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HÉNON TYPE EQUATIONS WITH ONE-SIDED EXPONENTIAL GROWTH

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Abstract. We deal with the following class of problems:

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u_+) + f(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1 is the unit ball in $\mathbb{R}^2$, g is a C^1-function in $[0,+\infty)$ which is assumed to be in the subcritical or critical growth range of Trudinger–Moser type and $f \in L^\mu(B_1)$ for some $\mu > 2$. Under suitable hypotheses on the constant λ, we prove existence of at least two solutions to this problem using variational methods. In case of f radially symmetric, the two solutions are radially symmetric as well.

1. Introduction

In this paper we study the solvability of problems of the type

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u_+) + f(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1}$$

where $\lambda, \alpha \geq 0$ and $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. Here we assume that g has the maximum growth which allows us to treat problem (1.1) variationally in

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suitable Sobolev spaces, due to the well-known Trudinger–Moser inequality (see [18], [28]), which, in two dimensions, is given by

$$\sup_{\substack{u\in H_0^1(B_1)\\ \|\nabla u\|_2=1}} \int_{B_1} e^{\beta u^2}\,dx \begin{cases} < +\infty & \text{if } \beta \le 4\pi, \\ = +\infty & \text{if } \beta > 4\pi. \end{cases} \tag{1.2}$$

Working with a Hénon type problem in $H^1_{0,\mathrm{rad}}(B_1) \subset H^1_0(B_1)$, we observe that the weight $|x|^\alpha$ changes this fact. Indeed, one has

$$\sup_{\substack{u\in H_{0,\mathrm{rad}}^1(B_1)\\ \|\nabla u\|_2=1}} \int_{B_1} |x|^\alpha e^{\beta u^2}\,dx \begin{cases} < +\infty & \text{if } \beta \le 2\pi(2+\alpha), \\ = +\infty & \text{if } \beta > 2\pi(2+\alpha), \end{cases} \tag{1.3}$$

see [3] and [8]. Motivated by (1.2)–(1.3), we say that g has subcritical growth at $+\infty$ if

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta, \tag{1.4}$$

and g has critical growth at $+\infty$ if there exists $\beta_0 > 0$ such that

$$\lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = 0 \quad \text{for all } \beta > \beta_0; \qquad \lim_{t\to+\infty} \frac{g(t)}{e^{\beta t^2}} = +\infty \quad \text{for all } \beta < \beta_0. \tag{1.5}$$

1.1. Hypotheses. Before stating our main results, we shall introduce the following assumptions on the non-linearity g:

(g_0) $g \in C(\mathbb{R}, \mathbb{R}^+)$, $g(s) = 0$ for all $s \le 0$.

(g_1) There exist s_0 and $M > 0$ such that

$$0 < G(s) = \int_0^s g(t)\,dt \le Mg(s) \quad \text{for all } s > s_0.$$

(g_2) $|g(s)| = o(|s|)$ when $|s| \to 0$.

Following the well-established notation in the present literature, we denote by $\lambda_1 < \lambda_2 \le \ldots \le \lambda_j \le \ldots$ the sequence of eigenvalues of $(-\Delta, H^1_0(B_1))$, and by ϕ_j a j^{th} eigenfunction of $(-\Delta, H^1_0(B_1))$.

We observe that, using assumption (g_0), one can see that ψ is a non-positive solution to (1.1) if and only if it is a non-positive solution to the linear problem

$$\begin{cases} -\Delta\psi = \lambda\psi + f(x) & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1. \end{cases} \tag{1.6}$$

In order to get such solutions to (1.6), let us assume that

(f_1) $f(x) = h(x) + t\phi_1(x)$, where $h \in L^\mu(B_1)$, $\mu > 2$ and $\int_{B_1} h\phi_1\,dx = 0$.

For that matter, the parameter t plays a crucial role. We shall use this hypothesis in the first theorem of this paper.

1.2. Statement of main results. We divide our results in four theorems. The first one deals with the first solution to the problem, which is non-positive and is obtained by a simple remark about a linear problem related to our equation. The other theorems concern the second solution and are considered depending on the growth conditions of the non-linearity. In the critical case, since the weight $|x|^\alpha$ has an important role in the estimate of the minimax levels, mainly because of the difference between the versions (1.2) and (1.3) of the Trudinger–Moser inequality, the variational setting and methods used in $H_0^1(B_1)$ and $H_{0,\mathrm{rad}}^1(B_1)$ are different and, therefore, are given in separate theorems.

THEOREM 1.1 (Linear problem). *Assume that $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$ and* (f_1) *holds, then there exists a constant $T = T(h) > 0$ such that*:

(a) If $\lambda < \lambda_1$, there exists $\psi_t < 0$, a solution to (1.6) and, consequently, to (1.1), for all $t < -T$.
(b) If $\lambda > \lambda_1$, there exists $\psi_t < 0$, a solution to (1.6) and, consequently, to (1.1), for all $t > T$.

Furthermore, if f is radially symmetric, then ψ_t is radially symmetric as well.

THEOREM 1.2 (Subcritical case). *Let $f \in L^\mu(B_1)$ with $\mu > 2$ be such that there exists a non-positive solution ψ to* (1.1). *Assume* (g_0)–(g_2), (1.4) *hold and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Then, problem* (1.1) *has a second solution. Furthermore, if ψ is radially symmetric, the second solution is also radially symmetric.*

THEOREM 1.3 (Critical case). *Let $f \in L^\mu(B_1)$ with $\mu > 2$ be such that there exists a non-positive solution ψ to* (1.1). *Assume* (g_0)–(g_2), (1.5) *hold and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. Furthermore suppose that for all $\gamma \geq 0$ there exists $c_\gamma \geq 0$ such that*

$$g(s)s \geq \gamma e^{(\beta_0 s^2)} h(s) \quad \text{for all } s > c_\gamma, \tag{1.7}$$

where $h\colon \mathbb{R} \to \mathbb{R}^+$ is a continuous function satisfying

$$\liminf_{s \to +\infty} \frac{\log(h(s))}{s} > 0. \tag{1.8}$$

Then, problem (1.1) *has a second solution.*

THEOREM 1.4 (Radial critical case). *Let $f \in L^\mu(B_1)$ with $\mu > 2$ be such that there exists a radial and non-positive solution ψ to* (1.1). *Assume* (g_0)–(g_2), (1.5), (1.7) *hold with $h\colon \mathbb{R} \to \mathbb{R}^+$ being a continuous function satisfying*

$$\liminf_{s \to +\infty} \frac{\log(h(s))}{s} \geq 4\beta_0 \|\psi\|_\infty, \tag{1.9}$$

where β_0 is given in (1.5). *If $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$, then problem* (1.1) *has a second solution which is radially symmetric.*

Example 1.5. For examples of non-linearities with critical growth satisfying the assumptions of Theorem 1.3, one can consider

$$g(t) = \begin{cases} e^{\beta_0 t^2 + K_0 t}(2\beta_0 t^3 + K_0 t^2 + 2t) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

with $h(s) = e^{K_0 s}$ and $K_0 > 0$. If $K_0 \geq 4\beta_0 \|\psi\|_\infty$, then g also satisfies the hypotheses of Theorem 1.4.

Remark 1.6. We notice that if $\lambda < \lambda_1$, we shall use the Mountain Pass Theorem in the proofs of Theorems 1.2–1.4. On the other hand, if $\lambda > \lambda_1$, we need to use the Linking Theorem.

Remark 1.7. We also must point out an important fact: hypotheses (1.8) and (1.9) are definitely stricter than the one used in [7], since they do not require the additional function h. What happens is that, as far as we know, it is not proven, nor we were able to prove, that the estimate given in Lemma 3.5 of [7] is uniform in $m = m(n)$. That estimate concerns the minimax level $c(k)$ and, although it was not clear in [7], this level actually depends also on the parameter n, which surely changes with the considered terms of the Moser functions $z_{k,r(n)}$. So, a uniform estimate in this level must be proven in order to conclude that the weak limit of the associated PS sequence is actually non-trivial, as established in Proposition 3.4 of [7].

In Theorems 1.2–1.4, we assume that f is such that (1.1) admits a non-positive solution ψ. Then, a second solution will be given by $u + \psi$, where u is a non-trivial solution to the following problem:

$$\begin{cases} -\Delta u = \lambda u + |x|^\alpha g(u + \psi)_+ & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{1.10}$$

This means that we will focus our attention on looking for a non-trivial solution to problem (1.10), considering each of the cases listed above. This translation has an advantage of working with a homogeneous problem, without a forcing term that could hinder the desired estimates on the minimax levels. On the other hand, the function ψ plays this role in complicating the estimates, but we could handle them by performing some delicate arguments involving the additional hypothesis (1.7), with h satisfying (1.8) or (1.9).

1.3. Backgrounds. We intend to establish a link between Ambrosetti–Prodi problems and Hénon type equations. The first ones begin with the celebrate paper by A. Ambrosetti and G. Prodi [1] in 1975. This kind of problems has been studied, explored and extended by an enormous number of authors.

For a brief review, we refer the reader to [12], [9], [22]. In short, it deals with non-homogeneous problems such as

$$\begin{cases} -\Delta u = g(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where "g jumps eigenvalues", meaning that the limits

$$\lim_{s\to-\infty} \frac{g(s)}{s} < \lim_{s\to+\infty} \frac{g(s)}{s}$$

form an interval that contains at least one eigenvalue of $(-\Delta, H_0^1(B_1))$. Moreover, the existence of multiple solutions depends heavily on a usual hypothesis regarding a suitable parametrization of the forcing term f, which is extensively considered in almost all Ambrosetti–Prodi problems starting from the work of J. Kazdan and F. Warner in [15].

Since our work deals with non-linearities in critical growth range, let us focus our discussion on more specifically related studies. As far as we know, the first paper that addressed an Ambrosetti–Prodi problem involving critical non-linearities was the work of D. YinBin [29]. There, the author considered a function that was superlinear both in $+\infty$ and $-\infty$ and, so, asymptotically jumping all the eigenvalues of $(-\Delta, H_0^1(\Omega))$. In [12], D. Figueiredo and J. Yang dealt with one-sided critical growth, which allowed to explore more jumping possibilities. Due to natural limitations on the techniques they used, the existence of multiple solutions was investigated in dimensions $N \geq 7$ only. In [6], M. Calanchi and B. Ruf extended their results and proved that the same problem has at least two solutions provided $N \geq 6$. They also added a lower order growth term to the non-linearity, which guaranteed the existence of solutions in lower dimensions as well.

If we turn our attention to problems on domains in $\mathbb{R}^2$, where the critical growth is well known to be exponential, we find the paper of M. Calanchi et al. [7], which considered the following problem:

$$\begin{cases} -\Delta u = \lambda u + g(u_+) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is bounded and smooth in $\mathbb{R}^2$ and g has a subcritical or critical Trudinger–Moser growth. They proved the existence of two solutions for some forcing terms f, depending on the usual parametrization $f(x) = h(x) + t\phi_1(x)$. This result was a natural extension of the ones found in [6], [12].

Here we consider a similar problem with the weight $|x|^\alpha$ of the non-linearity. This term is proper for Hénon equations, originally introduced by M. Hénon in [14], which stems from the study of rotating stellar structures. In [19], W. Ni worked with this kind of problems in the context of the elliptic equations. For an idea about the rapidly increasing literature on Hénon type problems, see [2], [3], [16], [26], [27] for a non-linearity of Sobolev type, and [25], [8] for a non-linearity with Trudinger–Moser growth.

We prove existence of at least two solutions to the Ambrosetti–Prodi problem for a Hénon type equation with exponential growth using variational methods. We get a first solution ψ using the Fredholm Alternative in a related linear equation. A second solution is obtained via the Mountain Pass Theorem, if $\lambda < \lambda_1$, or the Linking Theorem, if $\lambda > \lambda_1$. In order to use these well-known critical point results, we need to prove some geometric conditions satisfied by the functional associated to the problem and, for the critical case, we need to estimate the minimax levels, which have to lie below some appropriate constants.

Under certain condition, we can see that the first solution we obtained is radially symmetric, due to a celebrated result of R. Palais known as the Principle of Symmetric Criticality (see [20]). After that, we get a second solution that is radially symmetric as well. In the subcritical case, we obtain this second solution with almost the same arguments regardless the space we consider, $H^1_0(B_1)$ or $H^1_{0,\mathrm{rad}}(B_1)$. However, in the critical case, we must consider the Trudinger–Moser inequality as given in (1.2), when we work in $H^1_0(B_1)$, or as in (1.3), in $H^1_{0,\mathrm{rad}}(B_1)$, which will dramatically change the arguments. In the last case, the minimax levels have a higher upper boundedness, that should indicate an easier task, but, since we can only work with radial functions, it is impossible to follow the steps we give in the $H^1_0(B_1)$ framework.

1.4. Outline. This paper is organized as follows. In Section 2, we study a linear problem related to our equation. This is an important step to guarantee a first solution to the problem, which will be denoted by ψ and is non-positive. This case is almost identical to the previous results in [24], [7], [6], [12], but we give a little remark concerning existence of a radially symmetric solution when the forcing term is also radial. In Section 3, we introduce a variational framework and prove the boundedness of Palais–Smale sequences of the functional associated to problem (1.10). We also show that this functional satisfies the (PS) condition, in the subcritical case. In Section 4, we obtain geometric conditions for the functional in order to prove the existence of a second solution to the problem, considering the subcritical growth both in $H^1_0(B_1)$ and $H^1_{0,\mathrm{rad}}(B_1)$. In Sections 5 and 6, we consider the problem in the critical growth range and guarantee the existence of a second solution in $H^1_0(B_1)$ and $H^1_{0,\mathrm{rad}}(B_1)$, respectively. In these cases, we also show that geometric conditions are satisfied and we prove the boundedness of the minimax levels by appropriate constants depending on β_0 when we consider the functional defined in $H^1_0(B_1)$ and on β_0 and α when the functional is considered in $H^1_{0,\mathrm{rad}}(B_1)$.

2. Linear problem

In this section, we prove Theorem 1.1, for that we consider the linear problem

$$\begin{cases} -\Delta\psi = \lambda\psi + f(x) & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1. \end{cases} \tag{2.1}$$

It is easy to see that if f is such that this linear problem admits a non-positive solution, then it will also be a solution to problem (1.1). Considering f decomposed as in (f_1), we will see that the sign of the (unique) solution to (2.1) can be established depending on t. Moreover, we give an idea on how to obtain radial solutions as well.

Proof. Up to the point where we discuss the radial case, this proof follows exactly the same arguments found in [24], [7], [6], [12], but we bring it here for the sake of completeness.

Since $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$, we obtain, by the Fredholm Alternative, a unique solution ψ to (2.1) in $H_0^1(B_1)$. Using (f_1), we can write $\psi = \psi_t = \omega + s_t\phi_1$, with

$$\int_{B_1} \omega\phi_1 \, dx = 0 \quad \text{and} \quad s_t = \frac{t}{\lambda_1 - \lambda}.$$

We recall that $f \in L^\mu(B_1)$, with $\mu > 2$. Thus, by the elliptic regularity, we have $\omega \in C^{1,\nu}$ for some $0 < \nu < 1$. Then

$$\left\| \frac{\lambda_1 - \lambda}{t}\,\psi - \phi_1 \right\|_{C^1} = \left| \frac{\lambda_1 - \lambda}{t} \right| \|\omega\|_{C^1}.$$

Let $\varepsilon > 0$ be such that $v > 0$ for all $v \in C^{1,\nu}$ such that $\|v - \phi_1\|_{C^1} < \varepsilon$. Since we want $\psi < 0$, we must have

$$\frac{\lambda - \lambda_1}{t} > 0 \quad \text{and} \quad \frac{\lambda_1 - \lambda}{t}\,\|\omega\|_{C^1} < \varepsilon.$$

So there exists $T > 0$ such that for $\lambda < \lambda_1$ and $t < -T$ or for $\lambda > \lambda_1$ and $t > T$, we have $\psi < 0$.

Now we notice that ψ is a solution to (2.1) if only if it is a critical point of the functional $I\colon H_0^1(B_1) \to \mathbb{R}$, given by

$$I(\psi) := \frac{1}{2}\int_{B_1} |\nabla\psi|^2 \, dx - \lambda\frac{1}{2}\int_{B_1} \psi^2 \, dx - \int_{B_1} f(x)\psi \, dx.$$

When we restrict I to $H^1_{0,\mathrm{rad}}(B_1)$, we also obtain a critical point of this functional on this subspace. If f is radially symmetric, by the Principle of Symmetric Criticality of Palais (see [20]), we can see that all critical points on $H^1_{0,\mathrm{rad}}(B_1)$ are also critical points on $H_0^1(B_1)$. So, due to the fact that I admits only one critical point in the whole space, we get that ψ is also radially symmetric. This completes the proof of Theorem 1.1. □

3. Variational formulation

We consider a real C^1-functional Φ defined on a Banach space E. When looking for critical points of Φ it has become standard to assume the following compactness condition:

$(\mathrm{PS})_c$ any sequence (u_j) in E such that

$$\Phi(u_j) \to c \quad \text{and} \quad \Phi'(u_j) \to 0 \quad \text{in } E^* \tag{3.1}$$

has a convergent subsequence. A sequence that satisfies (3.1) is called a (PS) sequence (at level c).

We shall use two well-known critical point theorems, namely, the Mountain Pass and Linking Theorems, to $\lambda < \lambda_1$ and $\lambda_j < \lambda < \lambda_{j+1}$, respectively, both without the (PS) conditions in the case of critical growth range. Let us state them here, for the sake of completeness. For the proofs we refer the reader to [4], [5], [10], [17], [21].

Theorem 3.1 (Mountain Pass Theorem). *Let Φ be a C^1-functional on a Banach space E satisfying*:

(Φ1) *There exist constants $\rho, \delta > 0$ such that $\Phi(u) \geq \delta$ if $u \in E$ and $\|u\| = \rho$.*
(Φ2) *$\Phi(0) < \delta$ and $\Phi(v) < \delta$ for some $v \in E$ such that $\|v\| > \rho$.*

Consider $\Gamma := \{\eta \in C([0,1], E) : \eta(0) = 0$ and $\eta(1) = v\}$ and set

$$c = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} \Phi(\eta(t)) \geq \delta.$$

Then there exists a sequence (u_j) in E satisfying (3.1). *Moreover, if Φ satisfies $(\mathrm{PS})_c$, then c is a critical value for Φ.*

Theorem 3.2 (Linking Theorem). *Let Φ be a C^1-functional on a Banach space $E = E_1 \oplus E_2$ such that $\dim E_1$ is finite and*:

(Φ1) *There exist constants $\delta, \rho > 0$ such that $\Phi(u) \geq \delta$ if $u \in E_2$ and $\|u\| = \rho$.*
(Φ2) *There exists $z \in E_2$ with $\|z\| = 1$ and there exists $R > \rho$ such that*

$$\Phi(u) \leq 0 \quad \text{for all } u \in \partial Q,$$

where $Q := \{v + sz : v \in E_1, \|v\| \leq R$ and $0 \leq s \leq R\}$.

Consider $\Gamma := \{\eta \in C(\overline{Q}, E) : \eta(u) = u$ if $u \in \partial Q\}$ and set

$$c = \inf_{\eta\in\Gamma} \max_{u\in Q} \Phi(\eta(u)) \geq \delta.$$

Then there exists a sequence (u_j) in E satisfying (3.1). *Moreover, if Φ satisfies $(\mathrm{PS})_c$, then c is a critical value for Φ.*

We will denote $H = H_0^1(B_1)$ or $H = H_{0,\mathrm{rad}}^1(B_1)$, depending on which theorem we are considering, with the Dirichlet norm

$$\|u\| = \bigg(\int_{B_1} |\nabla u|^2\,dx\bigg)^{1/2} \quad \text{for all } u \in H.$$

Recalling that our efforts are searching for a non-trivial weak solution to problem (1.10), we define the functional $J_\lambda\colon H \to \mathbb{R}$ as

$$J_\lambda(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2\,dx - \frac{\lambda}{2}\int_{B_1} |u|^2\,dx - \int_{B_1} |x|^\alpha G(u+\psi)_+\,dx, \tag{3.2}$$

where $\alpha, \lambda \geq 0$ and $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. By (g_0), (g_1) and (1.4) or (1.5), we have that the functional J_λ is C^1 and we can see that its derivative is given by

$$\langle J'_\lambda(u), v\rangle = \int_{B_1} \nabla u \nabla v\,dx - \lambda\int_{B_1} uv\,dx - \int_{B_1} |x|^\alpha g(u+\psi)_+ v\,dx, \tag{3.3}$$

for all $v \in H$ and the critical points of J_λ are (weak) solutions to (1.10). We observe that $u = 0$ satisfies $J'_\lambda(0) = 0$, which corresponds to the negative solution ψ to (1.6). To find a second solution to (1.1) we shall look for critical points of the functional J_λ with critical values $c > 0$.

3.1. Palais–Smale condition. Initially, from (g_1), we can see that for every $\sigma > 0$ there exists $s_\sigma > 0$ such that

$$0 < G(s) \leq \frac{1}{\sigma}\,g(s)s \quad \text{for all } s \geq s_\sigma. \tag{3.4}$$

The proof of the Palais–Smale condition for the subcritical case is essentially standard. For the reader's convenience, we sketch the proof in the next lemma. We must point out that it is not necessary to suppose that $g(s)$ is $O(s^2)$ at $s = 0$ as required in [7]. The well-known techniques we use here can also be handled in case $\alpha = 0$ and for that it is sufficient to admit that $g(s) = o(s)$ at $s = 0$, following usual assumptions.

LEMMA 3.3. *Suppose* (g_0)–(g_2) *hold. Let* $(u_n) \subset H$ *be a* $(\mathrm{PS})_c$ *sequence of* J_λ. *Then* (u_n) *is bounded in* H.

PROOF. Let $(u_n) \subset H$ be a $(\mathrm{PS})_c$ sequence of J_λ, that is,

$$\bigg|\frac{1}{2}\,\|u_n\|^2 - \frac{\lambda}{2}\,\|u_n\|_2^2 - \int_{B_1} |x|^\alpha G(u_n+\psi)_+\,dx - c\bigg| \to 0 \tag{3.5}$$

and

$$\bigg|\int_{B_1} \nabla u_n \nabla v\,dx - \lambda\int_{B_1} u_n v\,dx - \int_{B_1} |x|^\alpha g(u_n+\psi)_+ v\,dx\bigg| \leq \varepsilon_n\|v\| \tag{3.6}$$

for all $v \in H$, where $\varepsilon_n \to 0$. By (3.4), let us take $s_0 > 0$ such that

$$G(s) \leq \frac{1}{4}\,g(s)s \quad \text{for all } s \geq s_0. \tag{3.7}$$

Using (3.5)–(3.7), we get

$$c + \frac{\varepsilon_n\|u_n\|}{2} \geq \frac{1}{4}\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n+\psi)_+\,dx - C_0.$$

Thus, we have

$$\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n+\psi)_+\,dx \leq C + \varepsilon_n\|u_n\| \quad \text{with } C>0 \tag{3.8}$$

and, consequently,

$$\int_{B_1} |x|^\alpha g(u_n+\psi)_+\,dx \leq C + \varepsilon_n\|u_n\|. \tag{3.9}$$

First, we consider the case $0 \leq \lambda < \lambda_1$. Using (3.6) and (3.8), we have

$$\varepsilon_n\|u_n\| \geq \left(1 - \frac{\lambda}{\lambda_1}\right)\|u_n\|^2 - (C + \varepsilon_n\|u_n\|).$$

Thus, (u_n) is a bounded sequence.

Now we consider the case: $\lambda_k < \lambda < \lambda_{k+1}$. It is convenient to decompose H into appropriate subspaces:

$$H = H_k \oplus H_k^\perp, \tag{3.10}$$

where H_k is the finite dimensional subspace spanned by first k eigenfunctions, precisely,

$$H_k = \langle \phi_1, \ldots, \phi_k\rangle. \tag{3.11}$$

This notation is standard when dealing with this framework of high order eigenvalues, and we will use it throughout this paper.

For all u in H, let us take $u = u^k + u^\perp$, where $u^k \in H_k$ and $u^\perp \in H_k^\perp$. We notice that

$$\int_{B_1} \nabla u \nabla u^k\,dx - \lambda\int_{B_1} u u^k\,dx = \|u^k\|^2 - \lambda\|u^k\|_2^2 \tag{3.12}$$

and

$$\int_{B_1} \nabla u \nabla u^\perp\,dx - \lambda\int_{B_1} u u^\perp\,dx = \|u^\perp\|^2 - \lambda\|u^\perp\|_2^2. \tag{3.13}$$

By (3.6), (3.12) and the characterization of λ_k, we can see that

$$\begin{aligned}
-\varepsilon_n\|u_n^k\| &\leq \int_{B_1} \nabla u_n \nabla u_n^k\,dx - \lambda\int_{B_1} u_n u_n^k\,dx - \int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n^k\,dx \\
&\leq \left(1 - \frac{\lambda}{\lambda_k}\right)\|u_n^k\|^2 - \int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n^k\,dx.
\end{aligned}$$

Therefore,

$$C\|u_n^k\|^2 \leq \varepsilon_n\|u_n^k\| - \int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n^k\,dx \quad \text{with } C>0. \tag{3.14}$$

Similarly, we get

$$C\|u_n^\perp\|^2 \le \varepsilon_n\|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n^\perp \, dx. \tag{3.15}$$

Then, since H_k is a finite dimensional subspace, by (3.9) and (3.14), we obtain

$$\begin{aligned} C\|u_n^k\|^2 &\le \varepsilon_n\|u_n^k\| + \|u_n^k\|_\infty \int_{B_1} |x|^\alpha g(u_n+\psi)_+ \, dx \\ &\le \varepsilon_n\|u_n^k\| + C\|u_n^k\|(C+\varepsilon_n\|u_n\|) \le C + C\|u_n\| + C\varepsilon_n\|u_n\|^2. \end{aligned} \tag{3.16}$$

Using (3.8), (3.9) and (3.15), we have

$$\begin{aligned} C\|u_n^\perp\|^2 \le{}& \varepsilon_n\|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n \, dx + \|u_n^k\|_\infty(C+\varepsilon_n\|u_n\|) \\ \le{}& \varepsilon_n\|u_n^\perp\| + \int_{B_1} |x|^\alpha g(u_n+\psi)_+ (u_n+\psi) \, dx \\ &+ \|\psi\|_\infty \int_{B_1} |x|^\alpha g(u_n+\psi)_+ \, dx + \|u_n^k\|_\infty(C+\varepsilon_n\|u_n\|) \\ \le{}& \varepsilon_n\|u_n^\perp\| + C + \varepsilon_n\|u_n\| \\ &+ \|\psi\|_\infty \int_{B_1} g(u_n+\psi)_+ \, dx + C\|u_n\|(C+\varepsilon_n\|u_n\|) \\ \le{}& C + C\|u_n\| + C\varepsilon_n\|u_n\|^2. \end{aligned} \tag{3.17}$$

By summing the inequalities in (3.17) and (3.16), we reach

$$\|u_n\|^2 \le C + C\|u_n\| + C\varepsilon_n\|u_n\|^2,$$

proving the boundedness of the sequence (u_n) as desired. □

Remark 3.4. In the proof of Lemma 3.3 there is no difference between assuming subcritical or critical growth or considering the radial case or not. So we can conclude that even in the case of critical growth, every Palais–Smale sequence is bounded.

In case of subcritical growth, we can obtain the $(\mathrm{PS})_c$ condition for all levels in $\mathbb{R}$.

Lemma 3.5. *Assume* (g_0)–(g_2) *and* (1.4) *hold. Then the functional* J_λ *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c \in \mathbb{R}$.

Proof. Let (u_n) be a $(\mathrm{PS})_c$ sequence. By Lemma 3.3, we know that (u_n) is bounded. So we consider a subsequence denoted again by (u_n) such that, for some $u \in H$, we have

$$\begin{aligned} u_n &\rightharpoonup u && \text{weakly in } H,\\ u_n(x) &\to u(x) && \text{almost everywhere in } B_1,\\ u_n &\to u && \text{strongly in } L^q(B_1) \text{ or } L^q_{\rm rad}(|x|^\alpha B_1) \text{ for all } q \geq 1, \end{aligned}$$

Notice that there is nothing else to prove if $\|u_n\| \to 0$. Thus, one may suppose that $\|u_n\| \geq k > 0$ for n sufficiently large.

It follows from (g_1) and [11, Lemma 2.1] that

$$\int_{B_1} |x|^\alpha G(u_n+\psi)_+\,dx \to \int_{B_1} |x|^\alpha G(u+\psi)_+\,dx. \tag{3.18}$$

We will prove that

$$\int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n\,dx \to \int_{B_1} |x|^\alpha g(u+\psi)_+ u\,dx. \tag{3.19}$$

In fact, we have

$$\begin{aligned} &\bigg|\int_{B_1} |x|^\alpha g(u_n+\psi)_+ u_n\,dx - \int_{B_1} |x|^\alpha g(u+\psi)_+ u\,dx\bigg| \\ &\leq \bigg|\int_{B_1} |x|^\alpha [g(u_n+\psi)_+ - g(u+\psi)_+]u\,dx\bigg| + \bigg|\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n-u)\,dx\bigg|. \end{aligned}$$

First, let us focus on the second integral in the left side of this last estimate. By (1.4), we get

$$\bigg|\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n-u)\,dx\bigg| \leq C\int_{B_1} e^{\beta u_n^2}|u_n-u|\,dx \quad \text{for all } \beta > 0.$$

Using the Hölder inequality, we obtain

$$\int_{B_1} e^{\beta u_n^2}|u_n-u|\,dx \leq \bigg(\int_{B_1} e^{q\beta(u_n/\|u_n\|)^2\|u_n\|^2}\,dx\bigg)^{1/q}\|u_n-u\|_{q'},$$

where $1/q+1/q' = 1$. We take $q > 1$ and by (1.4) and Lemma 3.3, we can choose β sufficiently small such that $q\beta\|u_n\|^2 \leq 4\pi$. Thus by the Trudinger–Moser inequality, we have

$$\int_{B_1} e^{\beta u_n^2}|u_n-u|\,dx \leq C_1\|u_n-u\|_{q'}. \tag{3.20}$$

Since $u_n \to u$ strongly in $L^{q'}$, one has

$$\bigg|\int_{B_1} |x|^\alpha g(u_n+\psi)_+(u_n-u)\,dx\bigg| \to 0.$$

It remains to prove that

$$\bigg|\int_{B_1} |x|^\alpha [g(u_n+\psi)_+ - g(u+\psi)_+]u\,dx\bigg| \to 0$$

as well. Indeed. Let $\varepsilon > 0$ be given. By analogous arguments used to prove (3.20), there exists C_2 such that

$$\|g(u_n+\psi)_+ - g(u+\psi)_+\|_{2,|x|^\alpha} \le C_2.$$

Consider $\xi \in C_0^\infty(B_1)$ such that $\|\xi - u\|_2 < \varepsilon/2C_2$. Now, since

$$\int_{B_1} |x|^\alpha |g(u_n+\psi)_+ - g(u+\psi)_+|\,dx \to 0,$$

for this ε, there exists n_ϵ such that

$$\int_{B_1} |x|^\alpha |g(u_n+\psi)_+ - g(u+\psi)_+|\,dx < \frac{\varepsilon}{2\|\xi\|_\infty}$$

for all $n \ge n_\varepsilon$. Therefore

$$\begin{aligned}\left|\int_{B_1} |x|^\alpha [g(u_n+\psi)_+ - g(u+\psi)_+] u\,dx\right| \\ &\le \int_{B_1} |x|^\alpha |g(u_n+\psi)_+ - g(u+\psi)_+||\xi|\,dx \\ &\quad + \int_{B_1} |x|^\alpha |g(u_n+\psi)_+ - g(u+\psi)_+||\xi - u|\,dx < \varepsilon\end{aligned}$$

for all $n \ge n_\epsilon$, as desired. Consequently, we conclude that (3.19) holds.

Now taking $v = u$ and $n \to \infty$ in (3.6), we have

$$\|u\|^2 = \lambda\|u\|_2^2 + \int_{B_1} |x|^\alpha g(u+\psi)_+ u\,dx.$$

On the other hand, if $n \to \infty$ in (3.6) with $v = u_n$,

$$\|u_n\|^2 \to \lambda\|u\|_2^2 + \int_{B_1} |x|^\alpha g(u+\psi)_+ u\,dx$$

again by (3.19). Consequently, $\|u_n\| \to \|u\|$ and so $u_n \to u$ in H. □

Remark 3.6. Notice that (3.20) holds because, since we are supposing (1.4), we can choose β small enough. This fact is not true if we assume (1.5). Thus, only in the subcritical case, we can conclude the $(\mathrm{PS})_c$ condition is satisfied for all $c \in \mathbb{R}$.

4. Proof of Theorem 1.2. Subcritical case

This section is devoted to the proof of Theorem 1.2. Here, we consider ψ radially symmetric and $H = H^1_{0,\mathrm{rad}}(B_1)$. In the case that ψ is not necessarily radial and $H = H^1_0(B_1)$, the proof uses the same arguments.

4.1. Geometric condition. First, we consider $\lambda < \lambda_1$. We will show that the hypotheses of the Mountain Pass Theorem hold for the functional J_λ.

Proposition 4.1. *Suppose that* $\lambda < \lambda_1$*,* (g_0)–(g_2) *and* (1.4) *hold. Then there exist* $\rho, a > 0$ *such that* $J_\lambda(u) \geq a$ *if* $\|u\| = \rho$*.*

Proof. Initially, we notice that from (g_0)–(g_2), we can see, for all $\varepsilon > 0$ and $\beta > 0$, if g satisfies (1.4) or $\beta > \beta_0$ if g satisfies (1.5), that there exists K_ε such that

$$G(u) \leq \varepsilon u^2 + K_\varepsilon u^q e^{\beta u^2} \quad \text{for all } q \geq 1. \tag{4.1}$$

From (4.1), with $q = 3$ and the variational characterization of the eigenvalues, we obtain

$$J_\lambda(u) \geq C\|u\|^2 - \varepsilon \int_{B_1} |x|^\alpha (u+\psi)_+^2 \, dx - K_\varepsilon \int_{B_1} |x|^\alpha e^{\beta (u+\psi)_+^2} (u+\psi)_+^3 \, dx.$$

Moreover, due to the Hölder inequality, we have

$$\int_{B_1} |x|^\alpha (u+\psi)_+^2 \, dx \leq \bigg(\int_{B_1} |x|^{2\alpha} \, dx \bigg)^{1/2} \bigg(\int_{B_1} (u+\psi)_+^4 \, dx \bigg)^{1/2} \leq C\|u\|^2$$

and

$$\int_{B_1} |x|^\alpha e^{\beta (u+\psi)_+^2} (u+\psi)_+^3 \, dx \leq C \bigg(\int_{B_1} e^{2\beta \|u\|^2 (u/\|u\|)^2} \, dx \bigg)^{1/2} \|u\|^3.$$

Let us choose δ small enough such that $\|u\|^2 < \delta$, $\beta\delta \leq 4\pi$, and

$$J_\lambda(u) \geq C\|u\|^2 - C\|u\|^3 > 0$$

for all $u \in H$ with $\|u\| < \min\{1, \delta^{1/2}\}$. The second constant comes from the Trudinger–Moser inequality. Therefore, take $\rho = \|u\|$ and $a = C\rho^2 - C\rho^3$. □

Proposition 4.2. *Suppose* (g_0)–(g_2) *and* (1.4) *hold. Then, there exists* $R > \rho$ *such that* $J_\lambda(R\phi_1) \leq 0$*, where* ϕ_1 *is a first eigenfunction of* $(-\Delta, H)$ (*with* $\phi_1 > 0$ *and* $\|\phi_1\| = 1$) *and* ρ *is given in Proposition* 4.1*.*

Proof. We fix $R_0 > \rho$ and $0 < r < 1$ such that

$$\phi_1(x) \geq \frac{2\|\psi\|_\infty}{R_0} \quad \text{almost everywhere in } B_r.$$

We observe that (3.4) gives us

$$G(t) \geq C_\sigma t^\sigma - D_\sigma \tag{4.2}$$

for $\sigma > 2$ and $C_\sigma, D_\sigma \geq 0$. Thus, we obtain

$$J_\lambda(R\phi_1) \leq \frac{R^2}{2} - R^\sigma C_\sigma \int_{B_r} |x|^\alpha \bigg(\phi_1 + \frac{\psi}{R} \bigg)_+^\sigma dx + \pi r^2 D_\sigma.$$

Let $R \geq R_0$. We estimate the last integral

$$\int_{B_r} |x|^\alpha \left(\phi_1 + \frac{\psi}{R}\right)_+^\sigma dx \geq \left(\frac{\|\psi\|_\infty}{R_0}\right)^\sigma \frac{2r^{2+\alpha}\pi}{2+\alpha} = \tau > 0.$$

It follows that

$$J_\lambda(R\phi_1) \leq \frac{R^2}{2} - C_\sigma R^\sigma \tau + \pi r^2 D_\sigma.$$

Since $\sigma > 2$, we can choose $R > \rho$ such that $J_\lambda(R\phi_1) \leq 0$, which is the desired conclusion. □

Next, we consider $\lambda_k < \lambda < \lambda_{k+1}$. Before we proof geometric conditions of the Linking Theorem for J_λ, we need to split $H^1_{0,\mathrm{rad}}(B_1)$ into two orthogonal subspaces as we have done with $H^1_0(B_1)$ in (3.10). Recalling the notation introduced in (3.11), we observe that since ϕ_1 is radially symmetric, we have $H_1 = H_1 \cap H^1_{0,\mathrm{rad}}(B_1)$. Now, we set

$$H^*_k = H_k \cap H^1_{0,\mathrm{rad}}(B_1) \quad \text{for all } k \in \mathbb{N}$$

and, analogously with $H^1_0(B_1)$, we can write

$$H^1_{0,\mathrm{rad}}(B_1) = \bigcup_{k=1}^{\infty} H^*_k. \tag{4.3}$$

Moreover, it is straightforward to prove that the spectrum of $(-\Delta, H^1_{0,\mathrm{rad}}(B_1))$ is a subsequence of (λ_k) that we will denote by $\lambda^*_1 = \lambda_1 \leq \lambda^*_2 \leq \lambda^*_3 \leq \ldots \leq \lambda^*_k \leq \ldots$, where $\lambda^*_j \geq \lambda_j$ for all $j = 1, 2, \ldots$

REMARK 4.3. In the proof of Lemma 3.3, when we consider the radial case, we use the decomposition given by (4.3).

For $\lambda_k < \lambda$, we consider the corresponding subspace H_k and we write $H^1_{0,\mathrm{rad}}(B_1)$ as

$$H = H^1_{0,\mathrm{rad}}(B_1) = H^*_l \oplus ((H^*_l)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)) \tag{4.4}$$

where $H^*_l \subset H_k$ with $l = \max\{j : H^*_j \subset H_k\}$. Since $H^*_l \subset H_k$, we notice that $(H^*_l)^\perp \cap H^1_{0,\mathrm{rad}}(B_1) \subset H^\perp_k$. This decomposition will allow us to use the same variational inequalities that characterize λ_k and λ_{k+1} in the $H^1_0(B_1)$ environment.

PROPOSITION 4.4. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$,* (g_0)–(g_2) *and* (1.4) *hold. Then there exist $\rho, a > 0$ such that $J_\lambda(u) \geq a$ for all $u \in (H^*_l)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)$ with $\|u\| = \rho$.*

PROOF. We use the variational characterization of λ_{k+1} and (4.4), and the proof of this proposition is similar to that of Proposition 4.1. □

Proposition 4.5. *Suppose that* $\lambda_k < \lambda < \lambda_{k+1}$, (g_0)–(g_2) *and* (1.4) *hold. There exists* $z \in (H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1) \subset H_k^\perp$ *and* $R > 0$ *such that* $R\|z\| > \rho$ *and* $J_\lambda(u) \le 0$ *for all* $u \in \partial Q$, *where* $Q := \{v \in H_l^* : \|v\| \le R\} \oplus \{sz : 0 \le s \le R\}$.

Proof. We fix $R_0 > \rho$ and choose $z \in (H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1) \subset H_k^\perp$ and $x_0 \in B_1$ such that

(a) $\|z\|^2 < \lambda/\lambda_k - 1$;

(b) $B_r(x_0) \subset B_1$ and $z(x) \ge (K + 2\|\psi\|_\infty/R_0)$ almost everywhere in $B_r(x_0)$

with $K > 0$ satisfying $\|v\|_\infty \le K\|v\|$ for all $v \in H_k$. This choice is possible because $(H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)$ contains unbounded functions and H_k has finite dimension.

We consider a usual split $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$\begin{aligned} Q_1 &= \{v \in H_l^* : \|v\| \le R\}, \\ Q_2 &= \{v + sz : v \in H_l^*, \|v\| = R \text{ and } 0 \le s \le R\}, \\ Q_3 &= \{v + Rz : v \in H_l^* \text{ and } \|v\| \le R\}. \end{aligned}$$

Let u be on Q_1, by (g_0) and characterization of λ_k, it follows that

$$J_\lambda(u) \le \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_k}\right)\|u\|^2 \le 0,$$

independently of $R > 0$.

For Q_2, from (g_0) and since $v \in H_l^*$ and $z \in (H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)$, we get

$$J_\lambda(v + sz) \le \frac{1}{2}\,\|v\|^2 + \frac{1}{2}\,s^2\|z\|^2 - \frac{\lambda}{2}\,\|v\|_2^2 \le \frac{1}{2}\,R^2\left(1 - \frac{\lambda}{\lambda_k} + \|z\|^2\right) < 0,$$

independently of $R > 0$.

Now for Q_3, using (4.2), let $R \ge R_0$ and so, we can see that

$$\begin{aligned} J_\lambda(v + Rz) &\le \frac{R^2}{2}\,\|z\|^2 - R^\sigma C_\sigma \int_{B_1} |x|^\alpha \left(z + \frac{\psi + v}{R}\right)_+^\sigma dx + \pi D_\sigma \\ &\le \frac{R^2}{2}\,\|z\|^2 - C_\sigma R^\sigma \tau + \pi D_\sigma, \end{aligned}$$

where

$$\tau = \left(\frac{\|\psi\|_\infty}{R_0}\right)^\sigma \frac{2\pi r^{2+\alpha}}{2+\alpha} > 0.$$

Since $\sigma > 2$, we finish the proof. □

4.2. Proof of Theorem 1.2. We have proved that J_λ satisfies geometric and compactness conditions required in the Mountain Pass Theorem when $\lambda < \lambda_1$ and in the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$. Thus there exists a non-trivial critical point for J_λ and thus a solution to (1.10). □

5. Proof of Theorem 1.3. Critical case in $H_0^1(B_1)$

It is well known that for non-linear elliptic problems involving critical growth some concentration phenomena may occur, due to the action of the non-compact group of dilations. For problems (1.2) and (1.3) there is loss of compactness at the limiting exponent $\beta = 4\pi$ and $\beta = 2\pi(2+\alpha)$ respectively. Thus, the energy functional J_λ fails to satisfy the $(\mathrm{PS})_c$ condition for certain levels c. Such a failure makes it difficult to apply the standard variational approach to this class of problems. Our proof here relies on a Brezis–Nirenberg type argument: We begin by proving the geometric condition of the Mountain Pass Theorem when $\lambda < \lambda_1$ and the geometric condition in the Linking Theorem if $\lambda_k < \lambda < \lambda_{k+1}$. In the second step we show that the minimax levels belong to the intervals where the (PS) condition holds for the functional J_λ.

5.1. Geometric conditions. Initially we consider $\lambda < \lambda_1$. We will prove the geometric condition of the Mountain Pass Theorem.

PROPOSITION 5.1. *Assume* (g_0)–(g_2), $\lambda < \lambda_1$ *and* (1.5) *hold. Then, there exist* $a, \rho > 0$ *such that* $J_\lambda(u) \geq a$ *if* $\|u\| = \rho$.

PROOF. We can now proceed analogously to the proof of Proposition 4.1. □

Now, let us introduce the so-called Moser sequence. That is, for each n, we define

$$z_n(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{for } 0 \leq |x| \leq 1/n, \\ \dfrac{\log(1/|x|)}{(\log n)^{1/2}} & \text{for } 1/n \leq |x| \leq 1. \end{cases} \tag{5.1}$$

It is known that $z_n \in H$, $\|z_n\| = 1$ for all n and $\|z_n\|_2 = O(1/(\log n)^{1/2})$. For details see [18].

In order to apply our techniques, we shall consider a suitable translation of Moser's functions in a region of B_1 far from the origin where the presence of $|x|^\alpha$ can be neglected. We begin by noticing that, from (1.8), we obtain ε_0 and s_0 such that

$$\frac{\log(h(s))}{s} \geq \varepsilon_0 \quad \text{for all } s > s_0. \tag{5.2}$$

Since $\psi \equiv 0$ on ∂B_1, we can fix $r > 0$, small enough, and x_0 sufficiently close to ∂B_1, such that

$$\|\psi\|_{\infty,r} := \|\psi|_{B_{r^2}(x_0)}\|_\infty \leq \frac{\varepsilon_0}{2\beta_0} \quad \text{and} \quad |x| \geq \frac{1}{2} \quad \text{in } B_{r^2}(x_0), \tag{5.3}$$

with $B_r(x_0) \subset B_1$.

Let us take the following family of functions:

$$z_n^r(x) = \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{1/2} & \text{for } 0 \le |x-x_0| \le r^2/n, \\ \dfrac{\log(r^2/|x-x_0|)}{(\log n)^{1/2}} & \text{for } r^2/n \le |x-x_0| \le r^2, \\ 0 & \text{for } |x-x_0| \ge r^2. \end{cases} \tag{5.4}$$

We notice that $\operatorname{supp} z_n^r = \overline{B_{r^2}(x_0)}$ and, for all $n \in \mathbb{N}$, one has

$$\|z_n^r\|^2 = \int_{B_1} \sum_{i=1}^{2} \left(\frac{\partial z_n^r}{\partial x_i}\right)^2 dx = 2\pi \, \frac{1}{2\pi \log n} \int_{r^2/n}^{r^2} \frac{1}{\widetilde{r}^2} \, \widetilde{r} \, d\widetilde{r} = 1, \tag{5.5}$$

where $\widetilde{r} = |x - x_0|$.

This sequence will be used to guarantee the existence of a minimax level lying under an appropriate constant which will allow us to recover the compactness properties for J_λ that are lost when dealing with critical growth ranges. Before that, we see in the next proposition that for large n we still have the same geometric condition proved in Proposition 4.2 with z_n^r taking the place of ϕ_1.

Proposition 5.2. *Suppose that* (g$_0$)–(g$_2$) *and condition* (1.5) *hold. Then, there exists $R_n = R(n) > \rho$ such that $J_\lambda(R_n z_n^r) \le 0$.*

Proof. We fix $R_0 > \rho$ and $N \in \mathbb{N}$, such that for all $n > N$, we have

$$z_n^r(x) \ge \frac{2\|\psi\|_\infty}{R_0} \quad \text{almost everywhere in } B_{r^2/n}(x_0).$$

Using (4.2), let us take $R > R_0$. We get

$$J_\lambda(R z_n^r) \le \frac{R^2}{2} - C_\sigma R^\sigma \tau + \frac{r^4}{n^2} \, \pi D_\sigma,$$

where

$$\tau = \tau(n,\alpha) = \left(\frac{\|\psi\|_\infty}{R_0}\right)^\sigma \frac{2\pi r^{(2(2+\alpha))}}{(2+\alpha) n^{2+\alpha}}.$$

Since $\sigma > 2$, we can choose $R_n \ge R_0$ such that $J_\lambda(R_n z_n^r) \le 0$ and the proof is complete. □

Now consider $\lambda_k < \lambda < \lambda_{k+1}$ and $H = H_0^1(B_1)$. Before proving the geometry of linking let us split the supports of the Moser sequence and of k eigenfunctions, thus simplifying the proof.

Let us take x_0 and $B_r(x_0)$ as in (5.3) and set $\zeta_r \colon B_1 \to \mathbb{R}$ be

$$\zeta_r(x) = \begin{cases} 0 & \text{in } B_{r^2}(x_0), \\ \dfrac{|x-x_0|^{\sqrt{r}} - r^{2\sqrt{r}}}{r^{\sqrt{r}} - r^{2\sqrt{r}}} & \text{in } B_r(x_0) \setminus B_{r^2}(x_0), \\ 1 & \text{in } B_1 \setminus B_r(x_0). \end{cases}$$

Define $\phi_j^r = \zeta_r \phi_j$ and consider the finite-dimensional subspace $H_k^r = [\phi_1^r, \dots, \phi_k^r]$.

Now we will prove the next result, which is a two-dimensional version of [13, Lemma 2].

LEMMA 5.3. *If $r \to 0$, then $\phi_j^r \to \phi_j$ in H for all $j = 1, \dots, k$. Moreover, for each r small enough, we have that there exists c_k such that*

$$\|v\|^2 \le (\lambda_k + \varrho_r c_k)\|v\|_2^2 \quad \text{for all } v \in H_k^r,$$

where $\lim_{r\to 0} \varrho_r = 0$.

PROOF. For $j \in \{1, \dots, k\}$ fixed, we have

$$\begin{aligned}
\|\phi_j - \phi_j^r\|^2 &= \int_{B_1} |\nabla(\phi_j - \phi_j^r)|^2\,dx \\
&= \int_{B_1} |\nabla\phi_j(1-\zeta_r) - \phi_j\nabla\zeta_r|^2\,dx \\
&\le \int_{B_r} |\nabla\phi_j|^2|1-\zeta_r|^2\,dx \\
&\quad + 2\int_{B_r\setminus B_{r^2}} |\nabla\phi_j|(1-\eta_r)|\phi_j||\nabla\zeta_r|\,dx + \int_{B_r\setminus B_{r^2}} |\phi_j|^2|\nabla\zeta_r|^2\,dx \\
&\le C_1\|\nabla\phi_j\|_\infty^2 \frac{r^{2+2\sqrt{r}}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} \\
&\quad + C_2\|\phi_j\|_\infty\|\nabla\phi_j\|_\infty \frac{r^{2+2\sqrt{r}+1/2}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} + C_3\|\phi_j\|_\infty^2 \frac{r^{2+2\sqrt{r}-1}}{(r^{\sqrt{r}} - r^{2\sqrt{r}})^2} \\
&\le C\,\frac{r}{(1-r^{\sqrt{r}})^2}.
\end{aligned} \tag{5.6}$$

It is straightforward to prove that

$$\lim_{r\to 0} \frac{r}{(1-r^{\sqrt{r}})^2} = 0.$$

Consequently, $\|\phi_j - \phi_j^r\| \to 0$.

Now, let us take $v_r \in H_k^r$ such that $\|v_r\|_2 = 1$. Notice that

$$v_r = \sum_{j=1}^k c_j\zeta_r\phi_j = \zeta_r\overline{v} \quad \text{where } \overline{v} = \sum_{j=1}^k c_j\phi_j \in H_k$$

and $\|v_r - \overline{v}\| = o(1)$ when $r \to 0$. Thus,

$$\begin{aligned}
\|v_r\|^2 &= (\|v_r\|^2 - \|\overline{v}\|^2) + \|\overline{v}\|^2 \le \widetilde{C}_k\widetilde{\varrho}_r + \lambda_k\|\overline{v}\|_2^2 \\
&\le \widetilde{C}_k\widetilde{\varrho}_r + \lambda_k[(\|\overline{v}\|_2^2 - \|v_r\|_2^2) + \|v_r\|_2^2] = c_k\varrho_r + \lambda_k,
\end{aligned}$$

as desired. □

As usual, we must continue by choosing an appropriate decomposition for H. Notice that for r small enough, we can split the space H as $H = H_k^r \oplus H_k^\perp$.

First of all, we need to do some estimates which will be used to prove the geometric conditions of the Linking Theorem. Since $\lambda > \lambda_k$, the previous lemma guarantees that there exists r small enough in order to have

$$\frac{\lambda}{\lambda_k + c_k \varrho_r} - 1 > 0. \tag{5.7}$$

Now we choose δ sufficiently small such that

$$\delta^2 < \frac{\lambda}{\lambda_k + c_k \varrho_r} - 1. \tag{5.8}$$

The next two propositions regard geometric conditions of the Linking Theorem using this non-orthogonal direct sum.

Proposition 5.4. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, (g_0)–(g_2) and (1.5) hold. Then there exist $\rho, a > 0$ such that $J_\lambda(u) \geq a$ if $u \in H_k^\perp$ with $\|u\| = \rho$.*

Proof. We can now proceed analogously to the proof of Proposition 4.4. □

For the next proposition, let us remark that, since

$$|\operatorname{supp} z_n^r \cap \operatorname{supp} v| = |\partial B_{r^2}(x_0)| = 0 \quad \text{for all } v \in H_k^r, \tag{5.9}$$

one has $z_n^r \in (H_k^r)^\perp$.

Proposition 5.5. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, (g_0)–(g_2) and (1.5) hold. For each n large enough, there exists $R_n = R(n) > 0$ such that*

$$J_\lambda(u) \leq 0 \quad \textit{for all } u \in \partial Q_r^\delta,$$

where $Q_r^\delta := \{v + s\delta z_n^r : v \in H_k^r,\ \|v\| \leq R_n \text{ and } 0 \leq s \leq R_n\}$.

Proof. Fix $R_0 > \rho$ and $N \in \mathbb{N}$ such that for all $n > N$, we have

$$z_n^r(x) \geq \frac{2\|\psi\|_{\infty,r}}{\delta R} \quad \text{in } B_{r^2/n}(x_0) \text{ for all } R > R_0. \tag{5.10}$$

Let us take $R > R_0$ and as in Proposition 4.5 we split ∂Q_r^δ as follows:

$$\begin{aligned} Q_1 &= \{v \in H_k^r : \|v\| \leq R\}; \\ Q_2 &= \{v + s\delta z_n^r : v \in H_k^r, \|v\| = R \text{ and } 0 \leq s \leq R\}; \\ Q_3 &= \{v + R\delta z_n^r : v \in H_k^r, \|v\| \leq R\}. \end{aligned}$$

If $v \in Q_1$, from Lemma 5.3, we obtain

$$J_\lambda(v) \leq \frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_k + c_k \varrho_r}\bigg)\|v\|^2 \quad \text{for all } v \in H_k^r.$$

Since $\lambda > \lambda_k$ and due to the choice of r in (5.7), we can take $C_1 > 0$ such that

$$J_\lambda(v) \leq -C_1\|v\|^2 \leq 0 \tag{5.11}$$

for all $v \in H_k^r$ independently of $R > 0$. Using (5.9), we observe that

$$J_\lambda(v + s\delta z_n^r) = J_\lambda(v) + J_\lambda(s\delta z_n^r) \quad \text{for all } v \in H_k^r. \tag{5.12}$$

For Q_2, using (5.5), and the choice of δ in (5.8), we get

$$J_\lambda(v+s\delta z_n^r)=J_\lambda(v)+J_\lambda(R\delta z_n^r)\le\frac{1}{2}R^2\bigg(1-\frac{\lambda}{\lambda_k+c_k\varrho_r}+\delta^2\bigg)<0,$$

independently of $R>0$.

For Q_3, using (4.2), (5.11) and (5.12), we obtain

$$\begin{aligned}J_\lambda(v+R\delta z_n^r)&\le J_\lambda(v)+J_\lambda(R\delta z_n^r)\\&\le\frac{\delta^2R^2}{2}-R^\sigma C_\sigma\int_{B_{r^2/n}(x_0)}|x|^\alpha\bigg(\delta z_n^r-\frac{\|\psi\|_{\infty,r}}{R}\bigg)_+^\sigma dx+\frac{\pi r^4}{n^2}D_\sigma.\end{aligned}$$

Using (5.10), we can choose n sufficiently large and $R_n\ge R_0$ such that

$$J_\lambda(v+\delta R_nz_n^r)\le\frac{R_n^2\delta^2}{2}-C_\sigma R_n^\sigma\tau+\frac{\pi r^4}{n^2}D_\sigma\le 0,$$

where

$$\tau=\tau(n,\alpha)=\bigg(\frac{\|\psi\|_{\infty,r}}{R_0}\bigg)^\sigma\frac{2\pi r^{2(2+\alpha)}}{(2+\alpha)n^{2+\alpha}}>0$$

and this completes the proof. □

5.2. Control of minimax levels. For the Mountain Pass case, we define the minimax level of J_λ by

$$\widetilde{c}=\widetilde{c}(n)=\inf_{\upsilon\in\Gamma}\max_{w\in\upsilon([0,1])}J_\lambda(w)\tag{5.13}$$

where $\Gamma=\{\upsilon\in C([0,1],H):\upsilon(0)=0\text{ and }\upsilon(1)=R_nz_n^r\}$, with R_n such that $J_\lambda(R_nz_n^r)\le 0$ as in Proposition 5.2.

Proposition 5.6. *Let $\widetilde{c}(n)$ be given as in* (5.13)*. Then there exists n large enough such that $\widetilde{c}(n)<2\pi/\beta_0$.*

Proof. We claim that there exists n such that

$$\max_{t\ge0}J_\lambda(tz_n^r)<\frac{2\pi}{\beta_0}.\tag{5.14}$$

Let us fix some constants that we shall use in this proof. We can assume, without loss of generality, that there exists $C_0>0$ such that

$$\varepsilon_0\le\frac{\log(h(s))}{s}\le C_0\tag{5.15}$$

for all s large enough, where ε_0 is given in (5.2).

Indeed, note that if h satisfies (1.8), there exists $\widetilde{h}$ such that $h(s)\ge\widetilde{h}(s)$ for all s large enough and

$$0<\liminf_{s\to+\infty}\frac{\log(\widetilde{h}(s))}{s}\le\limsup_{s\to+\infty}\frac{\log(\widetilde{h}(s))}{s}<+\infty.\tag{5.16}$$

By (5.3), we have

$$\|\psi\|_{\infty,r}\le\frac{\varepsilon_0}{2\beta_0}.\tag{5.17}$$

Finally, we consider γ given by (1.7) such that

$$\gamma > \frac{2^{2+\alpha}}{r^4\beta_0}\exp\left(\frac{C_0^2}{4\beta_0}\right). \tag{5.18}$$

Now suppose by contradiction that (5.14) is not true. So, for all n, this maximum is larger than or equal to $2\pi/\beta_0$ (it is indeed a maximum, in view of Proposition 5.2). Let $t_n > 0$ be such that

$$J_\lambda(t_n z_n^r) = \max_{t\geq 0} J_\lambda(t z_n^r). \tag{5.19}$$

Then

$$J_\lambda(t_n z_n^r) \geq \frac{2\pi}{\beta_0} \quad \text{for all } n \in \mathbb{N}, \tag{5.20}$$

and, consequently, from (5.5),

$$t_n^2 \geq \frac{4\pi}{\beta_0} \quad \text{for all } n \in \mathbb{N}. \tag{5.21}$$

Let us prove that $t_n^2 \to 4\pi/\beta_0$. From (5.19) we get

$$\frac{d}{dt}\left(J_\lambda(t z_n^r)\right)\bigg|_{t=t_n} = 0.$$

So we have

$$\begin{aligned} t_n^2 &\geq \int_{B_{r^2/n}(x_0)} |x|^\alpha g(t_n z_n^r + \psi)_+ t_n z_n^r \, dx \\ &\geq \int_{B_{r^2/n}(x_0)} |x|^\alpha g(t_n z_n^r + \psi)_+ (t_n z_n^r + \psi)_+ \, dx. \end{aligned}$$

Then, (1.7) and (5.3) imply that there exists s_0 large enough such that

$$t_n^2 \geq \frac{1}{2^\alpha}\gamma \int_{B_{r^2/n}(x_0)} h\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right)\exp\left(\beta_0\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right)^2\right) dx,$$

where we have taken n sufficiently large such that

$$\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n + \psi\right) \geq \left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_{\infty,r}\right) \geq s_0 \quad \text{in } B_{r^2/n}(x_0).$$

We still have

$$\begin{aligned} t_n^2 \geq \frac{1}{2^\alpha}\gamma \int_{B_{r^2/n}(x_0)} \exp\bigg[&-\left(\frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)}\right)^2 \\ &+ \beta_0\left((t_n/\sqrt{2\pi})\log^{1/2} n + \psi + \frac{\log[h((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)]}{2\sqrt{\beta_0}((t_n/\sqrt{2\pi})\log^{1/2} n + \psi)}\right)^2\bigg] dx \end{aligned}$$

and taking an even larger n, (5.16) shows that

$$t_n^2 \geq \frac{1}{2^\alpha}\gamma\frac{\pi r^4}{n^2}\exp\left(-\frac{C_0^2}{4\beta_0}\right)\exp\left(\beta_0\left(\frac{t_n}{\sqrt{2\pi}}\log^{1/2} n - \|\psi\|_{\infty,r} + \frac{\varepsilon_0}{2\beta_0}\right)^2\right).$$

By (5.17), we see that

$$t_n^2 \geq \frac{1}{2^\alpha}\exp\bigg(-\frac{C_0^2}{4\beta_0}\bigg)\gamma\,\frac{\pi r^4}{n^2}\exp\bigg(\beta_0\frac{t_n^2}{2\pi}\log n\bigg) = \frac{1}{2^\alpha}\exp\bigg(-\frac{C_0^2}{4\beta_0}\bigg)\gamma\pi r^4\exp\bigg(\bigg(\beta_0\frac{t_n^2}{2\pi}-2\bigg)\log n\bigg), \tag{5.22}$$

which implies that t_n is bounded. Moreover, (5.22) together with (5.21) give $t_n^2 \to 4\pi/\beta_0$. Letting $n\to\infty$ in (5.22), one gets

$$\gamma \leq \frac{2^{2+\alpha}}{r^4\beta_0}\exp\bigg(\frac{C_0^2}{4\beta_0}\bigg),$$

which contradicts to the choice of γ in (5.18). This contradiction happens because we are supposing $\widetilde{c}(n)\geq 2\pi/\beta_0$, so we conclude this proof. □

Now we can define the minimax level for the linking case as

$$\widehat{c} = \widehat{c}(n) = \inf_{\nu\in\Gamma}\max_{w\in Q_r^\delta} J_\lambda(\nu(w)) \tag{5.23}$$

where $\Gamma = \{\nu\in C(Q_r^\delta;H) : \nu(w) = w \text{ if } w\in\partial Q\}$.

Proposition 5.7. *Let $\widehat{c}(n)$ be given as in* (5.23)*. Then there exists n large enough such that $\widehat{c}(n) < 2\pi/\beta_0$.*

Proof. From (5.9) we see that we have split the support of the functions in H_k^r of the support of z_n^r. Therefore, we have

$$\begin{aligned}\widehat{c}(n) &\leq \max\{J_\lambda(v+tz_n^r) : v\in H_k^r,\ \|v\|\leq R_n \text{ and } t\geq 0\}\\ &= \max\{J_\lambda(v)+J_\lambda(tz_n^r) : v\in H_k^r,\ \|v\|\leq R_n \text{ and } t\geq 0\}\\ &\leq \max\{J_\lambda(v) : v\in H_k^r \text{ and } \|v\|\leq R_n\} + \max\{J_\lambda(tz_n^r); t\geq 0\}.\end{aligned}$$

By (5.11), we see that $J_\lambda(v)\leq 0$ for all $v\in H_k^r$. It follows that

$$\widehat{c}(n)\leq \max\{J_\lambda(tz_n^r) : t\geq 0\}.$$

From now on, we can proceed analogously to the proof of Proposition 5.6. □

5.3. Proof of Theorem 1.3. Let us take n such that $c(n) < 2\pi/\beta_0$, where $c(n) = \widetilde{c}(n)$, if $\lambda<\lambda_1$ or $c(n) = \widehat{c}(n)$, if $\lambda>\lambda_1$. Let us consider (u_m), a (PS)-sequence at level $c(n)$. Since it is bounded by Lemma 3.3, then, up to a subsequence, we may assume that $u_m \rightharpoonup u$ weakly in H, $u_m\to u$ strongly in $L^p(B_1)$ for all $p\geq 1$ and almost everywhere in B_1. Therefore, we notice that u is a solution to (1.10). Indeed, for each $v\in C_c^\infty(B_1)$ we have

$$0\leftarrow \langle J_\lambda'(u_m),v\rangle = \int_{B_1}\nabla u_m\nabla v\,dx - \lambda\int_{B_1}u_m v\,dx - \int_{B_1}|x|^\alpha g(u_m+\psi)_+ v\,dx.$$

To see this fact we use that

$$\int_{B_1} \nabla u_m \nabla v\, dx \to \int_{B_1} \nabla u \nabla v\, dx,$$
$$\int_{B_1} u_m v\, dx \to \int_{B_1} uv\, dx,$$
$$\int_{B_1} |x|^\alpha g(u_m+\psi)_+ v\, dx \to \int_{B_1} |x|^\alpha g(u+\psi)_+ v\, dx$$

(the last one is due to (3.8), and using Lemma 2.1 in [11] and (g_1)). Thus, we can easily conclude that $\langle J_\lambda'(u), v\rangle = 0$ for all $v \in C_c^\infty(B_1)$. Consequently u is a weak solution to (1.10). We still need to ensure that $u \not\equiv 0$.

Suppose, instead, that $u \equiv 0$. So, we must have $\|u_m\|_2 \to 0$ and, again using (3.8), (g_1) and Lemma 2.1 in [11], we have

$$\int_{B_1} |x|^\alpha G(u_m+\psi)_+\, dx \to \int_{B_1} |x|^\alpha G(u+\psi)_+\, dx = 0.$$

Moreover, since

$$c(n) = \lim_{m\to\infty} J_\lambda(u_m) = \frac{1}{2}\lim_{m\to\infty} \|u_m\|^2,$$

and $c(n) < 2\pi/\beta_0$, one can find $\delta > 0$ and m_0 such that

$$\|u_m\|^2 \le \frac{4\pi}{\beta_0} - \delta \quad \text{for all } m \ge m_0. \tag{5.24}$$

Consider a small $\varepsilon > 0$ and $p > 1$ (sufficiently close to 1) in order to have

$$p(\beta_0+\varepsilon)(4\pi/\beta_0 - \delta) \le 4\pi.$$

By (1.5), one can take $C > 0$ sufficiently large such that

$$g(s)^p \le e^{p(\beta_0+\varepsilon)s^2} + C \quad \text{for all } s \ge 0. \tag{5.25}$$

From the fact that $\langle J_\lambda'(u_m), u_m\rangle = \varepsilon_m \to 0$, we can see that

$$\|u_m\|^2 \le \lambda \|u_m\|_2^2 + \int_{B_1} |x|^\alpha g(u_m+\psi)_+ u_m\, dx + \varepsilon_m.$$

We need to estimate the integral on the right of this last inequality. The Hölder inequality and (5.25) give

$$\begin{aligned}\int_{B_1} |x|^\alpha g(u_m+\psi)_+ u_m\, dx &\le \bigg(\int_{B_1} (g(u_m+\psi)_+)^p\, dx\bigg)^{1/p} \bigg(\int_{B_1} |u_m|^{p'}\, dx\bigg)^{1/p'} \\ &\le \bigg[\bigg(\int_{B_1} \exp(p(\beta_0+\varepsilon)u_m^2\, dx\bigg)^{1/p} + C\pi^{1/p}\bigg] \|u_m\|_{p'}\end{aligned}$$

and then we have

$$\begin{aligned}\|u_m\|^2 &\le \lambda\|u_m\|_2^2 + \varepsilon_m \\ &\quad + \left[\left(\int_{B_1} e^{(p(\beta_0+\varepsilon)(4\pi/\beta_0-\delta)(u_m/\|u_m\|)^2)} dx\right)^{1/p} + C\pi^{1/p}\right]\|u_m\|_{p'} \\ &\le \lambda\|u_m\|_2^2 + \varepsilon_m + \left[\left(\int_{B_1} e^{(4\pi(u_m/\|u_m\|)^2)}\, dx\right)^{1/p} + C\pi^{1/p}\right]\|u_m\|_{p'}.\end{aligned}$$

Since the last integral in the estimates above is bounded (because of the Trudinger–Moser inequality), we get $\|u_m\| \to 0$. Hence $u_m \to 0$ in H and then $J_\lambda(u_m) \to 0 = c(n)$. This is impossible since $c(n) \ge a > 0$ for all n. Thus $u \not\equiv 0$ is the desired solution. □

6. Proof of Theorem 1.4. Critical case in $H^1_{0,\mathrm{rad}}(B_1)$

In this section, we treat the radial case with g having critical growth. In this case, a solution u to problem (1.10) is in $H = H^1_{0,\mathrm{rad}}(B_1)$, which will force us to change some calculations.

6.1. Geometric conditions. If $\lambda < \lambda_1$, we observe that the geometric conditions follow from Propositions 5.1 and 5.2, replacing z_n^r, given in (5.4), with z_n, given in (5.1). If $\lambda_k < \lambda < \lambda_{k+1}$, we use the same arguments developed in Proposition 5.4 in order to prove that 0 is a local minimum of J_λ in $((H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1))$. However, we cannot take the Moser sequence with support disjoint from the eigenfunctions, as we did in Proposition 5.5. Indeed, since now we work in an environment of radial functions, we need to use the sequence set in (5.1), instead of the one given in (5.4). This replacement brings some difficulties because we lose the advantage of being close to the boundary, where the interference of ψ could be considered negligible.

Initially, let $l = \max\{j : H_j^* \subset H_k\}$ and we set

$$T_l \colon H^1_{0,\mathrm{rad}}(B_1) \to (H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1)$$

as the orthogonal projection. We consider (5.1) and define

$$w_n(x) = T_l z_n(x). \tag{6.1}$$

Since $(H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1) \subset H_k^\perp$, by [23, Lemma 2], we have that the following estimates:

$$1 - \frac{A_k}{\log n} \le \|w_n\|^2 \le 1, \tag{6.2}$$

$$w_n(x) \ge \begin{cases} -\dfrac{B_k}{(\log n)^{1/2}} & \text{for all } x \in B_1, \\ \dfrac{(\log n)^{1/2}}{\sqrt{2\pi}} - \dfrac{B_k}{(\log n)^{1/2}} & \text{for all } x \in B_{1/n}, \end{cases} \tag{6.3}$$

where $B_{1/n} := \{x \in \mathbb{R}^2 : |x| < 1/n\}$ and $A_k, B_k > 0$ are such that

$$\|u\| \leq A_k \|u\|_2 \quad \text{and} \quad \|u\|_\infty \leq (B_k/B)\|u\|_2 \quad \text{for all } u \in H_k,$$

and B satisfies $\|z_n\|_2 \leq B/(\log n)^{1/2}$ for all $n \in \mathbb{N}$.

Proposition 6.1. *Suppose that $\lambda_k < \lambda < \lambda_{k+1}$, (g$_0$)–(g$_2$) and (1.5) hold. For each n large enough, there exists $R_n = R(n) > 0$ such that, if*

$$Q := \{v \in H_l^* : \|v\| \leq R_n\} \oplus \{s\delta w_n : 0 \leq s \leq R_n\},$$

with

$$\delta^2 \leq \frac{\lambda}{\lambda_k} - 1, \tag{6.4}$$

then $J_\lambda(u) \leq 0$ for all $u \in \partial Q$; moreover, $R_n \to \infty$ when $n \to \infty$.

Proof. Fix $R_0 > \rho$ and let us take $R > R_0$. As before, we split ∂Q as follows:

$$\begin{aligned} Q_1 &= \{v \in H_l^* : \|v\| \leq R\}; \\ Q_2 &= \{v + s\delta w_n : v \in H_l^*,\ \|v\| = R \text{ and } 0 \leq s \leq R\}; \\ Q_3 &= \{v + R\delta w_n : v \in H_l^*,\ \|v\| \leq R\}. \end{aligned}$$

Recalling that $G(s) \geq 0$ for all $s \in \mathbb{R}$ and considering $v \in Q_1$, we get

$$J_\lambda(v) \leq \left(1 - \frac{\lambda}{\lambda_k}\right)\|v\|^2 \leq 0, \tag{6.5}$$

independently of R. For Q_2, let us take δ satisfying (6.4). By (6.2), we obtain

$$J_\lambda(v + s\delta w_n) \leq \left(1 - \frac{\lambda}{\lambda_k} + \delta^2\right)\frac{R^2}{2} \leq 0,$$

independently of $R > 0$. For Q_3, we take $\delta > 0$ given in (6.4) and $v + R\delta w_n$ with $\|v\| \leq R$ and $v \in H_l^*$. Using (4.2), we obtain

$$J_\lambda(v + R\delta w_n) \leq \frac{1 + \delta^2}{2} R^2 - R^\sigma \int_{B_1} |x|^\alpha \left(\frac{-(\|v\|_\infty + \|\psi\|_\infty)}{R} + \delta w_n\right)_+^\sigma dx + D_\sigma.$$

Since $H_l^* \subset H_k$ has finite dimension, it follows that $\|v\|_\infty \leq C_k R$ for some $C_k > 0$. We can suppose that R_0, previously fixed, satisfies $\|\psi\|_\infty / R_0 \leq C_k$. Then, by (6.3) and considering $R > R_0$, we have

$$J_\lambda(v + R\delta w_n) \leq \frac{1 + \delta^2}{2} R^2 - \overline{C}_n R^\sigma + \pi D_\sigma,$$

where

$$\overline{C}_n = \left(-2C_k + \delta\left(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}}\right)\right)_+^\sigma \frac{2\pi}{(2+\alpha)n^{2+\alpha}} > 0.$$

Since $\sigma > 2$, the result is then proved by taking R_n large enough so that $\delta\|w_n\|R_n > \rho$ and, thus, we obtain

$$J_\lambda(v + R_n\delta w_n) \le \frac{1+\delta^2}{2} R_n - \overline{C}_n R_n^\sigma + \pi D_\sigma \le 0$$

and since $\overline{C}_n \to 0$ when $n \to \infty$, we can see that $R_n \to \infty$. This concludes the proof. □

6.2. Control of minimax levels. Here we can notice another important difference between the cases studied in $H_0^1(B_1)$ and in $H_{0,\mathrm{rad}}^1(B_1)$. Since, in the radial case, we need to work with radially symmetric functions whose supports have the center at $0 \in \mathbb{R}^2$, we cannot neglect the weight $|x|^\alpha$ as we did in the critical case. Actually, although the estimates are harder to obtain, it turns out to be an advantage because the environment of symmetric functions changes the boundedness of the minimax levels: in the critical case, we had that they were bounded by the constant $2\pi/\beta_0$. On the other hand, working in $H_{0,\mathrm{rad}}^1(B_1)$, this boundedness can be obtained by a greater value that depends on β_0 and α.

For the Mountain Pass problem, we define the minimax level of J_λ by

$$\overline{c} = \overline{c}(n) = \inf_{v\in\Gamma} \max_{w\in v([0,1])} J_\lambda(w), \tag{6.6}$$

where $\Gamma = \{v \in C([0,1], H) : v(0) = 0 \text{ and } v(1) = R_n z_n\}$, with R_n such that $J_\lambda(R_n z_n) \le 0$.

PROPOSITION 6.2. *Let $\overline{c}(n)$ be given in* (6.6). *Then there exists n large enough such that*

$$\overline{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}. \tag{6.7}$$

PROOF. We claim that exists n such that

$$\max_{t\ge0} J_\lambda(tz_n) < \frac{(2+\alpha)\pi}{\beta_0}. \tag{6.8}$$

First of all, let us fix some constants that we shall use in this proof. Analogously as we have done in Proposition 5.6, we can assume, without loss of generality, that

$$K_0 \le \frac{\log(h(s))}{s} \le C_0 \tag{6.9}$$

for all s large enough. By (1.9), K_0 can be taken large enough in order to have

$$\|\psi\|_\infty \le \frac{K_0}{2\beta_0}. \tag{6.10}$$

Finally, consider γ such that

$$\gamma > \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{C_0^2}{4\beta_0}\right). \tag{6.11}$$

We suppose, by contradiction, that (6.8) is not true. In the same way as in Theorem 1.3 with $\lambda < \lambda_1$, we conclude that

$$\gamma \leq \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{C_0^2}{4\beta_0}\right).$$

It contradicts (6.11), then we have that (6.7) holds. □

Remark 6.3. This last proposition explains why we had to assume (1.9) instead of (1.8). In the $H_0^1(B_1)$ environment, we could control the L^∞-norm of ψ by moving the supports of our Moser sequence far away from 0 and close enough to ∂B_1 so that $\|\psi\|_\infty$ would be sufficiently small and then it would not interfere in the estimates. Since this could not be done in the radial case, the interference was avoided by using (6.10).

Remark 6.4. We also point out the different choices we have made for the constant γ in (5.18) and (6.11), which explains, in part, the role of $|x|^\alpha$ in the radial case. Notice that γ in (5.18) must be greater than the one given in (6.11).

Now, for the linking problem, we can define the minimax level

$$\check{c} = \check{c}(n) = \inf_{\nu\in\Gamma} \max_{w\in\nu(Q)} J_\lambda(\nu(w)) \tag{6.12}$$

where $\Gamma = \{\nu \in C(Q,H) : \nu(w) = w \text{ if } w \in \partial Q\}$.

We remark that since we did not separate the supports of the eigenfunctions from the Moser sequence, the estimates done in Proposition 5.7 will not work in the radial case. Therefore, we need do handle more delicate arguments in order to achieve analogous results. This is done in the following proposition.

Proposition 6.5. *Let $\check{c}(n)$ be given as in* (6.12). *Then there exists n large enough such that*

$$\check{c}(n) < \frac{(2+\alpha)\pi}{\beta_0}.$$

Proof. Suppose by contradiction that for all n we have $\check{c}(n) \geq (2+\alpha)\pi/\beta_0$. We notice that

$$\check{c}(n) \leq \max\{J_\lambda(v + tw_n) : v \in H_l^* \text{ with } \|v\| \leq R_n, t \geq 0\}$$

and it follows that for each n there exist $v_n \in H_l^*$ and $t_n > 0$ such that

$$J_\lambda(v_n + t_n w_n) = \max\{J_\lambda(v + tw_n) : v \in H_l^* \text{ with } \|v\| \leq R_n, t \geq 0\}. \tag{6.13}$$

Therefore, we have

$$J_\lambda(v_n + t_n w_n) \geq \frac{(2+\alpha)\pi}{\beta_0} \quad \text{for all } n \in \mathbb{N}. \tag{6.14}$$

So, since $w_n \in ((H_l^*)^\perp \cap H^1_{0,\mathrm{rad}}(B_1))$, from (6.5) we obtain

$$t_n^2 \geq \frac{2(2+\alpha)\pi}{\beta_0} \quad \text{for all } n \in \mathbb{N}. \tag{6.15}$$

Let us assume the following claims whose proofs we give later:

CLAIM 6.6. *(v_n) and (t_n) are bounded sequences.*

CLAIM 6.7. *$t_n^2 \to \dfrac{2(2+\alpha)\pi}{\beta_0}$ in $\mathbb{R}$ and $v_n \to 0$ in H.*

Since $v_n \in H_l^*$, in view of Claim 6.7, we also get $\|v_n\|_\infty \to 0$. However, we have $v_n + t_n w_n \to \infty$ uniformly in $B_{1/n}$.

As we also have done in Proposition 6.2, let us observe that we can assume, without loss of generality, that

$$K_0 \leq \frac{\log(h(s))}{s} \leq C_0 \tag{6.16}$$

for all s large enough. From (1.9), K_0 can be taken large enough in order to have

$$\|\psi\|_\infty \leq \frac{K_0}{4\beta_0} \tag{6.17}$$

and, for n large enough,

$$\|v_n\|_\infty \leq \frac{K_0}{4\beta_0}.$$

Finally, consider γ such that

$$\gamma > \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right). \tag{6.18}$$

From (6.13), using the fact that the derivative of J_λ, restricted to $H_l^* \oplus \mathbb{R}w_n$ is zero at $v_n + t_n w_n$, we obtain

$$\|v_n+t_n w_n\|^2 - \lambda\|v_n+t_n w_n\|_2^2 - \int |x|^\alpha g(v_n+t_n w_n+\psi)_+(v_n+t_n w_n)\,dx = 0 \tag{6.19}$$

and we can conclude that

$$\gamma \leq \frac{(2+\alpha)^2}{\beta_0} \exp\left(\frac{4(2+\alpha)\pi B_k}{\sqrt{2\pi}} + \frac{C_0^2}{4\beta_0}\right),$$

which contradicts to the choice of γ in (6.18). This contradiction follows from the assumption $\check{c}(n) \geq (2+\alpha)\pi/\beta_0$ for all $n \in \mathbb{N}$, which concludes the proof. □

PROOF OF CLAIM 6.6. It is sufficient to prove that all subsequences of (t_n) and (v_n) have bounded subsequences. Let us suppose that this is not true. So, we can find subsequences, which by convenience we still denote by (t_n) and (v_n), respectively, such that all of their subsequences are unbounded. That means, we can assume that

$$t_{n_k} + \|v_{n_k}\| \to \infty \quad \text{for all subsequences } (n_k). \tag{6.20}$$

Therefore, one of the following two possibilities has to hold:

(i) either there exists a constant $C_0 > 0$ such that $t_n/\|v_n\| \geq C_0$, or
(ii) there are subsequences such that $t_n/\|v_n\| \to 0$.

Assume that (i) holds and using (6.20), we have that $t_n \to \infty$. Now we can see from (1.7) that

$$\begin{aligned} t_n^2 &\geq \int_{B_{1/n}} |x|^\alpha g(v_n + t_n w_n + \psi)_+ (v_n + t_n w_n + \psi)_+ \, dx \\ &\geq \gamma \int_{B_{1/n}} |x|^\alpha h(v_n + t_n w_n + \psi) \exp(\beta_0 (v_n + t_n w_n + \psi)^2) \, dx \end{aligned}$$

for n large enough and we can see from (1.9) that

$$h(s) \geq \widetilde{C} \tag{6.21}$$

for all s large enough. So we have

$$t_n^2 \geq \gamma \widetilde{C} \int_{B_{1/n}} |x|^\alpha \exp(\beta_0 (v_n + t_n w_n + \psi)^2) \, dx.$$

We notice that since H_l^* has finite dimension, we have that $\|v_n\|_\infty / t_n$ is bounded for all $x \in B_{1/n}$. We also know that $\|\psi\|_\infty / t_n$ is bounded. These facts together with (6.3), give us

$$\begin{aligned} v_n(x) &+ t_n w_n(x) + \psi(x) = t_n w_n(x) \bigg(1 + \frac{v_n(x) + \psi(x)}{t_n} \, \frac{1}{w_n} \bigg) \\ &\geq \frac{t_n}{\sqrt{2\pi}} \bigg((\log n)^{1/2} - \frac{\sqrt{2\pi} B_k}{(\log n)^{1/2}} \bigg) \bigg(1 - C \bigg(\frac{(\log n)^{1/2}}{\sqrt{2\pi}} - \frac{B_k}{(\log n)^{1/2}} \bigg)^{-1} \bigg) \\ &\geq \frac{t_n}{2} \, \frac{1}{\sqrt{2\pi}} \bigg((\log n)^{1/2} - \frac{\sqrt{2\pi} B_k}{(\log n)^{1/2}} \bigg) \end{aligned}$$

and taking n such that $(\log n)^{1/2} - \sqrt{2\pi} B_k / (\log n)^{1/2} \geq (1/2)(\log n)^{1/2}$, we obtain

$$v_n(x) + t_n w_n(x) + \psi(x) \geq \frac{t_n}{4\sqrt{2\pi}} \, (\log n)^{1/2}$$

and by (6.21) it follows that

$$\begin{aligned} t_n^2 &\geq \gamma \widetilde{C} \int_{B_{1/n}} |x|^\alpha \exp \bigg(\beta_0 \bigg(\frac{t_n}{4\sqrt{2\pi}} \, (\log n)^{1/2} \bigg)^2 \bigg) \, dx \\ &= \gamma \widetilde{C} \, \frac{2\pi}{2 + \alpha} \exp \bigg(\bigg(\beta_0 \frac{t_n^2}{32\pi} - (2 + \alpha) \bigg) \log n \bigg). \end{aligned}$$

Consequently, t_n must be bounded in case (i), which contradicts $t_n \to \infty$ in case (i).

So (ii) occurs. Since $\lim_{n \to \infty} t_n / \|v_n\| = 0$, by (6.20) we conclude that $\|v_n\| \to \infty$. By (6.19), we get

$$\|t_n w_n + v_n\|^2 \geq \int |x|^\alpha g(t_n w_n + v_n + \psi)_+ (t_n w_n + v_n)_+ \, dx.$$

Using (1.7) and (6.21), for n large enough, we have

$$\|t_n w_n + v_n\|^2 \geq \gamma \widetilde{C} \int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha e^{\beta_0 (t_n w_n + v_n + \psi)^2} \, dx.$$

Since we are supposing (ii), it follows that

$$(6.22)\qquad \begin{aligned} 1 &\geq \widetilde{C}\gamma \int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha \, \frac{e^{\beta_0 (t_n w_n + v_n + \psi)^2}}{\|t_n w_n + v_n\|^2}\, dx \\ &\geq \widetilde{C}\frac{\gamma}{2} \int_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} |x|^\alpha \, \frac{e^{\beta_0 (t_n w_n + v_n + \psi)^2}}{\|v_n\|^2}\, dx. \end{aligned}$$

On the other hand, we notice that

$$\begin{aligned} &\frac{t_n w_n + v_n + \psi}{\|v_n\|}\, \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}} \\ &= \frac{v_n}{\|v_n\|} + \frac{t_n}{\|v_n\|}\, w_n - \frac{t_n w_n + v_n}{\|v_n\|}\, \chi_{\{t_n w_n + v_n + \psi \leq c_\gamma\}} + \frac{\psi}{\|v_n\|}\, \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}}. \end{aligned}$$

Hence, we can see that

$$\frac{t_n w_n + v_n(x) + \psi}{\|v_n(x)\|}\, \chi_{\{t_n w_n + v_n + \psi \geq c_\gamma\}}(x) \to \widehat{v} \quad \text{a.e. in } H_0^1(B_1),$$

where $\widehat{v} \in H_k$, with $v_n/\|v_n\| \to \widehat{v}$ and $\|\widehat{v}\| = 1$. So using Fatou's Lemma in (6.22) and since we have assumed that $\|v_n\| \to \infty$, we reach a contradiction. So $\|v_n\|$ is bounded and, consequently, t_n is also bounded. □

Proof of Claim 6.7. First, we notice that for some appropriate subsequences we have $v_n \to v_0$ in H and $t_n \to t_0$ and since $z_n \rightharpoonup 0$ we get $w_n \rightharpoonup 0$ and $w_n \to 0$ for all $x \in B_1$. Then it follows

$$(6.23)\qquad v_n + t_n w_n \to v_0 \quad \text{a.e. in } B_1.$$

Moreover, in view of (6.19), we see that

$$(6.24)\qquad \int_{B_1} |x|^\alpha g(v_n + t_n w_n \psi)_+ (v_n + t_n w_n \psi)\, dx \leq \|v_n + t_n w_n\|^2 \leq C.$$

However, using [11, Lemma 2.1] and recalling (g_1), (6.23) and (6.24), we have

$$(6.25)\qquad \int_{B_1} |x|^\alpha G(v_n + t_n w_n + \psi)_+\, dx \to \int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, dx.$$

From (6.14) and (6.25) we can see that

$$(6.26)\qquad J_\lambda(v_0) + \frac{t_0^2}{2} \geq \frac{(2+\alpha)\pi}{\beta_0}$$

and, since $v_0 \in H_l^*$, in view of $J_\lambda(v_0) \leq 0$ we have

$$t_0^2 \geq \frac{2(2+\alpha)\pi}{\beta_0}.$$

Now we prove that $t_0^2 = 2(2+\alpha)\pi/\beta_0$. Let us suppose that this is not true. We have $t_0^2 > 2(2+\alpha)\pi/\beta_0$. Thus we can take small enough $\varepsilon > 0$ so that

$$t_n^2 > (1+\varepsilon)\, \frac{2(2+\alpha)\pi}{\beta_0}$$

for all large n. We consider

$$\varepsilon_n = \sup_{B_{1/n}} \frac{|v_n(x) + \psi(x)|}{t_n w_n(x)},$$

clearly we see that $\varepsilon_n \to 0$, which, together with (6.21), yields

$$\begin{aligned} C &\geq \gamma\widetilde{C} \int_{B_{1/n}} |x|^\alpha \exp[\beta_0(v_n + \psi + t_n w_n)^2]\, dx \\ &\geq \gamma\widetilde{C} \int_{B_{1/n}} |x|^\alpha \exp[\beta_0(-\varepsilon_n t_n w_n + t_n w_n)^2]\, dx. \end{aligned}$$

Using (1.7) and (6.24) and n large enough, we see that

$$\begin{aligned} C &\geq \gamma\widetilde{C}\, \frac{2\pi}{(2+\alpha)n^{2+\alpha}}\, e^{(\beta_0(1-\varepsilon_n)^2 t_n^2 [(\log n)^{1/2}/\sqrt{2\pi} - B_k/(\log n)^{1/2}]^2)} \\ &= \gamma\widetilde{C}\, \frac{2\pi}{(2+\alpha)e^{(2+\alpha)\log n}}\, e^{(\beta_0(1-\varepsilon_n)^2 t_n^2 [B_k^2/\log n - 2B_k/\sqrt{2\pi}])} e^{(\beta_0(1-\varepsilon_n)^2 t_n^2 \log n/(2\pi))}. \end{aligned}$$

We notice that $e^{(\beta_0(1-\varepsilon_n)^2 t_n^2 [B_k^2/\log n - 2B_k/\sqrt{2\pi}])} > C_1$ for n large enough and some $C_1 > 0$, due to the facts that $t_n^2 > 2\pi(2+\alpha)/\beta_0$ and $\varepsilon_n \to 0$. Thus, using $t_n^2 > (1+\varepsilon)2\pi(2+\alpha)/\beta_0$, we have

$$\begin{aligned} C &\geq C_1\gamma\widetilde{C}\, \frac{2\pi}{(2+\alpha)}\, e^{(\beta_0(1-\varepsilon_n)^2 t_n^2 \log n/(2\pi) - (2+\alpha)\log n)} \\ &\geq C_1\gamma\widetilde{C}\, \frac{2\pi}{(2+\alpha)}\, e^{((2+\alpha)\log n[(1-\varepsilon_n)^2(1+\varepsilon) - 1])} \to \infty, \end{aligned}$$

which is a contradiction. Consequently, we must have $t_0^2 = 2\pi(2+\alpha)/\beta_0$ as desired. So by (6.26) we get $J_\lambda(v_0) \geq 0$. But we know that $v_0 \in H_l^*$, so by (6.5), we have $J_\lambda(v_0) = 0$.

Now we must show that if $v_0 \in H_l^*$ and $J_\lambda(v_0) = 0$, then $v_0 = 0$ and we finish the proof of Claim 6.7. Consider $v_0 \in H_l^*$, then

$$\begin{aligned} 0 = J_\lambda(v_0) &= \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 - \int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, dx \\ &\leq \frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_k}\bigg)\|v_0\|^2 - \int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, dx \leq - \int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, dx. \end{aligned}$$

Since $G \geq 0$, we can see that $\int_{B_1} |x|^\alpha G(v_0 + \psi)_+\, dx = 0$. Thus

$$0 = J_\lambda(v_0) = \frac{1}{2}\|v_0\|^2 - \frac{\lambda}{2}\|v_0\|_2^2 \leq \frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_k}\bigg)\|v_0\|^2.$$

So we can see

$$\frac{1}{2}\bigg(1 - \frac{\lambda}{\lambda_k}\bigg)\|v_0\|^2 \geq 0$$

which gives us $v_0 = 0$. $\square$

6.3. Proof of Theorem 1.4. Let us take n such that $c(n) < (2+\alpha)\pi/\beta_0$, where $c(n) = \overline{c}(n)$, if $\lambda < \lambda_1$ or $c(n) = \check{c}$ if $\lambda > \lambda_1$. Consider u_m a (PS)-sequence at level $c(n)$, which is the minimax level is below $(2+\alpha)\pi/\beta_0$. Consider a $(\mathrm{PS})_c$ sequence (u_m), it is bounded by Lemma 3.3, so there exists a subsequence of (u_m) and $u \in H$ such that $u_m \rightharpoonup u$ weakly in H, $u_m \to u$ in $L^p_{\mathrm{rad}}(B_1)$ for all $p \geq 1$ and almost everywhere in B_1. In the same way as in Theorem 1.3, we conclude that u is a solution to (1.10) and $u \neq 0$. $\square$

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