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MULTIPLE POSITIVE SOLUTIONS FOR FRACTIONAL ELLIPTIC SYSTEMS INVOLVING SIGN-CHANGING WEIGHT

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Abstract. We study multiplicity results for positive solutions for a fractional elliptic system involving both concave-convex and critical growth terms. With the help of Nehari manifold and Ljusternik–Schnirelmann category, we investigate how the coefficient h of the critical nonlinearity affects the number of positive solutions to this problem and get a relationship between the number of positive solutions and the topology of the global maximum set of h.

1. Introduction and the main result

In this paper, we are concerned with the number of positive solutions to the following fractional elliptic system:

$$(\mathrm{E}_{f,g})\qquad \begin{cases} (-\Delta)^{s/2}u = f(x)|u|^{q-2}u + \dfrac{\alpha}{\alpha+\beta}\, h(x)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ (-\Delta)^{s/2}v = g(x)|v|^{q-2}v + \dfrac{\beta}{\alpha+\beta}\, h(x)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a bounded set in $\mathbb{R}^N$ with smooth boundary, $N > s$ with $s \in (0,2)$ fixed, $1 < q < 2$, $\alpha, \beta > 1$ satisfy $\alpha + \beta = 2^*_s = 2N/(N-s)$, 2^*_s is the fractional

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Sobolev critical exponent, and $(-\Delta)^{s/2}$ is the fractional Laplacian. Moreover, f, g, h are continuous functions.

In recent years, problems involving fractional operators have received a special attention since they have important applications in many sciences. We limit here ourselves with a non-exhaustive list of fields and papers in which these operators are used: obstacle problem [20], [26], optimization and finance [2], [13], phase transition [1], [28], material science [4], anomalous diffusion [18], [19], conformal geometry and minimal surfaces [5], [7], [8]. The list may continue with applications in crystal dislocation, soft thin films, multiple scattering, quasi-geostrophic flows, water waves, and so on. The interested reader may consult also references in the cited papers. Set $\alpha+\beta=p\le 2_s^*$, $f(x)\equiv g(x)$, $h(x)\equiv 1$ and $u=v$, then $(\mathrm{E}_{f,g})$ reduces to the following fractional elliptic equation with concave-convex nonlinearities:

$$(\mathrm{E}_\lambda)\qquad \begin{cases} (-\Delta)^{s/2}u=\lambda|u|^{q-2}u+|u|^{p-2}u & \text{in } \Omega,\\ u=0 & \text{on } \mathbb{R}^N\setminus\Omega. \end{cases}$$

Goyal and Sreenadh [16] studied the existence and multiplicity of positive solutions to (E_λ). Moreover, involving the Nehari manifold and Fibering maps, Chen and Deng [9] obtained the existence of multiple solutions to (E_λ) for the subcritical and critical cases. For the fractional Laplacian system with concave-convex nonlinearities, He, Squassina, and Zou [17] proved that $(\mathrm{E}_{\lambda,\mu})$ ($(\mathrm{E}_{f,g})$ with $f(x)\equiv\lambda$ and $g(x)\equiv\mu$) possesses at least two positive solutions when λ and μ are small enough. Similar results were achieved by Chen and Deng [10]. Their tool was the decomposition of the Nehari manifold.

There are several existence results for the following problem:

$$(1.1)\qquad \varepsilon^s(-\Delta)^{s/2}u+V(x)u=f(u),\quad x\in\mathbb{R}^N,$$

where ε is a positive parameter, f has a subcritical growth, V possesses a local minimum. For $\varepsilon=1$, we would like to cite [22], [24] for the existence of one positive solution imposing a global condition on V. For ε a small positive constant, several scholars established existence and concentration of positive solutions to (1.1), by imposing different conditions on V and f (see [23], [14], [29], [15]). In particular, with the help of Nehari manifold and Lusternik–Schnirelmann category, Figueiredo and Siciliano [15] obtained a relationship between the number of positive solutions and the topology of the minimum set of V.

An interesting question now is how the weight potential h of a critical term affects the number of positive solutions to $(\mathrm{E}_{f,g})$ involving critical nonlinearity and sign-changing weight potentials. Furthermore, we wonder if there is a similar relationship between the number of positive solutions to $(\mathrm{E}_{f,g})$ and the topology of the global maximum set of h as that in [15]. To state our main results, we introduce precise conditions on f,g and h:

(H_1) There exist a non-empty closed set $M = \{z \in \overline{\Omega} : h(z) = \max_{x\in\overline{\Omega}} h(x) = 1\}$ and a positive number $\rho > N-2$ such that $h(z) - h(x) = O(|x-z|^\rho)$ uniformly in $z \in M$ as $x \to z$.

(H_2) $f(z), g(z) > 0$ for $z \in M$.

REMARK 1.1. There are many examples of f, g and h that satisfy the hypotheses of our paper. For example, we assume that $0 \in M \subseteq \{x : |x| < 1\} \subseteq \Omega$ and define $h \in C^\infty(\mathbb{R}^3)$ by

$$h(x) := \begin{cases} 1 & \text{if } x \in M, \\ 1 - |x|^\rho & \text{if } x \in \{x : |x| < 1\} \setminus M, \\ 0 & \text{if } x \in \Omega \setminus \{x : |x| < 1\}. \end{cases}$$

Take $f, g \in C^\infty(\mathbb{R}^3)$ such that $f(z), g(z) > 0$ for $z \in M$. Then it is easy to check that f, g, h satisfy (H_1)–(H_2).

REMARK 1.2. Let $M_r = \{z \in \mathbb{R}^N : \operatorname{dist}(z, M) < r\}$ for $r > 0$. Then, by (H_1)–(H_2), there exist $C_0, r_0 > 0$ such that

$$f(z), g(z), h(z) > 0 \quad \text{for all } z \in M_{r_0} \subset \Omega$$

and

$$h(z) - h(x) \le C_0 |x-z|^\rho \quad \text{for all } x \in B_{r_0}(z) \subset \Omega$$

uniformly in $z \in M$, where $B_{r_0}(z) = \{x \in \mathbb{R}^N : |x - z| < r_0\}$.

Now let us state the main result of our work.

THEOREM 1.3. *Assume* (H_1)–(H_2) *hold. Then for each* $\delta < r_0$*, there exists* $\Lambda_\delta > 0$ *such that if* $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} < \Lambda_\delta$ $(q^* = 2_s^*/2_s^* - q)$, ($E_{f,g}$) *has at least* $\operatorname{cat}_{M_\delta}(M) + 1$ *distinct positive solutions, where* $f_+ = \max\{f, 0\}$, $g_+ = \max\{g, 0\}$*, and* cat *means the Lusternik–Schnirelmann category* (*see* [30]).

REMARK 1.4. Concerning regularity, one can get a priori estimate for the solutions to ($E_{f,g}$) and hence obtain, as in [3, Proposition 5.2], that $u, v \in C^\infty(\overline{\Omega})$ for $s = 1$, $u, v \in C^{0,s}(\overline{\Omega})$ if $0 < s < 1$ and $u, v \in C^{1,s-1}$ if $1 < s < 2$.

This paper is organized as follows: In Section 2, we introduce some notations and preliminaries. In Section 3, we give some technical results which are crucial to the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.1.

2. Notations and preliminaries

In this section, we collect preliminary facts for future reference. First of all, let us fix the standard notations which we will use in this paper. We denote the upper half-space in $\mathbb{R}_+^{N+1}$ by

$$\mathbb{R}_+^{N+1} := \{(x, y) : (x_1, \ldots, x_N, y) \in \mathbb{R}^{N+1},\ y > 0\}.$$

Denote the half cylinder with base Ω by $C_\Omega = \Omega \times (0,\infty) \subset \mathbb{R}^{N+1}_+$ and its lateral boundary by $\partial_L C_\Omega = \partial\Omega \times [0,\infty)$. We shall use C (C_i, $i = 1,2,\ldots$) to denote any positive constant.

Let φ_j, λ_j be the eigenfunctions and eigenvectors of $-\Delta$ in Ω with zero Dirichlet boundary data. The fractional Laplacian $(-\Delta)^{s/2}$ is defined in the space of functions

$$H_0^{s/2}(\Omega) := \bigg\{u = \sum_{j=1}^{\infty} a_j\varphi_j \in L^2(\Omega) : \|u\|_{H_0^{s/2}(\Omega)} = \bigg(\sum_{j=1}^{\infty} a_j^2 \lambda_j^{s/2}\bigg)^{1/2} < \infty\bigg\}$$

and $\|u\|_{H_0^{s/2}(\Omega)} = \|(-\Delta)^{s/4}u\|_{L^2(\Omega)}$. The dual space $H^{-s/2}(\Omega)$ is defined in the standard way as well as the inverse operator $(-\Delta)^{-s/2}$.

DEFINITION 2.1. We say that $(u,v) \in H_0^{s/2}(\Omega) \times H_0^{s/2}(\Omega)$ is a solution to $(\mathrm{E}_{f,g})$ if the identity

$$\begin{aligned}\int_\Omega &((-\Delta)^{s/4}u(-\Delta)^{s/4}\varphi_1 + (-\Delta)^{s/4}v(-\Delta)^{s/4}\varphi_2)\,dx \\ &= \int_\Omega (f(x)|u|^{q-2}u\varphi_1 + g(x)|v|^{q-2}v\varphi_2)\,dx \\ &\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega h(x)|u|^{\alpha-2}u|v|^\beta\varphi_1\,dx + \frac{\beta}{\alpha+\beta}\int_\Omega h(x)|u|^\alpha|v|^{\beta-2}v\varphi_2\,dx\end{aligned}$$

holds for all $(\varphi_1,\varphi_2) \in H_0^{s/2}(\Omega) \times H_0^{s/2}(\Omega)$.

Associated with $(\mathrm{E}_{f,g})$ we consider the energy functional

$$\begin{aligned}J_{f,g}(u,v) &:= \frac{1}{2}\int_\Omega (|(-\Delta)^{s/4}u|^2 + |(-\Delta)^{s/4}v|^2)\,dx \\ &\quad - \frac{1}{q}\int_\Omega (f(u_+)^q + g(v_+)^q)\,dx - \frac{1}{2_s^*}\int_\Omega h(u_+)^\alpha(v_+)^\beta\,dx,\end{aligned}$$

where $u_+ = \max\{u,0\}$ and $v_+ = \max\{v,0\}$. $J_{f,g}$ is well defined in $H_0^{s/2}(\Omega) \times H_0^{s/2}(\Omega)$, and moreover, the critical points of $J_{f,g}$ correspond to weak solutions of $(\mathrm{E}_{f,g})$.

To treat the nonlocal problem $(E_{f,g})$, we will study a corresponding extension problem, which allows us to investigate $(E_{f,g})$ by studying a local problem via classical variational methods. As in [6], we can define the extension operator and fractional Laplacian for functions in $H_0^{s/2}(\Omega)$.

DEFINITION 2.2. Given a function $u \in H_0^{s/2}(\Omega)$, we define its s-harmonic extension $\omega = E_s(u)$ to the cylinder C_Ω as a solution to the problem

$$\begin{cases}\operatorname{div}(y^{1-s}\nabla\omega) = 0 & \text{in } C_\Omega,\\ \omega = 0 & \text{on } \partial_L C_\Omega,\\ \omega = u & \text{on } \Omega\times\{0\},\end{cases}$$

and

$$(-\Delta)^{s/2}u(x) = -K_s \lim_{y\to 0^+} y^{1-s}\frac{\partial \omega}{\partial y}(x,y),$$

where K_s is a normalization constant.

The extension function ω belongs to the space $X_0^s(C_\Omega) = \overline{C_0^\infty(C_\Omega)}$ under the norm

$$\|\omega\|_{X_0^s(C_\Omega)} = \bigg(K_s \int_{C_\Omega} y^{1-s}|\nabla\omega|^2\,dx\,dy\bigg)^{1/2}.$$

The extension operator is an isometry between $H_0^{s/2}(\Omega)$ and $X_0^s(C_\Omega)$, namely,

$$\|\omega\|_{X_0^s(C_\Omega)} = \|u\|_{H_0^{s/2}(\Omega)}, \quad \text{for all } u \in H_0^{s/2}(\Omega). \tag{2.1}$$

With this extension, we can transform $(\mathrm{E}_{\lambda,\mu})$ into the following local problem:

$$(\widehat{\mathrm{E}}_{f,g}) \quad \begin{cases} -\mathrm{div}\,(y^{1-s}\nabla\omega_1) = 0, \quad -\mathrm{div}\,(y^{1-s}\nabla\omega_1) = 0 & \text{in } C_\Omega,\\ \omega_1 = \omega_2 = 0 & \text{on } \partial_L C_\Omega,\\ \dfrac{\partial\omega_1}{\partial\nu^s} = f(x)|\omega_1|^{q-2}\omega_1 + \dfrac{\alpha}{\alpha+\beta}\,h(x)|\omega_1|^{\alpha-2}\omega_1|\omega_2|^{\beta} & \text{on } C_\Omega\times\{0\},\\ \dfrac{\partial\omega_2}{\partial\nu^s} = g(x)|\omega_2|^{q-2}\omega_1 + \dfrac{\beta}{\alpha+\beta}\,h(x)|\omega_1|^{\alpha}|\omega_2|^{\beta-2}\omega_2 & \text{on } C_\Omega\times\{0\},\\ \omega_1 = u, \quad \omega_2 = v & \text{on } C_\Omega\times\{0\}, \end{cases}$$

where

$$\frac{\partial\omega_i}{\partial\nu^s} := -K_s \lim_{y\to 0^+} y^{1-s}\,\frac{\partial\omega_i}{\partial y}, \quad i = 1,2.$$

In the following, we will study $(\widehat{\mathrm{E}}_{f,g})$ in the framework of the Sobolev space $X = X_0^s(C_\Omega)\times X_0^s(C_\Omega)$ using the standard norm

$$\|(\omega_1,\omega_2)\|_X = \bigg(K_s\int_\Omega y^{1-s}(|\nabla\omega_1|^2 + |\nabla\omega_2|^2)\,dx\,dy\bigg)^{1/2}.$$

An energy solution to $(\widehat{\mathrm{E}}_{f,g})$ is a function $(\omega_1,\omega_2)\in X$ satisfying

$$\begin{aligned} K_s\int_{C_\Omega} y^{1-s}\nabla\omega_1\nabla\varphi_1\,dx\,dy + K_s\int_{C_\Omega} y^{1-s}\nabla\omega_2\nabla\varphi_2\,dx\,dy \\ = \int_{\Omega\times\{0\}} (f(x)|\omega_1|^{q-2}\omega_1\varphi_1 + g(x)|\omega_2|^{q-2}\omega_2\varphi_2)\,dx \\ + \frac{\alpha}{\alpha+\beta}\int_{\Omega\times\{0\}} h(x)|\omega_1|^{\alpha-2}\omega_1|\omega_2|^{\beta}\varphi_1\,dx \\ + \frac{\beta}{\alpha+\beta}\int_{\Omega\times\{0\}} h(x)|\omega_1|^{\alpha}|\omega_2|^{\beta-2}\omega_2\varphi_2\,dx, \end{aligned}$$

for all $(\varphi_1,\varphi_2)\in X$.

If (ω_1,ω_2) satisfies $(\widehat{\mathrm{E}}_{f,g})$, then the trace $(u,v)=(\omega_1(\,\cdot\,,0),\omega_2(\,\cdot\,,0))$ is a solution to $(\mathrm{E}_{f,g})$. The converse is also true. Therefore, both formulations are equivalent. We define the associated energy functional to $(\widehat{\mathrm{E}}_{f,g})$ by

$$I_{f,g}(\omega_1,\omega_2)=\frac{1}{2}\,\|(\omega_1,\omega_2)\|_X^2-\frac{1}{q}\int_{\Omega\times\{0\}}(f(x)(\omega_1)_+^q+g(x)(\omega_2)_+^q)\,dx \\ -\frac{1}{2_s^*}\int_{\Omega\times\{0\}}h(x)(\omega_1)_+^{\alpha}(\omega_2)_+^{\beta}\,dx,$$

where $(\omega_1)_+=\max\{\omega_1(x,0),0\}$ and $(\omega_2)_+=\max\{\omega_2(x,0),0\}$. Clearly, critical points of $I_{f,g}$ in X correspond to critical points of $J_{f,g}$ in $H_0^{s/2}(\Omega)\times H_0^{s/2}(\Omega)$.

In the following lemmas, we will list some relevant inequalities from [9], [17].

LEMMA 2.3. *For every $1\le r\le 2_s^*$, and every $\omega\in X_0^s(C_\Omega)$, it holds*

$$\bigg(\int_{\Omega\times\{0\}}|\omega|^r\,dx\bigg)^{2/r}\le C\int_{C_\Omega}y^{1-s}|\nabla\omega|^2\,dx\,dy,$$

for some positive constant C. Furthermore, the space $X_0^s(C_\Omega)$ is compactly embedded into $L^r(\Omega)$, for every $r<2_s^*$.

REMARK 2.4. When $r=2_s^*$, the best constant is denoted by $S(s,N)$, that is

$$S(s,N):=\inf_{\omega\in X_0^s(C_\Omega)\setminus\{0\}}\frac{\displaystyle\int_{C_\Omega}y^{1-s}|\nabla\omega|^2\,dx\,dy}{\bigg(\displaystyle\int_{\Omega\times\{0\}}|\omega|^{2_s^*}\,dx\bigg)^{2/2_s^*}}. \tag{2.2}$$

It is not achieved in any bounded domain and, for all $\omega\in X^s(\mathbb{R}_+^{N+1})$,

$$S(s,N)\bigg(\int_{\mathbb{R}^N\times\{0\}}|\omega|^{2_s^*}\,dx\bigg)^{1/2_s^*}\le\int_{\mathbb{R}_+^{N+1}}y^{1-s}|\nabla\omega|^2\,dx\,dy. \tag{2.3}$$

$S(s,N)$ is achieved for $\Omega=\mathbb{R}^N$ by functions ω_ε which are the s-harmonic extensions of

$$u_\varepsilon(x):=\frac{\varepsilon^{(N-s)/2}}{(\varepsilon^2+|x|^2)^{(N-s)/2}},\quad \varepsilon>0,\ x\in\mathbb{R}^N. \tag{2.4}$$

The constant $S(s,N)$ given in (2.2) takes the exact value

$$S(s,N)=\frac{2\pi^{s/2}\Gamma\Big(\dfrac{2-s}{2}\Big)\Gamma\Big(\dfrac{N+s}{2}\Big)\Big(\Gamma\Big(\dfrac{N}{2}\Big)\Big)^{s/N}}{\Gamma\Big(\dfrac{s}{2}\Big)\Gamma\Big(\dfrac{N-s}{2}\Big)(\Gamma(N))^{s/N}},$$

and it is achieved for $\Omega=\mathbb{R}^N$ by the functions $\omega_\varepsilon=E_s(u_\varepsilon)$.

We consider the following minimization problem:

$$S_{s,\alpha,\beta} := \inf_{(\omega_1,\omega_2)\in X\setminus\{(0,0)\}} \frac{\displaystyle\int_{C_\Omega} y^{1-s}(|\nabla\omega_1|^2 + |\nabla\omega_2|^2)\,dx\,dy}{\left(\displaystyle\int_{\Omega\times\{0\}} |\omega_1|^\alpha|\omega_2|^\beta\,dx\right)^{2/2_s^*}}. \tag{2.5}$$

From [17], we establish a relationship between $S(s,N)$ and $S_{s,\alpha,\beta}$.

LEMMA 2.5. *For the constants $S(s,N)$ and $S_{s,\alpha,\beta}$ introduced in* (2.2) *and* (2.5)*, it holds*

$$S_{s,\alpha,\beta} = \left(\left(\frac{\alpha}{\beta}\right)^{\beta/(\alpha+\beta)} + \left(\frac{\beta}{\alpha}\right)^{\alpha/(\alpha+\beta)}\right)S(s,N).$$

In particular, the constant $S_{s,\alpha,\beta}$ is achieved for $\Omega=\mathbb{R}^N$.

As $I_{f,g}$ is not bounded on X, we consider the behavior of $I_{f,g}$ on the Nehari manifold setting

$$N_{f,g} = \big\{(\omega_1,\omega_2)\in X\setminus\{(0,0)\} : I'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2)=0\big\}.$$

Clearly, $(\omega_1,\omega_2)\in N_{f,g}$ if and only if

$$\|(\omega_1,\omega_2)\|_X^2 = \int_{\Omega\times\{0\}} (f(x)(\omega_1)_+^q + g(x)(\omega_2)_+^q)\,dx + \int_{\Omega\times\{0\}} h(x)(\omega_1)_+^\alpha(\omega_2)_+^\beta\,dx.$$

On the Nehari manifold $N_{f,g}$, from Lemma 2.1 and the Young inequality, we have

$$\begin{aligned}
I_{f,g}(\omega_1,\omega_2) &= \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|(\omega_1,\omega_2)\|_X^2 \\
&\quad - \left(\frac{1}{q}-\frac{1}{2_s^*}\right)\int_{\Omega\times\{0\}} (f(x)(\omega_1)_+^q + g(x)(\omega_2)_+^q)\,dx \\
&\geq \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\|(\omega_1,\omega_2)\|_X^2 \qquad (2.6)\\
&\quad - \left(\frac{1}{q}-\frac{1}{2_s^*}\right)(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})C\|(\omega_1,\omega_2)\|_X^q \\
&\geq -(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})^{2/(2-q)}C, \qquad (2.7)
\end{aligned}$$

where C denotes positive constants (possibly different) independent of (ω_1,ω_2) in X. Let

$$\begin{aligned}
\psi_{f,g}(\omega_1,\omega_2) &:= I'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2) \qquad (2.8)\\
&= \|(\omega_1,\omega_2)\|_X^2 - \int_{\Omega\times\{0\}} (f(x)(\omega_1)_+^q + g(x)(\omega_2)_+^q)\,dx \\
&\quad - \int_{\Omega\times\{0\}} h(x)(\omega_1)_+^\alpha(\omega_2)_+^\beta\,dx.
\end{aligned}$$

Then, for $(\omega_1,\omega_2)\in N_{f,g}$,

$$\begin{aligned}
&\psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2)\\
&= (2-q)\|(\omega_1,\omega_2)\|_X^2 - (2_s^*-q)\int_{\Omega\times\{0\}} h(x)(\omega_1)_+^{\alpha}(\omega_2)_+^{\beta}\,dx \qquad (2.9)\\
&= (2-2_s^*)\|(\omega_1,\omega_2)\|_X^2 + (2_s^*-q)e\int_{\Omega\times\{0\}} (f(x)(\omega_1)_+^{q} + g(x)(\omega_2)_+^{q})\,dx. \qquad (2.10)
\end{aligned}$$

Similarly to the method used in [9], [17], we split $N_{f,g}$ into three parts:

$$\begin{aligned}
N_{f,g}^{+} &= \{(\omega_1,\omega_2)\in N_{f,g} : \psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2) > 0\};\\
N_{f,g}^{0} &= \{(\omega_1,\omega_2)\in N_{f,g} : \psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2) = 0\};\\
N_{f,g}^{-} &= \{(\omega_1,\omega_2)\in N_{f,g} : \psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2) < 0\}.
\end{aligned}$$

Then we have the following results.

LEMMA 2.6. *Suppose that* (ω_1^0,ω_2^0) *is a local minimizer for* $I_{f,g}$ *on* $N_{f,g}$. *Then, if* $(\omega_1,\omega_2)\notin N_{f,g}^0$, (ω_1,ω_2) *is a critical point of* $I_{f,g}$.

PROOF. If $(\omega_1^0,\omega_2^0)\in N_{f,g}$ is a local minimizer of $I_{f,g}$, then (ω_1^0,ω_2^0) is a non-trivial solution of the optimization problem

$$\text{minimize } I_{f,g}(\omega_1,\omega_2) \text{ subject to } \{(\omega_1,\omega_2) : \psi_{f,g}(\omega_1,\omega_2)=0\}.$$

Hence by the theory of multipliers, there exists $\theta\in\mathbb{R}$ such that

$$I'_{f,g}(\omega_1^0,\omega_2^0) = \theta\psi'_{f,g}(\omega_1^0,\omega_2^0).$$

This implies that $0 = I'_{f,g}(\omega_1^0,\omega_2^0)(\omega_1^0,\omega_2^0) = \theta\psi'_{f,g}(\omega_1^0,\omega_2^0)(\omega_1^0,\omega_2^0)$. Moreover, because of $(\omega_1^0,\omega_2^0)\notin N_{f,g}^0$, we get $\theta=0$, and so $I'_{f,g}(\omega_1^0,\omega_2^0)=0$. □

LEMMA 2.7. *There exists* $\Lambda_*>0$ *such that if*

$$\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0,\Lambda_*), \quad N_{f,g}^0=\emptyset.$$

PROOF. Suppose that $(\omega_1^0,\omega_2^0)\in N_{f,g}^0$, then from (2.9)–(2.10) and Lemma 2.1 we obtain

$$\|(\omega_1^0,\omega_2^0)\|_X^2 \le S_{s,\alpha,\beta}^{2_s^*/2}\|(\omega_1^0,\omega_2^0)\|_X^{2_s^*}$$

and

$$\|(\omega_1^0,\omega_2^0)\|_X^2 \le (\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})\,\frac{2_s^*-q}{2_s^*-2}\,S(s,N)^{q/2}\|(\omega_1^0,\omega_2^0)\|_X^{q}.$$

Thus we get

$$C_1 \le \|(\omega_1^0,\omega_2^0)\|_X \le C(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})^{1/(2-q)}C_2,$$

where $C_1,C_2>0$ and are independent of the choice of (ω_1^0,ω_2^0) and f,g. For $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)}$ sufficiently small, this is a contradiction. Hence, there exists $\Lambda_*>0$ such that if $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)}\in(0,\Lambda_*)$, $N_{f,g}^0=\emptyset$. □

In the sequel, we shall use Λ_* to denote different small parameters. By Lemma 2.4, for f,g with $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0,\Lambda_*)$, we write $N_{f,g} = N^+_{f,g} \cup N^-_{f,g}$ and define

$$\alpha^+_{f,g} = \inf_{(\omega_1,\omega_2)\in N^+_{f,g}} I_{f,g}(\omega_1,\omega_2); \qquad \alpha^-_{f,g} = \inf_{(\omega_1,\omega_2)\in N^-_{f,g}} I_{f,g}(\omega_1,\omega_2).$$

For each $(\omega_1,\omega_2) \in X$ with $\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx > 0$, set

$$t_{\max} = \left(\frac{(2-q)\|(\omega_1,\omega_2)\|^2_X}{(2^*_s-q)\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx}\right)^{1/(2^*_s-2)} > 0.$$

Then we have the following results.

LEMMA 2.8. *For each $(\omega_1,\omega_2) \in X$ with $\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx > 0$, we have the following*:

(a) *If $\int_{\Omega\times\{0\}}(f(x)(\omega_1)^q_+ + g(x)(\omega_2)^q_+)\,dx \le 0$, there is a unique $t^- > t_{\max}$ such that*

$$(t^-\omega_1, t^-\omega_2) \in N^-_{f,g} \quad \text{and} \quad I_{f,g}(t^-\omega_1,t^-\omega_2) = \sup_{t\ge 0} I_{f,g}(t\omega_1,t\omega_2).$$

(b) *If $\int_{\Omega\times\{0\}}(f(x)(\omega_1)^q_+ + g(x)(\omega_2)^q_+)\,dx > 0$, there are unique $0 < t^+ < t_{\max} > t^-$ such that $(t^+\omega_1,t^+\omega_2) \in N^+_{f,g}$, $(t^-\omega_1,t^-\omega_2) \in N^-_{f,g}$ and*

$$I_{f,g}(t^+\omega_1,t^+\omega_2) = \inf_{0\le t\le t_{\max}} I_{f,g}(t\omega_1,t\omega_2);\ I_{f,g}(t^-\omega_1,t^-\omega_2) = \sup_{t\ge 0} I_{f,g}(t\omega_1,t\omega_2).$$

LEMMA 2.9. *If $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0,\Lambda_*)$, then*

(a) $\alpha^+_{f,g} < 0$,

(b) $\alpha^-_{f,g} \ge \delta$ *for some* $\delta > 0$.

For the proofs of Lemmas 2.5–2.6, the readers are referred to [9], [17] for similar proofs.

REMARK 2.10. From Lemmas 2.5 and 2.6, it is easy to conclude that if (ω_1,ω_2) in $N^-_{f,g}$,

$$\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx > 0.$$

Next we establish that $I_{f,g}$ satisfies the $(\mathrm{PS})_c$-condition under some restriction on the level of $(\mathrm{PS})_c$-sequences.

LEMMA 2.11. *For $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0,\Lambda_*)$, $I_{f,g}$ satisfies the $(\mathrm{PS})_c$-condition for $c \in (-\infty, \alpha^+_{f,g} + s(K_sS_{s,\alpha,\beta})^{N/s}/(2N))$.*

PROOF. Let $\{(\omega_{1,n},\omega_{2,n})\} \subset X$ be a $(\mathrm{PS})_c$-sequence for $I_{f,g}$ and $c \in (-\infty, \alpha^+_{f,g} + s(K_sS_{s,\alpha,\beta})^{N/s}/(2N))$. Note (2.6), it is easy to obtain that $\{(\omega_{1,n},\omega_{2,n})\}$ is bounded in X. Thus, there exists a subsequence still denoted by $\{(\omega_{1,n},\omega_{2,n})\}$

and $(\omega_1,\omega_2)\in X$ such that $(\omega_{1,n},\omega_{2,n})\rightharpoonup(\omega_1,\omega_2)$ weakly in X. Furthermore, we get $I'_{f,g}(\omega_1,\omega_2)=0$ and

$$\int_{\Omega\times\{0\}}(f(x)(\omega_{1,n})_+^q+g(x)(\omega_{2,n})_+^q)\,dx = \int_{\Omega\times\{0\}}(f(x)(\omega_1)_+^q+g(x)(\omega_2)_+^q)\,dx+o(1);$$

$$\|(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\|_X^2=\|(\omega_{1,n},\omega_{2,n})\|_X^2-\|(\omega_1,\omega_2)\|_X^2+o(1).$$

Moreover, by the Brezis–Lieb Lemma, we obtain

$$\int_{\Omega\times\{0\}}h(x)(\omega_{1,n}-\omega_1)_+^\alpha(\omega_{2,n}-\omega_2)_+^\beta dx=\int_{\Omega\times\{0\}}h(x)(\omega_{1,n})_+^\alpha(\omega_{2,n})_+^\beta dx - \int_{\Omega\times\{0\}}h(x)(\omega_1)_+^\alpha(\omega_2)_+^\beta dx+o(1).$$

Since $I_{f,g}(\omega_{1,n},\omega_{2,n})=c+o(1)$ and $I'_{f,g}(\omega_{1,n},\omega_{2,n})=o(1)$, we deduce that

$$\frac{1}{2}\|(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\|_X^2 - \frac{1}{2_s^*}\int_{\Omega\times\{0\}}h(x)(\omega_{1,n}-\omega_1)_+^\alpha(\omega_{2,n}-\omega_2)_+^\beta\,dx=c-I_{f,g}(\omega_1,\omega_2)+o(1) \tag{2.11}$$

and

$$\begin{aligned} o(1)&=I'_{f,g}(\omega_{1,n},\omega_{2,n})(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\\ &=(I'_{f,g}(\omega_{1,n},\omega_{2,n})-I'_{f,g}(\omega_1,\omega_2))(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\\ &=\|(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\|_X^2\\ &\quad-\int_{\Omega\times\{0\}}h(x)(\omega_{1,n}-\omega_1)_+^\alpha(\omega_{2,n}-\omega_2)_+^\beta\,dx+o(1). \end{aligned} \tag{2.12}$$

Now we may assume that

$$\|(\omega_{1,n}-\omega_1,\omega_{2,n}-\omega_2)\|_X^2\to l \quad\text{and}\quad \int_{\Omega\times\{0\}}h(x)(\omega_{1,n}-\omega_1)_+^\alpha(\omega_{2,n}-\omega_2)_+^\beta\,dx\to l$$

as $n\to\infty$, for some $l\in[0,+\infty)$.

Suppose $l\neq 0$ and notice that $h\le 1$, using (2.5), (2.12) and passing to the limit as $n\to\infty$, we have $l\ge K_sS_{s,\alpha,\beta}l^{2/2_s^*}$, that is,

$$l\ge (K_sS_{s,\alpha,\beta})^{N/s}. \tag{2.13}$$

Then by (2.11)–(2.13) and $(\omega_1,\omega_2)\in N_{f,g}\cup\{(0,0)\}$, we have

$$c=I_{f,g}(\omega_1,\omega_2)+\bigg(\frac{1}{2}-\frac{1}{2_s^*}\bigg)l\ge \alpha_{f,g}^+ +\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s},$$

which contradicts the definition of c. Hence $l=0$, and the proof is completed. □

Next we obtain the existence of a local minimizer for $I_{f,g}$ on $N_{f,g}^+$.

Lemma 2.12. *For $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0, \Lambda_*)$, the functional $I_{f,g}$ has a minimizer $((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) \in N^+_{f,g}$ and it satisfies:*

(a) $I_{f,g}((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) = \alpha^+_{f,g}$;

(b) $((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})$ *is a positive solution of* $(\widehat{\mathrm{E}}_{f,g})$;

(c) $I_{f,g}((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) \to 0$ *as* $\|f_+\|_{L^{q^*}(\Omega)}, \|g_+\|_{L^{q^*}(\Omega)} \to 0$;

(d) $\|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|_X \to 0$ *as* $\|f_+\|_{L^{q^*}(\Omega)}, \|g_+\|_{L^{q^*}(\Omega)} \to 0$.

Proof. Note Lemmas 2.1–2.7, the proof of (a)–(b) is almost identical to the proof of [17, Proposition 6.1] and is omitted here for brevity. Now we give the proof of (c) and (d). It follows from (2.7) and Lemma 2.6 that

$$0 > I_{f,g}((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) \geq -(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})^{2/(2-q)}C.$$

We obtain $I_{f,g}((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) \to 0$ as $\|f_+\|_{L^{q^*}(\Omega)}, \|g_+\|_{L^{q^*}(\Omega)} \to 0$.

Now we show (d). By $((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g}) \in N^+_{f,g}$ and (2.10), we have

$$\begin{aligned}(2.14)\quad \|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|^2_X &\leq \frac{2^*_s - q}{2^*_s - 2} \int_{\Omega\times\{0\}} (f_+((\omega_1)^+_{f,g})^q + g_+((\omega_2)^+_{f,g})^q)\, dx \\ &\leq C(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)})\|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|^q_X.\end{aligned}$$

Since $I_{f,g}$ is coercive and bounded from below on $N_{f,g}$, $((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})$ is bounded in X and so by (2.14) we get

$$\|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|^{2-q}_X \leq C(\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)}).$$

Then

$$\|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|_X \to 0 \quad \text{as } \|f_+\|_{L^{q^*}(\Omega)}, \|g_+\|_{L^{q^*}(\Omega)} \to 0. \qquad \square$$

3. Some technical results

In this section, we shall provide some useful results which are crucial for the proof of Theorem 1.1. For $b > 0$, we define

$$J^b_\infty(\omega_1, \omega_2) = \frac{1}{2}\|(\omega_1, \omega_2)\|^2_X - \frac{b}{2^*_s}\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\, dx$$

and

$$N^b_\infty = \big\{(\omega_1, \omega_2) \in X \setminus \{(0,0)\} : (J^b_\infty)'(\omega_1, \omega_2)(\omega_1, \omega_2) = 0\big\}.$$

Then we have the following.

Lemma 3.1. *For each $(\omega_1, \omega_2) \in N^-_{f,g}$, we have*

(a) *There is a unique $t^b_{(\omega_1,\omega_2)}$ such that $(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2) \in N^b_\infty$ and*

$$\begin{aligned}\max_{t\geq 0} J^b_\infty(t\omega_1, t\omega_2) &= J^b_\infty(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2) \\ &= \frac{s}{N}\, b^{(s-N)/s}\bigg(\|(\omega_1, \omega_2)\|^{2^*}_X \Big/ \int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\, dx\bigg)^{(N-s)/s}.\end{aligned}$$

(b) *For $\mu \in (0,1)$, there is a unique $t^1_{(\omega_1,\omega_2)}$ such that $(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_2)$ in N^1_∞. Moreover,*

$$\begin{aligned}
&J^1_\infty(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_2)\\
&\le (1-\mu)^{-N/s}\bigg(I_{f,g}(\omega_1,\omega_2) + \frac{2-q}{2q}\,\mu^{q/(q-2)}C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{2/(2-q)}\bigg).
\end{aligned}$$

PROOF. (a) For each $(\omega_1,\omega_2) \in N^-_{f,g}$, let

$$\overline{h}(t) = J^b_\infty(t\omega_1, t\omega_2) = \frac{1}{2}\,t^2\|(\omega_1,\omega_2)\|^2_X - \frac{b}{2^*_s}\,t^{2^*_s}\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx.$$

Then due to Remark 2.2 and $2 < 2^*_s$, we have $\overline{h}(t) \to -\infty$ as $t \to \infty$,

$$\overline{h}'(t) = t\|(\omega_1,\omega_2)\|^2_X - bt^{2^*_s-1}\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx$$

and

$$\overline{h}''(t) = t\|(\omega_1,\omega_2)\|^2_X - b(2^*_s-1)t^{2^*_s-2}\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx.$$

Set

$$t^b_{(\omega_1,\omega_2)} = \bigg(\|(\omega_1,\omega_2)\|^2_X \Big/ \int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx\bigg)^{1/(2^*_s-2)} > 0.$$

Then $h'(t^b_{(\omega_1,\omega_2)}) = 0$, $(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2) \in N^b_\infty$ and $h''(t^b_{(\omega_1,\omega_2)}) = (2-2^*_s)$ $\cdot\|(\omega_1,\omega_2)\|^2_X < 0$. Hence there is a unique $t^b_{(\omega_1,\omega_2)}$ such that $(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2)$ in N^b_∞ and

$$\begin{aligned}
\max_{t\ge 0} J^b_\infty(t\omega_1, t\omega_2) &= J^b_\infty(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2)\\
&= \frac{s}{N}\,b^{(s-N)/s}\bigg(\|(\omega_1,\omega_2)\|^{2^*_s}_X \Big/\!\!\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx\bigg)^{(N-s)/s}.
\end{aligned}$$

(b) For $\mu \in (0,1)$, we have

$$\begin{aligned}
&\int_{\Omega\times\{0\}} (f_+(t^b_{(\omega_1,\omega_2)}\omega_1)^q_+ + g_+(t^b_{(\omega_1,\omega_2)}\omega_2)^q_+)\,dx\\
&\quad\le (\|f_+\|_{q^*} + \|g_+\|_{q^*})\,C\|(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2)\|^q_X\\
&\quad\le \frac{2-q}{2}\,((\|f_+\|_{q^*} + \|g_+\|_{q^*})C\mu^{-q/2})^{2/(2-q)}\\
&\qquad + \frac{q}{2}\,(\mu^{q/2}\|(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2)\|^q_X)^{2/q}\\
&\quad= \frac{2-q}{2}\,\mu^{q/(q-2)}C(\|f_+\|_{q^*} + \|g_+\|_{q^*})^{2/(2-q)} + \frac{q\mu}{2}\,\|(t^b_{(\omega_1,\omega_2)}\omega_1, t^b_{(\omega_1,\omega_2)}\omega_2)\|^2_X.
\end{aligned}$$

Then let $b = 1/(1-\mu)$, then by part (a),

$$\begin{aligned}
I_{f,g}(\omega_1,\omega_2) &= \max_{t\geq 0} I_{f,g}(t\omega_1,t\omega_2) \geq I_{f,g}\big(t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_1, t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_2\big) \\
&\geq \frac{1-\mu}{2}\,\big\|(t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_1, t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_2)\big\|^2_X \\
&\quad - \frac{1}{2^*_s}\,\big(t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\big)^{2^*_s}\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx \\
&\quad - \frac{2-q}{2q}\,\mu^{q/(q-2)}C(\|f_+\|_{q^*}+\|g_+\|_{q^*})^{2/(2-q)} \\
&= (1-\mu)J^{1/(1-\mu)}_\infty\big(t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_1, t^{1/(1-\mu)}_{(\omega_1,\omega_2)}\omega_2\big) \\
&\quad - \frac{2-q}{2q}\,\mu^{q/(q-2)}C(\|f_+\|_{q^*}+\|g_+\|_{q^*})^{2/(2-q)} \\
&= (1-\mu)^{N/s}\frac{s}{N}\left(\|(\omega_1,\omega_2)\|^{2^*_s}_X\Big/\int_{\Omega\times\{0\}} h(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx\right)^{(N-s)/s} \\
&\quad - \frac{2-q}{2q}\,\mu^{q/(q-2)}C(\|f_+\|_{q^*}+\|g_+\|_{q^*})^{2/(2-q)} \\
&= (1-\mu)^{N/s}J^1_\infty(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_2) \\
&\quad - \frac{2-q}{2q}\,\mu^{q/(q-2)}C(\|f_+\|_{q^*}+\|g_+\|_{q^*})^{2/(2-q)}.
\end{aligned}$$

This completes the proof. □

LEMMA 3.2. *Let $\{(\omega_{1,n},\omega_{2,n})\} \subset X$ be a nonnegative function sequence with*

$$\int_{\Omega\times\{0\}} (\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx = 1 \quad \text{and} \quad \|(\omega_{1,n},\omega_{2,n})\|^2_X \to K_sS_{s,\alpha,\beta}.$$

Then there exists a sequence $\{(y_n,\varepsilon_n)\} \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$\begin{aligned}
&(W_{1,n}(x,y), W_{2,n}(x,y)) \\
&\qquad := \big(E_s(\varepsilon_n^{(N-s)/2}\omega_{1,n}(\varepsilon_n x+y_n,0)), E_s(\varepsilon_n^{(N-s)/2}\omega_{2,n}(\varepsilon_n x+y_n,0))\big)
\end{aligned}$$

contains a convergent subsequence denoted again by $\{(W_{1,n}(x,y), W_{2,n}(x,y))\}$ such that

$$(W_{1,n}(x,y), W_{2,n}(x,y)) \to (W_1,W_2) \quad \text{in } X.$$

Moreover, we have $\varepsilon_n \to 0$ and $y_n \to y_0 \in \overline{\Omega}$ as $n \to \infty$.

PROOF. Let $Z_{n,1}(x) = \omega_{1,n}(x,0)$, $Z_{n,2}(x) = \omega_{2,n}(x,0)$, we have

$$\int_\Omega (Z_{n,1})^\alpha_+(Z_{n,2})^\beta_+\,dx = 1 \quad \text{and} \quad \|Z_{n,1}\|^2_{H^s_0(\Omega)} + \|Z_{n,1}\|^2_{H^s_0(\Omega)} \to K_sS_{\alpha,\beta}$$

as $n \to \infty$. By the proof of Lemma 2.2, we know that $\{Z_{n,1}\}$ and $\{Z_{n,2}\}$ are minimizing sequences for the critical Sobolev inequality in the form (2.2). Thus from [6, Theorem 3] and [21, Theorem 5] we deduce that there exist a sequence of points $\{y_n\} \subseteq \mathbb{R}^N$ and a sequence of numbers $\{\varepsilon_n\} \subset (0,\infty)$ such that $\widehat{Z}_{n,1}(x) =$

$\varepsilon_n^{(N-s)/2} Z_{n,1}(\varepsilon_n x + y_n) \to \widehat{Z}_1(x)$ and $\widehat{Z}_{n,2}(x) = \varepsilon_n^{(N-s)/2} Z_{n,2}(\varepsilon_n x + y_n) \to \widehat{Z}_2(x)$ in $H^s(\mathbb{R}^N)$ as $n \to \infty$. Moreover, we have $\varepsilon_n \to 0$ and $y_n \to y_0 \in \overline{\Omega}$ as $n \to \infty$. Denote $W_{1,n} = E_s(\widehat{Z}_{n,1})$, $W_{2,n} = E_s(\widehat{Z}_{n,2})$ and $W_1 = E_s(\widehat{Z}_1)$, $W_2 = E_s(\widehat{Z}_2)$. Then we obtain the result. $\square$

Next, we will use $\omega_\varepsilon = E_s(u_\varepsilon)$, the family of minimizers to inequality (2.2), where u_ε is given in (2.4). Let $\eta \in C^\infty(C_\Omega)$, $0 \le \eta(x,y) \le 1$ and for small fixed ρ_0,

$$\eta(x,y) = \begin{cases} 1 & \text{for } (x,y) \in B^+_{\rho_0/2} := \{(x,y) : |(x,y)| < \rho_0/2,\ y > 0\}, \\ 0 & \text{for } (x,y) \notin B^+_{\rho_0} := \{(x,y) : |(x,y)| < \rho_0,\ y > 0\}. \end{cases}$$

We take $\rho_0 < r_0$ small enough such that $\overline{B^+_{\rho_0}}(x-z,y) \subset \overline{C_\Omega}$ for all $z \in M$, where $\overline{B^+_{\rho_0}}(x-z,y) := \{(x,y) : |(x-z,y)| \le \rho_0,\ y \ge 0\}$.

For any $z \in M$, we define

$$v_{\varepsilon,z} = \eta(x-z,y)\,\omega_\varepsilon(x-z,y) = \eta(x-z,y) E_s(u_\varepsilon(x-z)).$$

From the same argument as in [17] we obtain that

$$\|v_{\varepsilon,z}\|^2_{X_0^s(C_\Omega)} = K_s \int_{\mathbb{R}^{N+1}_+} y^{1-s} |\nabla \omega_\varepsilon|^2 \,dx\,dy + O(\varepsilon^{N-s}), \tag{3.1}$$

$$\begin{aligned} \int_{\Omega\times\{0\}} (v_{\varepsilon,z})^{2_s^*}\,dx &= \int_{\mathbb{R}^N\times\{0\}} (\omega_\varepsilon)^{2_s^*}\,dx + O(\varepsilon^N) \\ &= \int_{\mathbb{R}^N} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2} \right)^N dx + O(\varepsilon^N). \end{aligned} \tag{3.2}$$

Then we have the following result.

LEMMA 3.3. *There exist $\varepsilon_0, \sigma(\varepsilon) > 0$ such that for $\varepsilon \in (0,\varepsilon_0)$ and $\sigma \in (0,\sigma(\varepsilon))$, we have*

$$\sup_{t\ge 0} I_{f,g}\big((\omega_1)^+_{f,g} + t\sqrt{\alpha} v_{\varepsilon,z}, (\omega_2)^+_{f,g} + t\sqrt{\beta} v_{\varepsilon,z}\big) < \alpha^+_{f,g} + \frac{s}{2N} (K_s S_{s,\alpha,\beta})^{N/s} - \sigma$$

uniformly in $z \in M$, where $((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})$ is a local minimum in Lemma 2.8. Furthermore, there exists $t_z^- > 0$ such that

$$\big((\omega_1)^+_{f,g} + t_z^- \sqrt{\alpha} v_{\varepsilon,z}, (\omega_2)^+_{f,g} + t_z^- \sqrt{\beta} v_{\varepsilon,z}\big) \in N^-_{f,g} \quad \textit{for all } z \in M.$$

PROOF. Since

$$\begin{aligned} & I_{f,g}\big((\omega_1)^+_{f,g} + t\sqrt{\alpha} v_{\varepsilon,z}, (\omega_2)^+_{f,g} + t\sqrt{\beta} v_{\varepsilon,z}\big) \\ &\quad = \frac{K_s}{2} \int_{C_\Omega} y^{1-s} \big(|\nabla((\omega_1)^+_{f,g} + t\sqrt{\alpha} v_{\varepsilon,z})|^2 \\ &\quad\quad + |\nabla((\omega_2)^+_{f,g} + t\sqrt{\beta} v_{\varepsilon,z})|^2 \big)\,dx\,dy \\ &\quad\quad - \frac{1}{q} \int_{\Omega\times\{0\}} \big(f(x)((\omega_1)^+_{f,g} + t\sqrt{\alpha}\eta(x-z,0) u_\varepsilon(x-z))^q_+ \end{aligned} \tag{3.3}$$

$$
\begin{aligned}
&+ g(x)((\omega_2)^+_{f,g} + t\sqrt{\beta}\eta(x-z,0)u_\varepsilon(x-z))^q_+\big)\,dx \\
&- \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)\big((\omega_1)^+_{f,g} + t\sqrt{\alpha}\eta(x-z,0)u_\varepsilon(x-z)\big)^\alpha_+ \\
&\cdot \big((\omega_2)^+_{f,g} + t\sqrt{\beta}\eta(x-z,0)u_\varepsilon(x-z)\big)^\beta_+\,dx \\
= &\,\frac{1}{2}\,\|((\omega_1)^+_{f,g},(\omega_2)^+_{f,g})\|^2_X + t^2\,\frac{\alpha+\beta}{2}\,\|v_{\varepsilon,z}\|^2_{X^s_0(C_\Omega)} \\
&+ tK_s\bigg(\int_{C_\Omega}\nabla(\omega_1)^+_{f,g}\nabla(\sqrt{\alpha}v_{\varepsilon,z})\,dx\,dy \\
&+ \int_{C_\Omega}\nabla(\omega_2)^+_{f,g}\nabla(\sqrt{\beta}v_{\varepsilon,z})\,dx\,dy\bigg) \\
&- \frac{1}{q}\int_{\Omega\times\{0\}}\big(f(x)((\omega_1)^+_{f,g} + t\sqrt{\alpha}\eta(x-z,0)u_\varepsilon(x-z))^q_+ \\
&+ g(x)((\omega_2)^+_{f,g} + t\sqrt{\beta}\eta(x-z,0)u_\varepsilon(x-z))^q_+\big)\,dx \\
&- \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)((\omega_1)^+_{f,g} + t\sqrt{\alpha}\eta(x-z,0)u_\varepsilon(x-z))^\alpha_+ \\
&\cdot \big((\omega_2)^+_{f,g} + t\sqrt{\beta}\eta(x-z,0)u_\varepsilon(x-z)\big)^\beta_+\,dx \\
\le &\, I_{f,g}((\omega_1)^+_{f,g},(\omega_2)^+_{f,g}) + t^2\,\frac{\alpha+\beta}{2}\,\|v_{\varepsilon,z}\|^2_{X^s_0(C_\Omega)} \\
&- \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)\big((\omega_1)^+_{f,g} + t\sqrt{\alpha}v_{\varepsilon,z}\big)^\alpha_+\big((\omega_2)^+_{\lambda,\mu} + t\sqrt{\beta}v_{\varepsilon,z}\big)^\beta_+\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^\beta_+\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} \alpha h(x)((\omega_1)^+_{f,g})^{\alpha-1}_+((\omega_2)^+_{f,g})^\beta_+(t\sqrt{\alpha}v_{\varepsilon,z})\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} \beta h(x)((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^{\beta-1}_+(t\sqrt{\beta}v_{\varepsilon,z})\,dx \\
= &\,\alpha^+_{f,g} + K(tv_{\varepsilon,z}),
\end{aligned}
$$

where

$$
\begin{aligned}
K(tv_{\varepsilon,z}) = &\, t^2\,\frac{\alpha+\beta}{2}\,\|v_{\varepsilon,z}\|^2_{X^s_0(C_\Omega)} \\
&- \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)\big((\omega_1)^+_{f,g} + t\sqrt{\alpha}v_{\varepsilon,z}\big)^\alpha_+\big((\omega_2)^+_{\lambda,\mu} + t\sqrt{\beta}v_{\varepsilon,z}\big)^\beta_+\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} h(x)((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^\beta_+\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} \alpha h(x)((\omega_1)^+_{f,g})^{\alpha-1}_+((\omega_2)^+_{f,g})^\beta_+(t\sqrt{\alpha}v_{\varepsilon,z})\,dx \\
&+ \frac{1}{2^*_s}\int_{\Omega\times\{0\}} \beta h(x)((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^{\beta-1}_+(t\sqrt{\beta}v_{\varepsilon,z})\,dx.
\end{aligned}
$$

In what follows we shall show that

$$\sup_{t\geq 0} K(tv_{\varepsilon,z}) < \frac{s}{2N}\,(K_s S_{s,\alpha,\beta})^{N/s} \quad \text{for } \varepsilon>0 \text{ small enough.}$$

It is easy to see that

$$\lim_{t\to 0} K(tv_{\varepsilon,z}) = 0 \quad \text{and} \quad \lim_{t\to+\infty} K(tv_{\varepsilon,z}) = 0.$$

Thus, for all ε sufficiently small, there exist $t_0>0$ and $t_1>0$ such that

$$K(tv_{\varepsilon,z}) < \frac{s}{2N}\,(K_s S_{s,\alpha,\beta})^{N/s} \quad \text{for all } t\in(0,t_0], \tag{3.4}$$

$$K(tv_{\varepsilon,z}) < \frac{s}{2N}\,(K_s S_{s,\alpha,\beta})^{N/s} \quad \text{for all } t\in[t_1,+\infty). \tag{3.5}$$

By the definition of $v_{\varepsilon,z}$, we get that

$$\begin{aligned}
&\int_{\Omega\times\{0\}} h(x)(v_{\varepsilon,z})^{2_s^*}\,dx \\
&= \int_{B_{r_0}(z)} (\eta(x-z,0)u_\varepsilon(x-z))^{2_s^*}\,dx = \int_{\mathbb{R}^N} h(x+z)(\eta(x,0)u_\varepsilon(x))^{2_s^*}\,dx \\
&= \int_{\mathbb{R}^N} h(x+z)\eta^{2_s^*}(x,0)(u_\varepsilon(x))^{2_s^*}\,dx = \int_{\mathbb{R}^N} h(x+z)\eta^{2_s^*}(x,0)\,\frac{\varepsilon^N}{(\varepsilon^2+|x|^2)^N}\,dx.
\end{aligned}$$

Thus, taking into account condition (H_1), we obtain

$$\begin{aligned}
0 &\leq \int_{\Omega\times\{0\}} (v_{\varepsilon,z})^{2_s^*}\,dx - \int_{\Omega\times\{0\}} h(x)(v_{\varepsilon,z})^{2_s^*}\,dx \\
&= \int_{\mathbb{R}^N} (v_{\varepsilon,z})^{2_s^*}\,dx - \int_{\mathbb{R}^N} h(x)(v_{\varepsilon,z})^{2_s^*}\,dx \\
&= \varepsilon^N\bigg(\int_{\mathbb{R}^N\setminus B_{\rho_0/2}} \frac{(1-h(x+z))\eta^{2_s^*}(x,0)}{(\varepsilon^2+|x|^2)^N}\,dx \\
&\quad + \int_{B_{\rho_0/2}} \frac{(1-h(x+z))}{(\varepsilon^2+|x|^2)^N}\,dx\bigg) \\
&\leq \varepsilon^N\bigg(\int_{\mathbb{R}^N\setminus B_{\rho_0/2}} \frac{1}{|x|^{2N}}\,dx + \int_{B_{\rho_0/2}} \frac{|x|^\rho}{(\varepsilon^2+|x|^2)^N}\,dx\bigg) \\
&\leq \varepsilon^N\bigg(C + C\int_0^{\rho_0/2} r^{\rho-N+1}\,d\rho\bigg) \leq C\varepsilon^N,
\end{aligned} \tag{3.6}$$

for all $z\in M$. From [11, Lemma 4.1], we see that there exist $C_1(\alpha), C_2(\beta)>0$ such that

$$\begin{aligned}
(a+b)^\alpha(c+d)^\beta &\geq a^\alpha c^\beta + a^\alpha d^\beta + b^\alpha c^\beta + b^\alpha d^\beta \\
&\quad + C_1(\alpha)a^{\alpha-1}bc^\beta + C_1(\alpha)a^{\alpha-1}bd^\beta + C_2(\beta)b^\alpha c^{\beta-1}d \\
&\quad + C_2(\beta)a^\alpha c^{\beta-1}d + C_1(\alpha)C_2(\beta)a^{\alpha-1}bc^{\beta-1}d
\end{aligned} \tag{3.7}$$

for any $a,b,c,d>0$. It follows from Remark 1.1 that $h(x)>0$ for all $x\in B_{r_0}(z)$. From (3.7) we obtain that

$$\begin{aligned}
&\int_{\Omega\times\{0\}} h(x)\big[\big((\omega_1)^+_{f,g}+t\sqrt{\alpha}v_{\varepsilon,z}\big)^\alpha_+\big((\omega_2)^+_{\lambda,\mu}+t\sqrt{\beta}v_{\varepsilon,z}\big)^\beta_+-((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^\beta_+\big]\,dx\\
&\quad-\int_{\Omega\times\{0\}} h(x)\big[\alpha((\omega_1)^+_{f,g})^{\alpha-1}_+((\omega_2)^+_{f,g})^\beta_+(t\sqrt{\alpha}v_{\varepsilon,z})\\
&\quad+((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^{\beta-1}_+(t\sqrt{\beta}v_{\varepsilon,z})\big]\,dx\\
&=\int_{B_{r_0}(z)\times\{0\}} h(x)\big[\big((\omega_1)^+_{f,g}+t\sqrt{\alpha}v_{\varepsilon,z}\big)^\alpha_+\big((\omega_2)^+_{\lambda,\mu}+t\sqrt{\beta}v_{\varepsilon,z}\big)^\beta_+\\
&\quad-((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^\beta_+\big]\,dx\\
&\quad-\int_{B_{r_0}(z)} h(x)\big[\alpha((\omega_1)^+_{f,g})^{\alpha-1}_+((\omega_2)^+_{f,g})^\beta_+(t\sqrt{\alpha}v_{\varepsilon,z})\\
&\quad+((\omega_1)^+_{f,g})^\alpha_+((\omega_2)^+_{f,g})^{\beta-1}_+(t\sqrt{\beta}v_{\varepsilon,z})\big]\,dx\\
&\geq\int_{B_{r_0}(z)\times\{0\}} h(x)\big[\alpha^{\alpha/2}\beta^{\beta/2}(tv_{\varepsilon,z})^{2^s_*}+C(tv_{\varepsilon,z})^{2^s_*-1}\big]\,dx\\
&=\int_{\Omega\times\{0\}} h(x)\big[\alpha^{\alpha/2}\beta^{\beta/2}(tv_{\varepsilon,z})^{2^s_*}+C(tv_{\varepsilon,z})^{2^s_*-1}\big]\,dx.
\end{aligned}$$

Thus we have

$$\begin{aligned}
K(tv_{\varepsilon,z})\leq t^2\,\frac{2^*_s}{2}\,\|v_{\varepsilon,z}\|^2_{X^s_0(C\Omega)}-\frac{\alpha^{\alpha/2}\beta^{\beta/2}}{2^*_s}t^{2^*_s}\int_{\Omega\times\{0\}}h(x)(v_{\varepsilon,z})^{2^*_s}\,dx\\
-C_1\int_{\Omega\times\{0\}}(v_{\varepsilon,z})^{2^*_s-1}\,dx
\end{aligned}\tag{3.8}$$

with some constant $C_1>0$. Note that

$$\begin{aligned}
\int_{\Omega\times\{0\}}(v_{\varepsilon,z})^{2^*_s-1}\,dx&=\int_{\Omega\times\{0\}}(\eta(x-z,0)u_\varepsilon(x-z))^{2^*_s-1}\,dx\\
&=\int_{B_{r_0}}\bigg[\frac{\eta(x,0)\varepsilon^{(N-s)/2}}{(\varepsilon^2+|x|^2)^{(N-s)/2}}\bigg]^{(N+s)/(N-s)}\,dx\\
&\geq\int_{B_{r_0/2}}\frac{\varepsilon^{(N+s)/2}}{\varepsilon^{N+s}(\varepsilon^2+|x|^2)^{(N+s)/2}}\,\varepsilon^N\,dx\\
&=C_2\varepsilon^{(N-s)/2}\int_0^{r_0/2}\frac{r^{N-1}}{(1+r^2)^{(N+s)/2}}\,dr=C_3\varepsilon^{(N-s)/2}
\end{aligned}\tag{3.9}$$

for some $C_2,C_3>0$. We from (3.1)–(3.9) obtain that

$$\begin{aligned}
K(tv_{\varepsilon,z})&\leq\frac{2^*_s}{2}\,t^2\|v_{\varepsilon,z}\|^2_{X^s_0(C\Omega)}\\
&\quad-\frac{\alpha^{\alpha/2}\beta^{\beta/2}}{2^*_s}\,t^{2^*_s}\int_{\Omega\times\{0\}}h(x)(v_{\varepsilon,z})^{2^*_s}\,dx-C_4\varepsilon^{(N-s)/2}
\end{aligned}\tag{3.10}$$

$$\le \frac{s}{2N}\bigg((\alpha+\beta)\|v_{\varepsilon,z}\|^2_{X_0^s(C_\Omega)}\bigg/\bigg(\int_{\Omega\times\{0\}}\alpha^{\alpha/2}\beta^{\beta/2}(v_{\varepsilon,z})^{2_s^*}\,dx\bigg)^{2/2_s^*}\bigg)^{N/s}$$
$$- C_4\varepsilon^{(N-s)/2}$$
$$= \frac{s}{2N}\bigg(\bigg(\bigg(\frac{\alpha}{\beta}\bigg)^{\beta/(\alpha+\beta)} + \bigg(\frac{\beta}{\alpha}\bigg)^{\alpha/(\alpha+\beta)}\bigg)$$
$$\cdot K_s\int_{\mathbb{R}^{N+1}_+} y^{1-s}|\nabla\omega_\varepsilon|^2\,dx\,dy + O(\varepsilon^{N-s})\bigg/$$
$$\bigg(\int_{\mathbb{R}^N}\bigg(\frac{\varepsilon}{\varepsilon^2+|x|^2}\bigg)^N dx + O(\varepsilon^N)\bigg)^{2/2_s^*}\bigg)^{N/s} - C_4\varepsilon^{(N-s)/2}$$
$$= \frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s} + O(\varepsilon^{N-s}) - C_4\varepsilon^{(N-s)/2} < \frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}$$

for ε sufficiently small and $t\in[t_0,t_1]$. From the compactness of $\overline{\Omega_r^-}$, it follows from (3.4)–(3.5) and (3.10) that there exist $\varepsilon_0,\sigma(\varepsilon)>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ and $\sigma\in(0,\sigma(\varepsilon))$, we have

$$\sup_{t\ge 0} I_{f,g}\big((\omega_1)^+_{f,g} + t\sqrt{\alpha}v_{\varepsilon,z}, (\omega_2)^+_{f,g} + t\sqrt{\beta}v_{\varepsilon,z}\big) < \alpha^+_{f,g} + \frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s} - \sigma$$

uniformly in $z\in M$. Arguing as the proof of [25, Lemma 4.4], we conclude that there exists $t_z^->0$ such that

$$\big((\omega_1)^+_{f,g} + t_z^-\sqrt{\alpha}v_{\varepsilon,z}, (\omega_2)^+_{f,g} + t_z^-\sqrt{\beta}v_{\varepsilon,z}\big) \in N^-_{f,g} \quad \text{for all } z\in M. \qquad \square$$

Lemma 3.4. *We have*

$$\inf_{(\omega_1,\omega_2)\in N^1_\infty} J^1_\infty(\omega_1,\omega_2) = \frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}.$$

Proof. Let $(\omega_1,\omega_2)\in N^1_\infty$, then

$$\|(\omega_1,\omega_2)\|^2_X = \int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx. \tag{3.11}$$

By (2.5), we see

$$KS_{s,\alpha,\beta}\bigg(\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx\bigg)^{2/2_s^*}$$
$$\le KS_{s,\alpha,\beta}\bigg(\int_{\Omega\times\{0\}} (\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx\bigg)^{2/2_s^*} \le \|(\omega_1,\omega_2)\|^2_X,$$

i.e.

$$\int_{\Omega\times\{0\}} h(x)(\omega_1)^\alpha_+(\omega_2)^\beta_+\,dx \le \bigg(\frac{1}{KS_{s,\alpha,\beta}}\bigg)^{2_s^*/2}\|(\omega_1,\omega_2)\|^{2_s^*}_X. \tag{3.12}$$

We from (3.11) and (3.12) deduce that

$$\|(\omega_1,\omega_2)\|^2_X \ge (K_sS_{s,\alpha,\beta})^{N/s}.$$

Then

$$J^1_\infty(\omega_1,\omega_2)=\frac{s}{2N}\,\|(\omega_1,\omega_2)\|^2_X\geq\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s},$$

and thus

$$\inf_{(\omega_1,\omega_2)\in N^1_\infty}J^1_\infty(\omega_1,\omega_2)\geq\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}. \tag{3.13}$$

Since

$$\max_{t\geq 0}\left(\frac{a}{2}\,t^2-\frac{b}{2^*_s}\,t^{2^*_s}\right)=\frac{s}{2N}\left(\frac{a}{b^{2/2^*_s}}\right)^{N/2}\quad\text{for any } a>0 \text{ and } b>0,$$

by (3.1)–(3.2) and (3.6), we deduce that

$$\begin{aligned}\sup_{t\geq 0}J^1_\infty(t\sqrt{\alpha}v_{\varepsilon,z},t\sqrt{\beta}v_{\varepsilon,z})&=\frac{s}{2N}\left(\frac{(\alpha+\beta)\displaystyle\int_{C_\Omega}|\nabla v_{\varepsilon,z}|^2\,dx\,dy}{\left(\alpha^{\alpha/2}\beta^{\beta/2}\displaystyle\int_{\Omega\times\{0\}}h(v_{\varepsilon,z})^{2^*_s}\,dx\right)^{2/2^*_s}}\right)^{N/s}\\&=\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}+O(\varepsilon^{N-s}).\end{aligned}$$

Then we obtain

$$\inf_{(\omega_1,\omega_2)\in N^1_\infty}J^1_\infty(\omega_1,\omega_2)\leq\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s},\quad\text{as }\varepsilon\to 0^+. \tag{3.14}$$

The desired result follows from (3.13) and (3.14). □

4. Proof of Theorem 1.1

In this section, we use the idea of category to get positive solutions to $(\mathrm{E}_{f,g})$ and give the proof of Theorem 1.1.

First let us recall the following two propositions related to the category.

Proposition 4.1 ([25, Theorem 2.1]). *Let R be a $C^{1,1}$ complete Riemannian manifold (modelled on a Hilbert space) and assume $F\in C^1(R,\mathbb{R})$ is bounded from below. Let $-\infty<\inf_R F<a<b<+\infty$. Suppose that F satisfies the* (PS)*-condition on the sublevel $\{u\in R: F(u)\leq b\}$ and that a is not a critical level for F. Then*

$$\#\,\{u\in F^a:\nabla F(u)=0\}\geq\mathrm{cat}_{F^a}(F^a),$$

where $F^a\equiv\{u\in R: F(u)\leq a\}$.

Proposition 4.2 ([12, Lemma 2.2]). *Let Q,Ω^+ and Ω^- be closed sets with $\Omega^-\subset\Omega^+$. Let $\phi\colon Q\to\Omega^+$, $\varphi\colon\Omega^-\to Q$ be two continuous maps such that $\phi\circ\varphi$ is homotopically equivalent to the embedding $j\colon\Omega^-\to\Omega^+$. Then $\mathrm{cat}_Q(Q)\geq\mathrm{cat}_{\Omega^+}(\Omega^-)$.*

The proof of Theorem 1.1 is based on Propositions 4.1 and 4.2. Due to Remark 2.2, we can define the continuous map $\Phi\colon X\setminus G\to\mathbb{R}^N$ by

$$\Phi(\omega_1,\omega_2):=\frac{\displaystyle\int_{\Omega\times\{0\}} x(\omega_1-(\omega_1)^+_{f,g})^\alpha_+(\omega_2-(\omega_2)^+_{f,g})^\beta_+\,dx}{\displaystyle\int_{\Omega\times\{0\}}(\omega_1-(\omega_1)^+_{f,g})^\alpha_+(\omega_2-(\omega_2)^+_{f,g})^\beta_+\,dx},$$

where

$$G=\bigg\{(\omega_1,\omega_2)\in X:\int_{\Omega\times\{0\}}(\omega_1-(\omega_1)^+_{f,g})^\alpha_+(\omega_2-(\omega_2)^+_{f,g})^\beta_+\,dx=0\bigg\}.$$

Lemma 4.3. *For each $0<\delta<r_0$, there exist $\Lambda_\delta,\delta_0>0$ such that if $(\omega_1,\omega_2)\in N^1_\infty$, $J^1_\infty(\omega_1,\omega_2)<s(K_sS_{s,\alpha,\beta})^{N/s}/(2N)+\delta_0$ and $\|f_+\|_{L^{q^*}(\Omega)}+\|g_+\|_{L^{q^*}(\Omega)}\in(0,\Lambda_\delta)$, then $\Phi(\omega_1,\omega_2)\in M_\delta$.*

Proof. Suppose the contrary. Then there exists a function sequence

$$\{(\omega_{1,n},\omega_{2,n})\}\subset N^1_\infty$$

such that

$$J^1_\infty(\omega_{1,n},\omega_{2,n})=\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}+o(1),$$
$$\|(f_n)_+\|_{L^{q^*}(\Omega)}+\|(g_n)_+\|_{L^{q^*}(\Omega)}=o(1),$$

and $\Phi(\omega_{1,n},\omega_{2,n})\notin M_\delta$ for all n. It is easy to see that $\{(\omega_{1,n},\omega_{2,n})\}$ is bounded in X. Furthermore, by Lemma 3.4, we have

$$\begin{aligned}\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}&\le\lim_{n\to\infty}J^1_\infty(\omega_{1,n},\omega_{2,n})=\lim_{n\to\infty}\frac{s}{2N}\,\|(\omega_{1,n},\omega_{2,n})\|^2_X\\&=\lim_{n\to\infty}\frac{s}{2N}\int_{\Omega\times\{0\}}h(x)(\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx\le\frac{s}{2N}\,(K_sS_{s,\alpha,\beta})^{N/s}.\end{aligned}\tag{4.1}$$

Define

$$\begin{aligned}(W_{1,n},W_{2,n})=\bigg((\omega_{1,n})_+\Big/\bigg(\int_{\Omega\times\{0\}}(\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx\bigg)^{1/(\alpha+\beta)},\\(\omega_{2,n})_+\Big/\bigg(\int_{\Omega\times\{0\}}(\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx\bigg)^{1/(\alpha+\beta)}\bigg),\end{aligned}$$

we see that $\int_{\Omega\times\{0\}}(W_{1,n})^\alpha_+(W_{2,n})^\beta_+\,dx=1$.

It follows from (4.1) and the definition of $S_{s,\alpha,\beta}$ that

$$\begin{aligned}K_sS_{s,\alpha,\beta}&\le\|(W_{1,n},W_{2,n})\|^2_X=\frac{\|(\omega_{1,n},\omega_{2,n})\|^2_X}{\displaystyle\int_{\Omega\times\{0\}}(\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx}\\&\le\frac{\|(\omega_{1,n},\omega_{2,n})\|^2_X}{\displaystyle\int_{\Omega\times\{0\}}h(x)(\omega_{1,n})^\alpha_+(\omega_{2,n})^\beta_+\,dx}=\|(\omega_{1,n},\omega_{2,n})\|^{2s/N}_X\le K_sS_{s,\alpha,\beta}.\end{aligned}$$

Hence we obtain

$$\lim_{n\to\infty}\|(W_{1,n},W_{2,n})\|_X^2 = K_s S_{s,\alpha,\beta} \tag{4.2}$$

and

$$\lim_{n\to\infty}\int_{\Omega\times\{0\}} h(x)(\omega_{1,n})_+^{\alpha}(\omega_{2,n})_+^{\beta}\,dx = \lim_{n\to\infty}\int_{\Omega\times\{0\}} (\omega_{1,n})_+^{\alpha}(\omega_{2,n})_+^{\beta}\,dx \tag{4.3}$$

By Lemma 3.2, there is a sequence $\{(y_n,\varepsilon_n)\}\in\mathbb{R}^N\times\mathbb{R}^+$ such that $\varepsilon_n\to 0$, $y_n\to y_0\in\overline{\Omega}$ and

$$\begin{aligned}(U_{1,n}(x,y),U_{2,n}(x,y)) = \big(&E_s(\varepsilon_n^{(N-s)/2}W_{1,n}(\varepsilon_n x+y_n)),\\ &E_s(\varepsilon_n^{(N-s)/2}W_{2,n}(\varepsilon_n x+y_n))\big)\to(U_1,U_2)\end{aligned}$$

in X as $n\to\infty$. Then by (4.1)–(4.3), we have

$$\begin{aligned}1 &= o(1)+\int_{\Omega\times\{0\}} h(x)(W_{1,n})_+^{\alpha}(W_{2,n})_+^{\beta}dx\\ &= \varepsilon^{-N}\int_{\Omega\times\{0\}} h(x)\Big(U_{1,n}\Big(\frac{x-y_n}{\varepsilon_n},0\Big)\Big)_+^{\alpha}\Big(U_{2,n}\Big(\frac{x-y_n}{\varepsilon_n},0\Big)\Big)_+^{\beta}dx+o(1)\\ &= h(y_0),\end{aligned}$$

as $n\to\infty$, which implies $y_0\in M$. Considering $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x)=x$ in Ω, we infer

$$\begin{aligned}\Phi(\omega_{1,n},\omega_{2,n}) &= \frac{\displaystyle\int_{\Omega\times\{0\}} x(\omega_{1,n}-(\omega_1)^+_{f_n,g_n})_+^{\alpha}(\omega_{2,n}-(\omega_2)^+_{f_n,g_n})_+^{\beta}\,dx}{\displaystyle\int_{\Omega\times\{0\}} (\omega_{1,n}-(\omega_1)^+_{f_n,g_n})_+^{\alpha}(\omega_{2,n}-(\omega_2)^+_{f_n,g_n})_+^{\beta}\,dx}\\ &= \frac{\displaystyle\int_{\mathbb{R}^N\times\{0\}} \varphi(x)(\omega_{1,n}-(\omega_1)^+_{f_n,g_n})_+^{\alpha}(\omega_{2,n}-(\omega_2)^+_{f_n,g_n})_+^{\beta}\,dx}{\displaystyle\int_{\mathbb{R}^N\times\{0\}} (\omega_{1,n}-(\omega_1)^+_{f_n,g_n})_+^{\alpha}(\omega_{2,n}-(\omega_2)^+_{f_n,g_n})_+^{\beta}\,dx}\end{aligned}$$

as $\|f_+\|_{L^{q^*}(\Omega)},\|g_+\|_{L^{q^*}(\Omega)}\to 0$

$$\begin{aligned}&= \int_{\mathbb{R}^N\times\{0\}}\varphi(\varepsilon_n x+y_n)\big|E_s(\varepsilon_n^{(N-s)/2}W_{1,n}(\varepsilon_n x+y_n))\big|^{\alpha}\\ &\quad\cdot\big|E_s(\varepsilon_n^{(N-s)/2}W_{1,n}(\varepsilon_n x+y_n))\big|^{\beta}\,dx\Big/\\ &\quad\int_{\mathbb{R}^N\times\{0\}}\big|E_s(\varepsilon_n^{(N-s)/2}W_{1,n}(\varepsilon_n x+y_n))\big|^{\alpha}\big|E_s(\varepsilon_n^{(N-s)/2}W_{1,n}(\varepsilon_n x+y_n))\big|^{\beta}\,dx\\ &\quad\to y_0\in M,\ \text{ as } n\to\infty,\end{aligned}$$

as $n\to\infty$, which is a contradiction. □

LEMMA 4.4. *There exists $\Lambda_\delta > 0$ small enough such that if*

$$\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta \quad \text{and} \quad (\omega_1, \omega_2) \in N^-_{f,g}$$

with

$$I_{f,g}(\omega_1, \omega_2) < \frac{s}{2N} (K_s S_{s,\alpha,\beta})^{N/s} + \frac{\delta_0}{2}$$

(δ_0 is given in Lemma 4.3), then $\Phi(\omega_1, \omega_2) \in M_\delta$.

PROOF. By Lemma 3.1, for $\mu \in (0,1)$, there is a unique $t^1_{(\omega_1,\omega_2)}$ such that $(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_2) \in N^1_\infty$ and

$$\begin{aligned} &J^1_\infty\big(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_2\big) \\ &\le (1-\mu)^{-N/s}\bigg(I_{f,g}(\omega_1,\omega_2) + \frac{2-q}{2q}\,\mu^{q/(q-2)} C(\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}})^{2/(2-q)}\bigg). \end{aligned}$$

Thus there exists $\Lambda_\delta > 0$ small enough such that if $\|f_+\|_{L^{q^*}} + \|g_+\|_{L^{q^*}} < \Lambda_\delta$ and

$$\begin{aligned} I_{f,g}(\omega_1,\omega_2) &< \frac{s}{2N} (K_s S_{s,\alpha,\beta})^{N/s} + \frac{\delta_0}{2}, \\ J^1_\infty(t^1_{(\omega_1,\omega_2)}\omega_1, t^1_{(\omega_1,\omega_2)}\omega_1) &\le \frac{s}{2N} (K_s S_{s,\alpha,\beta})^{N/s} + \delta_0. \end{aligned}$$

By Lemma 4.1 and $\|((\omega_1)^+_{f,g}, (\omega_2)^+_{f,g})\|_X \to 0$ as $\|f_+\|_{L^{q^*}}, \|g_+\|_{L^{q^*}} \to 0$, we complete the proof. □

Below we denote by $I_{N^-_{f,g}}$ the restriction of $I_{f,g}$ on $N^-_{f,g}$.

LEMMA 4.5. *Any sequence $\{(\omega_{1,n}, \omega_{2,n})\} \subset N^-_{f,g}$ such that*

$$I_{N^-_{f,g}}(\omega_{1,n}, \omega_{2,n}) \to c \in \bigg(-\infty, \alpha^+_{f,g} + \frac{s}{2N} (K_s S_{s,\alpha,\beta})^{N/s}\bigg)$$

and $I'_{N^-_{f,g}}(\omega_{1,n}, \omega_{2,n}) \to 0$ contains a convergent subsequence for all $\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)} \in (0, \Lambda_\delta)$.

PROOF. By hypothesis there exists a sequence $\{\theta_n\} \subset \mathbb{R}$ such that

$$I'_{f,g}(\omega_{1,n}, \omega_{2,n}) = \theta_n \psi'_{f,g}(\omega_{1,n}, \omega_{2,n}) + o(1),$$

where $\psi_{f,g}$ is defined in (2.8). Recall that $(\omega_{1,n}, \omega_{2,n}) \in N^-_{f,g}$ and so

$$\psi'_{f,g}(\omega_{1,n}, \omega_{2,n})(\omega_{1,n}, \omega_{2,n}) < 0.$$

If $\psi'_{f,g}(\omega_{1,n}, \omega_{2,n})(\omega_{1,n}, \omega_{2,n}) \to 0$, we from (2.9) and (2.10) obtain that there are two positive numbers C_1, C_2 independent of $(\omega_{1,n}, \omega_{2,n})$ such that

$$\begin{aligned} \|(\omega_{1,n}, \omega_{2,n})\|^2_X &\le C_1 \|(\omega_{1,n}, \omega_{2,n})\|^{2^*_s}_X + o(1), \\ \|(\omega_{1,n}, \omega_{2,n})\|^2_X &\le (\|f_+\|_{L^{q^*}(\Omega)} + \|g_+\|_{L^{q^*}(\Omega)}) C_2 \|(\omega_{1,n}, \omega_{2,n})\|^q_X + o(1). \end{aligned}$$

That is

$$\begin{aligned}\|(\omega_{1,n},\omega_{2,n})\|_X^2 &\geq C_1^{-2/(2_s^*-2)}+o(1),\\ \|(\omega_{1,n},\omega_{2,n})\|_X^2 &\leq (\|f_+\|_{L^{q^*}(\Omega)}+\|g_+\|_{L^{q^*}(\Omega)})^{2/(2-q)}C_2^{2/(2-q)}+o(1).\end{aligned}$$

If $\|f_+\|_{L^{q^*}(\Omega)}$ and $\|g_+\|_{L^{q^*}(\Omega)}$ are sufficiently small, this is impossible. Thus we may assume that $\psi'_{f,g}(\omega_{1,n},\omega_{2,n})(\omega_{1,n},\omega_{2,n}) \to l < 0$ as $n\to\infty$. Since $I'_{f,g}(\omega_{1,n},\omega_{2,n})(\omega_{1,n},\omega_{2,n}) = 0$, we conclude that $\theta_n \to 0$ and, consequently, $I'_{f,g}(\omega_{1,n},\omega_{2,n}) \to 0$. Using this information we have

$$\begin{aligned}&I_{f,g}(\omega_{1,n},\omega_{2,n}) \to c \in \bigg(-\infty, \alpha_{f,g}^+ + \frac{s}{2N}(K_sS_{s,\alpha,\beta})^{N/s}\bigg),\\ &I'_{f,g}(\omega_{1,n},\omega_{2,n}) \to 0,\end{aligned}$$

so by Lemma 2.7 the proof is over. □

Denote

$$\begin{aligned}c_{f,g} &:= \alpha_{f,g}^+ + \frac{s}{2N}(K_sS_{s,\alpha,\beta})^{N/s}-\sigma,\\ N_{f,g}^-(c_{f,g}) &:= \{(\omega_1,\omega_2)\in N_{f,g}^- : I_{f,g}(\omega_1,\omega_2) < c_{f,g}\}.\end{aligned}$$

Lemma 4.6. *Suppose* (H_1)–(H_2) *hold, and* $\|f_+\|_{L^{q^*}(\Omega)}+\|g_+\|_{L^{q^*}(\Omega)} \in (0,\Lambda_\delta)$, $I_{N_{f,g}^-}$ *has at least* $\mathrm{cat}_{M_\delta}(M)$ *critical points in* $N_{f,g}^-(c_{f,g})$.

Proof. For $z\in M$, by Lemma 3.3, we can define

$$F(z) = \big((\omega_1)_{f,g}^+ + t_z^-\sqrt{\alpha}v_{\varepsilon,z}, (\omega_2)_{f,g}^+ + t_z^-\sqrt{\alpha}v_{\varepsilon,z}\big) \in N_{f,g}^-(c_{f,g}).$$

By Lemma 4.5, $I_{N_{f,g}^-}$ satisfies the (PS)-condition on $N_{f,g}^-(c_{f,g})$. Moreover, it follows from Lemma 4.4 that $\Phi(N_{f,g}^-(c_{f,g}))\subset M_\delta$ for $\|f_+\|_{L^{q^*}}+\|g_+\|_{L^{q^*}} < \Lambda_\delta$. Define $\xi\colon [0,1]\times M\to M_\delta$ by

$$\xi(\theta,z) = \Phi\big((\omega_1)_{f,g}^+ + t_z^-\sqrt{\alpha}v_{(1-\theta)\varepsilon,z}, (\omega_2)_{f,g}^+ + t_z^-\sqrt{\beta}v_{(1-\theta)\varepsilon,z}\big) \in N_{f,g}^-(c_{f,g}).$$

Then straightforward calculations provide that

$$\xi(0,z) = \Phi\circ F(z) \quad\text{and}\quad \lim_{\theta\to1^-}\xi(\theta,z) = z.$$

Hence $\Phi\circ F$ is homotopic to the inclusion $j : M\to M_\delta$. By Propositions 4.1 and 4.2, $I_{f,g}$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N_{f,g}^-(c_{f,g})$. □

Lemma 4.7. *If* (ω_1,ω_2) *is a critical point of* $I_{N_{f,g}^-}$, *then it is a critical point of* $I_{f,g}$ *in* X.

Proof. Assume $(\omega_1,\omega_2)\in N_{f,g}^-$, then $I'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2) = 0$. On the other hand,

$$I'_{f,g}(\omega_1,\omega_2) = \theta\psi'_{f,g}(\omega_1,\omega_2) \tag{4.4}$$

for some $\theta\in\mathbb{R}$, where $\psi_{f,g}$ is defined in (2.8).

Remark that $(\omega_1,\omega_2)\in N^-_{f,g}$, and so $\psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2)<0$. Thus by (4.4),

$$0=\theta\psi'_{f,g}(\omega_1,\omega_2)(\omega_1,\omega_2),$$

which implies that $\theta=0$, consequently, $I'_{f,g}(\omega_1,\omega_2)=0$. □

Finally, we can give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. It follows from Lemmas 4.6 and 4.7 that $I_{f,g}$ admits at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $N^-_{f,g}$. Moreover, by Lemma 2.8 and $N^+_{f,g}\cap N^-_{f,g}=\emptyset$, $I_{f,g}$ has at least $\mathrm{cat}_{M_\delta}(M)+1$ critical points in X. Thus we obtain that $J_{f,g}$ has at least $\mathrm{cat}_{M_\delta}(M)+1$ nonnegative critical points. By the maximum principle [27], we complete the proof. □

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