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EXISTENCE OF MULTIPLE SOLUTIONS FOR A QUASILINEAR ELLIPTIC PROBLEM

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Abstract. In this paper we prove the existence of multiple solutions for a quasilinear elliptic boundary value problem, when the p-derivative at zero and the p-derivative at infinity of the nonlinearity are greater than the first eigenvalue of the p-Laplace operator. Our proof uses bifurcation from infinity and bifurcation from zero to prove the existence of unbounded branches of positive solutions (resp. of negative solutions). We show the existence of multiple solutions and we provide qualitative properties of these solutions.

1. Introduction

In this paper we study the existence of multiple solutions for the quasilinear elliptic boundary value problem

$$\begin{cases} \Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded and smooth domain, $1 < p < 2$, and $f\colon \mathbb{R} \to \mathbb{R}$ is a nonlinear function such that $f(0) = 0$ and

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(f_1) $|f(t)-f(s)| \le C_f|t-s|^{p-1}$, for all $s,t \in \mathbb{R}$,
(f_2) $f_p'(0) := \lim_{t\to 0} f(t)/|t|^{p-2}t > \lambda_1(p)$,
(f_3) $f_p'(\infty) := \lim_{|t|\to\infty} f(t)/|t|^{p-2}t > \lambda_1(p)$,
(f_4) there exists a positive number α such that $f(\alpha) \le 0 \le f(-\alpha)$,

where $C_f := \sup_{s\neq t} |f(s)-f(t)|/|s-t|^{p-1} \in \mathbb{R}$, and $\lambda_1(p)$ denotes the first eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We call $f_p'(0)$ the p-derivative at zero and $f_p'(\infty)$ the p-derivative at infinity. We point out that under hypotheses (f_2) and (f_3), both the$p$-derivatives at zero and at infinity can be arbitrarily greater than the eigenvalue $\lambda_1(p)$ (in particular, each one of them can hit a larger eigenvalue of (1.2)).

We prove that problem (1.1) has at least four nontrivial solutions, two of them are positive and the other two are negative. We also found some upper and lower bounds for the L^∞-norm of these solutions.

Theorem A. *If f satisfies* (f_1)–(f_4) *then problem* (1.1) *has at least four nontrivial solutions u_1, u_2, v_1 and v_2. Moreover, the solutions u_1 and u_2 are positive in Ω, and the solutions v_1 and v_2 are negative in Ω. In addition,*

$$\|u_2\|_{L^\infty} < \alpha < \|u_1\|_{L^\infty} \quad \textit{and} \quad \|v_2\|_{L^\infty} < \alpha < \|v_1\|_{L^\infty}.$$

Remarks 1.1. (a) The argument we present below allows to prove a more general result: if f satisfies (f_1)–(f_3), and

(f_4') there exist numbers $\alpha > 0$ and $\widetilde{\alpha} < 0$ such that $f(\alpha) \le 0 \le f(\widetilde{\alpha})$,

then problem (1.1) has at least four nontrivial solutions u_1, u_2, v_1 and v_2. Moreover, solutions u_1 and u_2 are positive on Ω, and solutions v_1 and v_2 are negative on Ω. In addition,

$$\|u_2\|_{L^\infty} < \alpha < \|u_1\|_{L^\infty} \quad \text{and} \quad \|v_2\|_{L^\infty} < |\widetilde{\alpha}| < \|v_1\|_{L^\infty}.$$

For the sake of simplicity, from now on we assume hypothesis (f_4) instead of (f_4') (i.e. $\widetilde{\alpha} = -\alpha$).

(b) We provide an example of a family of functions satisfying hypotheses (f_1)–(f_4'). Consider the following parameters: for $i = 1,2$ let us fix $M_i > 1$, $0 < a_i < \alpha_i < b_i$ and a couple of differentiable functions $h_i\colon [a_i,b_i] \to \mathbb{R}$ such that $h_i(\alpha_i) < 0$, $h_i(a_i) = M_i\lambda_1(p)a_i^{p-1}$, $h_i(b_i) = M_i\lambda_1(p)b_i^{p-1}$, and h_i' is bounded in (a_i,b_i). Let us define the continuous functions

$$g_i(t) = \begin{cases} M_i\lambda_1(p)t^{p-1} & \text{if } 0 \le t \le a_i \ \vee \ t \ge b_i,\\ h_i(t) & \text{if } a_i \le t \le b_i. \end{cases}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(t) = \begin{cases} g_1(t) & \text{if } t \geq 0, \\ -g_2(-t) & \text{if } t \leq 0. \end{cases}$$

We see that $f(0) = 0$ and it is not difficult to check that (f_2)–(f_4') hold (in particular, taking $\alpha_1 = \alpha_2$ condition (f_4) also holds). In order to verify that f satisfies condition (f_1), one can proceed as follows: consider the function $\Psi(t) = (1-t)^{p-1} + t^{p-1} - 1$, for $t \in [0,1]$, which has its maximum at the unique critical point $t = 1/2$. From this it follows that $\Psi(t) \geq \min\{\Psi(0), \Psi(1)\} = 0$. By using homogeneity arguments, one can establish that

$$|s^{p-1} - t^{p-1}| \leq |s-t|^{p-1}, \quad \text{for all } s,t \geq 0.$$

Using this fact and considering different cases for s and t, one can finally verify condition (f_1) holds true.

(c) Observe that if $p > 2$ and f satisfies condition (f_1), then $f' \equiv 0$ and f is a constant function. Hence f cannot satisfy conditions (f_2)–(f_4). Hypothesis (f_4), on the other hand, is needed to prove that branches of solutions to problem (1.3) (see below) cannot cross the asymptote $\|u\|_{L^\infty} = \alpha$ (see Lemma 3.1 in Section 3 below). This is why we assumed $1 < p < 2$ from the very beginning.

Our proof of Theorem A uses bifurcation from infinity and bifurcation from zero, applied to the problem

$$\begin{cases} \Delta_p u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda > 0$.

Theorem A is an extension to quasilinear equations of a result due to J. Cossio, S. Herrón and C. Vélez (see [5]) for the semilinear case. A key ingredient to extend the semilinear result to our situation is to prove that for problem (1.3) there exist unbounded branches of positive solutions (resp. of negative solutions) emanating from the bifurcation points $(\infty, \lambda_1/f_p'(\infty))$ and $(0, \lambda_1/f_p'(\infty))$ (see Theorem 4.3 and Theorem 4.8 in Section 4 below). Theorem 4.3 is very much inspired by a corresponding result in the semilinear case due to Ambrosetti and Hess (see [2] and [3, Section 4.4]), and by [1, Theorem 4.1]. Although our proof of Theorem 4.3 follows the ideas from [1], [2] and [3] our arguments have several differences with respect to these references, as will be better explained in Section 4. Theorem 4.8, on the other hand, essentially comes from the ideas by Del Pino and Manásevich in [9].

The existence of solutions to quasilinear elliptic problems like (1.3) has been widely investigated. Let us mention, besides [1] and [9], the papers [12], [11] and [8], the books [14] and [13], and the references therein. A. Ambrosetti et al.

in [1] showed the existence of an unbounded branch of positive solutions of problem (1.3) emanating from either zero or infinity when $f(u) \backsimeq u^{p-1}$ near 0 or near infinity; more precisely, when $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and there exist $m > 0$, $m_\infty > 0$, and $\varepsilon > 0$ such that

$$\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = m \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t) - m_\infty t^{p-1}}{t^{p-1-\varepsilon}} = 0.$$

They used a priori estimates and topological arguments. In [9], M. Del Pino and R. Manásevich proved that problem (1.1) has at least one nontrivial solution when

$$f_p^{'}(0) < \lambda_1(p) < f_p^{'}(\infty), \tag{1.4}$$

for $1 < p < \infty$. P. Drábek in [11], for $p > 2$, and S. Fučik et al. in [14], for $p > 1$, focus on the existence of solutions to problem (1.3) in the case when $f_p'(\infty)$ is not equal to an eigenvalue of $-\Delta_p$. By using topological arguments based on degree theory, they found conditions that allow to show that problem (1.3) has at least one solution for λ either below $\lambda_1(p)$ or between $\lambda_1(p)$ and $\lambda_2(p)$. In [12], Drabek et al. study a non-homogeneous version of problem (1.2) when parameter λ is near $\lambda_1(p)$. More recently, Del Pezzo and Quaas in [8] generalize the results from [9] to nonlocal problems involving fractional p-Laplacian operators. Contrary to conditions in [9], [12], [11], [14], and [8], here the p-derivative at zero and the p-derivative at infinity are both arbitrarily greater than the first eigenvalue of the p-Laplace operator.

Regarding quasilinear equations in the radially symmetric case, there has been a lot of research. We mention some works and refer the reader to references therein. For instance, J. Cossio and S. Herrón in [4] studied problem (1.1) when Ω is the unit ball in $\mathbb{R}^N$ and the p-derivative of the nonlinearity at zero is greater than $\mu_j(p)$, the j-radial eigenvalue of the p-Laplace operator, and the p-derivative at infinity is equal to the p-derivative at zero. When $p \geq 2$, they showed that problem (1.1) has $4j - 1$ radially symmetric solutions. In such a reference, the authors used bifurcation theory and the fact that in the radially symmetric case (1.1) reduces to an ordinary differential equation. J. Cossio, S. Herrón, and C. Vélez in [6] studied problem (1.1) in the radially symmetric case, when Ω is the unit ball in $\mathbb{R}^N$ and the problem is p-superlinear at the origin with $p > N \geq 2$. They proved that problem (1.1) has infinitely many solutions. The main tool that they used is the shooting method. M. Del Pino and R. Manásevich in [9] studied the existence of multiple nontrivial solutions for a quasilinear boundary value problem under radial symmetry; they extended the Global Bifurcation Theorem of P. Rabinowitz (see [21]) and proved the existence of nontrivial solutions for that kind of problems. In [15], García-Melián and Sabina de Lis study uniqueness for quasilinear problems in radially symmetric domains.

The paper is organized as follows: In Section 2 we recall several known and important results. Then, in Section 3 we establish some lemmas which will be used to prove Theorem A. We apply a nonlinear version of the strong maximum principle due to J.L. Vázquez (see [23]) to prove that if u is a weak solution to problem (1.3) (see the definition below for the precise meaning), then $\|u\|_{L^\infty} \neq \alpha$. We also apply an interpolation theorem due to A. Lê (see Theorem 2.3 below) to show that the function $(u,\lambda) \mapsto \|u\|_{L^\infty}$ is continuous, where (u,λ) is a solution of (1.3) (see Lemma 3.3 below for the precise statement). In Section 4 we prove Theorem A.

2. Preliminary results

In this section we summarize some important results which we will use to prove our theorem. From now on we will denote by $\|\cdot\|_{L^q}$ the norm in $L^q(\Omega)$, for $q \in [1,\infty]$, and we will denote by $\|\cdot\|_{W_0^{1,p}}$ the norm in the space $W_0^{1,p}(\Omega)$ given by $\|u\|_{W_0^{1,p}} = \|\nabla u\|_{L^p}$. Also, $\|\cdot\|_{C^{1,\gamma}}$ stands for the norm in the Hölder space $C^{1,\gamma}(\overline{\Omega})$, for $\gamma \in (0,1)$, and $\|\cdot\|_{C^1}$ stands for the norm in $C^1(\overline{\Omega})$.

Let us recall the definition of weak solution to problem (1.3). Given $\lambda > 0$, we say a function $u \in W_0^{1,p}(\Omega)$ solves (1.3) in the weak sense provided that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega \lambda f(u) v \, dx, \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{2.1}$$

We recall the following general regularity result, which is going to imply that a weak solution of (1.3) is essentially bounded (we took this result from [16], more precisely [16, Theorem 6.2.6], and adapt it to our context).

Theorem 2.1. *Let $\lambda > 0$ and $u \in W_0^{1,p}(\Omega)$ be a weak solution to* (1.3). *Assume that $\lambda f(u) \in L^1_{\mathrm{loc}}(\Omega)$, and there exist $\sigma \in [1, Np/(N-p))$, $r \in [1, N/(N-p))$, $c > 0$ and $a \in L^{r'}(\Omega)$, where $1/r + 1/r' = 1$, such that $a(x) \geq 0$, for almost every $x \in \Omega$, and $\lambda u(x) f(u(x)) \leq c|u(x)|^\sigma + a(x)|u(x)|$ for almost every $x \in \Omega$. Then $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty} \leq \eta$, where the constant $\eta > 0$ depends on N, p, σ, r, $\|a\|_{r'}$, c and $\|u\|_{L^{Np/(N-p)}}$.*

In our case, given $\lambda > 0$, if u is a weak solution of (1.3), hypothesis (f_1) implies that

$$|\lambda f(u)| \leq \lambda C_f |u|^{p-1} \in L^1_{\mathrm{loc}}(\Omega) \quad \text{and} \quad \lambda\, u\, f(u) \leq \lambda C_f |u|^p. \tag{2.2}$$

Hence, from Theorem 2.1, it follows that $u \in L^\infty(\Omega)$.

As pointed out by A. Lê in [17], from the results by E. DiBenedetto in [10] and by G. Lieberman in [18], it follows that there exists $\gamma \in (0,1)$ such that the inverse of the p-Laplace operator

$$L := (-\Delta_p)^{-1} \colon L^\infty(\Omega) \to C^{1,\gamma}(\overline{\Omega})$$

is a well-defined, continuous and compact mapping. Given $\lambda > 0$, if u is a weak solution of (1.3), condition (f_1) and Theorem 2.1 imply that $u \in L^\infty(\Omega)$ (as verified above), and then condition (f_1) implies $\lambda f(u) \in L^\infty(\Omega)$. Thus, $u = L(\lambda f(u)) \in C^{1,\gamma}(\overline{\Omega})$. From now on we will use this fact throughout the paper.

The following nonlinear version of the maximum principle, due to J.L. Vázquez (see [23, Theorem 5]) will be useful to prove Lemma 3.1 below. The hypothesis $\Delta_p u \in L^2_{\rm loc}(\Omega)$ in the following statement is understood in the sense of distributions.

Theorem 2.2. *Let $u \in C^1(\Omega)$ be such that $\Delta_p u \in L^2_{loc}(\Omega)$, $u \geq 0$ almost everywhere in Ω, $\Delta_p u \leq \xi(u)$ almost everywhere in Ω with $\xi\colon [0,\infty) \to \mathbb{R}$ continuous, nondecreasing, $\xi(0) = 0$ and either $\xi(s) = 0$ for some $s > 0$ or $\xi(s) > 0$ for all $s > 0$ but $\int_0^1 (s\,\xi(s))^{-1/p}\,ds = \infty$. Then if u does not vanish identically on Ω it is positive everywhere in Ω. Moreover, if $u \in C^1(\Omega \cup \{x_0\})$ for an $x_0 \in \partial\Omega$ and $u(x_0) = 0$, then $\partial u/\partial\overrightarrow{\nu}(x_0) > 0$, where $\overrightarrow{\nu}$ is an interior normal at x_0.*

In order to prove a continuity result (more precisely, Lemma 3.3 below), we will make use of the following interpolation inequality due to A. Lê (see [17, Corollary 1.3]).

Theorem 2.3. *There exist constants $c > 0$ and $0 < \theta < 1$ such that, for any $u \in C^{1,\gamma}(\overline{\Omega}) \cap W^{1,p}(\Omega)$, $\|u\|_{C^1} \leq c\,\|u\|^{1-\theta}_{C^{1,\gamma}}\,\|u\|^{\theta}_{W^{1,p}}$.*

3. Lemmas

We now establish some auxiliary results needed to prove our theorem in the next section. From now on we assume f satisfies conditions (f_1) to (f_4).

Lemma 3.1. *Assume that $\lambda > 0$ and u is a weak solution of the problem* (1.3). *Then $\|u\|_{L^\infty} \neq \alpha$.*

Proof. From Section 2, we already know that $u \in C^1(\overline{\Omega})$. We argue by contradiction: assume $\|u\|_{L^\infty} = \alpha$. Since $\lambda|f(u)| \leq \lambda C_f \|u\|^{p-1}_{L^\infty}$, it follows that $-\Delta_p u = \lambda f(u) \in L^2_{\rm loc}(\Omega)$.

We consider the function $\alpha - u \in C^1(\overline{\Omega})$, $\alpha - u \geq 0$ in Ω.

$$\Delta_p(\alpha - u) = {\rm div}(|\nabla(\alpha - u)|^{p-2}\nabla(\alpha - u)) = -{\rm div}(|\nabla u|^{p-2}\nabla u) \in L^2_{\rm loc}(\Omega).$$

Since $f(\alpha) \leq 0$, from (f_1) we see that

$$\begin{aligned} \Delta_p(\alpha - u) &= \lambda f(u) = \lambda f(\alpha - (\alpha - u)) \leq \lambda f(\alpha - (\alpha - u)) - \lambda f(\alpha) \\ &\leq \lambda |f(\alpha - (\alpha - u)) - f(\alpha)| \leq \lambda C_f |\alpha - u|^{p-1}. \end{aligned} \tag{3.1}$$

We now apply Theorem 2.2 to get the conclusion. Let us define $\xi\colon \mathbb{R}_+ \to \mathbb{R}$ by $\xi(s) = \lambda C_f s^{p-1}$. We see that ξ is continuous, increasing function, such that

$\xi(0) = 0$ and

$$\int_0^1 \frac{1}{(s\ \xi(s))^{1/p}}\,ds = c\int_0^1 \frac{1}{(s\ s^{p-1})^{1/p}}\,ds = c\ln s]_0^1 = +\infty.$$

Hence, Theorem 2.2 implies $\alpha - u > 0$ in Ω, i.e. $u < \alpha$ in Ω. Thus $\|u\|_{L^\infty} < \alpha$, which contradicts our initial assumption. $\square$

In the proof of Theorem A inequalities (3.2) of the following lemma will play an important role. These inequalities essentially come from the arguments leading to regularity results due to [10], [22] and [18].

Lemma 3.2. *For every $\lambda > 0$ there exist positive constants K_1 and K_2 depending on $|\Omega|$, N, C_f, p and λ, such that, if $u \in W_0^{1,p}(\Omega)$ is a weak solution of* (1.3)*, then*

$$\|u\|_{W_0^{1,p}} \le K_1\ \|u\|_{L^\infty}, \tag{3.2}$$

$$\|u\|_{L^\infty} \le K_2\ \|u\|_{W_0^{1,p}}. \tag{3.3}$$

Moreover, K_1 and K_2 are bounded if λ is bounded.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (1.3). Using the definition of weak solution and hipothesis (f_1) it follows that

$$\|u\|^p_{W_0^{1,p}} = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u\,dx = \int_\Omega \lambda u\,f(u)\,dx \le |\Omega|\lambda C_f\|u\|^p_{L^\infty}. \tag{3.4}$$

Defining $K_1 := (|\Omega|\lambda C_f)^{1/p}$, inequality (3.2) follows from (3.4).

Using (2.2) and a *boot-strap* argument (see, for instance, the proof of Theorem 6.2.6 in [16]) we get that there exists a positive constant $K := K(|\Omega|, N, C_f, p, \lambda)$, which is bounded when λ is bounded, such that

$$\|u\|_{L^\infty} \le K\,\|u\|_{L^{p_0}}, \tag{3.5}$$

where $p_0 = Np/(N-p)$ is the critical Sobolev exponent. Since $W_0^{1,p}(\Omega)$ is continuously embedded in $L^{p_0}(\Omega)$, we see that

$$\|u\|_{L^{p_0}} \le c_0\|u\|_{W_0^{1,p}}, \tag{3.6}$$

for a constant $c_0 > 0$. From (3.5) and (3.6) we get a constant $K_2 > 0$ satisfying inequality (3.3). The proof of Lemma 3.2 is complete. $\square$

Let us define

$$S = \{(u,\lambda) \in W_0^{1,p}(\Omega)\times\mathbb{R} : u \ne 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f(u))\}. \tag{3.7}$$

We will make use of the next lemma in the proof of Theorem A.

Lemma 3.3. *The function $\mathcal{N}_\infty\colon \overline{S} \subset W_0^{1,p}(\Omega)\times\mathbb{R} \to \mathbb{R}$ defined as $(u,\lambda) \mapsto \|u\|_{L^\infty}$ is continuous.*

Proof. We commence by observing that if $(u,\lambda)\in W_0^{1,p}(\Omega)\times\mathbb{R}$ is a limit point of S then $u=(-\Delta_p)^{-1}(\lambda f(u))$, and so $\|u\|_{L^\infty}$ is well-defined on all $\overline{S}$. Let us take $(u,\lambda_u),(v_n,\lambda_{v_n})\in\overline{S}$ such that $(v_n,\lambda_{v_n})\to(u,\lambda_u)$. Let us try to estimate $|\mathcal{N}_\infty(v_n,\lambda_{v_n})-\mathcal{N}_\infty(u,\lambda_u)|$. Observe that

$$\begin{aligned}\|v_n-u\|_{L^\infty}&=\|L(\lambda_{v_n}f(v_n))-L(\lambda_u f(u))\|_{L^\infty}\\&=\|\lambda_{v_n}^{1/(p-1)}L(f(v_n))-\lambda_u^{1/(p-1)}L(f(u))\|_{L^\infty}\\&\le\lambda_{v_n}^{1/(p-1)}\|L(f(v_n))-L(f(u))\|_{L^\infty}\\&\quad+|\lambda_{v_n}^{1/(p-1)}-\lambda_u^{1/(p-1)}|\,\|L(f(u))\|_{L^\infty}.\end{aligned}\tag{3.8}$$

Let us define

$$u^*=L(f(u))\quad\text{and}\quad {v_n}^*=L(f(v_n)).\tag{3.9}$$

In order to estimate $\|L(f(v_n))-L(f(u))\|_{L^\infty}$ we make use of Theorem 2.3. Indeed, since $u^*,v_n^*\in C^{1,\gamma}(\overline{\Omega})\cap W_0^{1,p}(\Omega)$, by using Theorem 2.3 and Poincaré's inequality we see that

$$\begin{aligned}\|L(f(v_n))-L(f(u))\|_{L^\infty}&\le\|L(f(v_n))-L(f(u))\|_{C^1}\\&=\|{v_n}^*-u^*\|_{C^1}\le c\|{v_n}^*-u^*\|_{C^{1,\gamma}}^{1-\theta}\|{v_n}^*-u^*\|_{W_0^{1,p}}^{\theta},\end{aligned}\tag{3.10}$$

with $0<\theta<1$. We claim that there exists $C>0$ such that

$$\|{v_n}^*-u^*\|_{C^{1,\gamma}}^{1-\theta}\le C.\tag{3.11}$$

To prove (3.11) we first show that there exists $M_1>0$ such that $u,v_n\in B_{M_1}^\infty$, the ball with radius M_1 centered at the origin in $L^\infty(\Omega)$. Since $v_n\to u$ in $W_0^{1,p}(\Omega)$, $\|v_n\|_{W_0^{1,p}}$, $\|v_n\|_{L^{p_0}}$, $\|u\|_{W_0^{1,p}}$, and $\|u\|_{L^{p_0}}$ are bounded by a constant. From Lemma 3.2 we have

$$\|u\|_{L^\infty}\le K\,\|u\|_{W_0^{1,p}}\quad\text{and}\quad\|v_n\|_{L^\infty}\le K\,\|v_n\|_{W_0^{1,p}},\tag{3.12}$$

where K denotes a positive constant. Thus, there exists $M_1>0$ such that

$$u,v_n\in B_{M_1}^\infty.\tag{3.13}$$

Combining (3.13) with the inequalities

$$\|f(u)\|_{L^\infty}\le C_f\|u\|_{L^\infty}^{p-1}\quad\text{and}\quad\|f(v_n)\|_{L^\infty}\le C_f\|v_n\|_{L^\infty}^{p-1},\tag{3.14}$$

we see that there exists $M_2>0$ such that

$$\|f(u)\|_{L^\infty}\le M_2\quad\text{and}\quad\|f(v_n)\|_{L^\infty}\le M_2.\tag{3.15}$$

As we mentioned above, in Section 2, from the regularity results the inverse of the p-Laplace operator

$$L:=(-\Delta_p)^{-1}\colon L^\infty(\Omega)\to C^{1,\gamma}(\overline{\Omega})\tag{3.16}$$

is a continuous and compact mapping. An immediate consequence of (3.15) and (3.16) is that there exists $M > 0$ such that

$$\|u^*\|_{C^{1,\gamma}} \leq M \quad \text{and} \quad \|{v_n}^*\|_{C^{1,\gamma}} \leq M. \tag{3.17}$$

Now (3.17) implies that there exists $C > 0$ such that

$$\|{v_n}^* - u^*\|_{C^{1,\gamma}}^{1-\theta} \leq C, \tag{3.18}$$

which proves (3.11). From (3.10), (3.11), and (3.18) we see that

$$\|L(f(v_n)) - L(f(u))\|_{L^\infty} \leq C\ \|{v_n}^* - u^*\|_{W_0^{1,p}}^{\theta}. \tag{3.19}$$

Since

$${v_n}^* = \frac{v_n}{{\lambda_{v_n}}^{1/(p-1)}} \quad \text{and} \quad u^* = \frac{u}{\lambda_u^{1/(p-1)}} \tag{3.20}$$

and $s \mapsto s^\theta$ is an increasing function, it follows that

$$\begin{aligned} \|L(f(v_n)) - L(f(u))\|_{L^\infty} &\leq C\ \|v_n {\lambda_{v_n}}^{-1/(p-1)} - u\lambda_u^{-1/(p-1)}\|_{W_0^{1,p}}^{\theta} \\ &= \frac{C}{\lambda_u^{\theta/(p-1)}\lambda_{v_n}^{\theta/(p-1)}} \|v_n\lambda_u^{1/(p-1)} - u\lambda_{v_n}^{1/(p-1)}\|_{W_0^{1,p}}^{\theta} \\ &= \frac{C}{\lambda_u^{\theta/(p-1)}\lambda_{v_n}^{\theta/(p-1)}} \|v_n(\lambda_u^{1/(p-1)} - \lambda_{v_n}^{1/(p-1)}) + \lambda_{v_n}^{1/(p-1)}(v_n - u)\|_{W_0^{1,p}}^{\theta} \\ &\leq \frac{C}{\lambda_u^{\theta/(p-1)}{\lambda_{v_n}}^{\theta/(p-1)}} \big(\|v_n\|_{W_0^{1,p}}|\lambda_u^{1/(p-1)} - \lambda_{v_n}^{1/(p-1)}| \\ &\quad + |\lambda_{v_n}|^{1/(p-1)}\|v_n - u\|_{W_0^{1,p}}\big)^{\theta}. \end{aligned} \tag{3.21}$$

Because the sequences $\{\|v_n\|_{W_0^{1,p}}\}$ and $\{\lambda_{v_n}\}$ are bounded, there exists C_1 such that

$$\|L(f(v_n)) - L(f(u))\|_{L^\infty} \leq C_1\big(|\lambda_u^{1/(p-1)} - \lambda_{v_n}^{1/(p-1)}| + \|v_n - u\|_{W_0^{1,p}}\big)^{\theta}. \tag{3.22}$$

From (3.8), (3.22), $\lambda_{v_n} \to \lambda_u$, and $v_n \to u$ in $W_0^{1,p}(\Omega)$ it follows that

$$|\mathcal{N}_\infty(v_n, \lambda_{v_n}) - \mathcal{N}_\infty(u, \lambda_u)| \to 0, \tag{3.23}$$

which proves the lemma. □

4. Proof Theorem A

Let f be a function satisfying the hypotheses (f_1)–(f_4). Because of the regularity theory recalled in Section 2 above, the problem of finding solutions $u \in C^{1,\gamma}(\overline{\Omega})$ to (1.3) is equivalent to find elements $u \in W_0^{1,p}(\Omega)$ such that

$$u = (-\Delta_p)^{-1}(\lambda f(u)). \tag{4.1}$$

We will prove that there are nontrivial solutions of (4.1) when $\lambda = 1$, i.e. four nontrivial solutions of (1.1).

Let $f^+\colon \mathbb{R}\to\mathbb{R}$ be defined as $f^+(t)=f(t)$ for $t\ge 0$, and $f^+(t)=0$ for $t<0$. Similarly, let $f^-\colon \mathbb{R}\to\mathbb{R}$ be defined as $f^-(t)=f(t)$ for $t\le 0$, and $f^-(t)=0$ for $t>0$. We observe that f can be written as

$$f(t)=f_p'(\infty)|t|^{p-2}t+g(t),$$

where $g(t)/|t|^{p-2}t\to 0$ as $|t|\to\infty$, and also

$$f(t)=f_p'(0)|t|^{p-2}t+\widehat{g}(t),$$

where $\widehat{g}(t)/|t|^{p-2}t\to 0$ as $t\to 0$. We have the following lemma.

Lemma 4.1. *For every $\lambda>0$ and $\tau\ge 0$ if $u\in W_0^{1,p}(\Omega)\setminus\{0\}$ satisfies*

$$u=(-\Delta_p)^{-1}(\lambda f^+(u)+\tau),$$

then $u\in C^{1,\gamma}(\overline{\Omega})$, $u>0$ on Ω and $\partial u/\partial\overrightarrow{n}<0$ (where $\overrightarrow{n}$ denotes the outer unit normal on $\partial\Omega$).

Remark 4.2. Taking $\tau=0$ in Lemma 4.1, we observe that, if $u\in W_0^{1,p}(\Omega)$ is a solution of

$$u=(-\Delta_p)^{-1}(\lambda f^+(u)) \tag{4.2}$$

and $\lambda>0$, then $u>0$ on Ω. Thus u satisfies (4.1), i.e. $(u,\lambda)\in S$. In a similar way, if $u\in W_0^{1,p}(\Omega)$ is a nontrivial solution of

$$u=(-\Delta_p)^{-1}(\lambda f^-(u)) \tag{4.3}$$

and $\lambda>0$, then $u<0$ on Ω. Thus u satisfies (4.1), i.e. $(u,\lambda)\in S$.

Proof. Using Theorem 2.1, hypothesis (f_1), and the well-definition of operator $L:=(-\Delta_p)^{-1}\colon L^\infty(\Omega)\to C^{1,\gamma}(\overline{\Omega})$, one can show $u\in C^{1,\gamma}(\overline{\Omega})$ (as we proved in Section 2 for the weak solutions of (1.3)). We claim $u\ge 0$ on Ω: if $u<0$ on a subdomain $D\subset\Omega$, because of the definition of f^+, $u=(-\Delta_p)^{-1}(\tau)$ on D. Since the p-Laplacian operator satisfies the maximum principle (see e.g. [16, Section 6.4]), $u\ge 0$ on D. This contradicts our assumption, and so $u\ge 0$ on Ω.

In order to conclude the proof, we apply again Theorem 2.2: let $\xi\colon[0,\infty)\to\mathbb{R}$ be defined as $\xi(s)=\lambda C_f s^{p-1}$. We see that ξ is a continuous, increasing function, such that $\xi(0)=0$ and

$$\int_0^1\frac{1}{(s\,\xi(s))^{1/p}}\,ds=+\infty.$$

Moreover, $\Delta_p u\le\xi(u)$ on Ω. The result follows from Theorem 2.2. $\square$

4.1. Bifurcation from infinity. We define

$$S^+ = \{(u,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : u \neq 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f^+(u))\},$$
$$S^- = \{(u,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : u \neq 0 \text{ and } u = (-\Delta_p)^{-1}(\lambda f^-(u))\}.$$

As we mentioned above, we use bifurcation theory (see [19], [20], [21] and [3]) to prove Theorem A. Let us recall that, in our framework, $(0,\lambda^*)$ is a *bifurcation point from zero* for equation $u = (-\Delta_p)^{-1}(\lambda f^+(u))$ if $(0,\lambda^*) \in \overline{S^+}$ or, equivalently, if there exists a sequence $\{(u_n,\lambda_n)\}_n$ in S^+ which converges to $(0,\lambda^*)$. Also, (∞,λ^*) or simply λ^* is a *bifurcation point from infinity* for equation $u = (-\Delta_p)^{-1}(\lambda f^+(u))$ if there exists a sequence $\{(u_n,\lambda_n)\}_n$ in S^+ such that $\lambda_n \to \lambda^*$ and $\|u_n\|_{W_0^{1,p}} \to \infty$ as $n \to \infty$. Similar definitions apply for equation $u = (-\Delta_p)^{-1}(\lambda f^-(u))$.

First we present an argument using bifurcation from infinity to show the existence of two one-sign solutions of (1.1). Secondly, we use bifurcation from zero to show the existence of two additional one-sign solutions. At the end of this section we include a bifurcation diagram which summarizes the arguments presented below.

Let us define $\Psi_+ \colon W_0^{1,p}(\Omega) \times \mathbb{R} \to W_0^{1,p}(\Omega)$ by

$$\Psi_+(z,\lambda) = \begin{cases} z - \|z\|^2_{W_0^{1,p}} (-\Delta_p)^{-1}\left[\lambda f^+\left(\dfrac{z}{\|z\|^2_{W_0^{1,p}}}\right)\right] & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

and Ψ_- in the same way, changing f^+ by f^-. Let us denote $i(\Psi_+(\,\cdot\,,\lambda),0)$ the index of $\Psi_+(\,\cdot\,,\lambda)$ with respect to zero. The following result will be used to prove the existence of two one-sign solutions for problem (1.1).

THEOREM 4.3. *The following assertions hold true.*

(a) $(0,\lambda_1/f_p'(\infty))$ *is the unique bifurcation point from zero for equation* $\Psi_+(z,\lambda) = 0$. *Moreover, there exists an unbounded connected component* Γ^+_∞ *of*

$$\Gamma^+ = \{(z,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : z \neq 0 \text{ and } \Psi_+(z,\lambda) = 0\},$$

emanating from the trivial solution of $\Psi_+(z,\lambda) = 0$ *at* $(0,\lambda_1/f_p'(\infty))$. *Analogously,* $(0,\lambda_1/f_p'(\infty))$ *is the unique bifurcation point from zero for equation* $\Psi_-(z,\lambda) = 0$. *Moreover, there exists an unbounded connected component* Γ^-_∞ *of*

$$\Gamma^- = \{(z,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : z \neq 0 \text{ and } \Psi_-(z,\lambda) = 0\},$$

emanating from the trivial solution of $\Psi_-(z,\lambda) = 0$ *at* $(0,\lambda_1/f_p'(\infty))$.

(b) $(\infty,\lambda_1/f_p'(\infty))$ *is the unique bifurcation point from infinity for* (4.2). *Moreover, there exists an unbounded connected component* Σ^+_∞ *of* S^+

bifurcating from $(\infty, \lambda_1/f_p'(\infty))$. *Analogously, the point* $(\infty, \lambda_1/f_p'(\infty))$ *is the unique bifurcation point from infinity for equation* (4.3). *Moreover, there exists an unbounded connected component* Σ_∞^- *of* S^- *bifurcating from* $(\infty, \lambda_1/f_p'(\infty))$.

Remark 4.4. As we mentioned in the introduction above, Theorem 4.3 is inspired by a corresponding result in the semilinear case due to Ambrosetti and Hess (see [2] and [3, Section 4.4]), and by Theorem 4.1 in [1] (see also [9]). The proof we present below closely follows the ideas from [2], [3] and [1], but our arguments have several differences with respect to these references. First, as expected, a lot of technicalities arise when trying to adapt the Δ-approach from [2] and [3] to the Δ_p nonlinear operator. Second, our hypotheses on f slightly differ from those in Theorem 4.1 of [1] (ours are a little less restrictive near infinity) and, in the proof presented in [1], several details are omitted. And third, our choice of functional spaces is different from both references. For the sake of completeness we include full details here.

In order to prove Theorem 4.3 we need the following lemmas.

Lemma 4.5. *Let* $J \subset \mathbb{R}^+$ *be a compact interval such that* $\lambda_\infty := \lambda_1/f_p'(\infty) \notin J$. *Then*:

(a) *There exists* $r > 0$ *such that* $u \neq (-\Delta_p)^{-1}(\lambda f^+(u))$ *for every* $\lambda \in J$ *and every* $u \in W_0^{1,p}(\Omega)$ *with* $\|u\|_{W_0^{1,p}} \geq r$.
(b) $i(\Psi_+(\,\cdot\,,\lambda), 0) = 1$ *for every* $\lambda < \lambda_\infty$.
(c) $(\infty, \lambda_1/f_p'(\infty))$ *is the only possible bifurcation point from infinity for equation* (4.2).

Proof. In order to prove a) we argue by contradiction. Assume there exist a sequence $\{\lambda_n\}_n \subset J$ and a sequence $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ such that $\|u_n\|_{W_0^{1,p}} \to +\infty$ and

$$u_n = (-\Delta_p)^{-1}(\lambda_n f^+(u_n)) \quad \text{for every } n \in \mathbb{N}. \tag{4.4}$$

Because of Lemma 4.1, $u_n > 0$ for every n. Dividing (4.4) by $\|u_n\|_{W_0^{1,p}}$ we get

$$\frac{u_n}{\|u_n\|_{W_0^{1,p}}} = (-\Delta_p)^{-1}\bigg(\frac{\lambda_n f_p'(\infty)u_n^{p-1} + \lambda_n g(u_n)}{\|u_n\|_{W_0^{1,p}}^{p-1}}\bigg) \quad \text{for every } n \in \mathbb{N}, \tag{4.5}$$

where $g(t)/|t|^{p-2}t \to 0$ as $t \to \infty$. What follows is a standard compactness argument. Indeed, since $\{u_n/\|u_n\|_{W_0^{1,p}}\}_n$ is a bounded sequence in $W_0^{1,p}(\Omega)$, there exists a subsequence, for which we keep the same notation, $\overline{v} \in W_0^{1,p}(\Omega)$

and $h \in L^p(\Omega)$ such that

$$
(4.6) \qquad \begin{cases} \dfrac{u_n}{\|u_n\|_{W_0^{1,p}}} \rightharpoonup \overline{v} & \text{weakly in } W_0^{1,p}(\Omega), \\ \dfrac{u_n}{\|u_n\|_{W_0^{1,p}}} \to \overline{v} & \text{strongly in } L^p(\Omega), \\ \dfrac{u_n(x)}{\|u_n\|_{W_0^{1,p}}} \to \overline{v}(x) & \text{a.e. } x \in \Omega, \\ \dfrac{u_n(x)}{\|u_n\|_{W_0^{1,p}}} \le h(x) & \text{a.e. } x \in \Omega. \end{cases}
$$

Now, let us verify $u_n^{p-1}/\|u_n\|^{p-1}_{W_0^{1,p}} \rightharpoonup \overline{v}^{\,p-1}$ and $g(u_n)/\|u_n\|^{p-1}_{W_0^{1,p}} \rightharpoonup 0$ weakly in $L^{p'}(\Omega)$, where $1/p + 1/p' = 1$. Let $\omega \in L^p(\Omega)$. Then, from (4.6),

$$
\frac{u_n^{p-1}(x)\omega(x)}{\|u_n\|^{p-1}_{W_0^{1,p}}} \to \overline{v}^{\,p-1}(x)\omega(x) \quad \text{a.e. } x \in \Omega \quad \text{and} \quad \frac{u_n^{p-1}\omega}{\|u_n\|^{p-1}_{W_0^{1,p}}} \le |h|^{p-1}\omega.
$$

Since $h \in L^p(\Omega)$, $|h|^{p-1} \in L^{p'}(\Omega)$. Hence, dominated convergence theorem implies that

$$
\int_\Omega \frac{u_n^{p-1}\omega}{\|u_n\|^{p-1}_{W_0^{1,p}}}\, dx \to \int_\Omega \overline{v}^{\,p-1}\omega\, dx \quad \text{as } n \to \infty.
$$

Since this holds true for every $\omega \in L^p(\Omega)$, Riesz representation theorem guarantees that $u_n^{p-1}/\|u_n\|^{p-1}_{W_0^{1,p}} \rightharpoonup \overline{v}^{p-1}$ weakly in $L^{p'}(\Omega)$. In order to verify that $g(u_n)/\|u_n\|^{p-1}_{W_0^{1,p}} \rightharpoonup 0$ weakly in $L^{p'}(\Omega)$, we take $\varepsilon > 0$ arbitrary and then, since $g(t)/|t|^{p-2}t \to 0$ as $t \to \infty$, there exists $M_\varepsilon > 0$ such that

$$
(4.7) \qquad t > M_\varepsilon \implies |g(t)| < \varepsilon t^{p-1}.
$$

Given $n \in \mathbb{N}$, we observe that

$$
(4.8) \quad \int_\Omega \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,\omega\, dx = \int_{|u_n|>M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,\omega\, dx + \int_{|u_n|\le M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,\omega\, dx.
$$

Regarding the first integral on the right-hand side of (4.8), from (4.7), Hölder inequality, and the continuity of the embedding, we get

$$
\begin{aligned}
(4.9) \quad \bigg| \int_{|u_n|>M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,\omega\, dx \bigg| &= \int_{|u_n|>M_\varepsilon} \frac{|g(u_n)|}{u_n^{p-1}} \frac{u_n^{p-1}}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,|\omega|\, dx \\
&\le \varepsilon \int_{|u_n|>M_\varepsilon} \frac{u_n^{p-1}}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,|\omega|\, dx \le \varepsilon \|\omega\|_{L^p} \bigg\| \frac{u_n^{p-1}}{\|u_n\|^{p-1}_{W_0^{1,p}}} \bigg\|_{L^{p'}} \le C\varepsilon \|\omega\|_{L^p}.
\end{aligned}
$$

With respect to the second integral on the right-hand side of (4.8), we have

$$\begin{aligned}(4.10)\quad &\bigg|\int_{|u_n|\le M_\varepsilon} \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\omega\,dx\bigg| \\ &\qquad = \|g\|_{L^\infty[0,M_\varepsilon]}\int_{|u_n|\le M_\varepsilon} \frac{|\omega|}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,dx \le \frac{\|g\|_{L^\infty[0,M_\varepsilon]}}{\|u_n\|^{p-1}_{W_0^{1,p}}}\|\omega\|_{L^1}.\end{aligned}$$

Since $\varepsilon > 0$ is fixed, $\|g\|_{L^\infty[0,M_\varepsilon]}$ is fixed. The right-hand side of (4.10) tends to zero as $n \to \infty$, because $\|u_n\|_{W_0^{1,p}} \to +\infty$. Thus, from (4.8)–(4.10), and the fact that $\omega \in L^p(\Omega)$ is arbitrary, we conclude $g(u_n)/\|u_n\|^{p-1}_{W_0^{1,p}} \rightharpoonup 0$ weakly in $L^{p'}(\Omega)$.

We then have that the argument on the right-hand side in (4.5) converges weakly to $\overline{\lambda} f_p'(\infty)\overline{v}$ in $L^{p'}(\Omega)$, for some $\overline{\lambda} \in J$. As $(-\Delta_p)^{-1}\colon L^{p'}(\Omega) \to W_0^{1,p}(\overline{\Omega})$ is compact, from (4.5) we get a further subsequence $\{u_n/\|u_n\|_{W_0^{1,p}}\}_n$ such that

$$\begin{aligned}(4.11)\quad \frac{u_n}{\|u_n\|_{W_0^{1,p}}} &= (-\Delta_p)^{-1}\bigg(\frac{\lambda_n f_p'(\infty)u_n^{p-1} + \lambda_n g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\bigg) \\ &\to (-\Delta_p)^{-1}(\overline{\lambda} f_p'(\infty)\overline{v})\end{aligned}$$

as $n \to \infty$, strongly in $W_0^{1,p}(\overline{\Omega})$. From (4.6) and (4.11) we conclude

$$(4.12)\qquad (-\Delta_p)^{-1}\left(\overline{\lambda} f_p'(\infty)\overline{v}\right) = \overline{v}.$$

Let us denote $v_n := u_n/\|u_n\|_{W_0^{1,p}}$ for each n. From (4.11) and (4.12), it follows that $v_n \to \overline{v}$ strongly in $W_0^{1,p}(\Omega)$. Since $\|u_n\|_{W_0^{1,p}} = 1$ for every n, it follows $\overline{v} \neq 0$. Therefore (4.12) means $\overline{\lambda} f_p'(\infty)$ is an eigenvalue of $-\Delta_p$ and $\overline{v}$ is an associated eigenfunction. This is absurd since $\overline{v} \ge 0$ (from (4.6)) and $\overline{\lambda} f_p'(\infty) \neq \lambda_1$ (since $\overline{\lambda} \in J$ and $\lambda_1/f_p'(\infty) \notin J$). This contradiction completes our proof of (a).

We now prove (b). Let $\lambda < \lambda_\infty$. Consider $J = [0,\lambda]$. For every $t \in [0,1]$ we have $t\lambda \in J$. From (a) there exists $r > 0$ such that

$$u - (-\Delta_p)^{-1}(t\lambda f^+(u)) \neq 0$$

for every $t\lambda \in J$ (i.e. for every $t \in [0,1]$) and every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}} \ge r$. For such an u, taking $z = u/\|u\|^2_{W_0^{1,p}}$, we get

$$z - \|z\|^2_{W_0^{1,p}}(-\Delta_p)^{-1}(t\lambda f^+\big(z/\|z\|^2_{W_0^{1,p}}\big)) \neq 0$$

for every $z \in W_0^{1,p}(\Omega)$ such that $\|z\|_{W_0^{1,p}} \le 1/r$. Hence, $\Psi_+(z,t\lambda) \neq 0$ for every $z \in W_0^{1,p}(\Omega)$ such that $0 < \|z\|_{W_0^{1,p}} \le 1/r$. Let us define the homotopy $H\colon W_0^{1,p}(\Omega) \times [0,1] \to W_0^{1,p}(\Omega)$ by $H(u,t) = \Psi_+(u,t\lambda)$. Using Leray–Schauder degree invariance under homotopies, we get

$$\deg(H(\,\cdot\,,1), B_{1/r}(0), 0) = \deg(H(\,\cdot\,,0), B_{1/r}(0), 0)$$

equivalently

$$\deg(\Psi_+(\,\cdot\,,\lambda), B_{1/r}(0), 0) = \deg(I, B_{1/r}(0), 0) = 1.$$

To prove (c), again we argue by contradiction. Assume there is a bifurcation point $\overline{\lambda}$ from ∞ such that $\overline{\lambda} \neq \lambda_\infty$. Let $J \subset \mathbb{R}^+$ be a compact interval such that $\overline{\lambda} \in J$ and $\lambda_\infty \notin J$. Then, there exists a sequence $\{(u_n, \lambda_n)\} \subset S^+$ such that $\|u_n\|_{W_0^{1,p}} \to +\infty$ and $\lambda_n \in J$ for large $n \in \mathbb{N}$. But this contradicts (a). $\square$

LEMMA 4.6. *The following assertions hold true*:

(a) *Let* $\lambda_\infty := \lambda_1/f_p'(\infty)$. *For every* $\lambda > \lambda_\infty$ *there exists* $R > 0$ *such that for all* $\tau \geq 0$ *and for every positive* $u \in W_0^{1,p}(\Omega)$, *with* $\|u\|_{W_0^{1,p}} \geq R$, *it holds that* $u \neq (-\Delta_p)^{-1}(\lambda f^+(u) + \tau)$.

(b) $i(\Psi_+(\,\cdot\,,\lambda), 0) = 0$ *for all* $\lambda > \lambda_\infty$.

PROOF. In order to prove (a) we argue by contradiction. Actually, our argument is similar to the one we used above when proving Lemma 4.5 part (a), but in this case it is more involved because of the τ-term. Assume there exist $\{\tau_n\}_n \subset [0,\infty)$ and a sequence $\{u_n\}_n \subset W_0^{1,p}(\Omega)$ of nonnegative functions such that $\|u_n\|_{W_0^{1,p}} \to \infty$ as $n \to \infty$ and

$$u_n = (-\Delta_p)^{-1}(\lambda f^+(u_n) + \tau_n) \quad \text{for every } n \in \mathbb{N}. \tag{4.13}$$

Since $f^+(t) = f_p'(\infty)|t|^{p-2}t + g(t)$, where $g(t)/|t|^{p-2}t \to 0$ as $t \to +\infty$, (4.13) can be written as

$$u_n = (-\Delta_p)^{-1}(\lambda f_p'(\infty)u_n^{p-1} + \lambda g(u_n) + \tau_n) \quad \text{for every } n \in \mathbb{N}. \tag{4.14}$$

Let $v_n = u_n/\|u_n\|_{W_0^{1,p}}$ for every $n \in \mathbb{N}$. Then v_n satisfies equation

$$v_n = (-\Delta_p)^{-1}\bigg(\lambda f_p'(\infty)v_n^{p-1} + \lambda\frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}}\bigg) \quad \text{for all } n \in \mathbb{N}. \tag{4.15}$$

We may assume (by passing to a subsequence) that either

(i) $\tau_n/\|u_n\|^{p-1}_{W_0^{1,p}} \to c \geq 0$ as $n \to \infty$, or

(ii) $\tau_n/\|u_n\|^{p-1}_{W_0^{1,p}} \to +\infty$ as $n \to \infty$.

Let us consider case (i). Assume first that $c = 0$. Since $\|v_n\|_{W_0^{1,p}} = 1$ for every $n \in \mathbb{N}$, we can suppose (by taking a subsequence) that there exists $\overline{v} \in W_0^{1,p}(\Omega)$ such that $v_n \rightharpoonup v$ (weakly) in $W_0^{1,p}(\Omega)$ and (4.6) holds true. Arguing as in the proof of Lemma 4.5,

$$\lambda f_p'(\infty)v_n^{p-1} \rightharpoonup \lambda f_p'(\infty)v^{p-1} \quad \text{and} \quad \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} \rightharpoonup 0 \quad \text{weakly in } L^{p'}(\Omega), \tag{4.16}$$

and, by our assumption that $c = 0$ in (i),

$$\frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}} \rightharpoonup 0 \quad \text{weakly in } L^{p'}(\Omega). \tag{4.17}$$

Since $(-\Delta_p)^{-1}\colon L^{p'}(\Omega) \to W_0^{1,p}(\Omega)$ is a compact operator, it follows from (4.15), (4.16) and (4.17)

$$v = (-\Delta_p)^{-1}\left(\lambda f_p'(\infty) v^{p-1}\right) \;\Leftrightarrow\; -\Delta_p v = \lambda f_p'(\infty) v^{p-1}. \tag{4.18}$$

Arguing as we did above after getting (4.12), we get that the nonnegative function $\overline{v}$ is also nonzero. Thus, (4.18) provides a contradiction since $\lambda > \lambda_1(p)/f_p'(\infty)$.

We now assume $\tau_n/\|u_n\|^{p-1}_{W_0^{1,p}} \to c > 0$ as $n \to \infty$. Let $\varepsilon \in (0, \lambda f_p'(\infty) - \lambda_1)$.

Claim 4.7. *there exists a weak positive supersolution $\omega \in W_0^{1,p}(\Omega)$ of problem*

$$\begin{cases} -\Delta_p \omega = (\lambda_1 + \varepsilon)\omega^{p-1} & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.19}$$

Proof of Claim 4.7. Let $\overline{\gamma} \in ((\lambda_1+\varepsilon)/\lambda, f_p'(\infty))$, so that

$$\lambda_1 + \varepsilon < \lambda\overline{\gamma} < \lambda f_p'(\infty). \tag{4.20}$$

We show that there exists a large $n \in \mathbb{N}$ such that

$$\int_\Omega \left(\lambda f_p'(\infty) v_n^{p-1} + \lambda \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}}\right)\phi\,dx \geq \int_\Omega \lambda\overline{\gamma} v_n^{p-1}\phi\,dx, \tag{4.21}$$

for every $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$. Let $\eta \in (0, (f_p'(\infty) - \overline{\gamma})/2)$. Since $g(t)/t^{p-1} \to 0$ as $t \to +\infty$, there exists $M_\eta > 0$ such that

$$\left|\frac{g(t)}{t^{p-1}}\right| < \eta, \quad \text{for all } t > M_\eta. \tag{4.22}$$

Since λ and M_η are fixed, $\lambda\|g\|_{L^\infty[0,M_\eta]}/\|u_n\|^{p-1}_{W_0^{1,p}} \to 0$ as $n \to \infty$. So, we can pick a large n, so that

$$\frac{-\lambda\|g\|_{L^\infty[0,M_\eta]}}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}} \geq \frac{c}{2}. \tag{4.23}$$

Given $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$, in order to obtain (4.21) we write

$$\begin{aligned} &\int_\Omega \left(\lambda f_p'(\infty) v_n^{p-1} + \lambda \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}}\right)\phi\,dx - \int_\Omega \lambda\overline{\gamma} v_n^{p-1}\phi\,dx \\ &\quad= \int_{\{u_n > M_\eta\}} \left(\lambda(f_p'(\infty) - \overline{\gamma}) v_n^{p-1} + \lambda\,\frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}}\right)\phi\,dx \\ &\qquad+ \int_{\{u_n \leq M_\eta\}} \left(\lambda\,\frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}}\right)\phi\,dx \\ &\qquad+ \int_{\{u_n \leq M_\eta\}} \lambda(f_p'(\infty) - \overline{\gamma}) v_n^{p-1}\phi\,dx \\ &\qquad+ \int_{\{u_n > M_\eta\}} \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}}\,\phi\,dx =: I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.24}$$

Observe $I_3 \geq 0$ and $I_4 \geq 0$. From (4.23) we get

$$I_2 \geq \int_{\{u_n \leq M_\eta\}} \frac{c}{2}\,\phi\,dx \geq 0.$$

Regarding I_1, observe that writing

$$\frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} = \frac{g(u_n)}{u_n^{p-1}} \frac{u_n^{p-1}}{\|u_n\|^{p-1}_{W_0^{1,p}}}$$

we get

$$I_1 = \int_{\{u_n > M_\eta\}} \bigg(\lambda(f_p'(\infty) - \overline{\gamma}) + \lambda \frac{g(u_n)}{u_n^{p-1}}\bigg) v_n^{p-1} \phi\,dx. \tag{4.25}$$

Using (4.22), (4.25) and our choice of η, we have

$$I_1 \geq \int_{\{u_n > M_\eta\}} \lambda\bigg(\frac{f_p'(\infty) - \overline{\gamma}}{2}\bigg) v_n^{p-1} \phi\,dx \geq 0.$$

We conclude (4.21) holds true. From (4.15), (4.21) and our choice of $\overline{\gamma}$ (namely, (4.20)), we get v_n is a supersolution of (4.19) for $n \in \mathbb{N}$ large, i.e. the following inequalities hold true in the weak sense

$$-\Delta_p v_n = \lambda f_p'(\infty) v_n^{p-1} + \lambda \frac{g(u_n)}{\|u_n\|^{p-1}_{W_0^{1,p}}} + \frac{\tau_n}{\|u_n\|^{p-1}_{W_0^{1,p}}} \geq \lambda\overline{\gamma} v_n^{p-1} > (\lambda_1 + \varepsilon) v_n^{p-1} \tag{4.26}$$

for large n. So, for such an n, v_n is a supersolution of (4.19). Moreover, since $v_n = u_n/\|u_n\|_{W_0^{1,p}}$, from (4.13) and Lemma 4.1 we conclude $v_n > 0$ on Ω and $\partial v_n/\partial \overrightarrow{n} < 0$ on $\partial\Omega$. This completes the proof of claim. □

Now, for every $t > 0$ and a positive eigenfunction ϕ_1 corresponding to λ_1, $t\phi_1$ is a subsolution of problem (4.19). Let v_n be a positive supersolution of (4.19). Using that $\partial v_n/\partial \overrightarrow{n} < 0$ and $\partial \phi_1/\partial \overrightarrow{n} < 0$ on $\partial\Omega$ (where $\overrightarrow{n}$ denotes the outer unit normal on $\partial\Omega$), one can prove there exists $t > 0$ such that $t\phi_1 \leq v_n$ on Ω. Using standard truncation and penalization techniques (see e.g. [7, Section 3], [13], the appendix in [15], or Section 4.5 in [16]), it can be proved the existence of a solution $\omega \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, of problem (4.19), such that $t\phi_1 \leq \omega \leq v_n$ in Ω. Thus ω is a positive eigenfunction corresponding to the eigenvalue $\lambda_1 + \varepsilon \neq \lambda_1$. This is a contradiction that shows case (i) above cannot actually occur.

Let us now consider case (ii). Let $\overline{\gamma}$ be as in (4.20). Arguing as in case i), from (4.15) it follows that, for $n \in \mathbb{N}$ sufficiently large, inequality $-\Delta_p v_n \geq \lambda\overline{\gamma} v_n^{p-1}$ holds true. Then, the same argument as presented in case i) follows, and we also get a contradiction. We have completed the proof of part (a).

We now prove (b). Let $\lambda > \lambda_\infty$. From (a), taking $\tau = t$, we know that for every $t \in [0,1]$ and every $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{W_0^{1,p}} \geq R$, $u \neq$

$(-\Delta_p)^{-1}(\lambda f^+(u) + t)$. Using again inversion $z = u/\|u\|^2_{W_0^{1,p}}$ and the homogeneity of $(-\Delta_p)^{-1}$, we observe that for every $t \in [0,1]$ and every $z \in W_0^{1,p}(\Omega)$ such that $0 < \|z\|_{W_0^{1,p}} \le 1/R$,

$$z \neq (-\Delta_p)^{-1}\big(\lambda \|z\|^{2(p-1)}_{W_0^{1,p}} f^+\big(z/\|z\|^2_{W_0^{1,p}}\big) + t\big). \tag{4.27}$$

Let $\varepsilon \in (0, 1/R)$. We now define homotopy $H : B_\varepsilon(0) \times [0,1] \to W_0^{1,p}(\Omega)$ as

$$H(z,t) = z - (-\Delta_p)^{-1}\big(\lambda \|z\|^{2(p-1)}_{W_0^{1,p}} f^+\big(z/\|z\|^2_{W_0^{1,p}}\big) + t\big) \quad \text{for every } z \neq 0,$$

and $H(0,t) := -(-\Delta_p)^{-1}(t)$. Using the same ideas we used above to prove part (a) of Lemma 4.5, it can be proved that H is actually continuous, and also that it is of the form identity $-$ compact.

Using the homotopy invariance property of Leray–Schauder degree, we obtain

$$\deg(H(\,\cdot\,,0), B_\varepsilon(0), 0) = \deg(H(\,\cdot\,,1), B_\varepsilon(0), 0).$$

On the other hand, $\deg(H(\,\cdot\,,0), B_\varepsilon(0), 0) = \deg(\Psi_+(\,\cdot\,,\lambda), B_\varepsilon(0), 0)$ and, from (4.27) and the definition of H,

$$\deg(H(\,\cdot\,,1), B_\varepsilon(0), 0) = 0. \qquad \square$$

Proof of Theorem 4.3. Lemmas 4.5 and 4.6 assert that $i(\Psi_+(\,\cdot\,,\lambda),0) = 1$ when $\lambda < \lambda_\infty$, and $i(\Psi_+(\,\cdot\,,\lambda),0) = 0$ when $\lambda > \lambda_\infty$. The fact that these two local degrees are different allows one to repeat the original arguments used by P. Rabinowitz to prove his global bifurcation theorem (see [19], [20], and [3, Sections 4.3 and 4.4]). $\square$

We now prove the existence of two solutions for problem (1.1). Since Σ^+_∞ bifurcates from $(\infty, \lambda_1/{f_p}'(\infty))$, there exist elements $(u,\lambda) \in \Sigma^+_\infty$ such that $\|u\|_{W_0^{1,p}}$ is arbitrarily large and λ is near $\lambda_1/f_p'(\infty)$. Hence, because of inequality (3.2) in Lemma 3.2, there exist elements $(u,\lambda) \in \Sigma^+_\infty$ such that $\mathcal{N}_\infty(u,\lambda) = \|u\|_{L^\infty} > \alpha$. Lemma 3.3 implies that $\mathcal{N}_\infty(\overline{\Sigma^+_\infty})$ is connected. Thus, Lemma 3.1 implies that

$$\|u\|_{L^\infty} > \alpha \quad \text{for all } (u,\lambda) \in \overline{\Sigma^+_\infty}. \tag{4.28}$$

Because of inequality (3.3) in Lemma 3.2,

$$\|u\|_{W_0^{1,p}} > (K_2)^{-1}\alpha \quad \text{for all } (u,\lambda) \in \overline{\Sigma^+_\infty} \cap (W_0^{1,p}(\Omega) \times [0,2]). \tag{4.29}$$

The constant K_2 in the previous inequality, technically, depends on λ. As it was pointed out in Lemma 3.2, K_2 is bounded if λ is bounded. Since $\lambda \in [0,2]$ in this case, the constant $K_2 > 0$ can be chosen independent of λ. Now we claim that there exists an element of the form $(u_1, 1) \in \overline{\Sigma^+_\infty}$. Let us argue by contradiction. Assume this is not true. Consider the cylinder

$$P = \{(u,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : \lambda \in [0,1],\ \|u\|_{W_0^{1,p}} \ge (K_2)^{-1}\alpha\}.$$

Hypothesis (f_2) implies that $\lambda_1/f_p'(\infty) < 1$. Therefore, from Theorem 4.3 it follows that $\operatorname{int} P \cap \overline{\Sigma_\infty^+} \neq \emptyset$. Also, since Σ_∞^+ corresponds to the unbounded connected component Γ_∞^+ of Γ^+, then $\operatorname{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P) \cap \overline{\Sigma_\infty^+} \neq \emptyset$. From (4.29) and our assumption, $\partial P \cap \overline{\Sigma_\infty^+} = \emptyset$. Thus, ∂P separates $\overline{\Sigma_\infty^+}$, i.e.

$$\overline{\Sigma_\infty^+} \subset \operatorname{int} P \cup \operatorname{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P),$$

which contradicts the connectedness of $\overline{\Sigma_\infty^+}$. This contradiction shows there exists $(u_1, 1) \in \overline{\Sigma_\infty^+}$. From Theorem 4.3, $u_1 \neq 0$, i.e. $(u_1, 1) \in \Sigma_\infty^+ \subset S^+$. As mentioned above, this means $u_1 > 0$ on Ω and u_1 satisfies (1.1). In a similar fashion we obtain a negative solution v_1. The previous argument shows these two solutions have $L^\infty(\Omega)$-norm greater than α.

4.2. Bifurcation from zero. First we state the following analogue of Theorem 4.3.

Theorem 4.8. *There exists an unbounded connected component Σ_0^+ of S^+ so that $(0, \lambda_1/f_p'(0))$ belongs to $\overline{\Sigma_0^+}$ and, if $(0,\lambda) \in \overline{\Sigma_0^+}$, then $\lambda = \lambda_1/f_p'(0)$. Also, there exists an unbounded connected component Σ_0^- of S^- such that $(0, \lambda_1/f_p'(0))$ in $\overline{\Sigma_0^-}$ and, if $(0,\lambda) \in \overline{\Sigma_0^-}$, then $\lambda = \lambda_1/f_p'(0)$.*

Remark 4.9. This result is essentially an adaptation of Lemma 3.1 in [9] to our case, and it can be proved either by following the arguments of [9, Theorem 1.1 and Lemma 3.1] or by using the same ideas we used above to prove Theorem 4.3.

We now prove the existence of two additional solutions for problem (1.1). Since $(0, \lambda_1/{f_p}'(0)) \in \overline{\Sigma_0^+}$, there exist elements $(u,\lambda) \in \Sigma_0^+$ such that $\|u\|_{W_0^{1,p}}$ is close to zero and λ is near $\lambda_1/f_p'(0)$. Hence, because of inequality (3.3) in Lemma 3.2, there exist elements $(u,\lambda) \in \Sigma_0^+$ such that $\mathcal{N}_\infty(u,\lambda) = \|u\|_{L^\infty} < \alpha$. From Lemma 3.3 it follows that $\mathcal{N}_\infty(\overline{\Sigma_0^+})$ is connected. Thus, Lemma 3.1 implies that

$$\|u\|_{L^\infty} < \alpha \quad \text{for all } (u,\lambda) \in \overline{\Sigma_0^+}. \tag{4.30}$$

Because of inequality (3.2) in Lemma 3.2,

$$\|u\|_{W_0^{1,p}} < K_1\alpha \quad \text{for all } (u,\lambda) \in \overline{\Sigma_0^+} \cap (W_0^{1,p}(\Omega) \times [0,2]). \tag{4.31}$$

Now we claim that there exists $(u_2, 1) \in \overline{\Sigma_0^+}$. Let us argue by contradiction. Assume this is not true. Define the cylinder

$$P = \{(u,\lambda) \in W_0^{1,p}(\Omega) \times \mathbb{R} : \lambda \in [0,1],\ \|u\|_{W_0^{1,p}} \leq K_1\alpha\}.$$

Hypothesis (f_2) implies that $\lambda_1/f_p'(0) < 1$. Therefore, from Theorem 4.8 it follows that $\operatorname{int} P \cap \overline{\Sigma_0^+} \neq \emptyset$. Also, the unboundedness of $\overline{\Sigma_0^+}$ implies $\operatorname{int}(W_0^{1,p}(\Omega) \times$

$\mathbb{R} \setminus P) \cap \overline{\Sigma_0^+} \neq \emptyset$. From (4.31) and our assumption, $\partial P \cap \overline{\Sigma_0^+} = \emptyset$. Thus, ∂P separates $\overline{\Sigma_0^+}$, i.e.

$$\overline{\Sigma_0^+} \subset \operatorname{int} P \cup \operatorname{int}(W_0^{1,p}(\Omega) \times \mathbb{R} \setminus P),$$

which contradicts the connectedness of $\overline{\Sigma_0^+}$. This contradiction shows there exists $(u_2, 1) \in \overline{\Sigma_0^+}$. From Theorem 4.8, $u_2 \neq 0$, i.e. $(u_2, 1) \in \Sigma_0^+ \subset S^+$. As mentioned above, this means $u_2 > 0$ on Ω and u_2 satisfies (1.1).

Arguing in a similar fashion with $\overline{\Sigma_0^-}$, the existence of a negative solution v_2 of (1.1) is obtained. From (4.30) (and its analogue for $\overline{\Sigma_0^-}$) we have $\|u_2\|_{L^\infty}, \|v_2\|_{L^\infty} < \alpha$.

We summarize the arguments presented above, in the following bifurcation diagram of Figure 1.

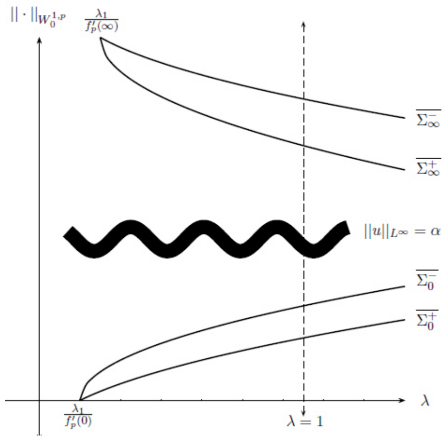

Figure 1. Bifurcation diagram for problem (1.3).

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