y \in \mathbb{R}^N, \\ &g(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon)(v_\varepsilon - v)\varphi = f(v_\varepsilon)(v_\varepsilon - v)\varphi && \text{for all } y \in \mathbb{R}^N. \end{aligned}$$

By the local compactness of the embedding $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ for any $q \geq N$, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} V_\varepsilon(y + y_\varepsilon)|v_\varepsilon|^{N-2} v_\varepsilon \varphi\, dy \to \int_{\mathbb{R}^N} V(x_0)|v|^{N-2} v\varphi\, dy.$$

By Lemma 1.1 and (F1), $\|f(v_\varepsilon)\|_N < \infty$, as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} g(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon)(v_\varepsilon - v)\varphi\, dy = \int_{\mathbb{R}^N} f(v_\varepsilon)(v_\varepsilon - v)\varphi\, dy \to 0.$$

So similarly to [4, Lemma 3], as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} |\nabla v_\varepsilon|^{N-2} \nabla v_\varepsilon \nabla \varphi\, dy \to \int_{\mathbb{R}^N} |\nabla v|^{N-2} \nabla v \nabla \varphi\, dy.$$

By (F1)–(F2), Lemma 1.1 and the compactness lemma of Strauss [35], as $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} g(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon)\varphi\, dy \to \int_{\mathbb{R}^N} f(v)\varphi\, dy.$$

Hence, v is a nontrivial solution to (2.13). Then, for a sufficiently large $R > 0$,

$$\liminf_{\varepsilon \to 0} \int_{B(y_\varepsilon, R)} |\nabla u_\varepsilon|^N \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^N = \frac{N}{2}\, L_{V(x_0)}(v).$$

Obviously, $L_{V(x_0)}(v) \geq E_{V(x_0)}$. Then, since $x_0 \in O$, we have $V(x_0) \geq m$ and

$$\liminf_{\varepsilon \to 0} \int_{B(y_\varepsilon, R)} |\nabla u_\varepsilon|^N \geq \frac{N}{2}\, E_m > 0,$$

which is a contradiction with (2.10) if d is small enough.

Step 2. Let $u_\varepsilon^1(y) = \varphi_\varepsilon(y - x_\varepsilon/\varepsilon)u_\varepsilon(y)$, $u_\varepsilon^2 = u_\varepsilon - u_\varepsilon^1$. We claim that, for small enough $d > 0$, $\Gamma_\varepsilon(u_\varepsilon^2) \geq 0$ and

$$\Gamma_\varepsilon(u_\varepsilon) \geq \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1) \quad \text{as } \varepsilon \to 0.$$

Obviously, $Q_\varepsilon(u_\varepsilon) = Q_\varepsilon(u_\varepsilon^2)$ and $Q_\varepsilon(u_\varepsilon^1) = 0$ for small enough $\varepsilon > 0$. For any $y \in \mathbb{R}^N$, $u_\varepsilon^1(y)u_\varepsilon^2(y) \geq 0$, then

$$\begin{aligned} |u_\varepsilon(y)|^N &= \big(|u_\varepsilon^1(y)|^2 + |u_\varepsilon^2(y)|^2 + 2u_\varepsilon^1(y)u_\varepsilon^2(y)\big)^{N/2} \\ &\geq (|u_\varepsilon^1(y)|^2 + |u_\varepsilon^2(y)|^2)^{N/2} \geq |u_\varepsilon^1(y)|^N + |u_\varepsilon^2(y)|^N, \end{aligned}$$

which implies that

$$\int_{\mathbb{R}^N} V_\varepsilon |u_\varepsilon|^N\, dy \geq \int_{\mathbb{R}^N} V_\varepsilon |u_\varepsilon^1|^N\, dy + \int_{\mathbb{R}^N} V_\varepsilon |u_\varepsilon^2|^N\, dy.$$

Meanwhile, it is easy to verify that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon^1|^N\, dy = \int_{\mathbb{R}^N} \varphi_\varepsilon^N\bigg(\,\cdot\, - \frac{x_\varepsilon}{\varepsilon}\bigg)|\nabla u_\varepsilon|^N\, dy + o(1),$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon^2|^N\, dy = \int_{\mathbb{R}^N} \bigg(1 - \varphi_\varepsilon\bigg(\,\cdot\, - \frac{x_\varepsilon}{\varepsilon}\bigg)\bigg)^N |\nabla u_\varepsilon|^N\, dy + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Obviously, for any $y \in \mathbb{R}^N$,

$$|\nabla u_\varepsilon(y)|^2 \geq \varphi_\varepsilon^2(y - x_\varepsilon/\varepsilon)|\nabla u_\varepsilon(y)|^2 + (1 - \varphi_\varepsilon(y - x_\varepsilon/\varepsilon))^2 |\nabla u_\varepsilon(y)|^2.$$

It follows that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^N \geq \int_{\mathbb{R}^N} |\nabla u_\varepsilon^1|^N + \int_{\mathbb{R}^N} |\nabla u_\varepsilon^2|^N + o(1).$$

Thus, we get

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon) \geq{}& \Gamma_\varepsilon(u_\varepsilon^1) + \Gamma_\varepsilon(u_\varepsilon^2) + o(1) \\ &- \int_{B_\varepsilon} \big(G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^1) - G(\varepsilon y, u_\varepsilon^2)\big)\, dy.\end{aligned} \tag{2.14}$$

By (F1)–(F2), there exists $\alpha \in (0, \alpha_n)$ such that, for any $\rho > 0$ with $\rho C_\alpha \in (0, 1/(2N))$ and fixed $q > N$, there exists $C > 0$ such that

$$|F(t)| \leq \rho \Psi_N(t) + C|t|^q, \quad t \in \mathbb{R}, \tag{2.15}$$

where Ψ_N and C_α are given in Lemma 1.2. By Lemma 1.2,

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \Psi_N(u_\varepsilon) = c < \infty.$$

Then, by $u_\varepsilon \to 0$ strongly in $L^q(B_\varepsilon)$ which has been proved in Step 1, we have

$$\begin{aligned}\limsup_{\varepsilon \to 0} &\int_{B_\varepsilon} \big(G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^1) - G(\varepsilon y, u_\varepsilon^2)\big)\, dy \\ &= \limsup_{\varepsilon \to 0} \left| \int_{B_\varepsilon} (F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2))\, dy \right| \\ &\leq \limsup_{\varepsilon \to 0} \int_{B_\varepsilon} (\rho \Psi_N(u_\varepsilon) + C|u_\varepsilon|^q)\, dy \leq c\rho.\end{aligned}$$

Since ρ is arbitrary, $\int_{B_e} (F(u_\varepsilon) - F(u_\varepsilon^1) - F(u_\varepsilon^2))\, dy = o(1)$ as $\varepsilon \to 0$. By (2.15), Lemma 1.2 and Sobolev's inequality, there exists $C > 0$ (independent of ε) such that

$$\begin{aligned}\Gamma_\varepsilon(u_\varepsilon^2) \geq P(u_\varepsilon^2) &\geq \frac{1}{N} \|u_\varepsilon^2\|_\varepsilon^N - \rho \int_{\mathbb{R}^N} \Psi_N(u_\varepsilon^2)\, dy - C\|u_\varepsilon^2\|_\varepsilon^q \\ &\geq \frac{1}{2N} \|u_\varepsilon^2\|_\varepsilon^N - C\|u_\varepsilon^2\|_\varepsilon^q.\end{aligned}$$

It follows from $q > N$ that $\Gamma_\varepsilon(u_\varepsilon^2) \geq \|u_\varepsilon^2\|_\varepsilon^N/(6N) \geq 0$ for $d > 0$ small. Therefore, the claim is true.

Step 3. Let $w_\varepsilon(y) := u_\varepsilon^1(y + x_\varepsilon/\varepsilon) = \varphi_\varepsilon(y) u_\varepsilon(y + x_\varepsilon/\varepsilon)$. Up to a subsequence, $w_\varepsilon \rightharpoonup w$ weakly in $W^{1,N}(\mathbb{R}^N)$, $w_\varepsilon \to w$ almost everywhere in $\mathbb{R}^N$. Now, we claim that

$$w_\varepsilon \to w \quad \text{strongly in } L^q(\mathbb{R}^N),$$

where q is given in (2.15). Assume by contradiction that there exists $r>0$ such that

$$\liminf_{\varepsilon\to 0}\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|w_\varepsilon-w|^q\,dy=2r>0.$$

Then, there exists $z_\varepsilon\in\mathbb{R}^N$ such that

$$\liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,1)}|w_\varepsilon-w|^q>r.$$

Obviously, $\{z_\varepsilon\}$ is unbounded. Without loss of generality, $\lim\limits_{\varepsilon\to 0}|z_\varepsilon|=\infty$. Then,

$$\liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,1)}|w_\varepsilon|^q\,dy\geq r,$$

i.e.

$$\liminf_{\varepsilon\to 0}\int_{B(z_\varepsilon,1)}\left|\varphi_\varepsilon(y)u_\varepsilon\left(y+\frac{x_\varepsilon}{\varepsilon}\right)\right|^q dy\geq r.$$

Similarly to [11], $|z_\varepsilon|\leq\beta/2\varepsilon$ for ε small enough. Assume that $\varepsilon z_\varepsilon\to z_0\in\overline{B(0,\beta/2)}$ and $\widetilde{w}_\varepsilon=w_\varepsilon(y+z_\varepsilon)\rightharpoonup\widetilde{w}$ weakly in $W^{1,N}(\mathbb{R}^N)$, almost everywhere in $\mathbb{R}^N$. Then $\widetilde{w}\not\equiv 0$ and as in Step 1, $\widetilde{w}$ satisfies

$$-\Delta_N\widetilde{w}(y)+V(x+z_0)|\widetilde{w}(y)|^{N-2}\widetilde{w}(y)=f(\widetilde{w}(y))\quad\text{in }\mathbb{R}^N.$$

Similarly to Step 1, we get a contradiction for $d>0$ small enough. Thus, $w_\varepsilon\to w$ strongly in $L^q(\mathbb{R}^N)$.

Step 4. By Step 3,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}G(\varepsilon x,u_\varepsilon^1)\,dx&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}G(\varepsilon x+x_\varepsilon,w_\varepsilon)\,dx\\&=\lim_{\varepsilon\to 0}\int_{O_\varepsilon-x_\varepsilon/\varepsilon}F(w_\varepsilon)\,dx=\int_{\mathbb{R}^N}F(w)\,dx.\end{aligned}$$

Then, similarly to [9, Proposition 4], there exist $U\in S_m$ and $y_\varepsilon\in\mathbb{R}^N$ such that $\lim\limits_{\varepsilon\to 0}|\varepsilon y_\varepsilon-x|=0$ and $\lim\limits_{\varepsilon\to 0}\|u_\varepsilon-\varphi_\varepsilon(\,\cdot\,-y_\varepsilon)U(\,\cdot\,-y_\varepsilon)\|_\varepsilon=0$. □

By Proposition 2.7, there exists $d_0>0$ small enough such that for any $d\in(0,d_0)$ there exist $\rho_d>0$, $\omega_d>0$ and $\varepsilon_d>0$ such that, for $\varepsilon\in(0,\varepsilon_d)$,

$$|\Gamma_\varepsilon'(u)|\geq\omega_d\quad\text{for }u\in\Gamma_\varepsilon^{E_m+\rho_d}\cap(X_\varepsilon^{d_0}\setminus X_\varepsilon^d).\tag{2.16}$$

Similarly to [11], by Proposition 2.5, we have

Proposition 2.8. *There exists $M_0>0$ such that for any $\delta>0$, there exist $\alpha_\delta>0$ and $\varepsilon_\delta<1$ such that for $\varepsilon\in(0,\varepsilon_\delta)$ and $t\in[0,T]$,*

$$\Gamma_\varepsilon(\gamma_\varepsilon^\delta(t))\geq E_m-\alpha_\delta\quad\textit{implies}\quad\gamma_\varepsilon^\delta(t)\in X_\varepsilon^{M_0\delta}.$$

By (2.16) and Proposition 2.8, with a deformation argument, similarly to [11], there exist $d>0$ and $\delta>0$ such that Γ_ε admits a Palais–Smale sequence in $\Gamma_\varepsilon^{D_\varepsilon^\delta}\cap X_\varepsilon^d$, where D_ε^δ is given in (2.9). Precisely, Proposition 2.9. *For sufficiently small $\varepsilon > 0$, there exists a sequence $\{u_{n,\varepsilon}\}_{n=1}^{\infty} \subset \Gamma_\varepsilon^{D_\varepsilon^\delta} \cap X_\varepsilon^d$ such that $|\Gamma'_\varepsilon(u_{n,\varepsilon})| \to 0$ as $n \to \infty$.*

In the following, by Proposition 2.9, we show the existence of a critical point of Γ_ε.

Proposition 2.10. *For sufficiently small $\varepsilon, d > 0$, Γ_ε has a nontrivial critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon^\delta}$.*

Proof. Let $\varepsilon > 0$ be fixed, small enough. By Proposition 2.9, there exists a sequence $\{u_{n,\varepsilon}\}_{n=1}^{\infty} \subset X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon^\delta}$ such that $|\Gamma'_\varepsilon(u_{n,\varepsilon})| \to 0$ as $n \to \infty$. Since X_ε^d is bounded, we can assume that $u_{n,\varepsilon} \rightharpoonup u_\varepsilon$ weakly in W_ε as $n \to \infty$. Similarly to [14, Proposition 3],

$$\lim_{R\to\infty} \sup_{n\ge 1} \int_{|x|\ge R} (|\nabla u_{n,\varepsilon}|^N + V_\varepsilon |u_{n,\varepsilon}|^N)\, dx = 0,$$

which immediately implies that $u_{n,\varepsilon} \to u_\varepsilon$ strongly in $L^r(\mathbb{R}^N)$ $(r \ge N)$ as $n \to \infty$. Moreover, by (F1)–(F2) and Lemma 1.1, $\sup \|f(u_{n,\varepsilon})\|_N < \infty$. Then, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} g(\varepsilon y, u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon)\varphi\, dy \to 0 \quad \text{as } n \to \infty.$$

Then, similarly to [28, Proposition 5.3], $u_{n,\varepsilon} \to u_\varepsilon$ strongly in W_ε as $n \to \infty$. Thus, $\Gamma'_\varepsilon(u_\varepsilon) = 0$ in W_ε and $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon^\delta}$. Obviously, $0 \notin X_\varepsilon^d$ for small enough $d > 0$. Thus $u_\varepsilon \not\equiv 0$. $\square$

Let us continue with the proof of Theorem 1.3. By Proposition 2.10, there exist $d > 0$ and $\varepsilon_0 > 0$ such that Γ_ε has a nontrivial critical point $u_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon^\delta}$ for $\varepsilon \in (0, \varepsilon_0)$. Since $f(t) = 0$ for $t \le 0$, we see that $u_\varepsilon \ge 0$.

Step 1. We prove that there exists $C > 0$ such that

$$\|u_\varepsilon\|_\infty < C \quad \text{uniformly for } \varepsilon \in (0, \varepsilon_0). \tag{2.17}$$

If (2.17) is true, then it follows from the Harnark inequality (see [37]) that $u_\varepsilon > 0$ in $\mathbb{R}^N$. Thus, from (V1) and (F1), it is easy to see that $\inf\limits_{\varepsilon\in(0,\varepsilon_0)} \|u_\varepsilon\|_\infty > 0$.

Now, we use the Moser iteration argument (see [30]) to prove (2.17). For any $L > 0$ and $\beta \ge 1$, define

$$u_{\varepsilon,L} = \min\{u_\varepsilon, L\} \quad \text{and} \quad v_\varepsilon = u_\varepsilon u_{\varepsilon,L}^{N(\beta-1)},$$

then $v_\varepsilon \in W_\varepsilon$. By $\Gamma'_\varepsilon(u_\varepsilon) = 0$ and (F1)–(F2), for any $\alpha \in (0, \alpha_N)$, there exists $C > 0$ such that u_ε satisfies

$$-\Delta_N u_\varepsilon \le C\Psi_N(u_\varepsilon) u_\varepsilon^{N-1}, \quad u_\varepsilon > 0, \quad x \in \mathbb{R}^N, \tag{2.18}$$

where Ψ_N is defined in Lemma 1.2. Using v_ε as a test function in (2.18), we get

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^N u_{\varepsilon,L}^{N(\beta-1)} + N(\beta-1)\int_{\mathbb{R}^N} |\nabla u_{\varepsilon,L}|^N u_{\varepsilon,L}^{N(\beta-1)} \leq C\int_{\mathbb{R}^N} \Psi_N(u_\varepsilon)u_\varepsilon^N u_{\varepsilon,L}^{N(\beta-1)}.$$

Set $w_{\varepsilon,L} = u_\varepsilon u_{\varepsilon,L}^{\beta-1}$. We have $\nabla w_{\varepsilon,L} = \nabla u_\varepsilon u_{\varepsilon,L}^{\beta-1} + (\beta-1)\nabla u_{\varepsilon,L}\nabla u_{\varepsilon,L}^{\beta-1}$. Then

$$\int_{\mathbb{R}^N} |\nabla w_{\varepsilon,L}|^N \leq C\beta^N \int_{\mathbb{R}^N} \Psi_N(u_\varepsilon) w_{\varepsilon,L}^N,$$

where $C > 0$ is independent of ε, L, β. Take some fixed $t > N$, then by (F2) and Lemma 1.1, we can choose α in Ψ_N small enough such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|\Psi_N(u_\varepsilon)\|_{L^{t/(t-N)}(\mathbb{R}^N)} < \infty.$$

So, $\|\nabla w_{\varepsilon,L}\|_N \leq C\beta\|w_{\varepsilon,L}\|_t$. Choosing some fixed $s > t$, by the Gagliardo–Nirenberg inequality (see [29, Proposiotion 8.12]), we have

$$\|w_{\varepsilon,L}\|_s \leq C(\|\nabla w_{\varepsilon,L}\|_N + \|w_{\varepsilon,L}\|_t) \leq C\beta\|w_{\varepsilon,L}\|_t,$$

where C depends only on s, t, N. Let $L \to \infty$, we get that

$$\|u_\varepsilon\|_{L^{s\beta}(\mathbb{R}^N)} \leq C^{1/\beta}\beta^{1/\beta}\|u_\varepsilon\|_{L^{t\beta}(\mathbb{R}^N)}, \tag{2.19}$$

where C depends only on s, t, N. Let $\kappa = s/t > 1$, $\beta = \kappa^n$, then by (2.19),

$$\|u_\varepsilon\|_{L^{t\kappa^{n+1}}(\mathbb{R}^N)} \leq C^{\kappa^{-n}}\kappa^{n\kappa^{-n}}\|u_\varepsilon\|_{L^{t\kappa^n}(\mathbb{R}^N)}. \tag{2.20}$$

By iterating (2.20), we can get that for any $n \geq 1$,

$$\|u_\varepsilon\|_{L^{t\kappa^{n+1}}(\mathbb{R}^N)} \leq C^{\sum_{i=0}^{n}\kappa^{-i}}\kappa^{\sum_{i=1}^{n} i\kappa^{-i}}\|u_\varepsilon\|_{L^t(\mathbb{R}^N)},$$

which implies that $\|u_\varepsilon\|_\infty \leq C\|u_\varepsilon\|_{L^t(\mathbb{R}^N)}$, where C does not depend on ε. Recalling that $u_\varepsilon \in X_\varepsilon^d$, we know that $\sup\|u_\varepsilon\|_t < \infty$. Therefore, the claim (2.17) is concluded.

Step 2. There exist $C, c > 0$ (independent of ε) and $y_\varepsilon \in \mathbb{R}^N$ such that

$$0 < w_\varepsilon(y) \leq C\exp(-c|y|) \quad \text{for } y \in \mathbb{R}^N,\ \varepsilon \in (0,\varepsilon_0), \tag{2.21}$$

where $w_\varepsilon(y) = u_\varepsilon(y+y_\varepsilon)$. By Proposition 2.7 and (2.17), for small enough $d > 0$ there exist $\{y_\varepsilon\} \subset \mathbb{R}^N$, $x \in \mathcal{M}$, $U \in S_m$ such that

$$\lim_{\varepsilon\to 0}|\varepsilon y_\varepsilon - x| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\|u_\varepsilon - U(\,\cdot - y_\varepsilon)\|_\varepsilon = 0.$$

Then, for any $\sigma > 0$, there exists $R > 0$ (independent of ε) such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \int_{\mathbb{R}^N\setminus B(0,R)} w_\varepsilon^N \leq \sigma.$$

Moreover, since $\Gamma_\varepsilon'(u_\varepsilon) = 0$ and $\{u_\varepsilon\}$ is bounded in $L^\infty(\mathbb{R}^N)$ uniformly for $\varepsilon \in (0,\varepsilon_0)$, there exists $C > 0$ (independent of ε) such that $-\Delta_N w_\varepsilon \leq Cw_\varepsilon^{N-1}$

in $\mathbb{R}^N$. Then by [37, Theorem 1.3], there exists $C > 0$ (independent of ε) such that $\sup_{B(y,1)} w_\varepsilon \le C\|w_\varepsilon\|_{L^N(B(y,2))}$ for any $y \in \mathbb{R}^N$. Then, $0 < w_\varepsilon(y) \le C\sigma^{1/N}$ for $\varepsilon \in (0, \varepsilon_0)$, $|y| \ge R + 2$. Thus, (2.21) follows from the maximum principle.

Step 3. Similarly to [40], by (2.21) it is easy to show that $Q_\varepsilon(u_\varepsilon) = 0$ for small $\varepsilon > 0$. Thus, u_ε is a critical point of P_ε and a solution to (2.2). Moreover, by (2.21), $u_\varepsilon(x) \to 0$ as $\varepsilon \to 0$ uniformly for $x \in \mathbb{R}^N \setminus O_\varepsilon$, which implies that u_ε is a solution to the original problem (2.1) for small enough $\varepsilon > 0$. By Lemma 2.3 and Step 1, $u_\varepsilon \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. Let $z_\varepsilon \in \mathbb{R}^N$ be such that $\|w_\varepsilon\|_\infty = w_\varepsilon(z_\varepsilon)$, then by Step 1 and (2.21), $\{z_\varepsilon\} \subset \mathbb{R}^N$ is bounded. Up to a subsequence, we can assume that $z_\varepsilon \to z_0$ as $\varepsilon \to 0$. Let $\widetilde{x_\varepsilon} = y_\varepsilon + z_\varepsilon$, then $\max_{\mathbb{R}^N} u_\varepsilon = u_\varepsilon(\widetilde{x_\varepsilon})$. Let $x_\varepsilon = \varepsilon y_\varepsilon + \varepsilon z_\varepsilon$, then $\max_{\mathbb{R}^N} v_\varepsilon = v_\varepsilon(x_\varepsilon)$ and $\lim_{\varepsilon\to 0} x_\varepsilon = \lim_{\varepsilon\to 0} \varepsilon y_\varepsilon = x \in \mathcal{M}$. Finally, it is easy to check that $v_\varepsilon(\varepsilon\,\cdot\,+x_\varepsilon) \to U(\,\cdot\,+z_0)$ strongly in W_ε as $\varepsilon \to 0$. This completes the proof.

3. Proof of Theorem 1.4

In this section, assume that f satisfies (F1) and (F4)–(F5), we consider the semiclassical states of (1.1) in the critical case. We use a truncation argument to prove Theorem 1.4. Similar arguments can also be found in [41].

3.1. The limit problem. We study the existence and properties of ground state solutions to the limiting problem (1.2) in the critical case. It was shown in [39] that (1.2) possesses a ground state solution by means of the following constraint minimization problem:

$$A := \inf\big\{T(u) : G(u) = 0,\, u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}\big\}, \tag{2.1}$$

where

$$T(u) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla u|^N\,dx \quad \text{and} \quad G(u) = \int_{\mathbb{R}^N}\left(F(u) - \frac{m}{N}\,|u|^N\right)dx.$$

If problem (2.1) admits a minimizer u, then there exists $\theta > 0$ such that $u(\,\cdot\,/\sqrt[N]{\theta})$ is indeed the ground state solution to (1.2). Following [5], [39], to prove the existence of the minimizer to (1.2), it is enough to prove that $A < 1/N$. For this goal, let

$$c := \inf_{u\in W^{1,N}(\mathbb{R}^N)\setminus\{0\}} \max_{t\ge 0} L_m(tu),$$

then it can be seen from [39] that $A \le c$. It follows from Lemma 3.2 (see below) that $A < 1/N$, then from [39] one has the following

Lemma 3.1. *Assume* (F1), (F_4) *and* (F_5) *hold with*

$$\beta_0 > \frac{e}{w_{N-1}} \cdot \frac{(N-2)!}{N^{N-2}}\,m. \tag{2.2}$$

Then (1.2) *admits a positive ground state solution. Moreover, the least energy* E_m *is obtained by a mountain pass value.*

Lemma 3.2. *There exists* $w \in W^{1,N}(\mathbb{R}^N)\setminus\{0\}$ *such that* $\max_{t\geq 0} L_m(tw) < 1/N$ *where*

$$L_m(u) = \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla u|^N + m\,|u|^N)\,dx - \int_{\mathbb{R}^N} F(u)\,dx.$$

Proof. Let us first remark a few facts: by (2.2) we can choose $r > 0$ such that

$$\beta_0 > \frac{Ne^{(N-2)!\,r^N m/N^{N-1}}}{w_{N-1}r^N}, \tag{2.3}$$

and considering the Moser sequence of functions

$$\widetilde{w}_n(x) := w_{N-1}^{-1/N}\begin{cases} (\log n)^{(N-1)/N} & \text{if } |x| \leq \dfrac{r}{n},\\ \dfrac{\log(r/|x|)}{(\log n)^{1/N}} & \text{if } \dfrac{r}{n} \leq |x| \leq r,\\ 0 & \text{if } |x| \geq r,\end{cases}$$

where w_{N-1} is the volume of the unit sphere in $\mathbb{R}^N$, it is readily seen that

$$\|\nabla\widetilde{w}_n\|_N = 1 \quad\text{and}\quad \|\widetilde{w}_n\|_N^N = \frac{1}{\log n}\bigg(\frac{(N-1)!}{N^N}\,r^N + o_n(1)\bigg).$$

Let

$$|||\widetilde{w}_n|||^N := \|\nabla\widetilde{w}_n\|_N^N + m\|\widetilde{w}_n\|_N^N = 1 + \frac{d_n(r)}{\log n}\,m,$$

where

$$d_n(r) := \frac{(N-1)!}{N^N}\,r^N + o_n(1) \quad\text{and}\quad o_n(1)\to 0, \quad \text{as } n\to+\infty.$$

Set $w_n := \widetilde{w}_n/\,|||\,\widetilde{w}_n|||$, then for n large enough,

$$w_n^{N/(N-1)}(x) \geq w_{N-1}^{-1/(N-1)}\bigg(\log n - \frac{d_n(r)}{N-1}\,m_i\bigg), \quad |x| \leq \frac{r}{n}. \tag{2.4}$$

Following the argument of Adimurthi [1] (see also [24]), we have

Claim. There exists $n \in \mathbb{N}$ such that $\max_{t\geq 0} L_m(tw_n) < 1/N$.

Indeed, assume by contradiction that

$$\max_{t\geq 0} L_m(tw_n) \geq \frac{1}{N}, \quad n\in\mathbb{N}.$$

As a consequence of (F_5), for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$sf(s) \geq (\beta_0-\varepsilon)\exp(\alpha_N s^{N/(N-1)}), \quad \text{for all } s \geq R_\varepsilon, \tag{2.5}$$

which implies that there exist $C_1, C_2 > 0$ such that

$$F(s) \geq C_1 s^{N+1} - C_2, \quad s \geq 0, \tag{2.6}$$

which yields $L_m(tw_n) \to -\infty$, as $t \to \infty$. Thus there exists $t_n > 0$ such that

$$L_m(t_n w_n) = \max_{t\geq 0} L_m(tw_n) \geq \frac{1}{N}, \tag{2.7}$$

which in turn gives

$$\frac{1}{N} \leq \frac{t_n^N}{N} - \int_{\mathbb{R}^N} F(t_n w_n) \leq \frac{t_n^N}{N},$$

thus $t_n \geq 1$.

Next we show that actually $\lim\limits_{n\to\infty} t_n = 1$. Observe that

$$t_n^N = \int_{\mathbb{R}^N} f(t_n w_n) t_n w_n \, dx, \tag{2.8}$$

and

$$t_n w_n = \frac{t_n}{|||\widetilde{w}_n|||} \frac{(\log n)^{(N-1)/N}}{w_{N-1}^{1/N}} \to +\infty, \quad \text{as } n \to \infty,\ x \in B_{r/n},$$

for n large enough. Then for n large enough, by (2.4) and (2.5), we have

$$\begin{aligned} t_n^N &\geq (\beta_0 - \varepsilon) \int_{B_{r/n}} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx \\ &\geq \frac{w_{N-1}}{N} \, r^N (\beta_0 - \varepsilon) \, e^{N t_n^{N/(N-1)} [\log n - d_n(r) m/(N-1)] - N \log n}, \end{aligned}$$

which implies that $\{t_n\}$ is bounded and also $\limsup\limits_{n\to\infty} t_n \leq 1$. Thus, Claim is proved.

Noting that $w_n \to 0$ almost everywhere in $\mathbb{R}^N$, by the Lebesgue dominated convergence theorem, as $n \to \infty$ one has

$$\int_{\{t_n w_n < R_\varepsilon\}} f(t_n w_n) t_n w_n \, dx \to 0$$

and

$$\int_{\{t_n w_n < R_\varepsilon\}} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx \to \frac{w_{N-1}}{N} \, r^N.$$

Then from (2.8) and (2.5) it follows that

$$\begin{aligned} t_n^N =& \int_{B_r} f(t_n w_n) t_n w_n \, dx \\ \geq& (\beta_0 - \varepsilon) \int_{B_r} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx + \int_{\{t_n w_n < R_\varepsilon\}} f(t_n w_n) t_n w_n \, dx \\ &- (\beta_0 - \varepsilon) \int_{\{t_n w_n < R_\varepsilon\}} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx \\ =& (\beta_0 - \varepsilon) \left[\int_{B_r} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx - \frac{w_{N-1}}{N} \, r^N + o_n(1) \right]. \end{aligned}$$

Let us estimate the term

$$\int_{B_r} \exp\left(\alpha_N (t_n w_n)^{N/(N-1)}\right) dx.$$

On one hand, it follows from (2.4) that

$$\int_{B_{r/n}} \exp\big(\alpha_N (t_n w_n)^{N/(N-1)}\big)\,dx \\ \geq \frac{w_{N-1}}{N}\, r^N\, e^{N t_n^{N/(N-1)}[\log n - d_n(r)m/(N-1)] - N\log n}.$$

Noting also that $t_n \geq 1$, we have

$$\liminf_{n\to\infty} \int_{B_{r/n}} \exp\big(\alpha_N (t_n w_n)^{N/(N-1)}\big)\,dx \geq \frac{w_{N-1}}{N}\, r^N\, e^{-(N-2)!/N^{N-1}\, r^N m}.$$

On the other hand, using the change of variable $s = re^{-|||\widetilde{w}_n|||(\log n)^{1/N}\,t}$,

$$\begin{aligned}
&\int_{B_r\setminus B_{r/n}} \exp\big(\alpha_N (t_n w_n)^{N/(N-1)}\big)\,dx \\
&= w_{N-1} r^N\, |||\, \widetilde{w}_n\, |||\, (\log n)^{1/N} \\
&\quad\cdot \int_0^{(\log n)^{(N-1)/N}/|||\widetilde{w}_n|||} e^{N\left[(t_n t)^{N/(N-1)} - |||\widetilde{w}_n|||(\log n)^{1/N}\ t\right]}\,dt \\
&\geq w_{N-1} r^N\, |||\, \widetilde{w}_n\, |||\, (\log n)^{1/N} \int_0^{(\log n)^{(N-1)/N}/|||\widetilde{w}_n|||} e^{-N|||\widetilde{w}_n|||(\log n)^{1/N} t}\,dt \\
&= \frac{w_{N-1}}{N}\, r^N \big(1 - e^{-N\log n}\big).
\end{aligned}$$

Then

$$\liminf_{n\to\infty} \int_{B_r} \exp\big(\alpha_N (t_n w_n)^{N/(N-1)}\big)\,dx \geq \frac{w_{N-1}}{N}\, r^N \big(e^{-(N-2)! r^N m/N^{N-1}} + 1\big),$$

which implies

$$1 = \lim_{n\to+\infty} t_n^N \geq (\beta_0 - \varepsilon)\, \frac{w_{N-1}}{N}\, r^N e^{-(N-2)! r^N m/N^{N-1}}.$$

Since ε is arbitrary, we have

$$\beta_0 \leq \frac{N e^{(N-2)! r^N m/N^{N-1}}}{w_{N-1} r^N},$$

which contradicts (2.3) and the proof is complete. □

With the same assumptions on f as in Theorem 1.4, G.Q. Zhang and J. Sun (see [39]) proved that there exists a radially symmetric positive ground state solution U to (1.2) (see also [5] for $N = 2$) and the least energy E_m is corresponding to a mountain path value. Moreover,

$$\begin{gathered}
\frac{1}{N}\int_{\mathbb{R}^N} |\nabla U|^N\,dx = L_m(U) = E_m < \frac{1}{N}, \\
\int_{\mathbb{R}^N} \left(F(U) - \frac{m}{N}\, U^N\right) dx = 0.
\end{gathered} \tag{2.9}$$

Let S_m be the set of positive ground state solutions U to (1.2) with $U(0) = \max_{x\in\mathbb{R}^N} U(x)$. Then $S_m \neq \phi$. Similarly to Proposition 2.1, we have the following result.

PROPOSITION 3.3. *Under the assumptions of Theorem* 1.4*,*

(a) *for any* $U \in S_m$*,* $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ $(\alpha \in (0,1))$ *is radially symmetric and* $\partial U/\partial r \leq 0$*,* $r = |x|$*;*
(b) S_m *is compact in* $W^{1,N}(\mathbb{R}^N)$*;*
(c) $0 < \inf\{\|U\|_\infty : U \in S_m\} \leq \sup\{\|U\|_\infty : U \in S_m\} < \infty$*;*
(d) *there exist* $C, c > 0$*, independent of* $U \in S_m$*, such that* $|D^\alpha U(x)| \leq C\exp(-c|x|)$*,* $x \in \mathbb{R}^N$ *for* $|\alpha| = 0, 1$*.*

To prove Proposition 3.3, we need the following convergence result, which is proved in [39] by a similar argument as in the case $N = 2$ in [5].

LEMMA 3.4 ([39]). *Assume that f satisfies the same assumptions as in Theorem* 1.4 *and let* $\{v_n\}$ *be a sequence in* $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ *such that*

$$\sup_n \|\nabla v_n\|_N^N = \rho < 1 \quad \textit{and} \quad \sup_n \|v_n\|_N^N < \infty.$$

Then, if $v_n \to v$ *weakly in* $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ *as* $n \to \infty$*, we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(v_n) = \int_{\mathbb{R}^N} F(v).$$

PROOF OF PROPOSITION 3.3. Obviously, (a) can be proved similarly to Proposition 2.1. Now, we show the compactness of S_m. First, we prove that S_m is bounded in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ similarly to [11]. It suffices to prove that $\{\|U\|_N : U \in S_m\}$ is bounded. Otherwise, there exists $\{U_j\} \subset S_m$ such that $\lambda_j = \|U_j\|_N \to \infty$ as $j \to \infty$. Let $\widetilde{U}_j(x) = U_j(\lambda_j x)$, then $\widetilde{U}_j$ satisfies $\|\widetilde{U}_j\|_N = 1$, $\|\nabla\widetilde{U}_j\|_N^N = NE_m < 1$ and

$$-\frac{1}{\lambda_j^N}\,\Delta\widetilde{U}_j + m\widetilde{U}_j = f(\widetilde{U}_j) \quad \text{in } \mathbb{R}^N. \tag{2.10}$$

Assume that $\widetilde{U}_j \to U_0$ weakly in $W^{1,N}_{\mathrm{rad}}(\mathbb{R}^N)$ and strongly in $L^{N+1}(\mathbb{R}^N)$, as we can see from [11], $U_0 \equiv 0$. By (F1)–(F2), for any $\delta > 0$ and $\alpha > \alpha_N$, there exists $C > 0$ such that $|tf(t)| \leq \delta\Psi_N(t) + C|t|^{N+1}$ for $t \in \mathbb{R}$, where Ψ_N is defined in Lemma 1.1. Noting that $\|\nabla\widetilde{U}_j\|_N^N = NE_m < 1$, by Lemma 1.1, we can choose α close to α_N such that $\sup_j \int_{\mathbb{R}^N} \|\Psi_N(\widetilde{U}_j)\| < \infty$. Then for some $c > 0$,

$$\limsup_{j\to\infty} \left| \int_{\mathbb{R}^2} \widetilde{U}_j f(\widetilde{U}_j) \right| \leq c\delta,$$

that is, $\int_{\mathbb{R}^N} \widetilde{U}_j f(\widetilde{U}_j) \to 0$ as $j \to \infty$. Thus, by (2.10) $\|\widetilde{U}_j\|_N \to 0$ as $j \to \infty$, which is a contradiction. Therefore, the claim is proved and S_m is bounded in

$W^{1,N}_{\rm rad}(\mathbb{R}^N)$. Second, assume $\{u_n\} \subset S_m$ and $u_n \to u$ weakly in $W^{1,N}_{\rm rad}(\mathbb{R}^N)$, then by Lemma 3.4,

$$\int_{\mathbb{R}^2} F(u_n) \to \int_{\mathbb{R}^2} F(u).$$

Due to $E_m > 0$, $u \not\equiv 0$. As we can see from Proposition 2.1, up to a subsequence, $u_n \to u$ strongly in $W^{1,N}_{\rm rad}(\mathbb{R}^N)$ and S_m is compact in $W^{1,N}_{\rm rad}(\mathbb{R}^N)$.

In the following, we give the L^∞-estimate of S_m. By (F1) and $E_m > 0$, $\inf\{\|u\|_\infty : u \in S_m\} > 0$ is obvious. Noting that S_m is compact, to prove $\sup\{\|u\|_\infty : u \in S_m\} < \infty$, it suffices to prove that for any $\{u_n\} \subset S_m$ with $u_n \to u$ strongly in $W^{1,N}_{\rm rad}(\mathbb{R}^N)$, we have $\sup\limits_n \|u_n\|_\infty < \infty$. First, we claim that

$$\sup_n \int_{\mathbb{R}^N} |f(u_n)|^N < \infty. \tag{2.11}$$

By (a), (b), (F1) and the radial lemma in [7], there exists $R > 0$ such that $|f(u_n(x))| \le |u_n(x)|^{N-1}$ for $|x| \ge R$ and any n. Then, by the Sobolev embedding theorem, we get

$$\sup_n \int_{|x|\ge R} |f(u_n)|^N < \infty. \tag{2.12}$$

Let $\alpha > \alpha_N$ be fixed, by (F1)–(F2), there exists $C > 0$ such that $0 < f^N(t) \le C\Psi_N(N^{1-1/N}t)$ for $t > 0$, where Ψ_N is defined in Lemma 1.1. Now, we prove that

$$\lim_{n\to\infty} \int_{|x|\le R} \big(\Psi_N(N^{1-1/N}u_n) - \Psi_N(N^{1-1/N}u)\big)\,dx = 0, \tag{2.13}$$

which immediately implies by Lemma 1.1 that

$$\sup_n \int_{|x|\le R} |f(u_n)|^N\,dx < \infty.$$

Thus, by (2.12), the claim (2.11) holds.

Now, we prove (2.13). Since $u_n \to u$ strongly in $L^q(B(0,R))$ for any $q \ge 1$, it is easy to show that (2.13) is equivalent to

$$\lim_{n\to\infty} \int_{|x|\le R} \big(\exp(N\alpha u_n^{N/(N-1)}) - \exp(N\alpha u^{N/(N-1)})\big)\,dx = 0. \tag{2.14}$$

Due to $u \in L^\infty(\mathbb{R}^N)$ and $u_n \to u$ strongly in $W^{1,N}_{\rm rad}(\mathbb{R}^N)$, there exists $c > 0$ such that

$$\begin{aligned}
&\int_{|x|\le R} \big|\exp(N\alpha u_n^{N/(N-1)}) - \exp(N\alpha u^{N/(N-1)})\big|\,dx \\
&\le c\int_{|x|\le R} \big(\exp(N\alpha|u_n^{N/(N-1)} - u^{N/(N-1)}|) - 1\big)\,dx \\
&\le c\int_{|x|\le R} |u_n^{N/(N-1)} - u^{N/(N-1)}| \exp(N\alpha|u_n^{N/(N-1)} - u^{N/(N-1)}|)\,dx.
\end{aligned}$$

Noting that there exists $C > 0$ such that $|a^{N/(N-1)} - b^{N/(N-1)}| \le b^{N/(N-1)} + C|a-b|^{N/(N-1)}$ for any $a, b \ge 0$ we get

$$
\begin{aligned}
(2.15)\qquad & \int_{|x|\le R} \big|\exp(N\alpha u_n^{N/(N-1)}) - \exp(N\alpha u^{N/(N-1)})\big|\,dx \\
&\le c\int_{|x|\le R} |u_n^{N/(N-1)} - u^{N/(N-1)}|\exp(CN\alpha|u_n - u|^{N/(N-1)})\,dx \\
&\le c\bigg(\int_{|x|\le R} |u_n^{N/(N-1)} - u^{N/(N-1)}|^{N+1}\,dx\bigg)^{1/(N+1)} \\
&\quad \times \bigg(\int_{|x|\le R} \exp(C(N+1)\alpha|u_n - u|^{N/(N-1)})\,dx\bigg)^{N/(N+1)}.
\end{aligned}
$$

Since $\|\nabla(u_n - u)\|_N \to 0$ as $n \to \infty$, it follows from Lemma 1.1 that

$$\sup_n \int_{|x|\le R} \exp(C(N+1)\alpha|u_n - u|^{N/(N-1)})\,dx < \infty.$$

Thus, (2.14) follows from (2.15) and $u_n \to u$ strongly in $L^q(B(0,R))$ for $q \ge 1$. Second, for any $r > 0$, similarly to Step 1 in Proposition 2.1, by Lemma 2.2, there exists $C = C(r)$ (independent of n) such that $\sup_n \|u_n\|_{L^\infty(B(0,r))} \le C$. It follows from the radial lemma [7] that $\sup_n \|u_n\|_\infty < \infty$.

Finally, by a classical comparison principle, there exist $c, C > 0$ such that

$$U(x) + |\nabla U(x)| \le C\exp(-c|x|), \quad x \in \mathbb{R}^N,$$

for any $U \in S_m$. $\square$

3.2. The truncated problem. Since we are concerned with positive solutions to (1.1), from now on, we can assume that $f(t) = 0$ for $t < 0$. By Proposition 3.3, there exists $\kappa > 0$ such that

$$\sup_{U\in S_m} \|U\|_\infty < \kappa. \tag{2.16}$$

For any $k > \max\limits_{t\in[0,\kappa]} f(t)$, define $f_k(t) = \min\{f(t), k\}$, for $t \in \mathbb{R}$. Consider the truncated problem

$$-\Delta_N u + V_\varepsilon(x)|u|^{N-2}u = f_k(u), \quad u \in W_\varepsilon, \tag{2.17}$$

whose corresponding limiting problem is

$$-\Delta_N u + m|u|^{N-2}u = f_k(u), \quad u \in W^{1,N}(\mathbb{R}^N). \tag{2.18}$$

Define

$$L_m^k(u) = \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + m|u|^N)\,dx - \int_{\mathbb{R}^N} F_k(u)\,dx, \quad u \in W^{1,N}(\mathbb{R}^N),$$

where $F_k(s) = \int_0^s f_k(t)\,dt$. Similarly to [41], f_k satisfies (F1)–(F3) in Theorem 1.3 for any $k > \max\limits_{t\in[0,\kappa]} f(t)$. Then it follows from [23] that, for any $k > \max\limits_{t\in[0,\kappa]} f(t)$, (2.18) admits one positive ground state solution. Denote by E_m^k the least energy of (2.18) and by S_m^k the set of positive ground state solutions U to (2.18) with $U(0) = \max_{x\in\mathbb{R}^N} U(x)$. Then $E_m^k \geq E_m$ and $S_m^k \neq \varnothing$. By Proposition 2.1, $S_m^k \subset W_r^{1,N}(\mathbb{R}^N)$ is compact in $W^{1,N}(\mathbb{R}^N)$. Due to $S_m \subset S_m^k$, thus $E_m^k \leq E_m$ and $E_m^k = E_m$ for $k > \max_{t\in[0,\kappa]} f(t)$.

Lemma 3.5. *For $k > \max_{t\in[0,\kappa]} f(t)$, we have $S_m^k = S_m$.*

Proof. Noting that $S_m \subset S_m^k$ for $k > \max_{t\in[0,\kappa]} f(t)$, it suffices to prove $S_m^k \subset S_m$ for $k > \max_{t\in[0,\kappa]} f(t)$. Let

$$G_k(u) = \int_{\mathbb{R}^N} \left(F_k(u) - \frac{m}{N}|u|^N \right) dx,$$

then it is easy to show that

$$E_m^k = \inf\big\{T(u) : G_k(u) = 0,\ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}\big\}. \tag{2.19}$$

For any $u_k \in S_m^k$, u_k is a minimizer of (2.19). By the definition of f_k and $E_m^k = E_m$, u_k satisfies $T(u_k) = E_m$ and $G(u_k) \geq 0$, where

$$G(u) = \int_{\mathbb{R}^N} \left(F(u) - \frac{m}{N}|u|^N \right) dx.$$

Meanwhile, it is easy to show that

$$E_m = \inf\big\{T(u) : G(u) = 0,\ u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}\big\}. \tag{2.20}$$

Now, we claim that $G(u_k) = 0$. Otherwise, if $G(u_k) > 0$, there exists $\theta \in (0,1)$ such that $G(\theta u_k) = 0$. However, $T(\theta u_k) = \theta^N E_m < E_m$, which is a contradiction. Thus, $G(u_k) = 0$, which implies that u_k is a minimizer of (2.20). Therefore, u_k is a ground state solution to (1.3), i.e., $u_k \in S_m$. $\square$

Proof of Theorem 1.4. By Lemma 3.5 we fix $k > \max_{t\in[0,\kappa]} f(t)$ with $S_m^k = S_m$. We consider the following truncated problem:

$$-\varepsilon^N \Delta_N v + V(x)|v|^{N-2}v = f_k(v), \quad v > 0,\ x \in \mathbb{R}^N. \tag{2.21}$$

Since f_k satisfies (F1)–(F3), it follows from Theorem 1.3 that for sufficiently small $\varepsilon > 0$, there exists a positive solution v_ε to (2.21), such that there exist $U \in S_m$ and a maximum point $x_\varepsilon \in \mathbb{R}^N$ of v_ε such that $\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0$ and $v_\varepsilon(\varepsilon\,\cdot\, + x_\varepsilon) \to U(\,\cdot\, + z_0)$ as $\varepsilon \to 0$ in $W^{1,N}(\mathbb{R}^N)$ for some $z_0 \in \mathbb{R}^N$. Let $w_\varepsilon(\,\cdot\,) = v_\varepsilon(\varepsilon\,\cdot\, + x_\varepsilon)$, then w_ε satisfies

$$-\Delta_N w_\varepsilon + V_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right) w_\varepsilon^{N-1} = f_k(w_\varepsilon), \quad w_\varepsilon \in W_\varepsilon.$$

Since $0 \leq f_k(t) \leq k$ for all $t \in \mathbb{R}$, we get that $\sup_\varepsilon \|w_\varepsilon\|_{L^\infty(B(0,2))} < \infty$ by Lemma 2.2. It follows from Lemma 2.3 that $w_\varepsilon(\,\cdot\,) \to U(\,\cdot\, + z_0)$ uniformly in $B_1(0)$. By

Proposition 3.3, $\|v_\varepsilon\|_\infty = w_\varepsilon(0) \le \kappa$ holds uniformly for sufficiently small $\varepsilon > 0$. Then, $f_k(v_\varepsilon(x)) \equiv f(v_\varepsilon(x))$, $x \in \mathbb{R}^N$, for sufficiently small $\varepsilon > 0$. $\square$

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