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CQ METHOD FOR APPROXIMATING FIXED POINTS OF NONEXPANSIVE SEMIGROUPS AND STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. We use the CQ method for approximating a common fixed point of a left amenable semigroup of nonexpansive mappings, an infinite family of strictly pseudo-contraction mappings and the set of solutions of variational inequalities for monotone, Lipschitz-continuous mappings in a real Hilbert space. Our results are a generalization of a result announced by Nadezhkina and Takahashi [N. Nadezhkina and W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings, SIAM J. Optim. 16 (2006), 1230–1241] and some other recent results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. By $\mathrm{ne}(C)$, we denote the set of all nonexpansive mappings of C into itself and by $\mathrm{Fix}(T)$, we denote the set of fixed points of T (i.e. $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$), it is well known that $\mathrm{Fix}(T)$ is closed and convex. Let $A\colon C \to H$ be a nonlinear

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operator. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \text{for all } y \in C. \tag{1.1}$$

The set of solutions of variational inequality (1.1) is denoted by $\mathrm{VI}(C, A)$, that is,

$$\mathrm{VI}(C, A) = \{x \in C : \langle Ax, y - x\rangle \geq 0 \text{ for all } y \in C\}.$$

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [5], [7], [9], [13], [25]–[28] and the references therein. We start with Korpelevich's extragradient method which was introduced by Korpelevich [9] in 1976. He proved that the sequence $\{x_n\}$ generated via the recursion

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = P_C(x_n - \lambda_n A y_n), \quad n \geq 0, \end{cases}$$

where P_C is the metric projection from $\mathbb{R}^n$ onto C, A is a monotone operator and λ is a constant, converges strongly to a solution of $\mathrm{VI}(C, A)$. Note that the setting of the problem is the Euclidean space $\mathbb{R}^n$.

Korpelevich's extragradient method has been extensively studied in the literature for solving a more general problem that consists of finding a common point that lies in the solution set of a variational inequality and the set of fixed points of a nonexpansive mapping. Especially, Nadezhkina and Takahashi [14] introduced the following iterative method which combines Korpelevich's extragradient method and the CQ method:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.2}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H.

Inspired by the ideas in Korpelevich [9], Nadezhkina and Takahashi [14], Lau et al. [11], Lau et al. [12], Katchang and Kumam [10], Piri [15], [16], Piri and Badali [18] and the references therein, we introduce some new iterative schemes based on Korpelevich's extragradient method (and the CQ method) for finding a common element of the set of solutions of the variational inequality for a monotone, Lipschitz-continuous mapping, the set of fixed points of an infinite family of strictly pseudo-contraction mappings and the set of fixed points of a left amenable semigroup of nonexpansive mappings. We obtain strong convergence theorems for the sequences generated by the corresponding processes. The results in this paper generalize, improve and unify some well-known convergence theorems in the literature.

2. Preliminaries

Let S be a semigroup and let $l^\infty(S)$ be the space of all bounded real valued functions defined on S with supremum norm. For $s \in S$ and $f \in l^\infty(S)$, we define elements $l(s)f$ and $r(s)f$ in $l^\infty(S)$ by

$$(l(s)f)(t) = f(st), \quad (r(s)f)(t) = f(ts), \quad \text{for all } t \in S.$$

Let X be a subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. X is said to be left invariant (resp. right invariant) if $l(s)(X) \subset X$ (resp. $r(s)(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu((l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $l^\infty(S)$ is amenable when S is a commutative semigroup (see [11]). A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_\alpha \|l(s)^*\mu_\alpha - \mu_\alpha\| = 0,$$

for each $s \in S$, where $l(s)^*$ is the adjoint operator of $l(s)$.

Let C be a closed convex subset of a Banach space E and let T be a mapping of C into itself. Then $\varphi = \{T(t) : t \in S\}$ is called a representation of S as nonexpansive mappings on C if $T(s) \in \mathrm{ne}(C)$ for each $s \in S$, $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $\mathrm{Fix}(\varphi)$ the set of common fixed points of φ, i.e.

$$\mathrm{Fix}(\varphi) = \bigcap_{t \in S} \{x \in C : T(t)x = x\},$$

by $l^\infty(S, E)$ the Banach space of all bounded mappings of S into a Banach space E with supremum norm, and by $l_c^\infty(S, E)$ the subspace of elements $f \in l^\infty(S, E)$ such that $f(S) = \{f(s) : s \in S\}$ is a relatively weakly compact subset of E. Let X be a subspace of $l^\infty(S)$ containing 1 such that for each $f \in l^\infty(S, E)$ and $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X. Then, for each $\mu \in X^*$ and $f \in l_c^\infty(S, E)$, let us define a continuous linear functional $\tau(\mu)f$ on E^* by

$$\tau(\mu)f \colon x^* \mapsto \mu\langle f(\,\cdot\,), x^*\rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. We know that if μ is a mean on X, then $\tau(\mu)f$ is contained in the closure of convex hull of $\{f(s) : s \in S\}$. We also know that for each $\mu \in X^*$, $\tau(\mu)$ is a bounded linear mapping of $l_c^\infty(S,E)$ into E such that for each $f \in l_c^\infty(S,E)$, $\|\tau(\mu)\| \leq \|\mu\|\|f\|$ (see [8]). Let $\varphi = \{T(t) : t \in S\}$ be a representation of S as nonexpansive mappings on C such that $T(\cdot)x \in l_c^\infty(S,E)$ for some $x \in C$. If for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X, then there exists a unique point x_0 of E such that $\mu\langle T(s)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$ (see [6] and [22]). We denote such a point x_0 by $T(\mu)x$.

Lemma 2.1 ([11]). *Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\varphi = \{T(s) : s \in S\}$ be a nonexpansive semigroup on H such that $\{T(s)x : s \in S\}$ is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \mapsto \langle T_t x, y^*\rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. Then*:

(a) *$T(\mu)$ is nonexpansive mapping from C into C.*
(b) *$T(\mu)x = x$ for each $x \in \mathrm{Fix}(\varphi)$.*
(c) *$T(\mu)x \in \overline{\mathrm{co}}\{T(s)x : s \in S\}$ for each $x \in C$.*

Notation 2.2.

(a) $\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence.
(b) $\omega_\omega\{x_n\} = \{x \in H : \exists\{x_{n_j}\} \subset \{x_n\}$ and $x_{n_j} \rightharpoonup x\}$.

Let C be a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\|x - y_0\| = d(x, C),$$

where $d(x,C) = \inf_{y \in C} \|x - y\|$. The number $d(x,C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x,C)\}.$$

This defines a mapping P_C from X into 2^C and it is called a metric (nearest point) projection onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

Lemma 2.3 ([24]). *Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let $x \in H$ and $y \in C$. Then, the following statements are equivalent:*

(a) *$y = P_C(x)$,*
(b) *$\langle x - y, y - z\rangle \geq 0$, for all $z \in C$.*
(c) *$\|x - y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2$.*

Lemma 2.4 ([23]). *Let H be a real Hilbert space. Then, for all $x, y \in H$,*

(a) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$,
(b) $\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.

Definition 2.5 ([2]). A mapping $T\colon C \to C$ is called λ-strictly pseudo-contractive of Browder and Petryshyn type if there exists a constant $\lambda \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \lambda\|(I-T)x - (I-T)y\|^2, \quad \text{for all } x, y \in C. \tag{2.1}$$

It is well known that the last inequality is equivalent to

$$\langle Tx - Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-\lambda}{2}\,\|(I-T)x - (I-T)y\|^2, \tag{2.2}$$

for all $x, y \in C$. If $\lambda = 1$, then T is called a pseudo-contractive mapping, that is,

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \quad \text{for all } x, y \in C. \tag{2.3}$$

This is equivalent to

$$\langle (I-T)x - (I-T)y, x-y\rangle \geq 0, \quad \text{for all } x, y \in C. \tag{2.4}$$

Lemma 2.6 ([2]). *Let $T\colon C \to H$ be a λ-strictly pseudo-contractive mapping. Define $S\colon C \to H$ by $S(x) = \delta I(x) + (1-\delta)T(x)$ for each $x \in C$. Then, as $\delta \in [\lambda, 1)$, T is a nonexpansive mapping such that* $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself, we define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I,\\
U_{n,n-1} &= \gamma_{n-1} S_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\;\;\vdots\\
U_{n,k} &= \gamma_k S_k U_{n,k+1} + (1-\gamma_k)I,\\
U_{n,k-1} &= \gamma_{k-1} S_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\;\;\vdots\\
U_{n,2} &= \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,\\
W_n = U_{n,1} &= \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{2.5}$$

where, $0 \leq \gamma_n \leq 1$, $S_n = \delta_n I + (1-\delta_n)T_n$ and $\gamma_n \leq \delta_n < 1$, for all $n \in \mathbb{N}$. We can obtain S_n is a nonexpansive mapping and $\mathrm{Fix}(S_n) = \mathrm{Fix}(T_n)$ by Lemma 2.6. Furthermore, we obtain W_n is a nonexpansive mapping. To establish our results, we need the following technical lemmas.

Lemma 2.7 ([21]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S_n\}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_n \leq b < 1$ for every $n \in \mathbb{N}$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

In view of the previous lemma, we will define

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}, \quad \text{for all } x \in C.$$

Lemma 2.8 ([21]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{S_n\}$ be an infinite family of nonexpansive mappings of C into itself and let $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_n \leq b < 1$ for every $n \in \mathbb{N}$. Then*

$$\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \neq \emptyset.$$

The following lemmas follow from Lemmas 2.6–2.8.

Lemma 2.9 ([4]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \emptyset$. Define $S_n = \delta_n I_n + (1-\delta_n)T_n$ and $0 < \lambda_n \leq \delta_n < 1$ and let $\{\gamma_n\}$ be a real sequence such that $0 < \gamma_n \leq b < 1$ for every $n \in \mathbb{N}$. Then*

$$\operatorname{Fix}(W) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) = \bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \neq \emptyset.$$

Lemma 2.10 ([3]). *Let C be a nonempty closed convex subset of a Hilbert space. Let $\{S_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \neq \emptyset$ and let $\{\gamma_n\}$ be a real sequence such that $0 < \gamma_n \leq b < 1$ for every $n \in \mathbb{N}$. If K is a bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. Consider the functional $r_a(\,\cdot\,, \{x_n\})\colon X \to \mathbb{R}$ defined by

$$r_a(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|, \quad \text{for all } x \in X.$$

The infimum of $r_a(\,\cdot\,, \{x_n\})$ over K is called an asymptotic radius of $\{x_n\}$ with respect to K and it is denoted by $r_a(K, \{x_n\})$. A point $x \in K$ is called an asymptotic center of the sequence $\{x_n\}$ with respect to K if

$$r_a(x, \{x_n\}) = \inf\{r_a(y, \{x_n\}) : y \in K\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to K is denoted by $C_a(K, \{x_n\})$. This set may be empty, a singleton, or infinite.

Lemma 2.11 ([1]). *Let X be a uniformly convex Banach space satisfying the Opial condition and K a nonempty closed convex subset of X. If a sequence $\{x_n\} \subset K$ converges weakly to a point x_0, then x_0 is an asymptotic center of $\{x_n\}$ with respect to K.*

A set-valued mapping $U\colon H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in U(x)$ and $g \in U(y)$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $U\colon H \to 2^H$ is maximal if the graph of $G(U)$ of U is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping U is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(U)$ implies that $f \in Ux$.

Lemma 2.12 ([19]). *Let A be a monotone mapping of C into H and let $N_C x$ be the normal cone to C at $x \in C$, that is, $N_C x = \{y \in H : \langle z - x, y\rangle \leq 0$ for all $z \in C\}$ and define*

$$Ux = \begin{cases} Ax + N_C x & \text{for } x \in C, \\ \emptyset & \text{for } x \notin C. \end{cases} \tag{2.6}$$

Then U is maximal monotone and $0 \in Ux$ if and only if $x \in \mathrm{VI}(C, A)$.

Notation 2.13. The open ball of radius r centered at 0 is denoted by B_r and for a subset D of H, by $\overline{\mathrm{co}}\, D$ we denote the closed convex hull of D. For $\varepsilon > 0$ and a mapping $T\colon D \to H$, we let $F_\varepsilon(T; D)$ be the set of ε-approximate fixed points of T, i.e. $F_\varepsilon(T; D) = \{x \in D : \|x - Tx\| \leq \varepsilon\}$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{f_n\}_{n=1}^\infty$ be a sequence of ψ_n-contraction self-mappings of C such that $\{f_n\}_{n=1}^\infty$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C. Let $\{T_n\}_{n=1}^\infty$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself. Let S be a semigroup and $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup of C into itself such that for all $n \in \mathbb{N}$, $T_n(\mathrm{Fix}(\varphi)) \subset \mathrm{Fix}(\varphi)$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, $t \mapsto \langle T_t x, y\rangle$ is an element of X for each $x, y \in C$ and $\{\mu_n\}_{n=0}^\infty$ is a left regular sequence of means on X. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{Fix}(\varphi) \cap \mathrm{VI}(C, A)$ be nonempty and bounded. Let $\{\zeta_n\}_{n=0}^\infty$, $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ be sequences such that $\{\zeta_n\}_{n=0}^\infty \subset [a, b]$ for some $a, b \in (0, 1/k)$, $\{\alpha_n\}_{n=0}^\infty \subset [0, c]$ for some $c \in [0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\{\beta_n\}_{n=0}^\infty \subset [0, 1)$, $\lim_{n\to\infty} \beta_n = 0$ and W_n be the mapping generated by $\{T_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ as in (2.5). Define sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$ and $\{z_n\}_{n=0}^\infty$ in C*

by the iteration algorithm

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) P_C(I - \zeta_n A) x_n, \\ z_n = \alpha_n f_n(y_n) + (1-\alpha_n) T(\mu_n) W_n P_C(x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + r_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3.1}$$

where, $r_n = \alpha_n \delta_n$ *and*

$$\delta_n = \sup\{\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_n - p\|] : p \in \mathcal{F}\}.$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *converge strongly to* $P_{\mathcal{F}} x_0$.

Proof. First we note that C_n is closed and Q_n is closed and convex for every $n \in \mathbb{N} \cup \{0\}$. As $C_n = \{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \le 0\}$, we also have C_n is convex for every $n \in \mathbb{N} \cup \{0\}$. As $Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \le 0\}$, we have $\langle x_n - z, x_n - x_0 \rangle \le 0$ for all $z \in Q_n$ and by Lemma 2.3, $x_n = P_{Q_n} x_0$. Put $t_n = P_C(x_n - \zeta_n A y_n)$ for every $n \in \mathbb{N} \cup \{0\}$. Next, we show that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Let $p \in \mathcal{F}$. From Lemma 2.3 and monotonicity of A, we have

$$\begin{aligned} \|t_n - p\|^2 &\le \|x_n - \zeta_n A y_n - p\|^2 - \|x_n - \zeta_n A y_n - t_n\|^2 \\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\zeta_n \langle A y_n, p - t_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\zeta_n [\langle A y_n - Ap, p - y_n\rangle \\ &\quad + \langle Ap, p - y_n\rangle + \langle A y_n, y_n - t_n\rangle] \\ &\le \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\zeta_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad - 2\langle x_n - y_n, y_n - t_n\rangle + 2\zeta_n\langle A y_n, y_n - t_n\rangle \\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\langle x_n - \zeta_n A y_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further, since $y_n = P_C(I - \zeta_n A) x_n$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned} &\langle x_n - \zeta_n A y_n - y_n, t_n - y_n\rangle \\ &\qquad = \langle x_n - \zeta_n A x_n - y_n, t_n - y_n\rangle + \langle \zeta_n A x_n - \zeta_n A y_n, t_n - y_n\rangle \\ &\qquad \le \langle \zeta_n A x_n - \zeta_n A y_n, t_n - y_n\rangle \le \zeta_n k \|x_n - y_n\| \|t_n - y_n\|. \end{aligned}$$

So, we have

$$\begin{aligned} \|t_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\quad + 2\zeta_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \end{aligned} \tag{3.2}$$

$$+ \zeta_n^2 k^2 \|x_n - y_n\|^2 + \|t_n - y_n\|^2$$
$$= \|x_n - p\|^2 + (\zeta_n^2 k^2 - 1)\|x_n - y_n\|^2 \leq \|x_n - p\|^2.$$

From $y_n = \beta_n x_n + (1-\beta_n)P_C(I - \zeta_n A)x_n$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\beta_n x_n + (1-\beta_n)P_C(I-\zeta_n A)x_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|P_C(I-\zeta_n A)x_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 = \|x_n - p\|^2. \end{aligned} \tag{3.3}$$

From $\zeta_n < 1/k$, $z_n = \alpha_n f_n(y_n) + (1-\alpha_n)T(\mu_n)W_n t_n$, Lemma 2.1 and relations (3.2) and (3.3), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|\alpha_n f_n(y_n) + (1-\alpha_n)T(\mu_n)W_n t_n - p\|^2 \\ &\leq [\alpha_n\|f_n(y_n) - p\| + (1-\alpha_n)\|T(\mu_n)W_n t_n - p\|]^2 \\ &\leq [\alpha_n\|f_n(y_n) - f_n(p)\| + \|f_n(p) - p\| + (1-\alpha_n)\|t_n - p\|]^2 \\ &\leq [\alpha_n\psi_n(\|y_n - p\|) \\ &\quad + \|f_n(p) - p\| + (1-\alpha_n)\|t_n - p\|]^2 \\ &\leq [\alpha_n\|y_n - p\| + \|f_n(p) - p\| + (1-\alpha_n)\|t_n - p\|]^2 \\ &\leq [\alpha_n\|x_n - p\| + \|f_n(p) - p\| + (1-\alpha_n)\|x_n - p\|]^2 \\ &\leq [\|x_n - p\| + \|f_n(p) - p\|]^2 \\ &\leq \|x_n - p\|^2 + \alpha_n[\|f_n(p) - p\|^2 + 2\|f_n(p) - p\|\|x_n - p\|] \\ &\leq \|x_n - p\|^2 + \alpha_n\delta_n = \|x_n - p\|^2 + r_n, \end{aligned} \tag{3.4}$$

for every $n \in \mathbb{N} \cup \{0\}$ and hence $p \in C_n$. So $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Next, we show by induction that

$$\mathcal{F} \subset C_n \cap Q_n, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

From $Q_0 = C$, we have $\mathcal{F} \subset C_0 \cap Q_0$. Suppose that $\mathcal{F} \subset C_n \cap Q_n$ for some $n \in \mathbb{N} \cup \{0\}$. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$, by Lemma 2.3, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \text{for all } z \in C_n \cap Q_n.$$

As $\mathcal{F} \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in \mathcal{F}$. This together with the definition of Q_{n+1} implies that $\mathcal{F} \subset Q_{n+1}$. Hence (3.9) holds. As in the proof of Theorem 3.1 in [16], we can prove that

$$\|x_0 - x_n\| \leq \|x_0 - u\|, \quad \text{for all } u \in \mathcal{F}, \tag{3.6}$$

and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

From $x_{n+1} \in C_n$, we have $\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + r_n$ and hence

$$\begin{aligned}\|z_n - x_n\|^2 &\le [\|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|]^2 \\ &\le 2\|z_n - x_{n+1}\|^2 + 2\|x_{n+1} - x_n\|^2 \le 4\|x_n - x_{n+1}\|^2 + 2r_n.\end{aligned}$$

Since $\lim_{n\to\infty} r_n = 0$, so from (3.7), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.8}$$

From $z_n = \alpha_n f_n(y_n) + (1-\alpha_n)T(\mu_n)W_n t_n$, (3.3), (3.2) and Lemma 2.1, we have

$$\begin{aligned}\|z_n - p\|^2 &= \|\alpha_n f_n(y_n) + (1-\alpha_n)T(\mu_n)W_n t_n - p\|^2 \\ &\le \alpha_n\|f_n(y_n) - p\|^2 + (1-\alpha_n)\|T(\mu_n)W_n t_n - p\|^2 \\ &\le \alpha_n[\|f_n(y_n) - f_n(p)\| + \|f_n(p) - p\|]^2 + (1-\alpha_n)\|t_n - p\|^2 \\ &\le \alpha_n\|y_n - p\|^2 + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|y_n - p\|] \\ &\quad + (1-\alpha_n)\|t_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_n - p\|] \\ &\quad + (1-\alpha_n)[\|x_n - p\|^2 + (\zeta_n^2 k^2 - 1)\|x_n - y_n\|^2].\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - y_n\|^2 &\le \frac{1}{(1-\alpha_n)(1-\zeta_n^2 k^2)}\big(\|x_n - p\|^2 - \|z_n - p\|^2 \\ &\qquad + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_n - p\|]\big) \\ &\le \frac{1}{(1-\alpha_n)(1-\zeta_n^2 k^2)}\big([\|x_n - p\| + \|z_n - p\|]\|x_n - z_n\| \\ &\qquad + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_n - p\|]\big) \\ &\le \frac{1}{(1-\alpha_n)(1-\zeta_n^2 k^2)}\big([2\|x_n - p\| + r_n]\|x_n - z_n\| \\ &\qquad + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_n - p\|]\big) \\ &\le \frac{1}{(1-\alpha_n)(1-\zeta_n^2 k^2)}\big([2\|x_0 - p\| + r_n]\|x_n - z_n\| \\ &\qquad + \alpha_n\|f_n(p) - p\|[\|f_n(p) - p\| + 2\|x_0 - p\|]\big).\end{aligned} \tag{3.9}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, so from (3.8) and (3.9), we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.10}$$

As A is k-Lipschitz-continuous, we have

$$\begin{aligned}\|y_n - t_n\| &= \|\beta_n x_n + (1-\beta_n)P_C(I - \zeta_n A)x_n - P_C(x_n - \zeta_n A y_n)\| \\ &\le \beta_n\|x_n - P_C(I - \zeta_n A)x_n\| \\ &\quad + (1-\beta_n)\|P_C(I - \zeta_n A)x_n - P_C(x_n - \zeta_n A y_n)\| \\ &\le \beta_n\|x_n - P_C(I - \zeta_n A)x_n\| + (1-\beta_n)\zeta_n k\|x_n - y_n\|\end{aligned}$$

$$\leq \beta_n[\|x_n - p\| + \|p - P_C(I - \zeta_n A)x_n\|] + (1 - \beta_n)\zeta_n k\|x_n - y_n\|$$
$$\leq 2\beta_n\|x_n - p\| + (1 - \beta_n)\zeta_n k\|x_n - y_n\|$$
$$\leq 2\beta_n\|x_0 - p\| + (1 - \beta_n)\zeta_n k\|x_n - y_n\|.$$

Since $\lim_{n\to\infty} \beta_n = 0$, from (3.10), we get

$$\lim_{n\to\infty} \|t_n - y_n\| = 0. \tag{3.11}$$

Noticing that $z_n = \alpha_n f_n(y_n) + (1 - \alpha_n)T(\mu_n)W_n t_n$, we have

$$z_n - y_n = \alpha_n(f_n(y_n) - y_n) + (1 - \alpha_n)[T(\mu_n)W_n t_n - y_n].$$

It follows that

$$(1 - c)\|T(\mu_n)W_n t_n - y_n\| \leq (1 - \alpha_n)\|T(\mu_n)W_n t_n - y_n\|$$
$$\leq \alpha_n\|f_n(y_n) - y_n\| + \|z_n - y_n\|$$
$$\leq \alpha_n[\|f_n(y_n) - f_n(p)\| + \|f_n(p) - p\| + \|p - y_n\|] + \|z_n - y_n\|$$
$$\leq \alpha_n[\psi(\|y_n - p\|) + \|f_n(p) - p\| + \|p - y_n\|] + \|z_n - y_n\|$$
$$\leq \alpha_n[\|y_n - p\| + \|f_n(p) - p\| + \|p - y_n\|] + \|z_n - y_n\|$$
$$\leq \alpha_n[2\|x_n - p\| + \|f_n(p) - p\|] + \|z_n - y_n\|$$
$$\leq \alpha_n[2\|x_0 - p\| + \|f_n(p) - p\|] + \|z_n - x_n\| + \|x_n - y_n\|.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, from (3.8) and (3.10), we get

$$\lim_{n\to\infty} \|T(\mu_n)W_n t_n - y_n\| = 0. \tag{3.12}$$

From Lemma 2.1, we have

$$\|x_n - T(\mu_n)W_n x_n\|$$
$$\leq \|x_n - y_n\| + \| y_n - T(\mu_n)W_n t_n\| + \|T(\mu_n)W_n t_n - T(\mu_n)W_n x_n\|$$
$$\leq \|x_n - y_n\| + \| y_n - T(\mu_n)W_n t_n\| + \|t_n - x_n\|$$
$$\leq \|x_n - y_n\| + \| y_n - T(\mu_n)W_n t_n\| + \|t_n - y_n\| + \|y_n - x_n\|.$$

It follows from (3.10), (3.11) and (3.12) that

$$\lim_{n\to\infty} \|x_n - T(\mu_n)W_n x_n\| = 0. \tag{3.13}$$

Set $D = \{y \in C : \|y - x_0\| \leq 2\|x_0 - p\|\}$, for $p \in \mathcal{F}$. We remark that D is a bounded closed convex set, from (3.2) and (3.6), $\{t_n\} \subset D$ and $\{x_n\} \subset D$, and it is invariant under φ and W_n. As it was proved in [11], [15], [18], we have

$$\limsup_{n\to\infty} \sup_{x\in D} \|T(\mu_n)x - T(t)T(\mu_n)x\| = 0, \quad \text{for all } t \in S. \tag{3.14}$$

We now claim that

$$\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0, \quad \text{for all } t \in S. \tag{3.15}$$

Let $t \in S$ and $\epsilon > 0$. As in the proof of Shioji and Takahashi [20, Lemma 1], there exists $\delta > 0$ such that

$$\overline{\text{co}}\, F_\delta(T(t); D) + B_\delta \subset F_\varepsilon(T(t); D). \tag{3.16}$$

Since $\{W_n t_n\} \subset D$, from (3.14) there exists $N_2 \in \mathbb{N}$ such that

$$T(\mu_n) W_n t_n \in F_\delta(T_t; D), \quad n \geq N_2. \tag{3.17}$$

Observe that

$$\begin{aligned}\|f_n(y_n) - T(\mu_n)W_n t_n\| &\\ &\leq \|f_n(y_n) - f_n(p)\| + \|f_n(p) - p\| + \|p - T(\mu_n)W_n t_n\| \\ &\leq \|y_n - p\| + \|f_n(p) - p\| + \|p - t_n\| \\ &\leq 2\|x_n - p\| + \|f_n(p) - p\| \leq 2\|x_0 - p\| + \|f_n(p)\| + \|p\|.\end{aligned} \tag{3.18}$$

Since $\{f_n(p)\}_{n=1}^{\infty}$ converges and $\lim\limits_{n\to\infty} \alpha_n = 0$, from (3.18), there exists $N_3 \in \mathbb{N}$ such that

$$\alpha_n(f_n(y_n) - T(\mu_n)W_n t_n) \in B_\delta, \quad n \geq N_3. \tag{3.19}$$

Observe that

$$\begin{aligned} z_n &= \alpha_n f_n(y_n) + (1 - \alpha_n) T(\mu_n) W_n t_n \\ &= \alpha_n(f_n(y_n) - T(\mu_n)W_n t_n) + T(\mu_n)W_n t_n. \end{aligned}$$

It follows from (3.17) and (3.19) that $z_n \in F_\varepsilon(T_t; D)$ for all $n \geq N = \max\{N_2, N_3\}$. Since $t \in S$ and $\varepsilon > 0$ are arbitrary, we get

$$\lim_{n\to\infty} \|z_n - T(t)z_n\| = 0, \quad \text{for all } t \in S. \tag{3.20}$$

Noticing that

$$\begin{aligned} \|x_n - T(t)x_n\| &\leq \|x_n - z_n\| + \|z_n - T(t)z_n\| + \|T(t)z_n - T(t)x_n\| \\ &\leq 2\|x_n - z_n\| + \|z_n - T(t)z_n\|, \end{aligned}$$

from (3.8) and (3.20), we get (3.15). Now we prove the weak ω-limit set of $\{x_n\}$, $\omega_\omega\{x_n\}$, is a subset of $\mathcal{F}$. Let $z \in \omega_\omega\{x_n\}$ and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Now, we prove that $z \in \text{Fix}(\varphi)$. Assume by contradiction that there exists $t \in S$ such that $z \neq T(t)z$. Since every Hilbert space satisfies the Opial condition, from (3.20) we have

$$\begin{aligned} \limsup_{j\to\infty} \|x_{n_j} - z\| &< \limsup_{j\to\infty} \|x_{n_j} - T(t)z\| \\ &\leq \limsup_{j\to\infty} (\|x_{n_j} - T(t)x_{n_j}\| + \|T(t)x_{n_j} - T(t)z\|) \\ &\leq \limsup_{j\to\infty} (\|x_{n_j} - T(t)x_{n_j}\| + \|x_{n_j} - z\|) \leq \limsup_{j\to\infty}(\|x_{n_j} - z\|) \end{aligned}$$

which derives a contradiction. Thus, we have $z \in \text{Fix}(\varphi)$. By our assumption, we have $T_i z \in \text{Fix}(\varphi)$ for all $i \in \mathbb{N}$ and then $W_n z \in \text{Fix}(\varphi)$, hence $T(\mu_n)W_n z = W_n z$.

As in the proof of Step 7 of [15, Theorem 3.1], we can show that $z \in \mathrm{Fix}(W)$. In terms of Lemma 2.9, we conclude that $z \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. As in the proof of Step 7 of [17, Theorem 3.1], we can show that $z \in \mathrm{VI}(C, A)$. Since $z \in \mathrm{Fix}(\varphi)$ and $z \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$; therefore, $z \in \mathcal{F}$. So, $\emptyset \neq \omega_\omega\{x_n\} \subset \mathcal{F}$. Since $x_n = P_{Q_n} x_0$ and $P_{\mathcal{F}} x_0 \subset \mathcal{F} \subset Q_n$, we have $\|x_n - x_0\| \leq \|x_0 - P_{\mathcal{F}} x_0\|$. By the lower semicontinuity of the norm, we have $\|w - x_0\| \leq \|x_0 - P_{\mathcal{F}} x_0\|$ for all $w \in \omega_\omega\{x_n\}$. However, since $\omega_\omega\{x_n\} \subset \mathcal{F}$, we must have $w = P_{\mathcal{F}} x_0$ for all $w \in \omega_\omega\{x_n\}$. Hence $x_n \rightharpoonup P_{\mathcal{F}} x_0$. To see that $x_n \to P_{\mathcal{F}} x_0$, we compute

$$\begin{aligned}
\|x_n - P_{\mathcal{F}} x_0\|^2 &= \|(x_n - x_0) + (x_0 - P_{\mathcal{F}} x_0)\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - P_{\mathcal{F}} x_0\rangle + \|x_0 - P_{\mathcal{F}} x_0\|^2 \\
&\leq 2\langle x_n - x_0, x_0 - P_{\mathcal{F}} x_0\rangle + 2\|x_0 - P_{\mathcal{F}} x_0\|^2 \\
&= -2\langle x_0 - x_n, x_0 - P_{\mathcal{F}} x_0\rangle + 2\|x_0 - P_{\mathcal{F}} x_0\|^2 \to 0.
\end{aligned}$$

That is, $\{x_n\}$ converges to $P_{\mathcal{F}} x_0$. It is easy to see that $\{y_n\}$ converges to $P_{\mathcal{F}} x_0$ and $\{z_n\}$ converges to $P_{\mathcal{F}} x_0$. □

Theorem 3.2. *Let C, $\{T_n\}_{n=1}^{\infty}$, S, φ, X, $\{\mu_n\}_{n=0}^{\infty}$, $\mathcal{F}$, $\{\zeta_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be as in Theorem 3.1. Define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ in C by the iteration algorithm*

$$\tag{3.21}
\begin{cases}
x_0 \in C, \\
y_n = \beta_n x_n + (1 - \beta_n) P_C (I - \zeta_n A) x_n, \\
z_n = \alpha_n y_n + (1 - \alpha_n) T(\mu_n) W_n P_C (x_n - \zeta_n A y_n), \\
C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0.
\end{cases}$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ converge strongly to $P_{\mathcal{F}} x_0$.

Proof. It suffices to replace f_n by I (identity mapping) for every $n \in \mathbb{N}$ in the proof of Theorem 3.1. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}_{n=1}^{\infty}$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ be sequences such that $\{\zeta_n\}_{n=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\}_{n=0}^{\infty} \subset [0, c]$ for some $c \in [0, 1)$ and W_n be the mapping generated by $\{T_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ as in (2.5). Define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ in C by the iteration* *algorithm*

$$
(3.22)\qquad \begin{cases} x_0 \in C, \\ y_n = P_C(I - \zeta_n A)x_n, \\ z_n = \alpha_n y_n + (1-\alpha_n)W_n P_C(x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases}
$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *converge strongly to* $P_{\mathcal{F}}x_0$.

Proof. It suffices to take $T(t) = I$, for all $t \in S$ in Theorem 3.1 and replace f_n by I (identity mapping) for every $n \in \mathbb{N}$ in the proof of Theorem 3.1. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H and A be a monotone and k-Lipschitz-continuous mapping of C into H. Let S be a semigroup and $\varphi = \{T(t) : t \in S\}$ be a nonexpansive semigroup of C into itself such that $\mathcal{F} = \mathrm{VI}(C, A) \cap \mathrm{Fix}(\varphi) \neq \emptyset$. Let X be a left invariant subspace of $L^\infty(S)$ such that $1 \in X$, $t \mapsto \langle T(t)x, y\rangle$ an element of X for each $x, y \in C$ and $\{\mu_n\}_{n=0}^{\infty}$ is a left regular sequence of means on X. Let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ be sequences such that $\{\zeta_n\}_{n=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\}_{n=0}^{\infty} \subset [0, c]$ for some $c \in [0, 1)$. Define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ in C by the iteration algorithm*

$$
(3.23)\qquad \begin{cases} x_0 \in C, \\ y_n = P_C(I - \zeta_n A)x_n, \\ z_n = \alpha_n y_n + (1-\alpha_n)T(\mu_n) P_C(x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases}
$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *converge strongly to* $P_{\mathcal{F}}\, x_0$.

Proof. It suffices to take $\beta_n = 0$ and $W_n = I$, for all $n \in \mathbb{N}$ in Theorem 3.1 and replace f_n by I for every $n \in \mathbb{N}$ in the proof of Theorem 3.1. □

Corollary 3.5 ([14, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and S be nonexpansive mappings of C into itself such that $\mathcal{F} = \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$ be sequences such that $\{\zeta_n\}_{n=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\}_{n=0}^{\infty} \subset [0, c]$ for some $c \in [0, 1)$.*

Define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$\begin{cases} x_0 \in C, \\ y_n = P_C(I - \zeta_n A)x_n, \\ z_n = \alpha_n y_n + (1-\alpha_n) S P_C(x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases} \tag{3.24}$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *converge strongly to* $P_{\mathcal{F}}x_0$.

Proof. It suffices to take $\beta_n = 0$ and $W_n = S$, for all $n \in \mathbb{N}$ in Theorem 3.1 and replace f_n by I for every $n \in \mathbb{N}$ in the proof of Theorem 3.1. □

Corollary 3.6 ([14, Theorem 4.1]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be a monotone and* k*-Lipschitz-continuous mapping of* C *into* H *such that* $\mathcal{F} = \mathrm{VI}(C, A) \neq \emptyset$. *Let* $\{\zeta_n\}_{n=0}^{\infty}$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *be sequences such that* $\{\zeta_n\}_{n=0}^{\infty} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$ *and* $\{\alpha_n\}_{n=0}^{\infty} \subset [0, c]$ *for some* $c \in [0, 1)$. *Define sequences* $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$\begin{cases} x_0 \in C, \\ y_n = P_C(I - \zeta_n A)x_n, \\ z_n = P_C(x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x. \end{cases} \tag{3.25}$$

Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *converge strongly to* $P_{\mathcal{F}}x_0$.

Proof. It suffices to take $\alpha_n = \beta_n = 0$ and $W_n = I$, for all $n \in \mathbb{N}$ and $T(t) = I$, for all $t \in S$ in Theorem 3.1. □

Example 3.7. Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a nonexpansive mapping of C into itself. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of ψ_n-contraction self-mappings of C such that $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent for any $x \in D$, where D is any bounded subset of C. Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself such that, for all $n \in \mathbb{N}$, $T_n(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and $\mathcal{F} = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \cap \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ be nonempty and bounded. Let $\{\zeta_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be sequences such that $\{\zeta_n\}_{n=0}^{\infty} \subset [a, b]$ for some $a, b \in (0, 1/k)$, $\{\alpha_n\}_{n=0}^{\infty} \subset [0, c]$ for some $c \in [0, 1)$,

$\lim_{n\to\infty} \alpha_n = 0$, $\{\beta_n\}_{n=0}^{\infty} \subset [0,1)$, $\lim_{n\to\infty} \beta_n = 0$ and W_n be the mapping generated by $\{S_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ as in (2.5). Define sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ in C by the iteration algorithm

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) P_C (I - \zeta_n A) x_n, \\ z_n = \alpha_n f_n(y_n) + (1-\alpha_n) \dfrac{2}{n^2+n} \displaystyle\sum_{k=0}^{n} W_n P_C (x_n - \zeta_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 + r_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3.26}$$

where $r_n = \alpha_n \delta_n$ and $\delta_n = \sup\{\|f_n(p) - p\| [\|f_n(p) - p\| + 2\|x_n - p\|] : p \in \mathcal{F}\}$. Then the sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ and $\{z_n\}_{n=0}^{\infty}$ converge strongly to $P_{\mathcal{F}} x_0$.

Proof. Let $S = \{0, 1, \ldots\}$ and $\varphi = \{T^n : n \in S\}$. For each $f = (x_0, x_1, \ldots)$ in $B(S)$, define

$$\mu_n = \frac{2}{n^2+n} \sum_{k=0}^{n} k x_k.$$

Then $\{\mu_n\}_{n=1}^{\infty}$ is a left regular sequence of means on $B(S)$. In fact, for $f \in B(S)$,

$$|\mu_n(f)| \le \frac{2}{n^2+n} \sum_{k=0}^{n} k|x_k| \le \frac{2}{n^2+n} \sum_{k=0}^{n} k\|f\| = \|f\|,$$

and

$$\mu_n(1) = \frac{2}{n^2+n} \sum_{k=0}^{n} k = 1.$$

It follows that $\|\mu_n\| = \mu_n(1) = 1$, i.e., μ_n is a mean on $B(S)$. Next, for each $f \in B(S)$ and $m \in S$,

$$\begin{aligned} |\mu_n(f) - \mu_n(l_m f)| &= \left| \frac{2}{n^2+n} \sum_{k=0}^{n} k x_k - \frac{2}{n^2+n} \sum_{k=0}^{n} k x_{k+m} \right| \\ &= \frac{2}{n^2+n} \left| \sum_{k=0}^{m} k x_k + \sum_{k=m+1}^{n} k x_k - \sum_{k=0}^{n-m} k x_{k+m} - \sum_{k=n-m+1}^{n} k x_{k+m} \right| \\ &= \frac{2}{n^2+n} \left| \sum_{k=0}^{m} k x_k + m \sum_{k=m+1}^{n} x_k - \sum_{k=n-m+1}^{n} k x_{k+m} \right| \\ &\le \frac{2\|f\|}{n^2+n} \left[\sum_{k=0}^{m} k + m \sum_{k=m+1}^{n} 1 - \sum_{k=n-m+1}^{n} k \right] \\ &= \frac{2\|f\|}{n^2+n} \left[\sum_{k=0}^{m} k + m \sum_{k=m+1}^{n} 1 - \sum_{k=n-m+1}^{n-m+m} k \right] \end{aligned}$$

$$= \frac{2\|f\|}{n^2+n}\left[2\sum_{k=0}^{m} k + 2m(n-m)\right] = \frac{2\|f\|}{n^2+n}\,[2mn + m - m^2],$$

for every $n \in \mathbb{N}$. So, we get $\lim_{n\to\infty} |\mu_n(f) - \mu_n(l_m f)| = 0$. Hence $\{\mu_n\}_{n=1}^{\infty}$ is a left regular sequence of means on $B(S)$. Further, for $x \in C$ and $y \in H$,

$$(\mu_n)_k \langle T^k x, y\rangle = \frac{2}{n^2+n}\sum_{k=0}^{m} k\langle T^k x, y\rangle = \left\langle \frac{2}{n^2+n}\sum_{k=0}^{m} kT^k x, y\right\rangle$$

and hence

$$T(\mu_n)x = \frac{2}{n^2+n}\sum_{k=0}^{m} kT^k x.$$

Therefore, the result follows from Theorem 3.1. □

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